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# Statistics on Lattice Walks and $q$ -Lassalle Numbers

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**Abstract.** This paper contains two results. First, I propose a  $q$ -generalization of a certain sequence of positive integers, related to Catalan numbers, introduced by Zeilberger, see Lassalle (2010). These  $q$ -integers are palindromic polynomials in  $q$  with positive integer coefficients. The positivity depends on the positivity of a certain difference of products of  $q$ -binomial coefficients.

To this end, I introduce a new inversion/major statistics on lattice walks. The difference in  $q$ -binomial coefficients is then seen as a generating function of weighted walks that remain in the upper half-plane.

**Résumé.** Cet document contient deux résultats. Tout d'abord, je vous propose un  $q$ -generalization d'une certaine séquence de nombres entiers positifs, liés à nombres de Catalan, introduites par Zeilberger (Lassalle, 2010). Ces  $q$ -integers sont des polynômes palindromiques à  $q$  à coefficients entiers positifs. La positivité dépend de la positivité d'une certaine différence de produits de  $q$ -coefficients binomial.

Pour ce faire, je vous présente une nouvelle inversion/major index sur les chemins du réseau. La différence de  $q$ -binomial coefficients est alors considérée comme une fonction de génération de trajets pondérés qui restent dans le demi-plan supérieur.

**Keywords:** lattice walks, statistics on words,  $q$ -integers

## 1 Introduction to Lassalle's Sequences and their $q$ -analogs.

Michel Lassalle [Las12] has discussed two related sequences of numbers  $A_k$  and  $\alpha_k$ .  $\{A_k\}$  is generated by the following recurrence:

$$A_n = (-1)^{n-1} C_n + \sum_{j=1}^{n-1} (-1)^{n-j-1} \binom{2n-1}{2j-1} A_j C_{n-j}, \quad A_1 = 1$$

The second sequence is

$$\alpha_n = \frac{2A_n}{C_n}$$

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He proved that both  $A_n$  and  $\alpha_n$  are positive integers and each sequence is increasing (and more). (It turns out that the second sequence is simply related to power sums of zeros of Bessel function  $J_0(z)$ ). It is intriguing to inquire whether there is a natural  $q$ -analog of these numbers. It may be generated by a  $q$ -analog of the above recurrence (with  $C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ , a  $q$ -Catalan):

$$A'_n(q) = (-1)^{n-1} q^{n-1} C_n(q) + \sum_{j=1}^{n-1} (-1)^{n-j-1} q^{n-j} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A'_j(q) C_{n-j}(q), \quad (1)$$

but a slightly renormalized version looks neater:

$$A_n(q) = (-1)^{n-1} q^{3-2n} C_n(q) + (-1)^n q^{3-2n} [2n-1]_q C_{n-1}(q) + \sum_{j=2}^{n-1} (-1)^{n-j-1} q^{2j-2n} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A_j(q) C_{n-j}(q) \quad (2)$$

with  $A_1(q) = 1$ . It turns out that  $A_n(q)$  are monic unimodal palindromic polynomials in  $q$  with positive integer coefficients. Here are some examples:

### Example 1

$$A_2(q) = 1$$

$$A_3(q) = 1 + q + q^2 + q^3 + q^4$$

$$A_4(q) = 1 + 2q + 3q^2 + 5q^3 + 6q^4 + 7q^5 + 8q^6 + 7q^7 + 6q^8 + 5q^9 + 3q^{10} + 2q^{11} + q^{12}$$

$$A_5(q) = 1 + 3q + 6q^2 + 12q^3 + 19q^4 + 29q^5 + 41q^6 + 54q^7 + 67q^8 + 80q^9 + 89q^{10} + 96q^{11} + 98q^{12} + 96q^{13} + 89q^{14} + 80q^{15} + 67q^{16} + 54q^{17} + 41q^{18} + 29q^{19} + 19q^{20} + 12q^{21} + 6q^{22} + 3q^{23} + q^{24}$$

The second Lassalle's sequence  $\alpha_k$  has the following  $q$ -analog:

$$\alpha_n = \frac{(1+q^n)A_n(q)}{C_n(q)} \quad (3)$$

And each of  $\alpha_n(q)$  is also a monic unimodal palindromic polynomial in  $q$  with positive integer coefficients. Here are examples of  $\alpha_n(q)$

### Example 2

$$\alpha_1(q) = 1 + q$$

$$\alpha_2(q) = 1$$

$$\alpha_3(q) = 1 + q$$

$$\alpha_4(q) = 1 + 2q + 2q^2 + 2q^3 + q^4$$

$$\alpha_5(q) = 1 + 3q + 5q^2 + 8q^3 + 9q^4 + 9q^5 + 8q^6 + 5q^7 + 3q^8 + q^9$$

The proof of positivity of  $\alpha_k$  relies (in addition to certain divisibility properties) on the positivity of

$$\binom{n}{k-1} \binom{n}{k} - \binom{n}{k-2} \binom{n}{k+1}$$

and Lassalle used the combinatorial interpretation of this difference of binomial coefficients as a generating function of the number of NSEW walks on a square lattice that start at the origin and finish at  $(2k - n - 1, 1)$  [GKS92].

Similarly, the positivity of  $\alpha_k(q)$  requires the positivity of

$$\left[ \begin{matrix} n \\ k-1 \end{matrix} \right]_q \left[ \begin{matrix} n \\ k \end{matrix} \right]_q - q^2 \left[ \begin{matrix} n \\ k-2 \end{matrix} \right]_q \left[ \begin{matrix} n \\ k+1 \end{matrix} \right]_q \tag{4}$$

However in this case a combinatorial interpretation has to be developed.

## 2 Introduction to $q$ -enumeration of Lattice walks

To understand the positivity of (4) combinatorially, i.e. as a generating function of certain weighted lattice walks, I first interpret the  $q$ -version of the generating function of all NSEW walks as a generating function of a certain (new) inversion statistics on lattice walks.

The total number of lattice walks from  $(0, 0)$  to  $(c, d)$  of length  $n$  is given by [DR84]

$$\binom{n}{\frac{1}{2}(n-c+d)} \binom{n}{\frac{1}{2}(n-c-d)}$$

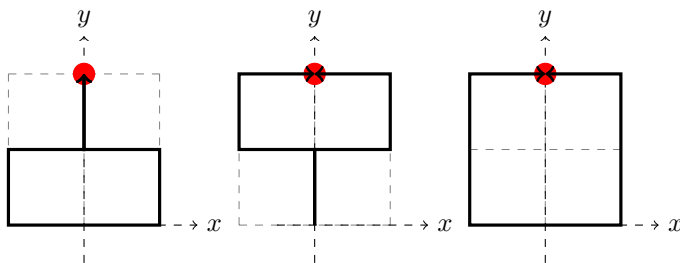
Think of a given walk as a word  $w$  composed of letters  $N, S, E, W$ . Then, the *walk inversion* statistics is defined

### Definition 1

$$\begin{aligned} \text{winv}N(w) &= \sum_{N \in w} \#S \text{ to the left of } N \\ \text{winv}W(w) &= \sum_{W \in w} \#S + \#N + \#E \text{ to the left of } W \\ \text{winv}E(w) &= \sum_{E \in w} \#S + \#N + 2\#W \text{ to the left of } E \\ \text{wpinv}(w) &= \text{winv}N(w) + \text{winv}W(w) + \text{winv}E(w) \end{aligned}$$

**Example 3** Here are the inversions of the walks (from left to right) on Fig. 1:

$$\begin{aligned} W N E N &= 2 \cdot 1 + 1 = 3 \\ E N W N &= 1 + 1 = 2 \\ N W N E &= 1 + (2 + 2 \cdot 1) = 5 \\ N E N W &= 1 + (2 + 1) = 4 \\ W N N E &= 2 + 2 \cdot 1 = 4 \\ E N N W &= 1 + 2 = 3 \end{aligned}$$



**Fig. 1:** Some walks from  $(0, 0)$  to  $(0, 2)$  with  $n = 4$  steps.

Denote the set of all lattice walks from  $(a, b)$  to  $(c, d)$  in  $n$  letters (steps) by  $P_n((a, b) | (c, d))$ . The  $q$ -analog of the walk enumeration formula is the following generating function.

**Proposition 1**

$$\left[ \begin{matrix} n \\ \frac{1}{2}(n - c + d) \end{matrix} \right]_q \left[ \begin{matrix} n \\ \frac{1}{2}(n - c - d) \end{matrix} \right]_q = \sum_{w \in P_n((0,0)|(c,d))} q^{winv(w)} \tag{5}$$

The following is true for the walks restricted to the upper half plane. Denote the set of walks starting at  $(0, 0)$  and ending at  $(c, d)$  in  $n$  steps and never going below the  $x$ -axis by  $P_n^+((0, 0) | (c, d))$ , then

**Proposition 2**

$$\left[ \begin{matrix} n \\ \frac{1}{2}(n + c + d) \end{matrix} \right]_q \left[ \begin{matrix} n \\ \frac{1}{2}(n + c - d) \end{matrix} \right]_q - q^{d+1} \left[ \begin{matrix} n \\ \frac{1}{2}(n + c + d) + 1 \end{matrix} \right]_q \left[ \begin{matrix} n \\ \frac{1}{2}(n + c - d) - 1 \end{matrix} \right]_q = \sum_{w \in P_n^+((0,0)|(c,d))} q^{winv(w)} \tag{6}$$

In Section 6 I will also introduce an analog of a major index,  $wmaj(w)$ , which is, conjecturally, equally distributed with  $winv(w)$  over lattice walks.

### 3 $q$ -Lassalle Numbers

The purpose of this section is to derive a bilinear recursion relations for  $A_k(q)$  and  $\alpha_k(q)$ , from which the positivity and integrality follow.

The strategy is to rewrite the recursion relation between  $A'(q)$  (as in (1)) as a difference equation. Then, using the  $q$ -difference equation for the generating function of  $q$ -Catalan, to obtain the  $q$ -difference equation for the generating function of the  $q$ -Lassalle numbers.

$$A'_n(q) = (-1)^{n-1} q^{n-1} C_n(q) + \sum_{j=1}^{n-1} (-1)^{n-j-1} q^{n-j} \left[ \begin{matrix} 2n-1 \\ 2j-1 \end{matrix} \right]_q A'_j(q) C_{n-j}(q) - \frac{(-1)^n q^n \left[ 2n \right]_q C_n(q)}{\left[ 2n \right]_q!} = \frac{q A'_n(q)}{\left[ 2n-1 \right]_q!} + \sum_{j=1}^{n-1} \frac{q A'_j(q)}{\left[ 2j-1 \right]_q!} \frac{(-1)^{n-j} q^{n-j} C_{n-j}(q)}{\left[ 2n-2j \right]_q!} \tag{7}$$

Introduce generating functions and the finite  $q$ -difference operator:

$$\begin{aligned}
 H(t; q) &= \sum_{k \geq 0} \frac{(-1)^k q^k C_k(q) t^{2k}}{[2k]_q!} \equiv \sum_{k \geq 0} \frac{(-1)^k q^k t^{2k}}{[k]_q! [k+1]_q!} \\
 P(t; q) &= \sum_{k \geq 1} \frac{q A'_k(q) t^{2k-1}}{[2k-1]_q!} \\
 D_q f(t) &= \frac{f(t) - f(qt)}{(1-q)t}
 \end{aligned}$$

then (7) is equivalent to

$$-D_q H(t, q) = P(t; q)H(t; q) \tag{8}$$

Recall Jackson’s basic  $q$ -Bessel function ([Ism82]):

$$J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n}{(q; q)_n (q^{\nu+1}; q)_n} \left(\frac{x}{2}\right)^{2n+\nu}, \quad \text{where } (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \tag{9}$$

For  $\nu = 1$

$$J_1^{(1)}(x; q) = \frac{1}{1-q} \sum_{n \geq 0} \frac{(-1)^n}{(q; q)_n (q^2; q)_n} \left(\frac{x}{2}\right)^{2n+1} = \sum_{n \geq 0} \frac{(-1)^n}{[n]_q! [n+1]_q!} \left(\frac{x}{2(1-q)}\right)^{2n+1}$$

So that

$$H(t; q) = \frac{1}{\sqrt{q}t} J_1^{(1)}(2(1-q)\sqrt{q}t; q), \text{ i.e.}$$

$q$ -Bessel function  $J_1^{(1)}(x; q)$  is a generating function for  $q$ -Catalan numbers.

$J_1^{(1)}(x; q)$  satisfies the following  $q$ -Bessel difference equation:

$$J_1^{(1)}(qx; q) - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) J_1^{(1)}(\sqrt{q}x; q) + \left(1 + \frac{x^2}{4}\right) J_1^{(1)}(x; q) = 0 \tag{10}$$

For  $H(t; q)$ , (10) translates to:

$$H(qt; q) - \left(1 + \frac{1}{q}\right) H(\sqrt{qt}; q) + \left(\frac{1}{q} + (1-q)^2 t^2\right) H(t; q) = 0 \tag{11}$$

Through the  $q$ -difference equation (8) this implies the following ungainly looking  $q$ -difference equation for  $P(t; q)$ :

$$\begin{aligned}
 &(1 + q + q(1-q)^2 t^2) q(1-q)\sqrt{qt}P(\sqrt{qt}; q) + q^2(1-q)^2 tP(t; q)\sqrt{qt}P(\sqrt{qt}; q) + \\
 &+ (1 + q + q(1-q)^2 t^2) q^2(1-q)^2 t^2 + q^2(1-q)^2 t^2 q(1-q)tP(t; q) \\
 &= (1 - q^2)tP(t; q) - q(1-q)(1 - q^2)t^2
 \end{aligned}$$

But collecting coefficients of  $t^{2n}$  on both sides makes things look better:

$$\begin{aligned} A'_1(q) &= 1 \\ A'_2(q) &= q^2 \\ \frac{[2]_q[s+1]_q}{[2s-1]_q!} A'_s(q) &= \frac{q^3 + q^{s+1}}{[2s-3]_q!} A'_{s-1}(q) + \sum_{k=2}^{s-2} \frac{q^{k+3} A'_k(q) A'_{s-k}(q)}{[2k-1]_q! [2s-2k-1]_q!} \end{aligned}$$

It's time to rescale, so let

$$A'_k(q) = q^{3k-4} A_k(q)$$

then the recursion relation for  $A_k(q)$  is:

$$A_s(q) = \frac{[2s-1]_q}{[2]_q [s+1]_q} \sum_{k=1}^{s-1} q^{k-1} \frac{[2s-2]_q!}{[2k-1]_q! [2s-2k-1]_q!} A_k(q) A_{s-k}(q) \tag{12}$$

Translating this recursion into that for  $\alpha_n(q)$  (as in (3)) produces:

$$\alpha_n(q) = \frac{1}{[2]_q [n]_q} \sum_{k=1}^{n-1} q^{k-1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \alpha_k(q) \alpha_{n-k}(q) \tag{13}$$

The ratio of the  $q$ -binomial coefficients can be rewritten as

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \frac{1}{[2]_q} \left( \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r \end{bmatrix}_q - q^2 \begin{bmatrix} n-1 \\ r-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_q \right)$$

Therefore, the proof depends on

- positivity and integrality of

$$c_{n,k}(q) = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^2 \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_q$$

- divisibility of  $c_{n,k}(q)$  and  $\alpha_k(q)$  by powers of  $[2]_q$ .

As may be seen from (6), the positivity and integrality of  $c_{n,k}(q)$  follows from its combinatorial interpretation as the generating function of *winv* statistics of upper half-plane lattice walks, namely the walks that start at  $(0, 0)$  and end up at  $(2k - n - 1, 1)$  in  $(n - 1)$  steps.

Therefore I continue with the lattice walk part of the story.

### 4 Walk Inversion Generating Function

In order to prove the walk inversion generating function formula, as in (5)

$$\begin{bmatrix} n \\ \frac{1}{2}(n-c+d) \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n-c-d) \end{bmatrix}_q = \sum_{w \in P_n((0,0)|(c,d))} q^{\text{winv}(w)}$$

I will need the following lemma. Denote the set of all walks of length  $n$  from  $(0, 0)$  to  $(c, d)$  with exactly  $k$  W steps by  $P_n((c, d); k)$ . Of course this fixes the number of S steps,  $r$  as well. The total number of steps in  $n$ ,  $(c + k)$  of which are E,  $k - W$ ,  $(d + r) - N$ , and  $r - S$ .

$$r = \frac{1}{2}(n - c - d) - k$$

As words, these walks are permutations of each other. Their total number is  $\frac{n!}{(c+k)!k!(d+r)!r!}$ .

**Lemma 1** *The generating function of the walk inversion statistics of  $n$ -step walks with  $k$  W steps is given by*

$$q^{k^2+ck} \frac{[n]_q!}{[c+k]_q! [\frac{1}{2}(n-c+d)-k]_q! [k]_q! [\frac{1}{2}(n-c-d)-k]_q!} = \sum_{w \in P_n((c,d);k)} q^{winv(w)} \quad (14)$$

Now, in general, to get to  $(c, d)$  from  $(0, 0)$  in  $n$  steps one might take just one W step, or two, etc. The maximal number of W steps (i.e. no S steps) is

$$\frac{1}{2}(n - c - d)$$

So that

**Lemma 2** *The walk inversion generating function is*

$$\sum_{w \in P_n((0,0)|(c,d))} q^{winv(w)} = \sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+ck} \frac{[n]_q!}{[c+k]_q! [\frac{1}{2}(n-c+d)-k]_q! [k]_q! [\frac{1}{2}(n-c-d)-k]_q!} \quad (15)$$

Now, consider how a walk can end up at  $(c, d)$  in  $n$  steps, i.e. where could it be at the previous step?

- either at  $(c - 1, d)$ , with the last step E; adding an E step changes  $winv$  by  $\#S + \#N + 2\#W = r + (d + r) + 2k = d + 2r + 2k = n - c$

The contribution of walks coming from the West is

$$q^{n-c} \sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+(c-1)k} \frac{[n-1]_q!}{[c-1+k]_q! [\frac{1}{2}(n-c-d)-k]_q! [k]_q! [\frac{1}{2}(n-c+d)-k]_q!}$$

- or at  $(c, d + 1)$  with the last step S; adding an S step does not change  $winv$ .

The contribution of the walks coming from the North:

$$\sum_{k=0}^{\frac{1}{2}(n-c-d)-1} q^{k^2+ck} \frac{[n-1]_q!}{[c+k]_q! [\frac{1}{2}(n-c+d)-k-1]_q! [k]_q! [\frac{1}{2}(n-c-d)-k-1]_q!}$$



- or at  $(c + 1, d)$  with the last step W; adding a W step changes  $winv$  by  $\#S + \#N + \#E = r + (d + r) + (k + c + 1) = d + 2r + c + k + 1 = n - 1 - k$

The contribution of walks coming from the East is

$$q^{n-1} \sum_{k=0}^{\frac{1}{2}(n-c-d)-1} q^{k^2+ck} \frac{[n-1]_q!}{[c+1+k]_q! [\frac{1}{2}(n-c+d)-k-1]_q! [k]_q! [\frac{1}{2}(n-c-d)-k-1]_q!}$$

- or at  $(c, d - 1)$  with the last step N; an addition of an N step with  $r$  prior S steps changes  $winv$  by  $\frac{1}{2}(n - c - d) - k$ .

The contribution of the walks coming from the South:

$$q^{\frac{1}{2}(n-c-d)} \sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+(c-1)k} \frac{[n-1]_q!}{[c+k]_q! [\frac{1}{2}(n-c-d)-k]_q! [k]_q! [\frac{1}{2}(n-c+d)-k]_q!}$$

The sum of these contributions gives a recursion that establishes (5).

## 5 Upper Half-Plane walks

Following the logic of walks reflection [GKS92], to every negative path from  $(0, 0)$  to  $(c, d)$  will associate a walk from  $(-2, 0)$  to  $(c, d)$  so that the change in  $winv$  is the same for every walk.

Here is the algorithm.

1. Separate each negative walk in two segments  $w = w_1 \cdot w_2$ , where  $\cdot$  means concatenation of words:
  - (a)  $w_1$ : the part of the walk that starts at  $(0, 0)$  and ends at  $(*, 0)$  (before it dips below the  $x$ -axis the first time)
  - (b)  $w_2$ : the rest of the walk that runs from  $(*, 0)$  to  $(c, d)$ ; Notice that  $w_2$  starts with S and necessarily has at least  $d + 1$  N steps. More precisely, if the walk has  $k$  S steps, it has  $d + k$  N steps.
2.  $\tilde{w}_1$ : move  $w_1$  down two steps, so that it starts at  $(0, -2)$  and ends at  $(*, -2)$ ;
3. attach  $\tilde{w}_2$  to the combined walk. The walk modified this way runs from  $(0, -2)$  to  $(c, d)$

Consider the simplest (negative) walk  $w = w_1 \cdot w_2$ : after dipping down below the  $x$ -axis it goes straight up to  $d$ , i.e. the part after reaching the  $x$ -axis ( $w_2$ ) looks like

$$w_2 = S \underbrace{NN \dots NN}_{d+1 \text{ times}}$$

Transform this walk into

$$\tilde{w}_2 = \underbrace{NN \dots NN}_{d+1 \text{ times}} S$$

Suppose that  $w_1$  contained  $k$  S steps. Then  $(\tilde{w} = \tilde{w}_1 \cdot \tilde{w}_2)$

$$winv(w) - winv(\tilde{w}) = (d + 1)(k + 1) - (d + 1)k = d + 1$$

Now consider a generic  $w$  with blocks of S and N letters:

$$w_2 = \underbrace{S \dots S}_{s_1 \text{ times}} \underbrace{N \dots N}_{n_1 \text{ times}} \underbrace{S \dots S}_{s_2 \text{ times}} \underbrace{N \dots N}_{n_2 \text{ times}} \dots \underbrace{S \dots S}_{s_r \text{ times}} \underbrace{N \dots N}_{n_r \text{ times}}$$

where

$$\sum_i^r n_i - s_i = d$$

By swapping the last S in the each string of  $s_i$  with the last N letter in  $n_i$  until  $d + 1$  letters N have been moved, transforms  $w_2$

$$\tilde{w}_2 = \underbrace{S \dots SS}_{s_1-1 \text{ times}} \underbrace{NN \dots NN}_{n_1 \text{ times}} S \underbrace{S \dots S}_{s_2-1 \text{ times}} \underbrace{N \dots N}_{n_2 \text{ times}} S \dots \underbrace{S \dots S}_{s_i-1 \text{ times}} \underbrace{N \dots N}_r \underbrace{S \dots N}_{n_i-r \text{ times}} \dots \underbrace{S \dots S}_{s_r \text{ times}} \underbrace{N \dots N}_{n_r \text{ times}}$$

with  $n_1 + n_2 + \dots + r = d + 1$

$$\begin{aligned} \text{winv}(w) - \text{winv}(\tilde{w}) &= \{n_1(s_1 + k) - n_1(s_1 - 1 + k)\} + \{n_2(s_1 + s_2 + k) - n_2(s_1 + s_2 - 1 + k)\} \\ &+ \dots + \{r(s_1 + \dots + s_i + k) - r(s_1 + \dots + s_i - 1 + k)\} \\ &+ \{(n_i - r)(s_1 + \dots + s_i + k) - (n_i - r)(s_1 + \dots + s_i + k)\} + \\ &+ \dots + \{n_r(s_1 + \dots + s_r) - n_r(s_1 + \dots + s_r)\} = \sum_i n_i + r = d + 1 \end{aligned}$$

$$\text{winv}(\tilde{w}) = q^{-(d+1)} \text{winv}(w)$$

So the total contribution of negative walks is

$$q^{d+1} P_n((0, -2) | (c, d)) = q^{d+1} \left[ \frac{1}{2} (n + 2 + c + d) \right]_q \left[ \frac{1}{2} (n - 2 + c - d) \right]_q$$

Hence (6).

## 6 Major Walk Index

Setting up the following order  $S > N > E > W$ , and with the usual definition of a descent set

$$\text{des}N(w) = \{i : S \text{ occurs as } i^{\text{th}} \text{ letter and N occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$\text{maj}N(w) = \sum_{i \in \text{des}N(w)} i$$

$$\text{des}E(w) = \{i : S \text{ or N occur as } i^{\text{th}} \text{ letter and E occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$\text{maj}E(w) = \sum_{i \in \text{des}E(w)} i$$

$$\text{des}W(w) = \{i : S, N, \text{ or E occur as } i^{\text{th}} \text{ letter and W occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$\text{maj}W(w) = \sum_{i \in \text{des}W(w)} i$$

one can give the following

**Definition 2**

$$wma_j(w) = ma_jN(w) + ma_jE(w) + ma_jW(w) + \#E \times \#W \tag{16}$$

Notice the unusual last term... Nevertheless, experiments show that  $winv$  and  $wma_j$  are equally distributed over all lattice walks as well as over upper half-plane walks.

**Conjecture 1** •

$$\sum_{w \in P_n((0,0)|(c,d))} q^{winv(w)} = \sum_{w \in P_n((0,0)|(c,d))} q^{wma_j(w)} \tag{17}$$

•

$$\sum_{w \in P_n^+((0,0)|(c,d))} q^{winv(w)} = \sum_{w \in P_n^+((0,0)|(c,d))} q^{wma_j(w)} \tag{18}$$

## 7 $q$ -Integers Associated with $q$ -super Ballot Numbers

Computer experiments indicate that there is a family of  $q$ -numbers related to several generalizations of  $q$ -Catalan. For instance, following [Ges92] define  $q$ -super Ballot numbers

$$B_{n,k,r}(q) = \frac{[k+2r]_q! [2n+k-1]_q!}{(k-1)_q! r_q! n_q! [n+k+r]_q!}$$

Then, I venture to make the following conjecture

**Conjecture 2** Define a new sequence  $A_{n,k,r}(q)$  with  $A_{1,k,r} = B_{0,k,r}(q)$  through the recurrence

$$(-1)^{n-1} A_{n,k,r}(q) = q^{n-1} B_{n,k,r}(q) + \sum_{j=1}^{n-1} (-1)^j q^{n-j-1} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A_{j,k,r}(q) B_{n-j,k,r}(q) \tag{19}$$

then the  $A_{n,k,r}(q)$  are polynomials in  $q$  with positive integers coefficients for all values of  $n, k, r > 0$ .

**Example 4**

$$\begin{aligned} A_{2,1,1}(q) &= 1 + 2q + 4q^2 + 4q^3 + 3q^4 + q^5 \\ A_{2,2,1}(q) &= 1 + 3q + 7q^2 + 11q^3 + 13q^4 + 12q^5 + 8q^6 + 4q^7 + q^8 \\ A_{2,2,2}(q) &= 1 + 3q + 9q^2 + 18q^3 + 33q^4 + 51q^5 + 72q^6 + 89q^7 + 100q^8 + 101q^9 + 93q^{10} + 77q^{11} \\ &\quad + 57q^{12} + 38q^{13} + 22q^{14} + 11q^{15} + 4q^{16} + q^{17} \\ A_{3,1,1}(q) &= 1 + 3q + 9q^2 + 17q^3 + 28q^4 + 38q^5 + 44q^6 + 43q^7 + 35q^8 + 24q^9 + 13q^{10} + 5q^{11} + q^{12} \\ A_{3,2,1}(q) &= 1 + 5q + 19q^2 + 51q^3 + 110q^4 + 199q^5 + 307q^6 + 412q^7 + 484q^8 + 499q^9 + 452q^{10} \\ &\quad + 358q^{11} + 245q^{12} + 143q^{13} + 69q^{14} + 26q^{15} + 7q^{16} + q^{17} \\ A_{3,2,2}(q) &= 1 + 5q + 22q^2 + 68q^3 + 181q^4 + 414q^5 + 848q^6 + 1567q^7 + 2652q^8 + 4134q^9 + 5980q^{10} \\ &\quad + 8058q^{11} + 10155q^{12} + 11997q^{13} + 13313q^{14} + 13892q^{15} + 13639q^{16} + 12597q^{17} + \\ &\quad + 10937q^{18} + 8913q^{19} + 6802q^{20} + 4845q^{21} + 3206q^{22} + 1958q^{23} + 1094q^{24} \\ &\quad + 552q^{25} + 247q^{26} + 95q^{27} + 30q^{28} + 7q^{29} + q^{30} \end{aligned}$$

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