



**HAL**  
open science

# Automatic choice of the threshold of a grain filter via Galton-Watson trees. Application to granite cracks detection

Romain Abraham, Maïtine Bergounioux, Pierre Debs

## ► To cite this version:

Romain Abraham, Maïtine Bergounioux, Pierre Debs. Automatic choice of the threshold of a grain filter via Galton-Watson trees. Application to granite cracks detection. 2016. hal-01337551v1

**HAL Id: hal-01337551**

**<https://hal.science/hal-01337551v1>**

Preprint submitted on 27 Jun 2016 (v1), last revised 10 Jan 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Automatic choice of the threshold of a grain filter via Galton-Watson trees. Application to granite cracks detection

Romain Abraham, Maitine Bergounioux, Pierre Debs

*Laboratoire MAPMO, CNRS, UMR 7349, Fédération Denis Poisson, FR 2964, Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France*

---

## Abstract

The goal of this paper is the presentation of post-processing method allowing to remove impulse noise in binary images, while preserving thin structures. We use a grain filter as in [5]. We propose a method to automatically determine the required threshold using Galton-Watson processes. We present numerical results using X-tomography images of a granite block (2D and 3D) to illustrate how we get rid of the noise while cracks are kept.

*Keywords:* Image processing, Galton-Watson, grain filters, cracks.

*2010 MSC:* 60J80, 68U10, 94A12

---

## 1. Introduction

A challenging issue in imaging processing is the identification of thin structures. It may be blood vessels, leaf veins or streets in a satellite or aerial image. From a mathematical point of view these structures have a zero Lebesgue measure and we have to use the Hausdorff measure instead. From a practical point of view, these structures may be not viewed once the discretization has been done. There are many methods to achieve this goal both from theoretical and numerical points of view (as the morphological skeletonization for example). However, they may be not very satisfactory since the images are very often corrupted with a high level of noise that, in addition, may not be a Gaussian one.

The goal of this paper is not the presentation of an image processing model that would identify the structures better than the existing ones but rather give a *post-processing* method allowing to remove any residual noise effect. Therefore,

---

*Email addresses:* [romain.abraham@univ-orleans.fr](mailto:romain.abraham@univ-orleans.fr) (Romain Abraham),  
[maitine.bergounioux@univ-orleans.fr](mailto:maitine.bergounioux@univ-orleans.fr) (Maitine Bergounioux),  
[pierre.debs@univ-orleans.fr](mailto:pierre.debs@univ-orleans.fr) (Pierre Debs)

we do not report on thin structures recovering techniques, and do not quote the related works.

The example we have in mind is the recovering of cracks inside a granite block (whose 2D slice is given by Figure 1.1) while removing the (impulse) noise. The right hand side image is the result of a pre-processing method that we present in Section 3, which allows to get rid of contours that are not cracks.

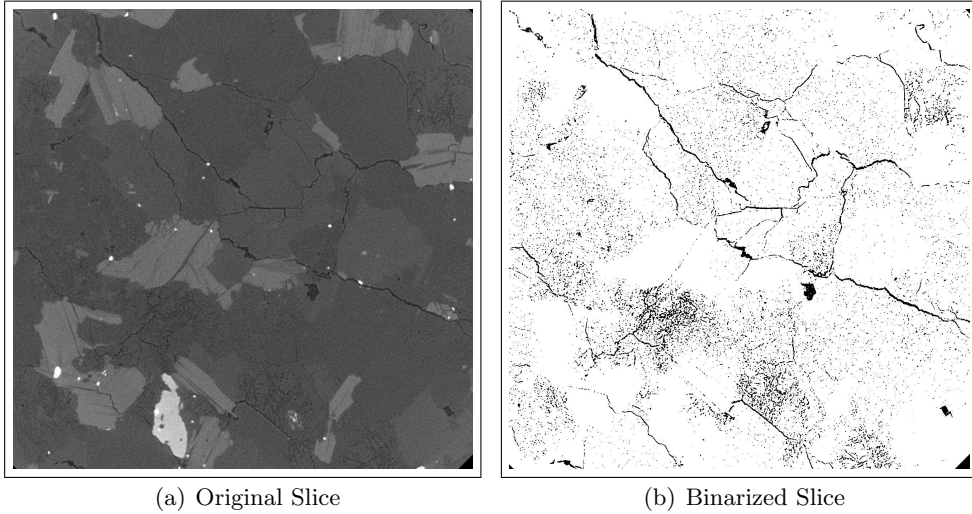


Figure 1.1: Original and binarized 2D slice of a 3D granite sample

There are many filters to address this problem. However, we focus on a *grain filter* [5]. Precisely, in what follows, we suppose that we have a binary image containing features we want to extract (namely thin structures as cracks). The image is corrupted by an impulse noise where every pixel is 1 with probability  $p$  and 0 with probability  $1 - p$  independently of the others. One feature that allows to distinguish cracks from the noise is that the cracks size is much larger than the connected components (that we define in the sequel) the noise may create. Therefore, a grain filter can be used in order to suppress the small connected components and keep the ones whose size is greater than some threshold  $a_0$ . The aim of this paper is to give a method to compute the threshold of the grain filter via a statistical hypothesis testing.

For that purpose, throughout the paper, we consider that the image contains only an impulse noise with parameter  $p$  (hypothesis  $\mathcal{H}_0$ ). Under  $\mathcal{H}_0$ , we denote by  $\mathcal{C}_0$  the connected component of the image that contains a given point, say the origin  $(0, 0)$ , and by  $\#\mathcal{C}_0$  its cardinal. We then fix a level  $\varepsilon > 0$  for the test and

find  $a_0$  such that, under  $\mathcal{H}_0$ ,

$$\mathbb{P}(\#\mathcal{C}_0 \geq a_0) \leq \varepsilon.$$

Keeping the connected components of size greater than  $a_0$ , we remove most of those due to the noise. The goal of this paper is to get an upper bound for the looked after probability, and hence to automatically compute the threshold  $a_0$ , by comparing the size of  $\mathcal{C}_0$  with the total population size of a Galton-Watson process.

We present the computation of the threshold in the next section and section 3 is devoted to the numerical realization (algorithms and numerical experimentation).

## 2. Computation of the threshold

### 2.1. Notations

If  $k < \ell$  are two integers, we denote by  $\llbracket k, \ell \rrbracket$  the set  $\{n \in \mathbb{Z}, k \leq n \leq \ell\}$ .

We work with the infinite norm on  $\mathbb{Z}^2$  i.e. if  $x_1 = (i_1, j_1)$ ,  $x_2 = (i_2, j_2)$  are two points of  $\mathbb{Z}^2$ , we set  $\|x_1 - x_2\| = \max(|i_1 - i_2|, |j_1 - j_2|)$ .

For  $x \in \mathbb{Z}^2$ ,  $\bar{B}(x)$  denotes the closed ball (with respect to the previous norm) centered at  $x$  with radius 1 and  $B(x)$  is the pointed ball  $\bar{B}(x) \setminus \{x\}$ . The elements of  $B(x)$  are called the neighbors of  $x$ .

For any finite subset  $A$  of  $\mathbb{Z}^2$ , we denote by  $\#A$  the cardinal of  $A$ .

**Definition 2.1.** *A set  $A \subset \mathbb{Z}^2$  is said to be connected if, for every  $x, y \in A$ , there exists a finite sequence  $(x_0, \dots, x_n)$  such that*

- $x_0 = x, x_n = y$ ;
- $\forall i \in \llbracket 0, n \rrbracket, x_i \in A$ ;
- $\forall i \in \llbracket 1, n \rrbracket, x_i \in B(x_{i-1})$ .

We fix a positive integer  $N$  and consider (for simplicity) the domain  $\Omega = \llbracket -N, N \rrbracket^2$ . A binary image on  $\Omega$  will be an application  $X : \Omega \mapsto \{0, 1\}$ . In all the paper, we always make the confusion between the binary image  $X$  and the set  $X^{-1}(1) = \{x \in \Omega, X(x) = 1\}$ , which is the set of the black pixels. In particular, we talk of the connected components of the image instead of the set  $X^{-1}(1)$ .

### 2.2. Galton-Watson process

A Galton-Watson process is a branching stochastic model introduced in 1873 by Francis Galton to describe population growth and is still the object of active research. We refer to [2] for a comprehensive exposition on this topic. We first give a formal definition of this process.

**Definition 2.2.** A Galton-Watson process is a stochastic process  $(Z_n)_{n \in \mathbb{N}}$  defined as follows: consider a probability measure  $\nu$  on the set of non-negative integers (called the offspring distribution) and a family  $(\xi_i^{(n)}, n \geq 0, i \geq 1)$  of i.i.d. random variables with distribution  $\nu$ . Then the process  $(Z_n)_{n \in \mathbb{N}}$  is defined recursively by

$$\begin{cases} Z_0 = 1, \\ Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)} \quad \text{for } n \geq 0 \end{cases}$$

with the convention that  $Z_{n+1} = 0$  if  $Z_n = 0$ .

To give an intuitive picture of that process  $(Z_n)_{n \in \mathbb{N}}$ , let us describe it informally as a population evolution model. We start with a single individual at generation 0. This individual gives birth to a random number  $\xi_1^{(0)}$  of offspring with distribution  $\nu$  which form generation 1. These offspring reproduce according to the same distribution  $\nu$  independently of each other and independently of the past and so on. Hence, the variable  $Z_n$  represents the population size at generation  $n$  whereas the variable  $\xi_i^{(n)}$  represents the number of offspring of the  $i$ -th individual of the  $n$ -th generation.

Of course, if the population dies out at some time  $n$ , then the population is zero for all further time: 0 is called an absorbing state for the process  $(Z_n)_{n \in \mathbb{N}}$ .

One of the initial questions raised by Francis Galton was the computation of the probability that the population dies out in finite time. This probability is called the *extinction probability* and is given by

$$\mathbb{P}(\exists n \in \mathbb{N}, Z_n = 0) = \lim_{n \rightarrow +\infty} \mathbb{P}(Z_n = 0).$$

To avoid trivialities, we suppose that  $\nu(0) > 0$  (if not, the extinction probability is 0) and that  $\nu(0) + \nu(1) < 1$  (if not, the extinction probability is clearly 1).

The answer to the question is given by the following result (see Chapter 1 Section 5 of [2])

**Proposition 2.1.** Let  $\xi$  be a random variable with distribution  $\nu$ . We denote by

$$\mu = \mathbb{E}[\xi] = \sum_{k=0}^{+\infty} k\nu(k)$$

the mean number of offspring, and by

$$F(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{+\infty} \nu(k)s^k \quad \text{for } s \in [-1, 1],$$

the generating function of  $\nu$ .

Then, the extinction probability of the process  $(Z_n)_{n \geq 0}$  is given by the smallest non-negative root  $q$  of the equation  $s = F(s)$ .

In particular, if  $\mu \leq 1$ ,  $q = 1$  (extinction arises almost surely) whereas if  $\mu > 1$ , then  $q < 1$  (the population has a positive probability for living forever).

The offspring distribution  $\nu$  and the Galton-Watson process  $(Z_n)_{n \geq 0}$  are called sub-critical (resp. critical, super-critical) if  $\mu < 1$  (resp.  $\mu = 1$ ,  $\mu > 1$ ).

### 2.3. Stochastic domination

Let  $m$  be the number of neighbors of one pixel. With our definition of  $B(x)$ , we have  $m = 8$  but the method remains valid for a general neighborhood. Let  $p \in (0, 1)$  and  $(Z_n)_{n \geq 0}$  a Galton-Watson process with the binomial  $\mathcal{B}(m, p)$  distribution as offspring distribution  $\nu$ , that is, for  $0 \leq k \leq m$ ,

$$\nu(k) = \binom{m}{k} p^k (1-p)^{m-k}.$$

According to Proposition 2.1, in order to obtain a sub-critical Galton-Watson process, we assume that  $mp < 1$ . We denote by  $\sharp Z$  the total population size:

$$\sharp Z = \sum_{n=0}^{+\infty} Z_n.$$

**Theorem 2.3.** *Under  $\mathcal{H}_0$ , the size  $\sharp \mathcal{C}_0$  of the connected component  $\mathcal{C}_0$  (and thus of any connected component) is stochastically dominated by  $\sharp Z$ , that is:*

$$\forall k \in \mathbb{N}, \quad \mathbb{P}(\sharp \mathcal{C}_0 \geq k) \leq \mathbb{P}(\sharp Z \geq k).$$

*Proof.* The main ideas of the proof are taken from [8] where the size of a connected component that arises in continuum percolation is compared to the size of a Galton-Watson process with Poisson offspring distribution.

Let  $(X_{(i,j)}, (i,j) \in \mathbb{Z}^2)$  be a family of i.i.d. Bernoulli random variables with parameter  $p$ . This models the noise of the image under  $\mathcal{H}_0$ . Remark that if the stochastic domination is true for an infinite image, it is a fortiori true for an image defined on  $\llbracket -N, N \rrbracket^2$ . If  $A$  is a subset of  $\mathbb{Z}^2$ , we denote by

$$X_A = \{(i,j) \in A, X_{(i,j)} = 1\}.$$

Then,  $\sharp X_A$  is distributed according to a binomial distribution with parameters  $p$  and  $\sharp A$ . Moreover, if  $A$  and  $B$  are disjoint subsets of  $\mathbb{Z}^2$ , then  $X_A$  and  $X_B$  are independent.

In order to label the points of some subset in a unique way, we define a total order (denoted by  $\leq$ ) on  $\mathbb{Z}^2$ . Let  $x_1, x_2 \in \mathbb{Z}^2$ . If  $\|x_1\| < \|x_2\|$  then we set  $x_1 \leq x_2$ .

To compare two points with the same norm, we label the  $8n$  points of norm  $n$  as on Figure 2.1.

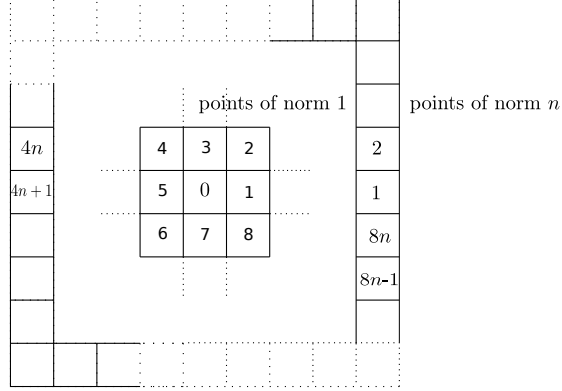


Figure 2.1: Labeling of points of norm  $n$

Then, if  $\|x_1\| = \|x_2\|$ , we say that  $x_1 \leq x_2$  if the label of  $x_1$  is less than the label of  $x_2$ .

We first construct the connected component  $\mathcal{C}_0$  recursively. We define a sequence  $(\tilde{W}_n)_{n \geq 0}$  of subsets of  $\mathbb{Z}^2$  such that  $\mathcal{C}_0 = \bigcup_{n \geq 0} \tilde{W}_n$  in the following way:

- $\tilde{W}_0 = \{(0, 0)\}$ .
- Suppose that  $\tilde{W}_0, \dots, \tilde{W}_n$  are constructed with  $\tilde{W}_n = \{x_1, \dots, x_k\}$  labeled in increasing order. Then, we set

$$\tilde{W}_{n+1} = \bigcup_{\ell=1}^k X_{C_\ell}$$

where

$$C_\ell = B(x_\ell) \setminus \left( \bigcup_{j=1}^{\ell-1} \bar{B}(x_j) \bigcup_{i=1}^n \bigcup_{x \in \tilde{W}_i} \bar{B}(x) \right),$$

with the convention  $\bigcup_{j=1}^0 \bar{B}(x_j) = \emptyset$ .

- If  $\tilde{W}_n = \emptyset$ , then we set  $\tilde{W}_k = \emptyset$  for every  $k > n$ .

Remark that, by construction, the  $C_\ell$  are pairwise disjoint.

In a second step, we consider a branching random walk  $(W_n)_{n \geq 0}$  on  $\mathbb{Z}^2$  (we refer to [10] for more details on branching random walks). This process takes its values in the set  $\bigcup_{n \geq 0} (\mathbb{Z}^2)^n$  of finite sequences of points of  $\mathbb{Z}^2$  (with the convention  $(\mathbb{Z}^2)^0 = \{\emptyset\}$ ) and is also defined recursively as follows:

- $W_0 = (0, 0)$ .
- Let us suppose that  $W_n$  is given and is non-empty, say  $W_n = (x_1^n, \dots, x_{k_n}^n)$ . For every  $i \leq k_n$ , we consider the 8 neighbors of  $x_i^n$  (ranked as on Figure 2.2)  $(y_1, \dots, y_8)$  and consider the sub-sequence  $W_{n+1}^i$  formed by taking each  $y_j$  with probability  $p$ , independently of the others (and independently of the other points  $x_j^n$ ,  $i \neq j$ ). Then  $W_{n+1}$  is the concatenation of all the sub-sequences  $(W_{n+1}^i, 1 \leq i \leq k_n)$ . Remark that a pixel may appear several times in the sequence  $W_{n+1}$ .

4	3	2
5	$x$	1
6	7	8

Figure 2.2: Labeling of the neighbors of a point  $x$

- If  $W_n = \emptyset$ , then  $W_{n+1} = \emptyset$ .

The subsequence  $W_{n+1}^i$  may be seen as the offspring of the pixel  $x_i$  and  $\#W_{n+1}^i$  is distributed according to a binomial  $\mathcal{B}(8, p)$  distribution. Therefore, the process  $(\#W_{n+1})_{n \geq 0}$  is distributed as a Galton-Watson process starting from a single individual and with offspring distribution  $\mathcal{B}(8, p)$ .

We now define a pruning procedure for the process  $(W_n)_{n \geq 0}$  which defines a set-valued process  $(\hat{W}_n)_{n \geq 0}$  as follows:

- $\hat{W}_0 = \{(0, 0)\}$ .
- Suppose that we have constructed  $\hat{W}_0, \hat{W}_1, \dots, \hat{W}_n$  where each set  $\hat{W}_i$  is composed of elements of  $W_i$ , with  $\hat{W}_n = \{\hat{x}_1^n, \dots, \hat{x}_k^n\}$  (ranked in increasing order). To every  $\hat{x}_i^n$  there corresponds a point  $x_{j_i}^n$  of  $W_n$  and we set

$$\hat{W}_{n+1}^i = W_{n+1}^{j_i} \setminus \left( \bigcup_{\ell=1}^{i-1} \bar{B}(\hat{x}_\ell^n) \bigcup_{r=1}^n \bigcup_{x \in \hat{W}_r} \bar{B}(x) \right)$$



and then

$$\hat{W}_{n+1} = \bigcup_{i=1}^k \hat{W}_{n+1}^i.$$

- If  $\hat{W}_n = \emptyset$ , then  $\hat{W}_{n+1} = \emptyset$ .

It is clear by construction (pruning procedure) that  $\#\hat{W}_n \leq \#W_n$  for every  $n \geq 0$  and that the process  $(\hat{W}_n)_{n \geq 0}$  is distributed as the process  $(\tilde{W}_n)_{n \geq 0}$  (since the recursive formulas are same), which ends the proof.  $\square$

#### 2.4. Computation of the threshold

Let  $(Z_n)_{n \geq 0}$  be a sub-critical Galton-Watson process with offspring distribution a binomial  $\mathcal{B}(m, p)$ . For this offspring distribution to be sub-critical, recall that we must suppose that

$$mp < 1. \quad (2.1)$$

We need an upper bound for the probability  $\mathbb{P}(\#Z \geq k)$ . For this purpose, we use Dwass formula that relates the total population size of any sub-critical (or critical) Galton-Watson process to the associated Lukasiewicz random walk, see [6]:

**Proposition 2.2.** *Let  $(Z_n)_{n \geq 0}$  be a critical or sub-critical Galton-Watson process with offspring distribution  $\nu$ . Let  $(X_n)_{n \geq 1}$  be a sequence of i.i.d. random variables with distribution  $\nu$ . We set for every  $n \geq 1$*

$$S_n = \sum_{i=1}^n X_i.$$

Then we have

$$\mathbb{P}(\#Z = n) = \frac{1}{n} \mathbb{P}(S_n = n - 1).$$

Recall that  $\nu$  is the binomial  $\mathcal{B}(m, p)$  distribution; then for every  $n \geq 1$ ,  $S_n$  is distributed according to a binomial  $\mathcal{B}(mn, p)$  distribution (as the sum of i.i.d  $\mathcal{B}(m, p)$  random variables). Therefore

$$\mathbb{P}(\#Z = n) = \frac{1}{n} \binom{mn}{n-1} p^{n-1} (1-p)^{mn-n+1}. \quad (2.2)$$

This exact formula allows us to compute the looked after threshold:

**Proposition 2.3.** *We set*

$$q = \frac{m^m p (1-p)^{m-1}}{(m-1)^{m-1}}, \text{ and } C = \sqrt{\left(\frac{m}{m-1}\right) \frac{e(1-p)}{pK^2} \frac{1}{1-q}} \quad (2.3)$$

with  $K = e^{\frac{3}{2}(1-\ln \frac{3}{2})}$ . Then, for every  $\varepsilon > 0$ , the threshold is given by

$$a_0 = \frac{\ln \frac{\varepsilon}{C}}{\ln q} \quad (2.4)$$

that is for every  $a \geq a_0$ , we have

$$\mathbb{P}(\#Z \geq a) \leq \varepsilon.$$

*Proof.* By Proposition 2.2 and Equation (2.2), we get for every  $a > 0$

$$\mathbb{P}(\#Z \geq a) \leq \sum_{n \geq a} \binom{mn}{n} p^{n-1} (1-p)^{mn-n+1}.$$

We now use the following bounds for the factorial function whose proof is postponed to the end of this section:

**Lemma 2.4.** *For every positive integer  $n$ , we have*

$$Kn^{n+\frac{1}{2}}e^{-n} \leq n! \leq n^{n+\frac{1}{2}}e^{-n}e \quad (2.5)$$

with  $K = \exp\left(\frac{3}{2}\left(1 - \ln \frac{3}{2}\right)\right)$ .

Using these bounds in the binomial coefficient, we get

$$\begin{aligned} \mathbb{P}(\#Z \geq a) &\leq \sum_{n \geq a} \frac{(mn)^{mn+\frac{1}{2}}e^{-mn}e}{K^2 n^{n+\frac{1}{2}}e^{-n} (mn-n)^{mn-n+\frac{1}{2}}e^{-mn+n}} p^{n-1} (1-p)^{mn-n+1} \\ &\leq \left(\frac{m}{m-1}\right)^{\frac{1}{2}} \frac{e(1-p)}{pK^2} \sum_{n \geq a} \left(\frac{m^m p(1-p)^{m-1}}{(m-1)^{m-1}}\right)^n n^{-\frac{1}{2}} \\ &\leq \left(\frac{m}{m-1}\right)^{\frac{1}{2}} \frac{e(1-p)}{pK^2} \sum_{n \geq a} q^n. \end{aligned}$$

Let us remark that the function  $x \mapsto x(1-x)^{m-1}$  reaches its maximum on the interval  $[0, 1]$  at  $x = \frac{1}{m}$ . As we supposed that  $p < \frac{1}{m}$ , then

$$q < \frac{m^m \frac{1}{m} \left(1 - \frac{1}{m}\right)^{m-1}}{(m-1)^{m-1}} = 1.$$

The previous sum is then finite and

$$\mathbb{P}(\#Z \geq a) \leq \left(\frac{m}{m-1}\right)^{\frac{1}{2}} \frac{e(1-p)}{pK^2} \frac{q^a}{1-q} := Cq^a.$$

Therefore (recall that  $q < 1$  which gives  $\ln q < 0$ ),

$$a \geq \frac{\ln \frac{\varepsilon}{C}}{\ln q} \Rightarrow Cq^a \leq \varepsilon \Rightarrow \mathbb{P}(\#Z \geq a) \leq \varepsilon.$$

□

*Proof of Lemma 2.4.* By the trapezoidal rule, as the logarithm function is concave, we have for every positive integer  $k$

$$\frac{\ln(k+1) + \ln k}{2} \leq \int_k^{k+1} \ln t \, dt.$$

This implies

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{\ln(k+1) + \ln k}{2} &\leq \int_1^n \ln t \, dt \Leftrightarrow \ln n! - \frac{\ln n}{2} \leq n \ln n - n + 1 \\ &\Leftrightarrow \ln n! \leq \left(n + \frac{1}{2}\right) \ln n - n + 1 \\ &\Leftrightarrow n! \leq n^{n+\frac{1}{2}} e^{-n} e \end{aligned}$$

which gives the upper bound.

Still using the concavity of the logarithm function, its graph is always below its tangent which is given at point  $k$  by the equation

$$y = \frac{1}{k}x - 1 + \ln k.$$

This yields

$$\begin{aligned} \int_{k-1/2}^{k+1/2} \ln t \, dt &\leq \frac{\ln k + 1/2k + \ln k - 1/2k}{2} = \ln k \\ &\Rightarrow \int_{3/2}^{n+1/2} \ln t \, dt \leq \sum_{k=2}^n \ln k = \ln n! \\ &\Rightarrow (n+1/2) \ln(n+1/2) - n - \frac{3}{2} \ln \frac{3}{2} + 1 \leq \ln n! \\ &\Rightarrow (n+1/2) \ln n + (n+1/2) \ln(1+1/2n) - n - \frac{3}{2} \ln \frac{3}{2} + 1 \leq \ln n! \end{aligned}$$

To get the lower bound, it suffices to check that

$$(n+1/2) \ln(1+1/2n) \geq 1/2$$

which comes immediately from the inequality

$$\forall x \geq 0, \ln(1+x) \geq x - \frac{x^2}{2}.$$

□

### 3. Application

#### 3.1. Methodology

As already mentioned, the application we have in mind is the identification of the cracks that appear in granite samples. More precisely, the sample has been imaged<sup>1</sup> via X-tomography process and we have to deal with a 3D image which size is 1287 x 1287 x 99. For more details on the geological context one can refer to [7, 9]. We present the analysis on a 2D slice for sake of simplicity (see Figure 1.1 (a)).

We first briefly present the preprocessing method:

1. Images are pre-processed with the method of [3, 4]. More precisely, we use a variational model that allows to split the image in three components denoted  $v$ ,  $u$  and  $w$ :
  - the  $v$  component is a smooth (continuous) function which distributional derivative is of bounded variation: it represents the image dynamic;
  - the  $u$  component is a bounded variation function and ideally, should be piecewise constant. It is called a *cartoon* component: the image contours are given by the jump set of  $u$ ;
  - the  $w$  component is a  $L^2$  function that represents the details and/or the noise.

The main inconvenient of variational methods is their dependence to some parameters. The result of the above decomposition depends on the choice of such parameters. We refer to [4] to get tuning rules for the two parameters  $\lambda$  and  $\mu$  to be used in this context. From a practical point of view we choose  $\mu \simeq 1.2\lambda$ .

The preprocessing method is robust enough with respect to the choice of parameters and we can get rid of most noise (involved in the  $w$  component). So we focus on the cartoon component  $u$  that involves all the contours. However, some noise still remains. The challenge is to distinguish the cracks that are thin *structures* from the noise (which is *unstructured*). However, it will be still quite difficult to recover micro-cracks.

2. Next, the  $u$  component has to be binarized. This is a delicate step since the threshold parameter is not easy to find. We decided to classical choose  $s = m(u) - \sigma(u)$  where  $m$  and  $\sigma$  are the mean value and the standard deviation respectively of  $u$ .

---

<sup>1</sup>We thank Olivier Rozenbaum, ISTO, Université d'Orléans, CNRS, BRGM

3. Last, we use the thresholded (binary) image to implement the method. We look for the connected components of the image and keep those which size is less than the value  $a$  given by (2.3) and (2.4). We assume that  $m$  (the number of neighbors in the chosen neighborhood) and  $\varepsilon$  (the test level) are given. A typical choice of  $\varepsilon$  is the inverse of the number of connected components as, by independence, the mean number of false positives would be 1 in this case. However, we decided to choose a smaller value of  $\varepsilon$  which improves the accuracy of the method, namely the inverse of the image size.

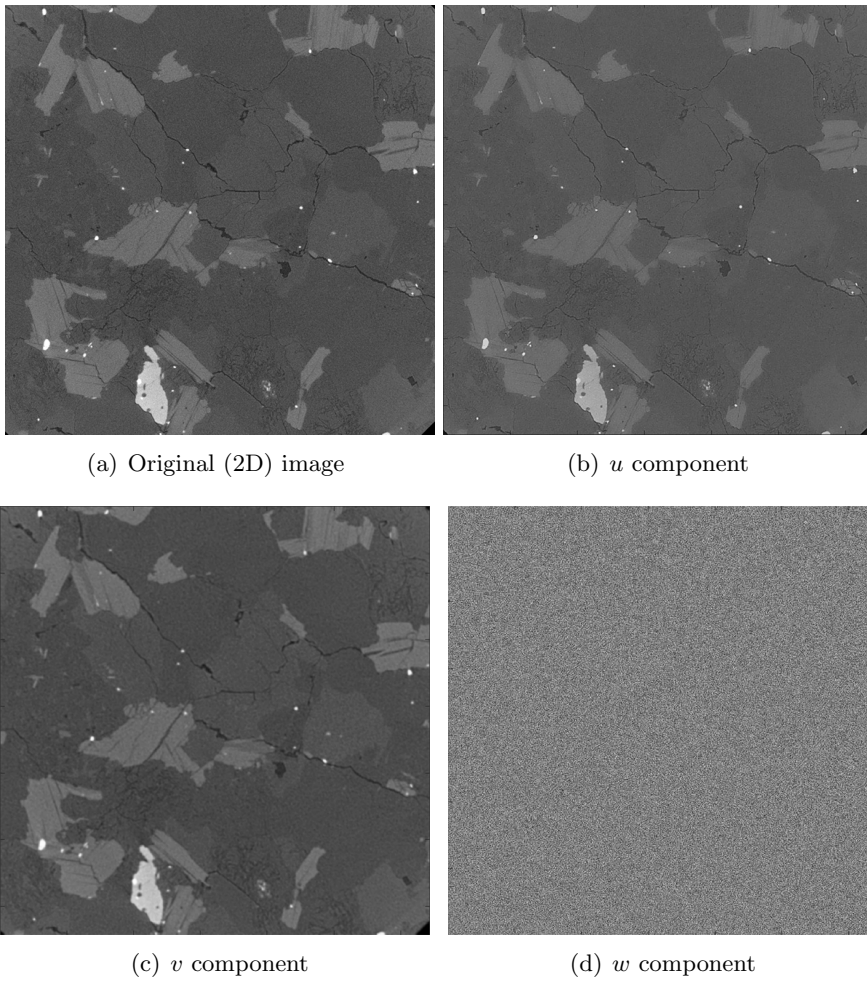


Figure 3.1: Image decomposition with  $\lambda = 5$  and  $\mu = 7$

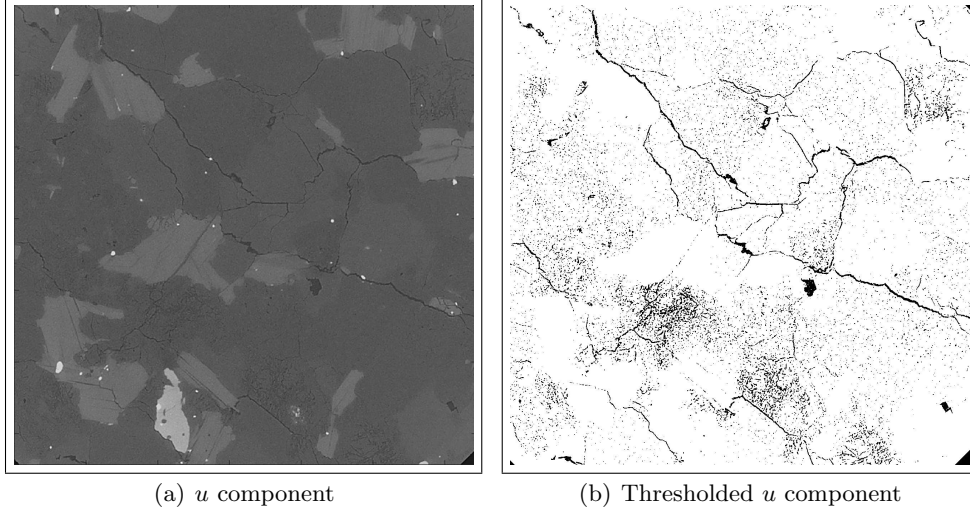


Figure 3.2: Threshold with  $s = m(u) - \sigma(u)$

### 3.2. Algorithms

The generic algorithm writes:

---

#### Algorithm 1

---

**Given data:** cartoon image  $u$  (function of bounded variation) given by the preprocessing decomposition,  $\varepsilon \simeq 1/M$  where  $M = (2N + 1)^2$  is the size of  $u$ ,  $m = 3^2 - 1 = 8$  for 2D images and  $m = 3^3 - 1 = 26$  or 3D images (using  $3 \times 3 \times 3$  neighborhoods.)

1. Binarization of  $u$ :  $s = m(u) - \sigma(u)$  where  $m(u)$  is the mean value of  $u$  and  $\sigma(u)$  the standard deviation. Define the binarized image  $\bar{u}$  by:

$$\bar{u} = \begin{cases} 0 & \text{if } u > s \\ 1 & \text{otherwise} \end{cases}$$

Compute

$$p \simeq \frac{\text{Size}(\bar{u} = 1)}{M}.$$

Compute the threshold value  $a$  with (2.3) and (2.4). Verify that  $m < 1/p$  otherwise set  $m = \min(1/p, 8)$  (2D case) or  $\min(1/p, 26)$  (3D case)

2. Find the connected components of  $\bar{u}$  (and their size)

3. Suppress the connected components of  $\bar{u}$  whose size is less than  $a$  to get the significant contours  $\bar{u}^*$

---

The probability  $p$  estimate is an important issue (that also determines the  $m$  value). We first estimate it using the number of black pixels of the binary image divided by its total size. This rough method gives an upper bound for  $p$  that is too large an estimate. Indeed, the number of black pixels includes both the noise and the cracks, whereas we only want to count the noisy pixels. So we adopt a *multi-scale* like strategy to perform successive updates of  $p$  once we have performed a more accurate selection of cracks pixels.

---

**Algorithm 2** Algorithm with iterative update of  $p$

---

**Given data:** cartoon image  $u$  (function of bounded variation) given by the preprocessing decomposition -  $\varepsilon \simeq 1/M$  where  $M$  is the size of  $u$ .

Binarization of  $u$  to get  $\bar{u}$ .

Find the connected components of  $\bar{u}$  (and their size)

0. **Initialization:**  $k = 0$ ,  $\bar{u}_0 = 0$ .

1. **Iteration k**

Compute  $\bar{u} - \bar{u}_k$  and

$$p_k \simeq \frac{\text{Size}((\bar{u} - \bar{u}_k) = 1)}{M} .$$

Compute the threshold value  $a_k$  with (2.3) and (2.4).

Suppress the connected components of  $\bar{u}$  which size are less than  $a_k$ : we get  $\bar{u}_{k+1}$

2. **Stopping criterion.** Stop for instance when the sequence  $(p_k)$  becomes stationary.

---

### 3.3. 2D numerical experimentation

#### 3.3.1. 2D results for $\lambda = 5$ and $\mu = 7$

The cartoon component has been binarized using  $m(u) - \sigma(u)$  as a thresholding parameter (see Figure 3.2). We present in Figure 3.3 the result of the first iteration of Algorithm 2:

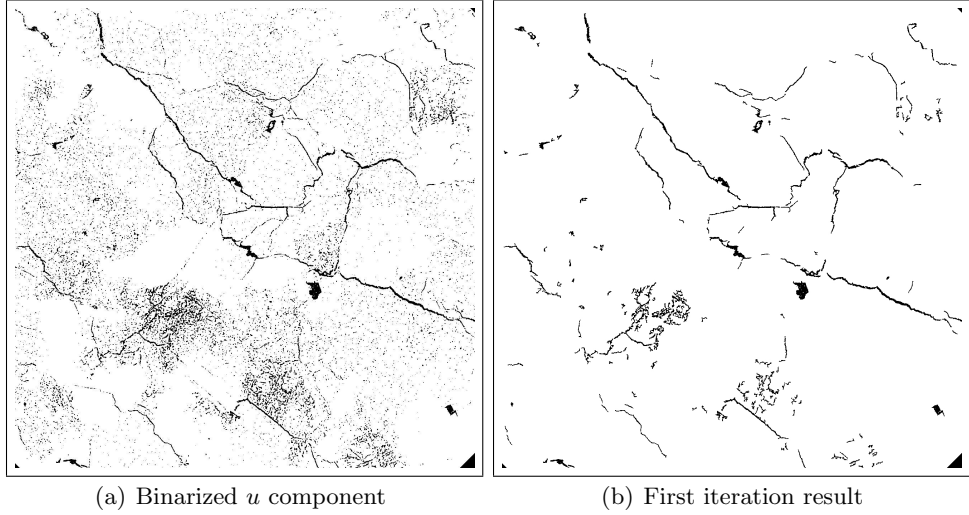


Figure 3.3: Results for iteration 1 (Algorithm 1):  $\varepsilon = 10^{-4}$  and  $m = 8$ . The probability estimate is  $p = 0.0514$ .

We report in Table 3.1 the evolution of  $p$  estimate. We see that the iterative process becomes stationary very quickly (5 iterations).

Iteration	1	2	3	4	5	6
$p_k$	0.0514	0.0304	0.0246	0.0224	0.0217	0.0217
$a_k$	39.27	18.16	14.70	13.51	13.18	13.18

Table 3.1:  $p_k$  and  $a_k$  values -  $m = 8$ ,  $\varepsilon = 10^{-4}$

Next figure shows the evolution of cracks identification:



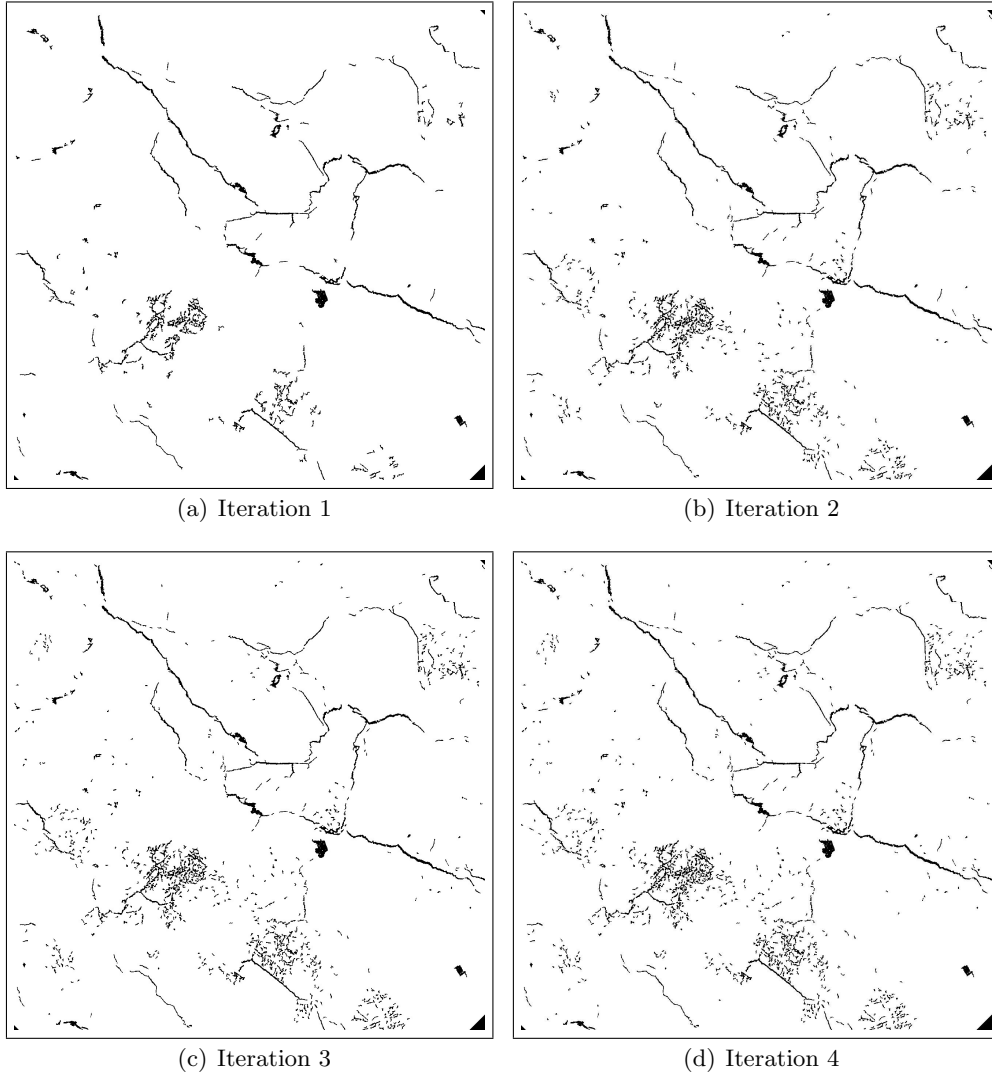


Figure 3.4: Iterations

### 3.3.2. Sensitivity to the preprocessing parameters $\lambda$ and $\mu$

We mentioned before that the preprocessing was robust with respect to the choice of  $\lambda$  and  $\mu$  parameters. These parameters should allow to put much noise as possible in the component  $w$ . They have to be large enough, but not too much to avoid the storage of too many contours in  $w$ . Though it is not our concern here, we briefly report on the influence of the pre-processing using another pair of parameters  $(\lambda, \mu)$ . For more details, one can refer to [3, 4]. A test has been

done with  $\lambda = 1, \mu = 2$  and  $\lambda = 5, \mu = 7$  (we used the last pair for numerical tests). Table 3.2 gives the evolution of  $p$  estimate with respect of the different parameters.

Iteration $k$	1	2	3	4	5
$p_k$ for $\lambda = 5, \mu = 7$	0.0514	0.0304	0.0246	0.0224	0.0217
$p_k$ for $\lambda = 1, \mu = 2$	0.0936	0.0847	0.0820	0.0812	0.0812

Table 3.2:  $p_k$  values -  $m = 8, \varepsilon = 10^{-4}$

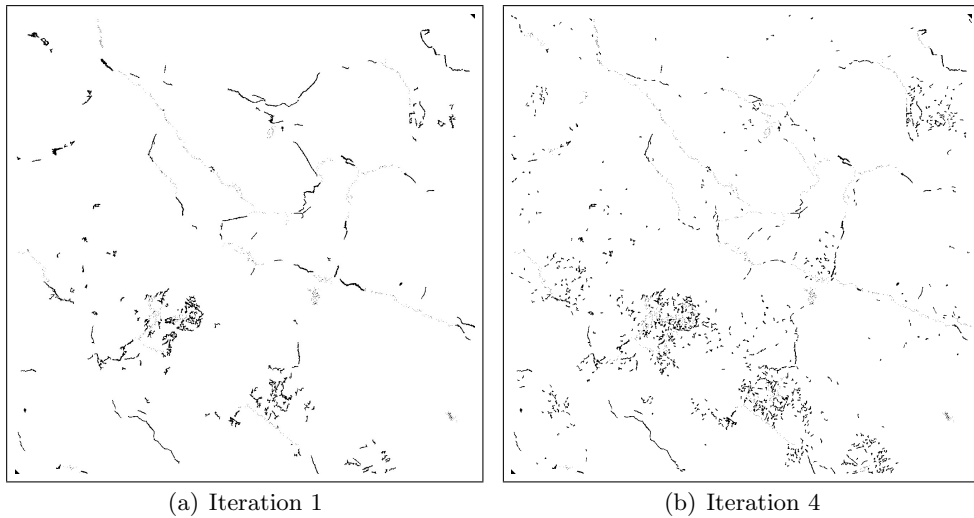
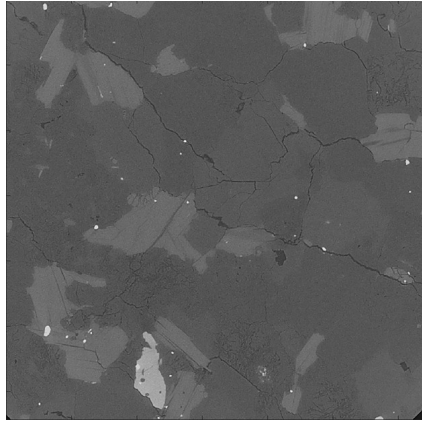
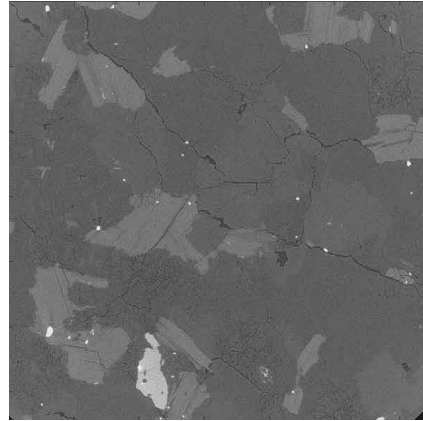


Figure 3.5: Difference of solutions for the two different preprocessing parameter pairs.

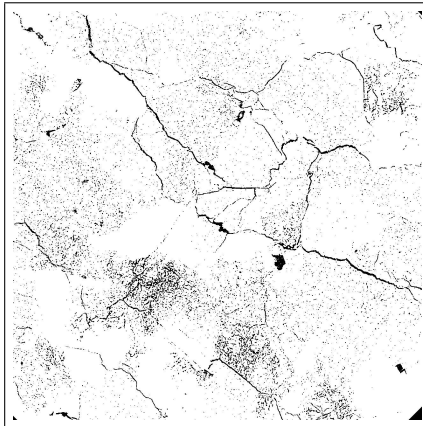
We observe that it is preferable to choose large parameters so that we get rid of much noise as possible. However, the difference of solutions gives a good snapshot of the micro cracks so that a suitable strategy may be the use of different preprocessing (with different parameters) to achieve a complete recovering of the cracks.



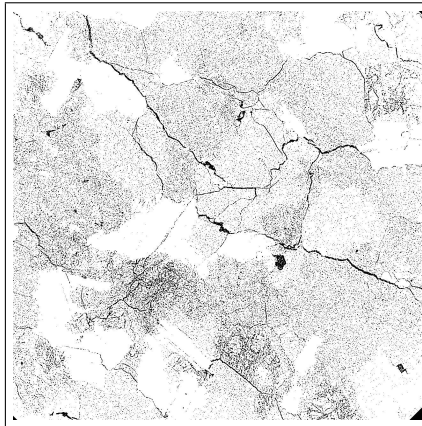
(a)  $u$  component for  $\lambda = 5, \mu = 7$



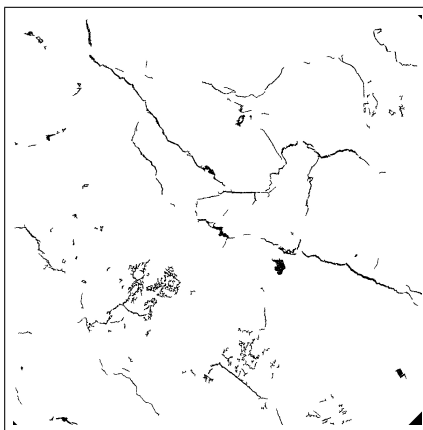
(b)  $u$  component for  $\lambda = 1, \mu = 2$



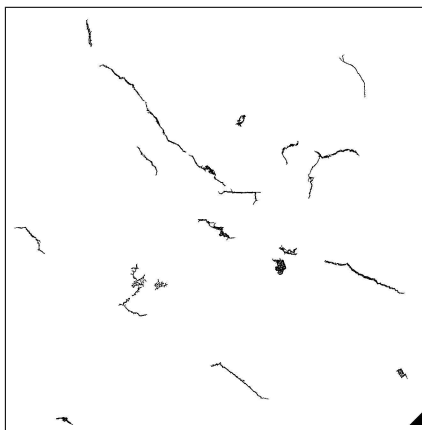
(c) Binarized  $u$  component for  $\lambda = 5, \mu = 7$



(d) Binarized  $u$  component for  $\lambda = 1, \mu = 2$



(e) Solution for  $\lambda = 5, \mu = 7$



(f) Solution for  $\lambda = 1, \mu = 2$

Figure 3.6: Comparison of solutions for two different preprocessing parameter pairs -  $\varepsilon = 10^{-4}$  and  $m = 8$ .

### 3.4. 3D - results

Though, we have presented the method with 2D images (whose domains are  $\llbracket -N, N \rrbracket^2$ ) we may extend the result to 3D stacks. Recall that we consider a 3D stack which size is 1287 x 1287 x 99. Next figure shows the solution

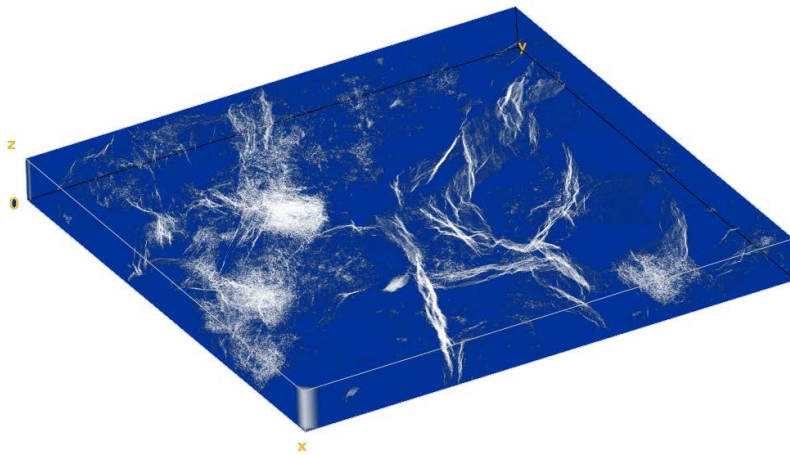


Figure 3.7: Solution

## 4. Conclusion

The automatic threshold we propose seems to be performing and we are able to recover a lot of information from the binary images. Though, we have presented the method on a 2D material image, it can be applied to any binary images arising from appropriate pre-processings. One may think on as angiography images for example.

However, we could say that the size of the connected components is rather a crude criterion for distinguishing cracks for noise and some other geometric characteristic (such as the diameter of the sets or the ratio between the diameter and “the width” whatever this means) would better discriminate our features. The key point is that we also want to get an upper bound for the tail probability  $\mathbb{P}(\#Z \geq k)$  that we indeed could have for the total population size but which is not obvious in general (see for instance [1] for recent works in this direction).

Another point that we did not really focus on is the estimation of the parameter  $p$  as it was not the central point of this work. Our method (especially after iterations) seems rather satisfactory but always overestimates this probability. Some more sophisticated methods have been developed in the literature any may be used to improve the results.

## References

- [1] Louigi Addario-Berry, Luc Devroye, and Svante Janson. Sub-gaussian tail bounds for the width and height of conditioned galton-watson trees. *Ann. Probab.*, 41(2):1072–1087, 03 2013. doi: 10.1214/12-AOP758. URL <http://dx.doi.org/10.1214/12-AOP758>.
- [2] K. B. Athreya and P. E. Ney. *Branching processes*. Dover Publications, Inc., Mineola, NY, 2004. ISBN 0-486-43474-5. Reprint of the 1972 original [Springer, New York; MR0373040].
- [3] M. Bergounioux. Mathematical Analysis of a Inf-Convolution Model for Image Processing. *J. Optim. Theory Appl.*, 168(1):1–21, 2016. ISSN 0022-3239. doi: 10.1007/s10957-015-0734-8. URL <http://dx.doi.org/10.1007/s10957-015-0734-8>.
- [4] M. Bergounioux. Second order decomposition model for image processing : numerical experimentation. to appear, Radons Series, Degruyter, 2016. URL <https://hal.archives-ouvertes.fr/hal-01077648>.
- [5] Vicent Caselles and Pascal Monasse. Grain filters. *J. Math. Imaging Vision*, 17(3):249–270, 2002. ISSN 0924-9907. doi: 10.1023/A:1020715626538. URL <https://dx-doi-org.budistant.univ-orleans.fr/10.1023/A:1020715626538>.
- [6] M. Dwass. The total progeny in a branching process. *J. Appl. Probab.*, 6: 682–686, 1969.
- [7] A Maurício, C Figueiredo, MF Pereira, C Alves, M Bergounioux, and O Rozenbaum. Assessment of stone heritage decay by x-ray computed microtomography: I-a case study of portuguese braga granite. *Microscopy and Microanalysis*, 21(S6):160–161, 2015.
- [8] Mathew D. Penrose. Conitnuum percolation and euclidian minimal spanning tree in high dimensions. *Annal of Applied Probability*, 6(2):528–544, 1996.
- [9] Olivier Rozenbaum, Maitine Bergounioux, António Maurício, Carlos Figueiredo, Carlos Alves, and Luc Barbanson. Versatile three-dimensional denoising and segmentation method of x-ray tomographic images: applications to geomaterials characterizations. preprint, 2016.
- [10] Zhan Shi. *Branching random walks*. Springer International Publishing, 2015. ISBN 978-3-319-25371-8. École d’Été de Probabilités de Saint-Flour XLII - 2012.