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# THE ROTATING NORMAL FORM OF BRAIDS IS REGULAR

JEAN FROMENTIN

ABSTRACT. Defined on Birman–Ko–Lee monoids, the rotating normal form has strong connections with the Dehornoy's braid ordering. It can be seen as a process for selecting between all the representative words of a Birman–Ko–Lee braid a particular one, called *rotating* word. In this paper we construct, for all  $n \ge 2$ , a finite-state automaton which recognizes rotating words on n strands, proving that the rotating normal form is regular. As a consequence we obtain the regularity of a  $\sigma$ -definite normal form defined on the whole braid group.

### 1. INTRODUCTION

Originally, the group  $B_n$  of *n*-strand braids was defined as the group of isotopy classes of *n*-strand geometric braids. An algebraic presentation of  $B_n$  was given by E. Artin in [1]:

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \begin{array}{cc} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \ge 2\\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$
(1)

An *n*-strand braid is an equivalence class consisting of (infinitely many) words in the letters  $\sigma_i^{\pm 1}$ . The standard correspondence between elements of the presented group  $B_n$  and geometric braids consists in using  $\sigma_i$  as a code for the geometric braid where only the *i*th and the (i + 1)st strands cross, with the strand originally at position (i + 1) in front of the other.



FIGURE 1. Interpretation of a word in the letters  $\sigma_i^{\pm 1}$  as a geometric braid diagram.

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In 1998, J.S. Birman, K.H. Ko, and S.J. Lee [3] introduced and investigated for each n a submonoid  $B_n^{+*}$  of  $B_n$ , which is known as the *Birman–Ko–Lee* monoid. The name *dual braid monoid* was subsequently proposed because several numerical parameters obtain symmetric values when they are evaluated on the positive braid monoid  $B_n^+$  and on  $B_n^{+*}$ , a correspondence that was extended to the more general context of Artin–Tits groups by D. Bessis [2] in 2003. The dual braid monoid  $B_n^{+*}$  is the submonoid of  $B_n$ generated by the braids  $a_{i,j}$  with  $1 \leq i < j \leq n$ , where  $a_{i,j}$  is defined by  $a_{i,j} = \sigma_i \cdots \sigma_{j-1} \sigma_j \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}$ . In geometrical terms, the braid  $a_{i,j}$  corresponds to a crossing of the *i*th and *j*th strands, both passing behind the (possible) intermediate strands.



FIGURE 2. In the geometric braid  $a_{1,4}$ , the strands 1 and 4 cross under the strands 2 and 3.

**Remark.** In [3], the braid  $a_{i,j}$  is defined to be  $\sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ , corresponding to a crossing of the *i*th and *j*th strands, both passing in **front** of the (possible) intermediate strands. The two definitions lead to isomorphic monoids. Our choice is this of [13] and has connections with Dehornoy's braid ordering:  $B_{n-1}^{+*}$  is an initial segment of  $B_n^{+*}$ .

By definition,  $\sigma_i$  equals  $a_{i,i+1}$  and, therefore, the positive braid monoid  $B_n^+$  is included in the monoid  $B_n^{+*}$ , a proper inclusion for  $n \ge 3$  since the braid  $a_{1,3}$  does not belong to the monoid  $B_3^+$ .

For  $n \ge 2$ , we denote by  $A_n$  the set  $\{a_{p,q} \mid 1 \le p < q \le n\}$ . If p and q are two integers of  $\mathbb{N}$  satisfying  $p \le q$ , we denote by [p,q] the interval  $\{p, ..., q\}$  of  $\mathbb{N}$ . The interval [p,q] is said to be *nested* in the interval [r,s] if the relation  $r holds. The following presentation of the monoid <math>B_n^{+*}$  is given in [3].

**Proposition 1.1.** The monoid  $B_n^{+*}$  is presented by generators  $A_n$  and relations:

$$a_{p,q}a_{r,s} = a_{r,s}a_{p,q}$$
 for  $[p,q]$  and  $[r,s]$  disjoint or nested, (2)

$$a_{p,q}a_{q,r} = a_{q,r}a_{p,r} = a_{p,r}a_{p,q} \quad \text{for } 1 \leq p < q < r \leq n.$$

$$(3)$$

Since [2] and [3] it is known that the dual braid monoid  $B_n^{+*}$  admits a Garside structure whose simple elements are in bijection with the noncrossing partitions of n. In particular, there exists a normal form associated with this Garside structure, the so-called greedy normal form.

The rotating normal form is another normal form on  $B_n^{+*}$ , and was introduced in [12] and [13]. Roughly speaking, for every braid  $\beta \in B_n^{+*}$  the

rotating normal form picks up a unique representative word on the letters  $A_n$ among all of these representing  $\beta$ . It can be seen as a map  $r_n$  from the dual braid monoid  $B_n^{+*}$  to the set of words  $A_n^*$ . The language of all *n*-rotating words, denoted by  $R_n$ , is then the image of  $B_n^{+*}$  under the map  $r_n$ . We recall that the rotating normal form is a dual version of the alternating normal form introduced by P. Dehornoy in [8] and itself building on S. Burckel's normal form defined in [4]

The aim of this paper is to construct for all  $n \ge 2$  an explicit finite-state automaton which recognizes the language  $R_n$ , implying that the language of *n*-rotating words is regular. Following [13] and [14] we can define, from the rotating normal form, a  $\sigma$ -definite normal form defined on the whole braid group. The corresponding language is then proven to be regular.

The paper is divided as follows. In section 2 we recall briefly the construction of the rotating normal form and its useful already known properties. In the third section we describe the left reversing process on dual braid monoids. In section 4 we give a syntactical characterization of *n*-rotating normal words. In the fifth section we construct, for each  $n \ge 2$ , a finite-state automaton which recognizes the language  $R_n$  of *n*-rotating normal words. Section 5 is devoted to establish the regularity of the  $\sigma$ -definite normal form. In last section we prove that the rotating normal form is not right-automatic.

#### 2. The rotating normal form

The main ingredient used to define the rotating normal form is the Garside automorphism  $\phi_n$  of  $B_n^{+*}$  defined by  $\phi_n(\beta) = \delta_n \beta \delta_n^{-1}$  where  $\delta_n = a_{1,2} a_{2,3} \dots a_{n-1,n}$  is the Garside braid of  $B_n^{+*}$ . In terms of Birman–Ko–Lee generators, the map  $\phi_n$  can be defined by:

$$\phi_n(a_{p,q}) = \begin{cases} a_{p+1,q+1} & \text{for } q \leq n-1, \\ a_{1,p+1} & \text{for } q = n. \end{cases}$$
(4)

Geometrically,  $\phi_n$  should be viewed as a rotation, which makes sense provided braid diagrams are drawn on a cylinder rather than on a plane rectangle.



FIGURE 3. Rolling up the usual braid diagram helps us to visualize the symmetries of the braids  $a_{p,q}$ . On the resulting cylinder,  $a_{p,q}$  naturally corresponds to the chord connecting vertices p and q. With this representation,  $\phi_n$  acts as a clockwise rotation of the marked circles by  $2\pi/n$ .

For  $\beta$  and  $\gamma$  in  $B_n^{+*}$ , we say that  $\gamma$  is a *right divisor* of  $\beta$ , if there exists a dual braid  $\beta'$  of  $B_n^{+*}$  satisfying  $\beta = \beta' \gamma$ .

**Definition 2.1.** For  $n \ge 3$  and  $\beta$  a braid of  $B_n^{+*}$ , the maximal braid  $\beta_1$  lying in  $B_{n-1}^{+*}$  that right-divides the braid  $\beta$  is called the  $B_{n-1}^{+*}$ -tail of  $\beta$ .

Using basic Garside properties of the monoid  $B_n^{+*}$  we obtain the following result (Proposition 2.5 of [13]) which allows us to express each braid of  $B_n^{+*}$  as a unique finite sequence of braids lying in  $B_{n-1}^{+*}$ .

**Proposition 2.2.** Assume  $n \ge 3$ . For each non-trivial braid  $\beta$  of  $B_n^{+*}$  there exists a unique sequence  $(\beta_b, ..., \beta_1)$  of braids of  $B_{n-1}^{+*}$  satisfying  $\beta_b \ne 1$  and

$$\beta = \phi_n^{b-1}(\beta_b) \cdot \dots \cdot \phi_n(\beta_2) \cdot \beta_1, \tag{5}$$

for each  $k \ge 1$ , the  $B_{n-1}^{+*}$ -tail of  $\phi_n^{b-k}(\beta_b) \cdot \dots \cdot \phi_n(\beta_{k+1})$  is trivial. (6)

Under the above hypotheses, the sequence  $(\beta_b, ..., \beta_1)$  is called the  $\phi_n$ -splitting of the braid  $\beta$ . It is shown in [13] that Condition (6) can be replaced by:

for each 
$$k \ge 1$$
,  $\beta_k$  is the  $B_{n-1}^{+*}$ -tail of  $\phi_n^{b-k}(\beta_b) \cdot \dots \cdot \phi_n(\beta_{k-1}) \cdot \beta_k$ . (7)



FIGURE 4. The  $\phi_6$ -splitting of a braid of  $B_6^{+*}$ . Starting from the right, we extract the maximal right divisor that keeps the sixth strand unbraided, then extract the maximal right divisor that keeps the first strand unbraided, etc.

**Example 2.3.** Consider the braid  $\beta = a_{1,2}a_{2,3}a_{1,2}a_{2,3}$  of  $B_3^{+*}$ . Using relations (3) on the underlined factors we obtain:

$$\beta = a_{1,2}a_{2,3}a_{1,2}a_{2,3} = a_{1,2}a_{2,3}a_{1,3}a_{1,2} = a_{1,2}a_{1,3}a_{1,2}a_{1,2}.$$

We decompose  $\beta$  as  $\phi_3(\gamma_1) \cdot \beta_1$  with  $\gamma_1 = \phi_3^{-1}(a_{1,2}a_{1,3}) = a_{1,3}a_{2,3}$  and  $\beta_1 = a_{1,2}a_{1,2}$ . As the word  $a_{1,2}a_{1,3}$  is alone in its equivalence class, the braid  $\phi_3(\gamma_1) = a_{1,2}a_{1,3}$  is not right-divisible by  $a_{1,2}$  and so its  $B_2^{+*}$ -tail is trivial. The braid  $\phi_3(\gamma_1)$  is exactly the one of (6) for n = 3 and k = 1. Considering  $\gamma_1$  instead of  $\beta$  we obtain  $\gamma_1 = \phi_3(\gamma_2) \cdot \beta_2$  with  $\gamma_2 = \phi_3^{-1}(a_{1,3}a_{2,3}) = a_{2,3}a_{1,2}$  and  $\beta_2 = 1$ . As the word  $a_{1,3}a_{2,3}$  is also alone in its equivalence class, the braid  $\phi_3(\gamma_2) = a_{1,3}a_{2,3}$  is not right-divisible by  $a_{1,2}$  and so its  $B_2^{+*}$ -tail is trivial. The braid  $\phi_3(\gamma_2)$  is the one of (6) for n = 3 and k = 2.

We decompose the braid  $\gamma_2$  as  $\phi_3(\gamma_3) \cdot \beta_3$  with  $\gamma_3 = \phi_3^{-1}(a_{2,3}) = a_{1,2}$  and  $\beta_3 = a_{1,2}$ . As the braid  $\phi_3(\gamma_3) = a_{2,3}$  is not right-divisible by  $a_{1,2}$ , its  $B_2^{+*}$ -tail is trivial. The braid  $\phi_3(\gamma_3)$  it the one of (6) for n = 3 and k = 3. Eventually we express  $\gamma_3$  as  $\phi_3(\gamma_4) \cdot \beta_4$  with  $\gamma_4 = 1$  and  $\beta_4 = a_{1,2}$ . The process ends as the remaining braid  $\gamma_4$  is trivial. We conclude that the  $\phi_3$ -splitting of  $\beta$  is  $(\beta_4, \beta_3, \beta_2, \beta_1) = (a_{1,2}, a_{1,2}, 1, a_{1,2}^2)$ .

Before giving the definition of the rotating normal we fix some definitions about words.

**Definition 2.4.** Assume  $n \ge 2$ . A word on the alphabet  $A_n$  is an  $A_n$ -word. A word on the alphabet  $A_n^{\pm} = A_n \sqcup A_n^{-1}$  is an  $A_n^{\pm}$ -word. The braid represented by the  $A_n^{\pm}$ -word w is denoted by  $\overline{w}$ . For w, w' two  $A_n^{\pm}$ -words, we say that w is equivalent to w', denoted by  $w \equiv w'$  if  $\overline{w} = \overline{w'}$  holds. The empty word is denoted by  $\varepsilon$ .

The *n*-rotating normal form is an injective map  $r_n$  from  $B_n^{+*}$  to the set of  $A_n$ -words defined inductively using the  $\phi_n$ -splitting.

**Definition 2.5.** For  $\beta \in B_2^{+*}$ , we define  $r_2(\beta)$  to be the unique  $A_2$ -word  $a_{1,2}^k$  representing  $\beta$ . The rotating normal form of a braid  $\beta \in B_n^{+*}$  with  $n \ge 3$  is:

 $r_n(\beta) = \phi_n^{b-1}(r_{n-1}(\beta_b)) \cdot \dots \cdot \phi_n(r_{n-1}(\beta_2)) \cdot r_{n-1}(\beta_1),$ 

where  $(\beta_b, ..., \beta_1)$  is the  $\phi_n$ -splitting of  $\beta$ . A word w is said to be *n*-rotating if it is the *n*-rotating normal form of a braid of  $B_n^{+*}$ .

As the *n*-rotating normal form of a braid of  $B_{n-1}^{+*}$  is equal to its (n-1)rotating normal form we can talk, without ambiguities, about the *rotating normal form* of a dual braid.

**Example 2.6.** We reconsider the braid  $\beta$  of Example 2.3. We know that the  $\phi_3$ -splitting of  $\beta$  is  $(a_{1,2}, a_{1,2}, 1, a_{1,2}^2)$ . Since  $r_2(1) = \varepsilon$ ,  $r_2(a_{1,2}) = a_{1,2}$  and  $r_2(a_{1,2}^2) = a_{1,2}^2$  we obtain:

$$r_3(\beta) = \phi_3^3(a_{1,2}) \cdot \phi_3^2(a_{1,2}) \cdot \phi_3(\varepsilon) \cdot a_{1,2}^2 = a_{1,2}a_{1,3}a_{1,2}a_{1,2}.$$

Some properties of the rotating normal form have been established in [13]. Connections, established in [12] and [13], between the rotating normal form and the braid ordering introduced by P. Dehornoy in [6] are based on these properties.

We finish this section with some already known or immediate properties about  $\phi_n$ -splittings and *n*-rotating words.

**Definition 2.7.** For every non-empty word w, the last letter of w is denoted by  $w^{\#}$ . Assume  $n \ge 2$ . For each non-trivial braid  $\beta$  in  $B_n^{+*}$ , we define the *last letter* of  $\beta$ , denoted  $\beta^{\#}$ , to be the last letter in the rotating normal form of  $\beta$ .

**Lemma 2.8** (Lemma 3.2 of [13]). Assume  $n \ge 3$  and let  $(\beta_b, ..., \beta_1)$  be the  $\phi_n$ -splitting of a braid of  $B_n^{+*}$ .

- (i) For  $k \ge 2$ , the letter  $\beta_k^{\#}$  is of type  $a_{..,n-1}$  unless  $\beta_k = 1$ .
- (ii) For  $k \ge 3$  and for k = b, we have  $\beta_k \ne 1$ .

The fact that  $\beta_b$  is not trivial is a direct consequence of the definition of  $\phi_n$ -splitting. As, for  $k \ge 2$ , the braid  $\beta' = \phi_n(\beta_{k+1}^{\#})\beta_k$  is a right divisor of  $\phi_n^{b-k}(\beta_b) \cdot \ldots \cdot \beta_k$ , it must satisfy some properties. In particular, if  $\beta_{k+1}^{\#} = a_{p-1,n-1}$  holds then the  $B_{n-1}^{+*}$ -tail of  $\phi_n(a_{p,n}\beta_k)$  is trivial by (6).

**Definition 2.9.** For  $n \ge 3$  and p, r, s in [1, n - 1], we say that a letter  $a_{r,s}$  is an  $a_{p,n}$ -barrier if the relation r holds.

There exists no  $a_{p,n}$ -barrier with  $n \leq 3$  and the only  $a_{p,4}$ -barrier is  $a_{1,3}$ , which is an  $a_{2,4}$ -barrier. By definition, if the letter x is an  $a_{p,n}$ -barrier, then in the presentation of  $B_n^{+*}$  there exists no relation of the form  $a_{p,n} \cdot x = y \cdot a_{p,n}$  allowing one to push the letter  $a_{p,n}$  to the right through the letter x: so, in some sense, x acts as a barrier.

The following result guarantees that the rotating normal form of a braid satisfying some properties must contain an  $a_{p,n}$ -barrier.

**Lemma 2.10** (Lemma 3.4 of [13]). Assume  $n \ge 4$ ,  $p \in [2, n-2]$  and let  $\beta$  be a braid of  $B_{n-1}^{+*}$  such that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(a_{p,n}\beta)$  is trivial. Then the rotating normal form of  $\beta$  is not the empty word and it contains an  $a_{p,n}$ -barrier.

It turns out that the above mentioned braid property are easily checked for entries of a  $\phi_n$ -splitting.

**Lemma 2.11** (Lemma 3.5 of [13]). Assume  $n \ge 3$  and let  $(\beta_b, ..., \beta_1)$  be the  $\phi_n$ -splitting of some braid of  $B_n^{+*}$ . Then, for each k in [2, b-1] such that  $\beta_{k+1}^{\#}$  is not  $a_{n-2,n-1}$  (if any), the rotating normal form of  $\beta_k$  contains  $a \phi_n(\beta_{k+1}^{\#})$ -barrier.

#### 3. Left reversing for dual braid monoid

The left-reversing process was introduced by P. Dehornoy in [7]. It is a powerful tool for the investigation of division properties in some monoids as stated by Proposition 3.6.

**Definition 3.1.** A monoid M defined by a presentation  $\langle S | R \rangle^+$  is *left-complemented* if there exists a map  $f : S \times S \to S^*$  satisfying:

$$R = \{ f(x, y)x = f(y, x)y \,|\, (x, y) \in S^2 \}$$

together with  $f(x, x) = \varepsilon$  for all  $x \in S$ .

As the relation x = x is always true for  $x \in S$  we say that M is leftcomplemented even if x = x does not occur in R for  $x \in S$ .

 $\mathbf{6}$ 

The monoid  $B_3^{+*}$  with presentation of Proposition 1.1 is left-complemented with respect to the map f given by

$$f(a_{1,2}, a_{2,3}) = f(a_{1,2}, a_{1,3}) = a_{1,3},$$
  

$$f(a_{2,3}, a_{1,2}) = f(a_{2,3}, a_{1,3}) = a_{1,2},$$
  

$$f(a_{1,3}, a_{1,2}) = f(a_{1,3}, a_{2,3}) = a_{2,3}.$$

However the monoid  $B_4^{+*}$  with presentation of Proposition 1.1 is not leftcomplemented. Indeed there is no relation of type ...  $a_{1,3} = ... a_{2,4}$ . It follows that the words  $f(a_{1,3}, a_{2,4})$  and  $f(a_{2,4}, a_{1,3})$  are not well-defined.

In general for  $1 \leq p < r < q < s \leq n$ , the words  $f(a_{p,q}, a_{r,s})$  and  $f(a_{r,s}, a_{p,q})$  are not defined for the presentation of  $B_n^{+*}$  given in Proposition 1.1. In order to obtain a left-complemented presentation of  $B_n^{+*}$  we must exhibit some extra relations.

For example, the relation  $a_{2,3}a_{1,4}a_{1,3} \equiv a_{3,4}a_{1,2}a_{2,4}$  holds and so we can consider  $f(a_{1,3}, a_{2,4})$  to be  $a_{2,3}a_{1,4}$ . However the relation  $a_{1,4}a_{2,3}a_{1,3} \equiv a_{3,4}a_{1,2}a_{2,4}$  is also satisfied and so  $f(a_{1,3}, a_{2,4}) = a_{1,4}a_{2,3}$  is an other valid choice.

**Lemma 3.2.** For  $n \ge 2$ , the map  $f_n : A_n \times A_n \to A_n^*$  defined by:

$$f_n(a_{p,q}, a_{r,s}) = \begin{cases} \varepsilon & \text{for } a_{p,q} = a_{r,s}, \\ a_{p,s} & \text{for } q = r, \\ a_{s,q} & \text{for } p = r \text{ and } q > s, \\ a_{r,p} & \text{for } q = s \text{ and } p > r, \\ a_{r,q}a_{p,s} & \text{for } p < r < q < s, \\ a_{s,q}a_{r,p} & \text{for } r < p < s < q, \\ a_{r,s} & \text{otherwise.} \end{cases}$$

provides a structure of left-complemented monoid to  $B_n^{+*}$ .

*Proof.* Direct computations using Proposition 1.1 establish  $f_n(x, y) \cdot x \equiv f_n(y, x) \cdot y$  for all  $(x, y) \in A_n^2$ .

Assume  $n \ge 2$ . As illustrated above, the characterization of the map  $f_n$  from the presentation of  $B_n^{+*}$  is not well-defined: many choices are possible. Our map  $f_n$  admits the following remarkable property: for every letters  $a_{p,q}$  and  $a_{r,s}$  of  $A_n$ , the last letter of the word  $f_n(a_{p,q}, a_{r,s})$  is of the form  $a_{t,s}$ , with  $t \le r$ , whenever q < s. This property will be use in the sequel, for example in the proof of Lemma 4.6.

**Definition 3.3.** Assume  $n \ge 2$ . For w and w' two  $A_n^{\pm}$ -words, we say that w left-reverses in one step to w', denoted by  $w \curvearrowright^1 w'$ , if we can obtain w' from w substituting a factor  $xy^{-1}$  (with  $x, y \in A_n$ ) by  $f_n(x, y)^{-1} f_n(y, x)$ . We say that w left-reverses to w', denoted by  $w \curvearrowright w'$ , if there exists a sequence  $w = w_1, ..., w_{\ell} = w'$  of  $A_n^{\pm}$ -words satisfying  $w_k \curvearrowright^1 w_{k+1}$  for  $k \in [1, \ell - 1]$ .

**Example 3.4.** The word  $u = a_{1,2}a_{2,3}a_{1,2}a_{1,3}^{-1}$  left-reverses to  $a_{2,3}a_{2,3}$  as the following left reversing sequence shows (reversed factor are underlined)

$$a_{1,2}a_{2,3}\underline{a_{1,2}a_{1,3}^{-1}} \curvearrowright^1 a_{1,2}\underline{a_{2,3}a_{1,3}^{-1}}a_{2,3} \curvearrowright^1 \underline{a_{1,2}a_{1,2}^{-1}}a_{2,3}a_{2,3} \curvearrowright^1 a_{2,3}a_{2,3},$$

which is denoted by  $a_{1,2}a_{2,3}a_{1,2}a_{1,3}^{-1} \cap a_{2,3}a_{2,3}$ .

A consequence of Lemma 1.1 of [7] is the following definition of left denominator and left numerator of an  $A_n^{\pm}$ -word.

**Definition 3.5.** Assume  $n \ge 2$ . For w an  $A_n^{\pm}$ -word, we denote by D(w) and N(w) the  $A_n$ -words, if there exist, such that  $w \curvearrowright D(w)^{-1} N(w)$  holds. If such words exist, then they are unique, hence the notations D(w) and N(w) are unambiguous. The word N(w) is the *left numerator* of w while the word D(w) is its *left denominator*.

Reconsidering Example 3.4, we obtain that the left denominator of u is  $D(u) = \varepsilon$  and that its left numerator is  $N(u) = a_{2,3}a_{2,3}$ .

Assume  $n \ge 2$ . By Example 8 of [10] based on [3], the monoid  $B_n^{+*}$  is a Garside monoid and  $f_n$  is a *left lcm selector*. A consequence of Lemma 4.3 of [10] is that for every  $w \in A_n^{\pm}$ , the words N(w) and D(w) exist. We obtain also the following result:

**Proposition 3.6.** Assume  $n \ge 2$ . For w an  $A_n$ -word and  $a_{p,q}$  in  $A_n$ , the braid  $\overline{w}$  is right-divisible by  $a_{p,q}$  if and only if  $D(w \cdot a_{p,q}^{-1})$  is empty.

Since the left denominator of  $u = a_{1,2}a_{2,3}a_{1,2}a_{1,3}^{-1}$  is empty, the braid  $a_{1,3}$  right-divides the braid  $a_{1,2}a_{2,3}a_{1,2}$ .

## 4. CHARACTERIZATION OF ROTATING NORMAL WORDS

The aim of this section is to give a syntactical characterization of n-rotating words among  $A_n$ -words.

**Lemma 4.1.** Assume  $n \ge 3$ . For  $\beta$  in  $B_{n-1}^{+*}$  there is equivalence between

- (i) an  $A_n$ -word representing  $\beta$  contains an  $a_{p,n}$ -barrier,
- (ii) every  $A_n$ -word representing  $\beta$  contains an  $a_{p,n}$ -barrier.

*Proof.* Clearly (*ii*) implies (*i*). For (*i*)  $\Rightarrow$  (*ii*) it is sufficient to prove that relations of Proposition 1.1 preserve  $a_{p,n}$ -barriers. For relation (2) this is immediate since it conserves involved letters. If one of the three words  $a_{r,s}a_{s,t}, a_{s,t}a_{r,s}$  and  $a_{r,t}a_{r,s}$  of relations (3) contains an  $a_{p,n}$ -barrier then the two others also, as illustrated by the following chord diagram.



A letter  $a_{r,s}$  is an  $a_{p,n}$ -barrier if and only if its chord intersects properly that of the letter  $a_{p,n}$ .

As a consequence, containing an  $a_{p,n}$ -barrier is not a word property but a braid one.

**Lemma 4.2.** Assume that  $n \ge 3$ . A non-empty braid  $\beta \in B_n^{+*}$  has trivial  $B_{n-1}^{+*}$ -tail if and only if there exists a unique letter in  $A_n$  that rightdivides  $\beta$ , and this letter is of the form  $a_{...n}$ .

*Proof.* If  $a_{p,n}$  is the only letter right-divising  $\beta$ , then the  $B_{n-1}^{+*}$ -tail of  $\beta$  is certainly trivial. Conversely, assume that the  $B_{n-1}^{+*}$ -tail of  $\beta$  is trivial. Hence, no letter  $a_{p,q}$  with q < n right-divides  $\beta$ . Assume, for a contradiction, that two distinct letters  $a_{p,n}$  and  $a_{q,n}$  with p < q < n right-divide  $\beta$ . The braid  $\beta$  is then right-divisible by their lcm

$$a_{p,n} \lor a_{q,n} = a_{p,q} a_{q,n} = a_{q,n} a_{p,n} = a_{p,n}, a_{p,q},$$

hence by  $a_{p,q}$ , which is impossible since the  $B_{n-1}^{+*}$ -tail of  $\beta$  is trivial.

We conclude that, under some hypotheses, the last letter of a word is a braid invariant.

**Definition 4.3.** For  $n \ge 3$  and  $p, q \in [2, n-1]$ , we say that an *n*-rotating word w is an  $a_{p,n}$ -ladder lent on  $a_{q-1,n-1}$  if there exist a decomposition

$$w = v_0 x_1 v_1 \ldots v_{h-1} x_h v_h,$$

a strictly increasing sequence j(0), ..., j(h), with j(0) = p and j(h) = n - 1, and a sequence i(1), ..., i(h) such that:

(i) for each  $k \leq h$ , the letter  $x_k$  is  $a_{i(k),j(k)}$  with i(k) < j(k-1) < j(k),

(*ii*) for each k < h, the word  $v_k$  contains no  $a_{j(k),n}$ -barrier,

(*iii*) the last letter of  $v_h$  is  $a_{q-1,n-1}$ .



FIGURE 5. An  $a_{2,6}$ -ladder lent on  $a_{3,5}$  (the last letter). The gray line starts at position 1 and goes up to position 5 using the bar of the ladder. The empty spaces between bars in the ladder are represented by a framed box. In such boxes the vertical line representing the letter  $a_{i,j}$  does not cross the gray line. The bar of the ladder are represented by black thick vertical lines.

Condition (i) is equivalent to saying that for each  $k \leq h$ , the letter  $x_k$  is an  $a_{j(k-1),n}$ -barrier of type  $a_{\dots,j(k)}$ .

An immediate adaptation of Proposition 3.9 of [13] is:

**Lemma 4.4.** Assume  $n \ge 4$ ,  $p \in [2, n-2]$  and let  $\beta$  be a braid of  $B_{n-1}^{+*}$  such that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta)$  is trivial and such that  $\beta$  contains an  $a_{p,n}$ -barrier. Then the rotating normal form of  $\beta$  is an  $a_{p,n}$ -ladder lent of  $\beta^{\#}$ .

In order to obtain a syntactical characterization of *n*-rotating words we want a local version of condition (6) characterizing a  $\phi_n$ -splitting. The following result is the first one in this way.

**Proposition 4.5.** Assume  $n \ge 4$ . For  $\beta \in B_{n-1}^{+*}$  and  $p \in [2, n-2]$  there is equivalence between:

(i) the  $B_{n-1}^{+*}$ -tail of  $\phi_n(a_{p,n}\beta)$  is trivial,

(ii) the  $B_{n-1}^{**}$ -tail of  $\phi_n(\beta)$  is trivial and  $\beta$  contains an  $a_{p,n}$ -barrier,

Our proof of Proposition 4.5 rests on the following Lemma.

**Lemma 4.6.** Assume  $n \ge 3$ . For u an  $A_{n-1}$ -word and  $p \in [1, n-1]$ , the left denominator  $D(ua_{p,n}^{-1})$  is not empty. More precisely, their exists  $q \in [1, p]$  such that  $D(ua_{p,n}^{-1})^{-1}$  begins with  $a_{q,n}^{-1}$ .

*Proof.* Assume that  $w_1, ..., w_\ell$  is a reversing sequence from the word  $w_1 = ua_{p,n}^{-1}$  to the word  $D(w_1)^{-1}N(w_1)$ . For  $k \in [1, \ell]$  we denote by  $y_k^{-1}$  the leftmost negative letter in  $w_k$ , if such a negative letter exists in  $w_k$ . We will show by induction on  $k \in [1, \ell]$  the existence of a non-increasing sequence r(k) such that  $y_k = a_{r(k),n}$ .

By construction, we have r(1) = p and  $y_1 = a_{r(1),n}$ . Each reversing step consists in replacing a factor  $xy^{-1}$  of  $w_k$  by  $f_n(x,y)^{-1}f_n(y,x)$ . For  $k \in [1, \ell - 1]$ , if the reversed factor of  $w_k$  does not contain the leftmost occurence of  $y_k^{-1}$  in  $w_k$  then,  $y_{k+1}$  equals  $y_k$  and so r(k+1) = r(k). Assume now that the reversed factor contains the leftmost negative letter  $y_k$  of  $w_k$ and that  $y_k$  is equal to  $a_{r(k),n}$ . In particular, their exits a positive letter xsuch that the reversed factor is of the form  $xy_k^{-1} = xa_{r(k),n}^{-1}$ . As all letters of  $w_k$  on the left of the leftmost negative letter of  $w_k$  must belong to  $A_{n-1}$ , the letter x is of the form  $a_{i,j}$  with  $1 \leq i < j < n$ . Lemma 3.2 implies

$$f_n(x, y_k) = f_n(a_{i,j}, a_{r(k),n}) = \begin{cases} a_{i,n} & \text{for } j = r(k), \\ a_{r(k),j}a_{i,n} & \text{for } i < r(k) < j, \\ a_{r(k),n} & \text{otherwise,} \end{cases}$$

which gives in particular

$$xy_k^{-1} = a_{i,j}a_{r(k),n}^{-1} \curvearrowright \begin{cases} a_{i,n}^{-1} \cdots & \text{for } i < r(k) \leq j, \\ a_{r(k),n}^{-1} \cdots & \text{otherwise.} \end{cases}$$
(8)

It follows that  $y_{k+1}$  is equal to  $a_{r(k+1),n}$  with  $r(k+1) \leq r(k)$ . Eventually we obtain that  $ua_{p,n}^{-1}$  left reverses to  $a_{r(\ell),n}^{-1} \cdots$  with  $r(\ell) \leq r(1) = p$  and so the desired property on  $D(ua_{p,n})$  holds.

Proof of Proposition 4.5. Let us first assume (i). As the the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta)$  is a right-divisor of  $\phi_n(a_{p,n}\beta)$  the first statement of (ii) holds and the second statement of (ii) is Lemma 2.10.

We now prove  $(ii) \Rightarrow (i)$ . By Lemma 4.2 and Condition (ii), there exists a unique letter in  $A_n$  that right-divises  $\phi_n(\beta)$  and it is of type  $a_{..,n}$ . It follows that the last letter  $\beta^{\#}$  of  $\beta$  is of type  $a_{..,n-1}$ . We denote by w the rotating normal form of  $\beta$ . Let  $a_{r,s}$  be an  $A_n$ -letter different from  $\beta^{\#}$ . It remains to show that  $a_{r,s}$  cannot be a right divisor of  $a_{p,n}\beta$ .

Assume first  $s \leq n-1$ . Then the letter  $a_{r,s}$  belongs to  $A_{n-1}$ . As  $\beta$  lies in  $B_{n-1}^{+*}$ , Lemma 4.2 implies that the braid  $a_{r,s}$  could not be a right-divisor of  $\beta$ . Hence, by Proposition 3.6 the word  $D(w a_{r,s}^{-1})$  must be non-empty. As the reversing of an  $A_{n-1}^{\pm}$ -word is also an  $A_{n-1}^{\pm}$ -word, there exists a letter  $a_{t,t'}$ with t' < n such that:

$$a_{p,n} w a_{r,s}^{-1} \cap a_{p,n} a_{t,t'}^{-1} \cdots,$$

holds. Clearly, the braid  $a_{t,t'}$  is not a right divisor of  $a_{p,n}$  (since we have t' < n). Therefore, by Proposition 3.6, the left denominator of  $a_{p,n} w a_{r,s}^{-1}$  is not empty, and we conclude that  $a_{r,s}$  is not a right divisor of  $a_{p,n}\beta$ .

Assume now s = n. Hypotheses on  $\beta$  plus Lemma 4.4 imply that w is an  $a_{p,n}$ -ladder lent on  $\beta^{\#}$ . Following Definition 4.3, we write

$$w = v_0 x_1 v_1 \cdots v_{h-1} x_h v_h.$$

By Lemma 4.6, there exist two maps  $\eta$  and  $\mu$  from  $\mathbb{N}$  to itself such that

$$wa_{r,n}^{-1} = w_h a_{\eta(h),n}^{-1} \curvearrowright w'_h a_{\mu(h),n}^{-1} \cdots \curvearrowright \dots \curvearrowright w_0 a_{\eta(0),n}^{-1} \cdots \curvearrowright w'_0 a_{\mu(0),n}^{-1} \cdots,$$
  
where for all  $k \in [0, h]$ ,

$$w_k = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k,$$
  
 $w'_k = v_0 x_1 v_1 \cdots v_{k-1} x_k.$ 

By construction  $w_0$  is  $v_0$  while  $w'_0$  is the empty word and  $\eta(h)$  is r. For  $k \in [0, h]$ , Lemma 4.6 with  $u = v_k$  and  $p = \eta(k)$  imply that the word

$$w_k a_{\eta(k),n}^{-1} \cdots = w'_k v_k a_{\eta(k),n}^{-1} \cdots$$

left-reverses to  $w'_k a^{-1}_{\mu(k),n} \cdots$  with  $\mu(k) \leq \eta(k)$ . Then, for  $k \neq 0$ , Lemma 4.6 (with  $u = v_{k-1}$  and  $p = \mu(k)$ ) implies that the word

$$w'_k a^{-1}_{\mu(k),n} \cdots = w_{k-1} x_k a^{-1}_{\mu(k),n} \cdots$$

left-reverses to  $w_{k-1} a_{\eta(k-1),n}^{-1} \cdots$  with  $\eta(k-1) \leq \mu(k)$ . Using an inductive argument on k = h, ..., 0 we then obtain:

$$\mu(0) \leqslant \eta(0) \leqslant \mu(1) \leqslant \dots \leqslant \mu(h) \leqslant \eta(h) = r.$$
(9)

Following Definition 4.3 we write  $x_k = a_{i(k),j(k)}$ . We will now prove for all  $k \in [0, h - 1]$ ,

$$\mu(k+1) \leqslant j(k+1) \Rightarrow \eta(k) < j(k). \tag{10}$$

Let  $k \in [0, h-1]$  and assume  $\mu(k+1) \leq j(k+1)$ . Definition 4.3 (i) guarantees the relation i(k+1) < j(k) < j(k+1). For the case where  $\mu(k+1) \leq i(k+1)$  we have:

$$\eta(k) \leqslant \mu(k+1) \leqslant i(k+1) < j(k),$$

and we are done in this case. The remaining case is  $\mu(k+1) > i(k+1)$ . By relation (8), with i = i(k+1), j = j(k+1) and  $r = \mu(k+1)$  (satisfying the relation  $i < r \leq j$ ) we obtain

$$x_{k+1} a_{\mu(k+1),n}^{-1} = a_{i(k+1),j(k+1)} a_{\mu(k+1),n}^{-1} \cap a_{i(k+1),n}^{-1} v,$$

for some  $A_n^{\pm}$ -word v. In particular, we have  $\eta(k) = i(k+1) < j(k)$  and (10) is established.

For k = h - 1 the left-hand member of (10) is satisfied since j(h) is equal to n-1 and

$$\mu(h) \leqslant \eta(h) = r \leqslant n - 1$$

holds by definition of r. Properties (9) and (10) imply  $\mu(k) < j(k)$  for all  $k \in [0, h-2]$ . In particular we have  $\mu(0) < j(0) = p$  together with  $wa_{r,n}^{-1} \sim a_{\mu(0),n}^{-1} \cdots$ . As  $a_{\mu(0),n}$  cannot be a right divisor of  $a_{p,n}$  it follows that the left denominator of  $a_{p,n} w a_{r,n}^{-1}$  is also non-empty and so that  $a_{r,n}$  is not a right divisor of  $a_{p,n}\beta$ .

As the reader can see, the case p = n - 1 is excluded from Proposition 4.5. This case is treated by the following result.

**Proposition 4.7.** Assume  $n \ge 3$ . For  $\beta$  a non-trivial braid of  $B_{n-1}^{**}$  there is equivalence between:

- (i) the  $B_{n-1}^{+*}$ -tail of  $\phi_n(a_{n-1,n}\beta)$  is trivial, (ii) the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta)$  is trivial.

*Proof.* As a right divisor of  $\phi_n(\beta)$  is also a right of  $\phi_n(a_{n-1,n}\beta)$ , the implication  $(i) \Rightarrow (ii)$  is clear. Let us now prove  $(ii) \Rightarrow (i)$ . For  $x \in A_{n-1}$ and y, z in  $A_n$ , the only relations in  $B_n^{+*}$  of the form  $a_{n-1,n} x = y z$  with  $(y,z) \neq (a_{n-1,n},x)$  are commutations relations, for which x is of type  $a_{p,q}$ with p < q < n-1. Hence, if w is a representative of  $\beta$ , the relations in the dual braid monoid  $B_{n-1}^{**}$  allow only transforming the word  $a_{n-1,n} w$  into words of the form  $u a_{n-1,n} v$ . The  $A_{n-2}$ -word u represents a dual braid that commute with  $a_{n-1,n}$  and uv represents  $\beta$ . By Lemma 4.2 the only letter in  $A_n$  that right-divides  $\phi_n(\beta)$  is of type  $a_{\dots,n}$ , hence any representative word of  $\beta$  must end with a letter of type  $a_{\dots n-1}$ . In particular v is not an  $A_{n-2}$ word and so v is not empty and it admits only  $\beta^{\#}$  as last letter. We have then established that every representative word of  $a_{n-1,n}\beta$  ends with the letter  $\beta^{\#}$  which is of type  $a_{\dots,n-1}$ . Therefore, the  $B_{n-1}^{+*}$ -tail of  $\phi_n(a_{n-1,n}\beta)$  is trivial. 

**Theorem 4.8.** Assume  $n \ge 3$ . A finite sequence  $(\beta_b, ..., \beta_1)$  of braids in  $B_{n-1}^{+*}$  is the  $\phi_n$ -splitting of a braid of  $B_n^{+*}$  if and only if: (i) for  $k \ge 3$  and k = b, the braid  $\beta_k$  is not trivial,

(ii) for  $k \ge 2$ , the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta_k)$  is trivial,

(iii) if, for  $k \ge 3$ , we have  $\beta_k^{\#} \ne a_{n-2,n-1}$ , then  $\beta_{k-1}$  contains an  $\phi_n(\beta_k^{\#})$ -barrier.

*Proof.* Let  $(\beta_b, ..., \beta_1)$  be the  $\phi_n$ -splitting of some braid of  $B_n^{+*}$ . Condition (i) is Lemma 2.8.(ii). Condition (6) implies that the  $B_{n-1}^{+*}$ -tail of

$$\phi_n^{b-k}(\beta_b) \cdot \ldots \cdot \phi_n(\beta_{k+1})$$

is trivial for  $k \ge 1$ . In particular the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta_{k+1})$  must be trivial for  $k \ge 1$ , which implies (*ii*). Condition (*iii*) is Lemma 2.11.

Conversely, let us prove that a sequence  $(\beta_b, ..., \beta_1)$  of braids of  $B_{n-1}^{+*}$  satisfying (i), (ii) and (iii) is the  $\phi_n$ -splitting of some braid of  $B_n^{+*}$ . Condition (i)implies that  $\beta_b$  is not trivial. For  $k \ge 2$  we denote by  $\gamma_k$  the braid

$$\gamma_k = \phi_n^{b-k}(\beta_b) \cdot \ldots \cdot \phi_n(\beta_{k+1}) \cdot \beta_k$$

For  $k \ge 3$ , and for  $k \ge 2$  whenever  $\beta_2 \ne 1$ , we first prove

 $\beta_k^{\#}$  is the only  $A_n$ -letter that right-divides  $\gamma_k$ . (11)

We note that Condition (i) guarantees the existence of  $\beta_k^{\#}$  for  $k \ge 3$ . For k = b, Condition (ii) implies that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta_b)$  is trivial. Hence, by Lemma 4.2 the only  $A_{n-1}$ -letter that right-divides  $\beta_b$  is  $\beta_b^{\#}$ . Since any right divisors of a braid of  $B_{n-1}^{+*}$  lies in  $B_{n-1}^{+*}$ , we have established (11) for k = b.

Assume (11) holds for  $k \ge 4$ , or  $k \ge 3$  whenever  $\beta_2 \ne 1$ , and let us prove it for k-1. By Condition (*ii*), there exists p such that  $\beta_k^{\#}$  is  $a_{p-1,n-1}$ . Conditions (*ii*) and (*iii*) imply that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta_{k-1})$  is trivial and that  $\beta_{k-1}$  contains an  $a_{p,n}$ -barrier whenever p < n-1 holds. Hence, by Proposition 4.5 (for p < n-1) and Proposition 4.7 (for p = n-1), the  $B_{n-1}^{+*}$ -tail of the braid  $\phi_n(a_{p,n}\beta_{k-1})$  is trivial. Using Lemma 4.2, we obtain that  $\phi_n(a_{p,n}\beta_{k-1})$  is right-divisible by a unique  $A_n$ -letter. It follows that the unique  $A_n$ -letter right-divising  $a_{p,n}\beta_{k-1}$  is  $\beta_{k-1}^{\#}$ .

We denote by  $ua_{p-1,n-1}$  and v two  $A_n$ -words representing  $\gamma_k$  and  $\beta_{k-1}$ respectively. The braid  $\gamma_{k-1}$  is then represented by  $\phi_n(u)a_{p,n}v$ . Let y be an  $A_n$ -letter different from  $\beta_{k-1}^{\#}$ . As y is not a right divisor of  $a_{p,n}\beta_{k-1}$ , Proposition 3.6 implies the existence of an  $A_n$ -letter x different from  $a_{p,n}$ satisfying

$$\phi_n(u)a_{p,n}vy^{-1} \frown \phi_n(u)a_{p,n}x^{-1}\cdots$$

The word  $\phi_n(u)a_{p,n}$  represents  $\phi_n(\gamma_k)$ . By induction hypothesis x is not a right divisor of  $\phi_n(\gamma_k)$ . Then Proposition 3.6 implies that  $D(\phi_n(u)a_{p,n}x^{-1})$  is not empty. It follows  $D(\phi_n(u)a_{p,n}vy^{-1}) \neq \varepsilon$  and so, always by Proposition 3.6, the letter y is not a right divisor of  $\gamma_{k-1}$ . Eventually we have established (11) for  $k \geq 3$ .

A direct consequence of (11) and Condition (*ii*) is that the only  $A_n$ -letter right-dividing  $\phi_n(\gamma_k)$  is of type  $a_{..,n}$  and so, by Lemma 4.2 the  $B_{n-1}^{+*}$ -tail of the braid  $\gamma_k$  is trivial for  $k \ge 3$  and for k = 2 whenever  $\beta_k \ne 1$ . It remains to establish that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\gamma_2)$  is also trivial whenever  $\beta_2$ is trivial. Assume  $\beta_2 = 1$ . Condition (*iii*) implies  $\beta_3^{\#} = a_{n-2,n-1}$ . By (11),  $a_{n-2,n-1}$  is the only  $A_n$ -letter that right-divides  $\gamma_3$ . Since  $\gamma_2 = \phi_n(\gamma_3)$  holds, the letter  $\phi_n^2(a_{n-2,n-1}) = a_{1,n}$  is the only letter right-dividing  $\phi_n(\gamma_2)$ . In particular the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\gamma_2)$  is trivial.  $\square$ 

Conditions (i), (ii) and (iii) are easy to check if the braids  $\beta_1, ..., \beta_b$  are given by their rotating normal forms.

**Corollary 4.9.** Assume  $n \ge 3$  and let  $(w_b, ..., w_1)$  be a finite sequence of  $A_{n-1}$ -words. The  $A_n$ -word

$$\phi_n^{b-1}(w_b) \cdot \dots \cdot \phi_n(w_2) \cdot w_1, \tag{12}$$

is n-rotating if the following conditions are satisfied:

(i) for  $k \ge 1$ , the word  $w_k$  is (n-1)-rotating,

(ii) for  $k \ge 3$ , the word  $w_k$  ends with  $a_{p-1,n-1}$  for some p,

(iii) the word  $w_2$  is either empty (except for b = 2) or ends with  $a_{p-1,n-1}$ for some p.

(iv) if, for  $k \ge 3$ , the word  $w_k$  ends with  $a_{p-1,n-1}$  with  $p \ne n-1$  then the word  $w_{k-1}$  contains an  $a_{p,n}$ -barrier.

*Proof.* Assume that  $(w_b, ..., w_1)$  satisfies Conditions (i)-(iv) and let us prove that the word w defined in (12) is rotating.

We denote by  $\beta_i$  (resp.  $\beta$ ) the braid represented by  $w_i$  (resp. w). By Condition (i) and Definition 2.5, the word w is rotating if and only if  $(\beta_b, ..., \beta_1)$  is a  $\phi_n$ -splitting. Conditions (*ii*) and (*iii*) imply Condition (*i*) of Theorem 4.8. Theorem 4.8.(iii) is a consequence of (ii) and (iv).

We now prove Condition (ii) of Theorem 4.8. Let k be in [2, b]. If the  $B_{n-1}^{*}$ -tail of  $\phi_n(\beta_k)$  is not trivial, then there exists  $a_{p,q}$ , with  $1 \leq p < p$ q < n, that right-divides  $\phi_n(\beta_k)$ . As  $\beta_k$  lies in  $B_{n-1}^{**}$ , we must have  $p \neq 1$  and therefore  $\beta_k$  is right-divisible by  $a_{p-1,q-1}$  with  $q-1 \leq n-2$ . The  $B_{n-1}^{+*}$ -tail of  $w_k$  is then not trivial. Since the word  $w_k$  is (n-1)-rotating, its last letter must come from its  $B_{n-1}^{+*}$ -tail. Hence  $w_k$  must end with a letter  $a_{i,j}$ satisfying  $j \leq n-2$ , which is in contradiction with Conditions (*ii*) and (*iii*). 

We conclude applying Theorem 4.8.

It is not true that any decomposition of an n-rotating word as in (12) satisfies Conditions (i) - (iv) of Corollary 12. However we have the following result.

**Proposition 4.10.** For  $n \ge 3$  and every n-rotating word w, there exists a unique sequence  $(w_b, ..., w_1)$  of (n-1)-rotating words such that w decomposes as in (12) and Conditions (ii) - (iv) of 4.9 hold.

*Proof.* By definition of a rotating normal word and by Lemma 2.8 such a sequence exists. Let us prove now the unicity. Assume w is a n-rotating normal word and that  $(w_b, ..., w_1)$  and  $(w'_c, ..., w'_1)$  are two different sequences of (n-1)-rotating normal words satisfying Conditions (*ii*) and (*iii*) of Corollary 4.9. Let k be the minimal integer satisfying  $w_k \neq w'_k$ . Since the sum of the word lengths of the two sequences are the same, we have  $k \leq \min\{b, c\}$ .

Without loss of generality, we may assume that  $w'_k$  is a proper suffix of  $w_k$ , *i.e.*,  $w_k = u \cdot w'_k$ . Let x be the last letter of  $w'_{k+1}$  or the last letter of  $w'_{k+2}$  if  $w_{k+1}$  is empty. By Conditions (*ii*) and (*iii*) of Corollary 4.9, the letter x is equal to  $a_{p-1,n-1}$  for some p and  $w_k$  admits either  $\phi_n(a_{p-1,n-1})w'_k = a_{p,n}w'_k$  or  $\phi_n^2(a_{p-1,n-1})w'_k = a_{1,p+1}w'_k$  as suffix. The first case is impossible since  $w_k$  is an  $A_{n-1}$ -word. The second case may occur only for k = 1 and  $w'_2 = \varepsilon$ . In this case  $w'_2$  is empty and so Condition (*iv*) of Corollary 4.9 implies that the last letter of  $w'_3$ , which is x, is equal to  $a_{n-2,n-1}$ . This implies that  $w_k$  admits  $a_{1,n}u$  as suffix, which is also impossible since  $w_k$  is an  $A_{n-1}$ -word.  $\Box$ 

A direct consequence of Corollary 4.9 and Proposition 4.10 is

**Theorem 4.11.** For  $n \ge 3$ , an  $A_n$ -word w is rotating if and only if it can be expressed as in (12) subject to Conditions (i) – (iv) of Corollary 4.9.

#### 5. Regularity

In this section we will show that the language of *n*-rotating words, denoted by  $R_n$ , is regular, *i.e.*, there exists a finite-state automaton recognizing the language of *n*-rotating words. As the rotating normal form is defined using right division it is more natural for an automaton to read words from the right. For  $w = x_0 \cdot \ldots \cdot x_k$  an  $A_n$ -word we will denote by  $\Pi(w)$  the mirror word  $x_k \cdot \ldots \cdot x_0$ . By Theorem 1.2.8 of [11] the language  $R_n$  is regular if and only if the language  $\Pi(R_n)$  is. In this section we will construct an automaton recognizing  $\Pi(R_n)$ .

For us a *finite-state automaton* is a 6-tuple  $(S \cup \{\otimes\}, A, \mu, Y, i, \otimes)$  where  $S \cup \{\otimes\}$  is the finite set of *states*, A is a finite *alphabet*,

$$\mu: (S \cup \{\otimes\}) \times A \to S \cup \{\otimes\}$$

is the transition function,  $Y \subseteq S$  is the set of accepting states and *i* is the *initial state*. In this paper each automaton is equipped with an undrawn dead state  $\otimes$  and each state except the dead one is accepting, *i.e.*, Y = S always hold.

To characterize  $\mathcal{A}$  it is enough to describe  $\mu$  on  $(s, x) \in S \times A$  instead of  $(S \cup \{\otimes\}) \times A$ . For example an automaton recognizing the language  $R_2 = \Pi(R_2)$  is  $\mathcal{A}_2 = (\{1, \otimes\}, \{a_{1,2}\}, \mu_2, \{1\}, 1, \otimes)$  with  $\mu_2(1, a_{1,2}) = 1$ . The corresponding automaton diagram is:



The horizontal arrow points to the initial state.

We will now construct an automaton  $\mathcal{A}_3$  recognizing the language  $\Pi(R_3)$ .

**Proposition 5.1.** An  $A_3$ -word  $x_b^{e_b} \cdot ... \cdot a_{1,3}^{e_3} a_{2,3}^{e_2} a_{1,2}^{e_1}$  where

$$x_{k} = \begin{cases} a_{1,2} & \text{if } k \equiv 1 \mod 3, \\ a_{2,3} & \text{if } k \equiv 2 \mod 3, \\ a_{1,3} & \text{if } k \equiv 3 \mod 3. \end{cases}$$

is rotating if and only if  $e_k \neq 0$  for all  $k \ge 3$ .

*Proof.* The 2-rotating words are powers of  $a_{1,2}$ . Let w be the word of the statement. Defining  $w_k$  to be  $a_{1,2}^{e_k}$ , we obtain

$$w = \phi_n^{b-1}(w_b) \cdot \dots \cdot \phi_n(w_2) \cdot w_1$$

As each non-empty word  $w_k$  ends with  $a_{1,2}$ , Condition (iv) of Corollary 4.9 is vacuously true. Hence the word w is rotating if and only if it satisfies Conditions (ii) and (iii) of Corollary 4.9, *i.e.*,  $e_k$  is not 0 for  $k \ge 3$ .

As a consequence the following automaton recognizes the language  $\Pi(R_3)$ :



We now explain the principle of our construction of an automaton  $\mathcal{A}_n$ recognizing the language  $\Pi(R_n)$  for  $n \ge 4$ . Following the Condition (i) of Corollary 4.9, the construction will be done by induction on  $n \ge 3$ . Quite natural it will be more complicated for  $n \ge 4$  than for n = 2 and n = 3since Condition (iv) of Corollary 4.9 is not empty for  $n \ge 4$  and must be checked. As suggested by Condition (ii) the language  $R_n^*$  (see Definition 5.2) of *n*-rotating words ending with a letter  $a_{\dots,n}$  is in the core of the process. Hence, instead of directly construct the automaton  $\mathcal{A}_n$  we will construct an automaton  $\mathcal{A}_n^*$  recognizing the language  $\Pi(R_n^*)$ . As the language  $\Pi(R_n)$  is the concatenation of  $\Pi(R_{n-1})$  and  $\Pi(R_n^*)$ , the automaton  $\mathcal{A}_n$  is naturally obtained from  $\mathcal{A}_{n-1}$  and  $\mathcal{A}_n^*$ .

Assume that the automaton  $\mathcal{A}_{n-1}^*$  recognizing  $\Pi(R_{n-1}^*)$  is given. In order to check Condition (iv) of Corollary 4.9, we must modify  $\mathcal{A}_{n-1}^*$  to store information about the existence of  $a_{\dots n}$ -barriers in the involved word. A duplication of states contained in  $\mathcal{A}_{n-1}^*$  together with some standard modifications on the transition function allow us to store one bit of information. Since there is exactly n-3 types of  $a_{\dots n}$ -barriers we must multiply the number of states of  $\mathcal{A}_{n-1}^*$  by at most  $2^{n-3}$  in order to obtain an automaton  $\mathcal{B}_{n-1}^0$ recognizing  $\Pi(R_{n-1}^*)$  and detecting barriers.

Then, for  $k \in [1, n-1]$  we construct an automaton  $\mathcal{B}_{n-1}^k$  recognizing the language  $\Pi(\phi_n^k(R_{n-1}^*))$  by applying  $\phi_n^k$  to  $\mathcal{B}_{n-1}^0$ . Eventually we obtain  $\mathcal{A}_n^*$  by plugging cyclically automata  $\mathcal{B}_{n-1}^0, ..., \mathcal{B}_{n-1}^{n-1}$ . Transitions between automata  $\mathcal{B}_{n-1}^k$  and  $\mathcal{B}_{n-1}^{k+1}$  will be done in respect with Condition (*iv*) of Corollary 4.9 thanks to informations about encountered barriers stored in  $\mathcal{B}_{n-1}^k$ .

**Definition 5.2.** For  $n \ge 2$ , we denote by  $R_n^*$  the language of *n*-rotating words that are empty or end with a letter of type  $a_{p,n}$  for some *p*.

As explained above, we will first construct by induction on  $n \ge 3$  an automaton  $\mathcal{A}_n^*$  for the language  $\Pi(R_n^*)$ .

**Definition 5.3.** A partial automaton is a 6-tuple  $P = (S \cup \{\otimes\}, A, \mu, S, I, \otimes)$ where S, A and  $\mu$  are defined as for an automaton and  $I : A \to S \cup \{\otimes\}$  is the *initial* map. The closure of a partial automaton P is the automaton

$$\mathcal{A}(P) = (S \cup \{\circ, \otimes\}, A, \mu^c, S \cup \{\circ\}, \circ, \otimes)$$

whose transition function is given by

$$\mu^{c}(s,x) = \begin{cases} I(x) & \text{if } s = \circ, \\ \mu(s,x) & \text{otherwise.} \end{cases}$$

In the sequel, we will plug partial automata using their initial maps. A partial automaton is represented as an automaton excepted for the initial map I: for each  $x \in A$  we draw an arrow attached to the state I(x) and labelled x. We say that a partial automaton recognizes a given language if its closure does.



FIGURE 6. The partial automaton  $P_3$  and the corresponding closure, which recognizes the language  $\Pi(R_3^*)$ .

We will now show how to construct by induction a partial automaton  $P_n^*$  recognizing  $\Pi(R_n^*)$  for  $n \ge 3$ . For n = 3 this is already done by Figure 6. For the sequel we assume that  $n \ge 4$  and that

$$P_{n-1}^* = (S_{n-1}^* \cup \{\otimes\}, A_{n-1}, \mu_{n-1}^*, S_{n-1}^*, I_{n-1}^*, \otimes)$$

is a given partial automaton which recognizes the language  $\Pi(R_{n-1}^*)$ .

We construct a partial automaton  $P_n^0$  that recognize  $\Pi(R_{n-1}^*)$  and store informations about the encountered barriers. We define  $S_n^0$  to be the set

$$S_n^0 = \{0\} \times S_{n-1}^* \times \mathcal{P}(\{a_{2,n}, ..., a_{n-2,n}\}).$$

A state in  $S_n^0$  is then written (0, s, m). The integer 0 is used to identify this particular partial automaton among those that will eventually constitute  $P_n^*$ . The set m stores informations on encountered barriers.

For  $a_{i,j} \in A_{n-1}$  we denote by  $bar(a_{i,j})$  the set of letters  $a_{p,n}$  such that  $a_{i,j}$  is an  $a_{p,n}$ -barrier, *i.e.*,

$$\operatorname{bar}(a_{i,j}) = \{a_{p,n} \mid i$$

**Definition 5.4.** We define  $P_n^0 = (S_n^0 \cup \{\otimes\}, A_{n-1}, \mu_n^0, S_n^0, I_n^0, \otimes)$  to be the partial automaton defined by: for all  $x \in A_{n-1}$ ,

$$I_{n}^{0}(x) = \begin{cases} (0, I_{n-1}^{*}(x), \operatorname{bar}(x)) & \text{if } I_{n-1}^{*}(x) \neq \otimes, \\ \otimes & \text{if } I_{n-1}^{*}(x) = \otimes. \end{cases}$$

and for all  $(0, s, m) \in S_n^0$  and for all  $x \in A_{n-1}$ ,

$$\mu_n^0((0,s,m),x) = \begin{cases} (0,\mu_{n-1}^*(s,x),m \cup \operatorname{bar}(x)) & \text{if } \mu_{n-1}^*(s,x) \neq \otimes, \\ \otimes & \text{if } \mu_{n-1}^*(s,x) = \otimes. \end{cases}$$

**Proposition 5.5.** The partial automaton  $P_n^0$  recognizes the language  $\Pi(R_{n-1}^*)$ . Moreover an accepted  $A_n$ -word  $\Pi(w)$  contains an  $a_{p,n}$ -barrier if and only if  $P_n^0$  has state (0, s, m) with  $a_{p,n} \in m$  after reading  $\Pi(w)$ .

*Proof.* We denote by  $\mathcal{A}$  and  $\mathcal{A}'$  the closure of  $P_{n-1}^*$  and  $P_n^0$  respectively. Assume  $w = w_1 \cdot \ldots \cdot w_\ell$  is an  $A_n$ -word of length  $\ell$ . For all  $k \in [1, \ell]$ , we denote by  $s_k$  (resp.  $s'_k$ ) the state of the automaton  $\mathcal{A}$  (resp.  $\mathcal{A}'$ ) after reading the k-th letter of  $\Pi(w)$ , *i.e.*, the letter  $w_{\ell-k+1}$ . By construction of  $\mu_n^0$ , for all  $k \in [1, \ell]$ , we have

$$s'_{k} = \begin{cases} \otimes & \text{if } s_{k} = \otimes, \\ (0, s_{k}, m_{k}) & \text{otherwise (for a certain set } m_{k}) \end{cases}$$

In particular  $s'_{\ell} \neq \otimes$  if and only if  $s_{\ell} \neq \otimes$ . Hence both automata accept the word  $\Pi(w)$  or not and so  $P_n^0$  recognizes  $\Pi(R_{n-1}^*)$  since  $P_{n-1}^*$  does.

Let us prove the result on  $m_{\ell}$  whence  $\Pi(w)$  is accepted. By an immediate induction of  $k \in [1, \ell]$ , the set  $m_k$  contains  $\operatorname{bar}(w_{\ell}) \cup \ldots \cup \operatorname{bar}(w_{\ell-k+1})$ . The set  $m_{\ell}$  is then  $\operatorname{bar}(w_1) \cup \ldots \cup \operatorname{bar}(w_{\ell})$ . We conclude by definition of  $\operatorname{bar}()$ , that w contains an  $a_{p,n}$ -barrier if and only if  $a_{p,n}$  belongs to  $m_{\ell}$ .  $\Box$ 

As the only  $a_{p,4}$ -barrier in  $A_4$  is  $a_{1,3}$ , the partial automaton  $P_4^0$  is obtained from  $P_3^*$  by connecting edges labelled  $a_{1,3}$  to a copy of  $P_3$ , as illustrated on Figure 7.

We now construct n-1 twisted copies of  $P_n^0$  using the word homomorphism  $\phi_n$ . For  $t = (0, s, m) \in S_n^0$  and  $k \in [1, n-1]$ , we define  $\Phi_n^k(t)$  to be



FIGURE 7. The partial automaton  $P_4^0$ . Obsolete transitions from  $P_3^*$  are in gray. New added transitions are dashed. The right partial automaton is  $P_4^0$  without inaccessible states.

the state (k,s,m). We also define  $S_{n}^{k}$  to be  $\Phi_{n}^{k}\left(S_{n}^{0}\right)$  and

$$P_n^k = \left(S_n^k \cup \{\otimes\}, \phi_n^k(A_{n-1}), \mu_n^k, S_n^k, I_n^k, \otimes\right)$$

to be the partial automaton given by  $I_n^k(\phi_n^k(x)) = \Phi_n^k\left(I_n^0(x)\right)$  and

$$\mu_n^k((k, s, m), \phi_n^k(x)) = \Phi_n^k \left( \mu_n^0((0, s, m), x) \right),$$

with the convention  $\Phi_n^k(\otimes) = \otimes$ . In other words,  $P_n^k$  is obtained from  $P_n^0$  by replacing the letter x by  $\phi_n^k(x)$  and the state (0, s, m) by (k, s, m). By Proposition 5.5, we obtain immediately that  $P_n^k$  recognizes the language  $\phi_n^k(\Pi(R_n^*))$  and store informations about encountered barriers.

We can now construct the partial automaton  $P_n^*$  by plugging cyclically n partial automata  $P_n^k$  for  $k \in [0, n-1]$ . Transition between two adjacent partial automata is done using initial maps accordingly to Condition (iv) of Corollary 4.9.

**Definition 5.6.** We define  $P_n^* = (S_n^* \cup \{\otimes\}, A_n, \mu_n^*, S_n, I_n^*, \otimes)$ , with  $S_n^* = S_n^0 \sqcup \ldots \sqcup S_n^k$  to be the partial automaton given by

$$I_n^*(x) = \begin{cases} I_n^1(x) & \text{if } x = a_{p,n} \text{ with } p \neq 1, \\ I_n^2(x) & \text{if } x = a_{1,n}, \\ \otimes & \text{otherwise,} \end{cases}$$

and with transition function

$$\mu_{n}^{*}((k, s, m), \phi_{n}^{k}(x)) = \begin{cases} \mu_{n}^{k}((k, s, m), \phi_{n}^{k}(x)) & \text{if } x \in A_{n-1}, \\ I_{n}^{k+1}(\phi_{n}^{k}(x)) & \text{if } x = a_{n-1,n} \\ I_{n}^{k+1}(\phi_{n}^{k}(x)) & \text{if } x = a_{p,n} \text{ with } 2 \leqslant p \leqslant n-2 \\ & \text{and } a_{p,n} \in m, \\ \otimes & \text{otherwise,} \end{cases}$$

with the convention  $I_n^n = I_n^0$ .

We summarize the construction of the partial automaton  $P_n^*$  on the following diagram.



An arrow labelled  $T_k$  represents the set of transitions  $\mu_n^*((k, s, m), \phi_n^k(a_{p,n}))$ .

**Lemma 5.7.** The partial automaton  $P_n^*$  recognizes the language  $\Pi(R_n^*)$ .

*Proof.* Let  $\mathcal{A}^*$  be the closure of  $P_n^*$  and w be a non-empty  $A_n$ -word. There exists a unique sequence  $(w_b, ..., w_1)$  of  $A_{n-1}$ -words such that  $w_b \neq \varepsilon$ , w is equal to

$$\phi_n^{b-1}(w_b) \cdot \ldots \cdot \phi_n(w_2) \cdot w_1$$

and for all *i*, the word  $\phi_n^i(w_i)$  is the maximal suffix of  $\phi_n^{b-1}(w_b) \cdot \ldots \cdot \phi_n^i(w_i)$ belonging to  $\phi_n^i(A_{n-1}^*)$ . By definition of  $I_n^*$ , the word  $\Pi(w)$  is accepted by  $P_n^*$ only if *w* ends with a letter  $a_{p,n}$  for some *p*.

We assume now that w is such a word. Thus the first integer j such that  $w_j$  is non-empty is 2 or 3. More precisely, we have j = 2 if p > 1 and j = 3 if p = 1. In both cases, after reading the first letter of  $\Pi(w)$ , the automaton  $\mathcal{A}^*$  is in state  $s \in S_n^j$ . The automaton reaches a state outside of  $S_n^j$  if it goes to the state  $\otimes$  or if it reads a letter outside of  $\phi_n^{j-1}(A_{n-1})$ , *i.e.*, a letter of  $\phi_n^j(\Pi(w_{j+1}))$ . This is a general principle: after reading a letter of  $\phi_n^{i-1}(\Pi(w_i))$  the automaton  $\mathcal{A}$  is in state (t, s, m) with  $t = i \mod n$ . By construction of  $P_n^t$ , the word  $\phi_n^{i-1}(\Pi(w_i))$  provides an accepted state if and only if  $w_i$  is a word of  $R_{n-1}$ .

At this point we have shown that  $\Pi(w)$  is accepted by  $\mathcal{A}$  only if w is empty or if w satisfies  $w^{\#} = a_{p,n}$  together with Conditions (i), (ii) and (iii) of Corollary 4.9. Let i be in [j, k-1] and assume that  $\mathcal{A}$  is in an acceptable state (t, s, m) with  $t = i \mod n$  after reading the word

$$\Pi(\phi_n^{i-1}(w_i)\cdot\ldots\cdot\phi_n(w_2)\cdot w_1).$$

We denote by x the letter  $w_{i+1}^{\#}$ . By construction of  $w_{i+1}$ , the letter x does not belong to  $\phi_n^{i-1}(A_{n-1})$  and so  $x = \phi_n^i(a_{p,n})$  for some p. By definition of  $\mu_n^*$ we have  $\mu_n^*((t, s, m), \phi_n^i(a_{p,n})) \neq \otimes$  if and only if p = n - 1 or if  $p \in [2, n - 2]$ together with  $a_{p,n} \in m$ . As, by construction of  $P_n^t$ , we have  $a_{p,n} \in m$  if and only if  $w_i$  contains an  $a_{p,n}$ -barrier, Condition (*iv*) of Corollary 4.9 is satisfied.

Eventually, by Corollary 4.9, the word  $\Pi(w)$  is accepted by  $\mathcal{A}$  if and only if w belongs to  $R_n^*$ .



FIGURE 8. Partial automaton recognizing the language  $\Pi(R_4^*)$ .

Assume that an automaton  $\mathcal{A}_{n-1} = (S_{n-1} \cup \{\otimes\}, A_{n-1}, \mu_{n-1}, S_{n-1}, i, \otimes)$ recognizing the language  $\Pi(R_{n-1})$  for  $n \ge 4$  is given. Plugging  $\mathcal{A}_{n-1}$  to the partial automaton  $P_n^* = (S_n^* \cup \{\otimes\}, A_n, \mu_n^*, S_n^*, I_n^*)$  we construct an automaton  $\mathcal{A}_n = (S_n \cup \{\otimes\}, A_n, \mu_n, S_n, i, \otimes)$  defined by  $S_n = S_{n-1} \sqcup S_n^*$  and

$$\mu_n(s,x) = \begin{cases} \mu_{n-1}(s,x) & \text{if } s \in S_{n-1} \text{ and } x \in A_{n-1}, \\ I_n^*(x) & \text{if } s \in S_{n-1} \text{ and } x \in A_n \setminus A_{n-1}, \\ \mu_n^*(s,x) & \text{if } s \in S_n^*. \end{cases}$$

**Proposition 5.8.** If  $\mathcal{A}_{n-1}$  recognizes  $\Pi(R_{n-1})$ , the automaton  $\mathcal{A}_n$  recognizes the language  $\Pi(R_n)$ .

*Proof.* Let w be an  $A_n$ -word,  $w_1$  be the maximal suffix of w which is an  $A_{n-1}$ -word and w' be the corresponding prefix. By Corollary 4.9, the word w is rotating if and only if  $w_1$  and w' are. By construction, the automaton  $\mathcal{A}_n$  is in acceptable state after reading  $\Pi(w_1)$  if and only if  $w_1$  is an (n-1)-rotating word. Hence w is accepted only if  $w_1$  is rotating. Assume this is



FIGURE 9. Automaton  $\mathcal{A}_4$  for the language  $\Pi(R_4)$ .

the case. By Lemma 5.7 the automaton  $\mathcal{A}_n$  is in an acceptable state after reading  $\Pi(w')$  if and only if the word w' is rotating. Eventually the word  $\Pi(w)$  is accepted by  $\mathcal{A}$  if and only if  $w_1$  and w' are both rotating, which is equivalent to w is rotating.

By Proposition 5.8, the language  $\Pi(R_n)$  is regular and so we obtain:

**Theorem 5.9.** The language of n-rotating words  $R_n$  is regular.

# 6. Application to $\sigma$ -definite representative

It it well known since [6] that Artin's braid groups are orderable, by an ordering that enjoys many remarkable properties [9]. The existence of this ordering rests on the property that every non-trivial braid admits a  $\sigma$ -definite representative, *i.e.*, a braid word w in standard Artin generator  $\sigma_1, \sigma_2, \ldots$  in which the generator  $\sigma_i$  with highest index i occurs only positively (no  $\sigma_i^{-1}$ ), in which case w is called  $\sigma_i$ -positive, or only negatively (no  $\sigma_i$ ), in which case w is called  $\sigma_i$ -negative. Thanks to the word homomorphism defined by  $a_{p,q} \mapsto \sigma_p \ldots \sigma_{q-1} \sigma_q \sigma_{q-1}^{-1} \ldots \sigma_p^{-1}$ , the notions of  $\sigma$ -definite,  $\sigma_i$ -positive and  $\sigma_i$ -negative words extend naturally to  $A_n^{\pm}$ -words. In [13] a particular  $\sigma$ -definite representative is obtained for every non-trivial braid from its rotating normal form.

We say that a braid has index  $k \ge 2$  if it belongs to  $B_k$  and not to  $B_{k-1}$  with the convention  $B_1 = \{1\}$ . Each representative, in standard Artin generator, of a braid of index  $k \ge 2$  must contain a letter  $\sigma_{k-1}$  or a letter  $\sigma_{k-1}^{-1}$ . Hence such a braid is either  $\sigma_{k-1}$ -positive or  $\sigma_{k-1}$ -negative.

First we recall how uniquely express a braid as a quotient.

**Proposition 6.1** (Proposition 6.1 of [13]). Assume  $n \ge 3$ . Each braid  $\beta$  of  $B_n$  admits a unique expression  $\delta_n^{-t} w$  where t is a non-negative integer, w is an n-rotating normal word, and the braid  $\overline{w}$  is not left-divisible by  $\delta_n$  unless t is zero.

A direct adaptation to dual braid context of Proposition 4.4 of [14] gives:

**Proposition 6.2.** Assume  $n \ge 3$  and  $\beta$  is a braid of  $B_n^{+*}$ . Let t be a positive integer and  $(w_b, ..., w_1)$  the  $\phi_n$ -splitting of  $\beta$ . If  $t \ge b-1$  holds then the quotient  $\delta_n^{-t}\beta$  is represented by the  $\sigma_{n-1}$ -negative  $A_n^{\pm}$ -word

$$\delta_n^{-t+b+1} w_b \,\delta_n^{-1} \,w_{b-1} \,\delta_n^{-1} \dots w_2 \,\delta_n^{-1} \,w_1, \tag{13}$$

where  $\delta_n$  is written as the word  $a_{1,2} a_{2,3} \dots a_{n-1,n}$ . Moreover if t < b-1 then the quotient  $\delta_n^{-t} \beta$  is not  $\sigma_{n-1}$ -negative.

Proposition 6.2 gives specific  $\sigma$ -definite representative to each  $\sigma_{n-1}$ -negative braid of  $B_n$ . Unfortunately this representative word is not unique.

**Definition 6.3.** We denote by  $\mathbb{R}_n^{\sigma^{-1}}$  the words of the form (13) such that the braid  $\phi_n^{b-1}(w_b) \cdot \ldots \cdot \phi_n(w_2) \cdot w_1$  is not left-divisible by  $\delta_n$ .

Assume  $(w_b, ..., w_1)$  is a sequence of  $A_{n-1}$ -words satisfying Conditions (i)-(iv) of Corollary 4.9. In section 5 we have explicitly constructed by induction a finite-state automaton recognizing the mirror of the word  $\phi_n^{b-1}(w_b) \cdots w_1$ . This construction rests on verification of conditions of Corollary 4.9 and on the detection of transition between words  $\phi_n^{i-1}(w_i)$  and  $\phi_n^i(w_{i+1})$ . Thus, if we add a letter **\$** to the alphabet  $A_n$  we can construct a finite-state automaton recognizing the mirrors of words  $w_b \$ w_{b-1} \$ ... \$ w_2 \$ w_1$ .

Since  $A_n$  contains no negative letters, the word  $\delta_n^{-1}$  can play the role of the letter **\$**. Hence their exists a finite-state automaton recognizing the reverse

of the word

$$w_b \, \delta_n^{-1} \, w_{b-1} \, \delta_n^{-1} \dots w_2 \, \delta_n^{-1} \, w_1$$

We then obtain that mirror of words of (13) constitute a regular language.

**Proposition 6.4.** For  $n \ge 2$ , the language  $R_n^{\sigma^{-1}}$  is regular.

Proof. The result is immediate for n = 2. Assume  $n \ge 3$ . We denote by  $W_n$  the set of words as in (13). As explained previously the language  $W_n$  is regular. Let  $\mathcal{B}_n$  be a finite-state automaton recognizing  $W_n$  and let w be a word as in (13). Following Section 9.2 of [11] we can modify  $\mathcal{B}_n$  such that it memorize the maximal simple element head(w) of  $\mathcal{B}_n^{+*}$  that left-divides the dual braid  $\phi_n^{b-1}(w_b) \cdot \ldots \cdot \phi_n(w_2) \cdot w_1$ . Accepting only words such that head(w) is different from  $\delta_n$  we obtain an finite-sate automaton which recognize the language  $\mathbb{R}_n^{\sigma^{-1}}$ .

Since the inverse of  $\sigma_{n-1}$ -positive braid is a  $\sigma_{n-1}$ -negative braid, each braid of index n admits a representative either in  $\mathbf{R}_n^{\sigma^{-1}}$  or in its inverse  $\mathbf{R}_n^{\sigma^+} = \{w^{-1} | w \in \mathbf{R}_n^{\sigma^{-1}}\}$ . As  $\mathbf{R}_n^{\sigma^{-1}}$  is regular the language  $\mathbf{R}_n^{\sigma^+}$  is also regular. Defining two regular languages  $\mathbf{R}_2^{\sigma^{-1}} = \{a_{1,2}^k | k < 0\}$  and  $\mathbf{R}_2^{\sigma^+} = \{a_{1,2}^k | k > 0\}$  we immediately obtain:

**Proposition 6.5.** Assume  $n \ge 2$ . Then each braid of  $B_n$  admits a unique  $\sigma$ -definite representative lying in

$$S_n = \{\varepsilon\} \sqcup \bigsqcup_{k=2}^n \left( R_k^{\sigma^{-1}} \sqcup R_k^{\sigma^+} \right).$$
(14)

Since the union of regular languages is also regular, we obtain:

**Theorem 6.6.** For  $n \ge 2$  the language  $S_n$  of  $\sigma$ -definite representative of braids of  $B_n$  is regular.

## 7. Automaticity

Using syntactical characterization of rotating words we have proved that the language of *n*-rotating words is regular. For W a finite-state automaton, we denote by L(W) the language recognized by W.

From [5] and [11] we introduce the following definition:

**Definition 7.1.** Let M be a monoid. A right automatic structure, resp. left automatic structure, on M consists of a set A of generators of M, a finite-state automaton W over A, and finite-state automata  $M_x$  over (A, A), for  $x \in A \cup \{\varepsilon\}$ , satisfying the following conditions:

(i) the map  $\pi: L(W) \to M$  is surjective,

(*ii*) for  $x \in A \cup \{\varepsilon\}$ , we have  $(u, v) \in L(M_x)$  if and only if  $\overline{ux} = \overline{y}$ , resp.  $\overline{xu} = \overline{y}$ , and both u and v are elements of L(W).

A consequence of Theorem 2.3.5 of [11] is that automaticity for groups is equivalent to *fellow traveller property*. As pointed in [5] the situation is more complicated for monoids. But with Definition 3.11 of [5] for fellow traveller property on monoids we have, Proposition 3.12 of [5], that right (resp. left) automaticity of monoids implies right (resp. left) fellow traveller property of monoids.

As pointed by the anonymous referee the rotating normal form is not right automatic (or even asynchronously right automatic) for  $n \ge 4$ . For all integers  $k \ge 0$ , we define two words

$$w = (a_{2,3}a_{1,2}a_{1,3})^k a_{1,2}^{3k}$$
, and  $w' = (a_{1,3}a_{2,3}a_{1,2})^k a_{1,4}a_{2,3}^{3k}$ .

The  $\phi_4$ -splitting of w is (w) and w is 3-rotating by Proposition 5.1 and so 4rotating. The  $\phi_4$ -splitting of w' is  $(\underbrace{a_{2,3}, \ldots, a_{2,3}}_{3k+1}, 1, a_{2,3}^{3k})$ . Using Corollary 4.9

$$3k+$$

we have that w' is 4-rotating.

We observe that  $a_{2,3}a_{1,2}a_{1,3}a_{1,2}^3 \equiv a_{1,3}a_{2,3}a_{1,2}a_{2,3}^3 = \delta_3^3$ . Since by [3] the braid  $\delta_3^3$  is in the center of  $B_3^{+*}$  we have  $w \equiv \delta_3^{3k}$  and

$$w' \equiv (a_{1,3}a_{2,3}a_{1,2})^k a_{2,3}^k a_{1,4} \equiv \delta_3^{3k} a_{1,4} \equiv w a_{1,4}.$$

Finally, w and w' end with 3k copies of two different letters of  $A_4$ , hence the (asynchronous) fellow traveller property is falsified. As a consequence we obtain that the *n*-rotating normal form is not automatic for n = 4 (and for n > 4 as well).

At this time we don't know if the rotating normal form is left automatic. We think that contrary to the previous example, if w is an *n*-rotating word and x is an  $A_n$ -letter then the existence of barrier in x prevents migration to far on the right of x during the computation of rotating normal form of the braid represented by xw.

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