The massive Feynman propagator on asymptotically Minkowski spacetimes

Christian Gérard, Michal Wrochna

The massive Feynman propagator
on asymptotically Minkowski spacetimes

Christian GÉRARD & Michał WROCHNA

ABSTRACT. We consider the massive Klein-Gordon equation on asymptotically Minkowski spacetimes, in the sense that the manifold is $\mathbb{R}^{1+d}$ and the metric approaches that of Minkowski space at infinity in a short-range way (jointly in time and space variables). In this setup we define Feynman and anti-Feynman scattering data and prove the Fredholm property of the Klein-Gordon operator with the associated Atiyah-Patodi-Singer type boundary conditions at infinite times. We then construct a parametrix (with compact remainder terms) for the Fredholm problem and prove that it is also a Feynman parametrix in the sense of Duistermaat and Hörmander.

1. INTRODUCTION & SUMMARY

1.1. **Introduction.** In the analysis of the free Klein-Gordon equation

$$(1.1) \qquad (\partial_t^2 - \Delta_{\mathrm{x}} + m^2)u(t,\mathrm{x}) = 0 \ \text{ on } \mathbb{R}_t \times \mathbb{R}_{\mathrm{x}}^d,$$

an essential elementary fact is that the Klein-Gordon operator $P_{\mathrm{free}} = \partial_t^2 - \Delta_{\mathrm{x}} + m^2$ possesses four distinguished inverses, namely the operators that multiply the Fourier transform of distributions by multiples of respectively

$$\frac{1}{(\tau \pm \mathrm{i}0)^2 - (k^2 + m^2)}, \quad \frac{1}{\tau^2 - (k^2 + m^2) \pm \mathrm{i}0},$$

and which are called the retarded/advanced, resp. Feynman/anti-Feynman propagator (we use the notation $\xi = (\tau, k)$ for the covariables corresponding to $x = (t,\mathrm{x}) \in \mathbb{R}^{1+d}$).

The retarded and advanced propagators are intimately linked to the Cauchy problem for (1.1), and can also be equivalently defined as the unique operators that solve the retarded/advanced problem

$$(1.2) \qquad P_{\mathrm{free}}u = f, \ \ \mathrm{supp}\,u \subset (\mathrm{supp} f) \pm C_+,$$

where C_+ is the forward lightcone. The well-posedness of the advanced/retarded problem (1.2) is a fact that generalizes to setups such as the Klein-Gordon operator

$$P := -\Box_g + V$$

on a globally hyperbolic[1] spacetime (M, g) and with potential $V \in C^\infty(M;\mathbb{R})$ (see e.g. [BGP]), and in consequence, this provides a natural and unambiguous definition of advanced and retarded propagators in a broad range of situations.

On the other hand, it is hardly obvious how the Feynman and anti-Feynman propagators generalize. This is a problem central to Quantum Field Theory on

[1]Let us recall that (M, g) is globally hyperbolic if it admits a Cauchy surface, i.e., a smooth hypersurface that is intersected by every inextensible, non-spacelike (i.e. causal) curve exactly once.

curved spacetimes in view of the role played by the Feynman propagator in interacting QFTs on Minkowski space.

A groundbreaking advance in that respect was provided by the work of Duistermaat and Hörmander [DH], who proved the existence and uniqueness modulo smooth terms of *Feynman* and *anti-Feynman parametrices*, i.e. inverses of P modulo smooth (but not necessarily compact) terms, distinguished by a specific structure of the singularities of the Schwartz kernel, as described by its *wave front set*. For later reference, let us recall the precise formulation:

Definition 1.1. *We say that G_{F} is a Feynman parametrix if the operators $1 - G_{\mathrm{F}} P$ and $1 - P G_{\mathrm{F}}$ have smooth Schwartz kernel and*

$$(1.3) \qquad \mathrm{WF}'(G_{\mathrm{F}}) = (\mathrm{diag}_{T^*M}) \cup \bigcup_{s \leq 0} (\Phi_s(\mathrm{diag}_{T^*M}) \cap \pi^{-1} \mathcal{N}),$$

where Φ_s is the bicharacteristic flow acting on the left component of diag_{T^*M} (i.e., the diagonal in $(T^*M \times T^*M) \setminus o$), and $\pi : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is the projection to the left component.

Above, $\mathrm{WF}'(G_{\mathrm{F}})$ stands for the primed wave front set of G_{F}, i.e. it is the image of the wave front set of the Schwartz kernel of G_{F} by the map $(x, \xi, x', \xi') \mapsto (x, \xi, x', -\xi')$. We refer to [Hö] for the definition and the basic properties of the wave front set of a distribution. We recall that the *bicharacteristic flow* Φ_t is the Hamilton flow of $p(x, \xi) = \xi \cdot g^{-1}(x) \xi$ restricted to the *characteristic set* $\mathcal{N} = p^{-1}(\{0\})$ (understood as a subset of $T^*M \setminus o$, where o is the zero section of the cotangent bundle), see [Hö].

Although not directly applicable in QFT, where actual inverses and bi-solutions are needed rather than parametrices, the results of Duistermaat and Hörmander were successfully adapted by Radzikowski in the study of so-called *Hadamard states* and of their two-point functions [Ra]. This has triggered important developments, culminating in rigorous treatments of perturbative interacting QFT on curved spacetimes [BF1, HW1, HW2, Da], where the role of the Feynman propagator is played by a 'time-ordered expression' obtained from Hadamard two-point functions, see e.g. [HW3, KM] for recent reviews. Without going into details, let us point out that this gives a notion of Feynman propagators that satisfy (1.3) and which are inverses of P (in the sense that when composed with P acting on test functions, they give the identity), and are constructed using methods that are *local* or *asymptotic* in time[2]. However, these are non-unique even if possible asymptotic symmetries of (M, g) are implemented. For instance, in a space-time that is asymptotic to Minkowski space as $t \to +\infty$ and $t \to -\infty$, there are at least two such Feynman propagators, one constructed from scattering data at $t = +\infty$ and the other one from $t = -\infty$ data [GW4].

A dramatically different perspective was proposed recently by Gell-Redman, Haber and Vasy [GHV] (basing on earlier developments including [BVW, HV1, HV2, Va1, Va2], cf. [VW] for the proof of (1.3) in that setting), who showed in the case of asymptotically Minkowski spacetimes (and assuming $V = 0$) that a Feynman parametrix can be obtained as the solution of a *global* problem of the form $Pu = f$, with u and f being in carefully chosen Hilbert spaces of distributions, and u being the unknown. This problem can be solved modulo finite dimensional, smooth terms (these can even be proved to vanish under extra assumptions), and therefore this yields a Feynman parametrix in a much stronger sense that in the work of Duistermaat & Hörmander, i.e. than in Def. 1.1, as it is the generalized

---

[2]A notable exception is a global construction that is shown to work for static and cosmological space-times in [BF3] and which is based on an improvement of a simple argument from spectral theory discussed e.g. in [FV1] (cf. [FMR] for the analogue in the case of the Dirac equation); its outcome is however highly non-unique.

inverse of a *Fredholm operator*[3]. Furthermore, it bears much more resemblance to elliptic inverses than the retarded and advanced propagators do, due to its positivity properties [Va3]. One can also argue that its definition is a highly canonical one (possibly modulo finite dimensional choices) as it relates directly to the bicharacteristic flow.

Moreover, a recent work of Bär and Strohmaier that treats the Dirac operator on compact globally hyperbolic spacetimes with space-like boundary [BS1] achieves to set up a Fredholm problem that is in many ways similar to that of Gell-Redman, Haber and Vasy. In Bär and Strohmaier's setting, the problem is formulated by imposing boundary conditions that are analogous to Atiyah-Patodi-Singer ones in the Riemannian case. Interestingly, they prove a Lorentzian analogue of the Atiyah-Patodi-Singer theorem [APS1, APS2] and relate the index to quantities of direct physical interest (in particular the so-called chiral anomaly), explaining also the relation to particle creation on curved space-times [BS2] (see also the works of Gibbons [Gi1, Gi2] and Gibbons and Richer [GR] for earlier related developments).

In the present paper, our main aim is to set up a Fredholm problem on a class of spacetimes similar to that considered in [GHV], but for the massive Klein-Gordon equation instead of the wave equation (i.e. for $V \neq 0$). On the other hand, we use an approach that is more closely related to the method of [BS1] and that in fact can be thought of as its non-compact, infinite time generalization, at least if one disregards distinct features of the Dirac and Klein-Gordon equations. In order to understand better the relation of the so-obtained Feynman propagator with the time-ordered expressions used usually in interacting QFT, we also construct a rather explicit parametrix with remainder terms that are at both compact and smooth.

1.2. **Main result.** We are primarily interested in the class of *asymptotically Minkowski spacetimes*, in the sense that (M, g) is a Lorentzian manifold (without boundary) with $M = \mathbb{R}^{1+d}$ and such that:

(aM)
$$g_{\mu\nu}(x) - \eta_{\mu\nu} \in S_{\mathrm{std}}^{-\delta}(\mathbb{R}^{1+d}), \ \delta > 1,$$

$(\mathbb{R}^{1+d}, g)$ is globally hyperbolic,

$(\mathbb{R}^{1+d}, g)$ has a time function $\tilde{t}$ such that $\tilde{t} - t \in S_{\mathrm{std}}^{1-\epsilon}(\mathbb{R}^{1+d}), \ \epsilon > 0,$

where $\eta_{\mu\nu}$ is the Minkowski metric and $S_{\mathrm{std}}^{\delta}(\mathbb{R}^{1+d})$ stands for the class of smooth functions f such that, denoting $\langle x \rangle = (1 + |x|)^{\frac{1}{2}}$,

$$\partial_x^\alpha f \in O(\langle x \rangle^{\delta - |\alpha|}), \ \alpha \in \mathbb{N}^{1+d}.$$

This way, g decays to the flat Minkowski metric simultaneously in time and in the spatial directions in a short-range[4] way. In a similar vein the potential is required to satisfy $V(y) - m^2 \in S_{\mathrm{std}}^{-\delta}(\mathbb{R}^{1+d}), \ m > 0$. Note that the definition (aM) covers a similar class of spacetimes to those considered in [BVW, GHV] (the latter are also called asymptotically Minkowski spacetimes therein), but strictly speaking they are not exactly the same: in our setup for instance (M, g) is globally hyperbolic, which is not clear from the outset in [BVW, GHV].

The main idea in the formulation of the Fredholm problem is to consider 'boundary conditions' that select asymptotic data which account for propagation of singularities within only one of the two connected components $\mathcal{N}^\pm$ of the characteristic set $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ of P. While in [BS1] there is indeed a boundary at finite times (consisting of the union of two time slices), here we need to consider infinite times

---

[3]Let us recall that a bounded operator is called Fredholm if the dimension of its kernel and cokernel are finite.

[4]This corresponds to the assumption $\delta > 1$.

instead, so boundary conditions are not to be understood literally as they are rather specified at the level of scattering data.

Let us illustrate how one can separate solutions according to $\mathcal{N}^\pm$, starting with the example of the free Klein-Gordon operator $P_{\text{free}} = \partial_t^2 - \Delta_{\text{x}} + m^2$. First, for $t, s \in \mathbb{R}$ let us denote by $\mathscr{U}_{\text{free}}(t, s)$ the Cauchy evolution propagator of P_{free}, i.e. the operator that maps $\mathscr{U}_{\text{free}}(t, s) :$ *Cauchy data at time $s \mapsto$ Cauchy data at time* t. Then $\mathscr{U}_{\text{free}}(t, s)$ is generated by a Hamiltonian which is selfadjoint in the energy space of Cauchy data, and its spectral projections to the half-lines $\mathbb{R}^\pm$ are given by

$$(1.4) \qquad c_{\text{free}}^{\pm,\text{vac}} := \frac{1}{2} \begin{pmatrix} \mathbf{1} & \pm\sqrt{-\Delta_{\text{x}} + m^2} \\ \pm\sqrt{-\Delta_{\text{x}} + m^2} & \mathbf{1} \end{pmatrix}.$$

The two operators[5] $c_{\text{free}}^{\pm,\text{vac}}$ project to Cauchy data of solutions propagating with wave front set in $\mathcal{N}^\pm$; equivalently, this corresponds to splitting the Cauchy evolution in terms of the two groups $e^{\pm it\sqrt{\Delta+m^2}}$. Now with our assumptions, P is close to P_{free} at infinity, and therefore it makes sense to use the splitting (1.4) on the level of scattering data.

In order to define scattering data in the setting of asymptotically Minkowski spaces we first make a change of variables by means of a diffeomorphism χ (see Subsect. 4.3), which allows to put the metric in the form

$$\chi^* g = -\hat{c}^2(t, \text{x})dt^2 + \hat{h}(t, \text{x})d\text{x}^2,$$

where $\hat{c}$ tends to 1 for large $|x|$, while $\hat{h}$ tends to some asymptotic metrics $\hat{h}_{\text{in/out}}$ depending on the sign of t. In these coordinates, a convenient choice of Cauchy data is $\varrho_s u := (u, -i\hat{c}^{-1}\partial_t u)\restriction_{t=s}$. On the other hand, the natural reference dynamics in this problem (at both future and past infinity) is that of the free Klein-Gordon operator P_{free}. Let us fix $t = 0$ as our reference time. We define the *Feynman* and *anti-Feynman scattering data maps*:

$$\varrho_{\text{F}} := \text{s}-\lim_{t_\pm \to \pm\infty} \left(c_{\text{free}}^{+,\text{vac}}\mathscr{U}_{\text{free}}(0, t_+)\varrho_{t_+} + c_{\text{free}}^{-,\text{vac}}\mathscr{U}_{\text{free}}(0, t_-)\varrho_{t_-} \right),$$

$$\varrho_{\overline{\text{F}}} := \text{s}-\lim_{t_\pm \to \pm\infty} \left(c_{\text{free}}^{+,\text{vac}}\mathscr{U}_{\text{free}}(0, t_-)\varrho_{t_-} + c_{\text{free}}^{-,\text{vac}}\mathscr{U}_{\text{free}}(0, t_+)\varrho_{t_+} \right),$$

as appropriate strong operator limits. We abbreviate the Sobolev spaces $H^m(\mathbb{R}^d)$ by H^m and denote by $\mathscr{U}(t, s)$ the Cauchy evolution propagator of P. Our main result can be stated as follows.

Theorem 1.2. *Assume* (aM) *and let $m \in \mathbb{R}$. Consider the Hilbert space*

$$(1.5) \quad \mathcal{X}_{\text{F}}^m := \left\{ u \in (\chi^{-1})^* \left(C^0(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m) \right) : \ Pu \in \mathcal{Y}^m, \ \varrho_{\overline{\text{F}}}u = 0 \right\},$$

where $\mathcal{Y}^m := (\chi^{-1})^* \left(\langle t \rangle^{-\gamma} L^2(\mathbb{R}; H^m) \right)$ and $\frac{1}{2} < \gamma < \frac{1}{2} + \delta$. Then $P : \mathcal{X}_{\text{F}}^m \to \mathcal{Y}^m$ is Fredholm of index

$$(1.6) \qquad \text{ind } P|_{\mathcal{X}_{\text{F}}^m \to \mathcal{Y}^m} = \text{ind}(c_{\text{free}}^{-,\text{vac}}W_{\text{out}}^{-1} + c_{\text{free}}^{+,\text{vac}}W_{\text{in}}^{-1}),$$

where $W_{\text{out/in}}^{-1} = \lim_{t_\pm \to \pm\infty} \mathscr{U}_{\text{free}}(0, t_\pm)\mathscr{U}(t_\pm, 0)$. In particular the index is independent on m. Furthermore, there exists a Feynman parametrix $G_{\text{F}} : \mathcal{Y}^m \to \mathcal{X}_{\text{F}}^m$ such that $\mathbf{1} - PG_{\text{F}}$ and $\mathbf{1} - G_{\text{F}}P$ are compact operators.

Note that the space $\mathcal{X}_{\text{F}}^m$ is a closed subspace of the Hilbert space

$$\mathcal{X}^m = \{ u \in (\chi^{-1})^* \left(C^0(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m) \right) : \ Pu \in \mathcal{Y}^m \}$$

equipped with the norm $\|u\|_{\mathcal{X}^m}^2 = \|\varrho_0(\chi^{-1})^*u\|_{\mathcal{E}^m} + \|Pu\|_{\mathcal{Y}^m}^2$, where $\mathcal{E}^m$ is the energy space, see Def. 3.4.

---

[5]The operators $c_{\text{free}}^{\pm,\text{vac}}$ also have the interpretation of being the covariances (acting on Cauchy data) of the Minkowski vacuum state, see e.g. the discussion in [GW4].

As pointed out in [BS1], the condition $\varrho_{\overline{\mathrm{F}}} u = 0$ can be seen as an analogue of the Atiyah-Patodi-Singer boundary condition (even though this is less evident here as we do not consider the Dirac equation). Furthermore, one could equally well consider the *anti-APS* boundary condition $\varrho_{\mathrm{F}} u = 0$, which leads to an 'anti-Feynman' counterpart of Theorem 1.2 — interestingly, just as in [BS1], this differs from the Riemannian case where one boundary condition is preferred over the other. On the other hand, we also prove that the kernel (null space) of $P : \mathcal{X}_{\mathrm{F}}^m \to \mathcal{Y}^m$ consists of smooth functions, and that G_{F} satisfies a positivity condition $\mathrm{i}(G_{\mathrm{F}} - G_{\mathrm{F}}^*) \geq 0$ reminiscent of the positivity of the spectral measure in the limiting absorption principle. As pioneered in [BS1] and [BVW, Va3], this shows once again a striking similarity to the elliptic case.

On the side note it is worth mentioning that our results can also be adapted to asymptotically static space-times with compact Cauchy surface, for in that setting, decay in the spatial directions becomes irrelevant and consequently one can use directly the time decay estimates from [GW4] instead of the estimates considered here.

Furthermore, a large part of the arguments of Bär and Strohmaier [BS2] can be applied to our case, and thus allow one to interpret the index of $P : \mathcal{X}_{\mathrm{F}}^m \to \mathcal{Y}^m$ in terms of particle creation.

1.3. **Outline of proofs.** An important step in our proof is a procedure derived in detail in [GOW, GW4], which allows us to reduce the whole analysis to the case when the Klein-Gordon operator is of the form

$$(1.7) \qquad P = \partial_t^2 + r(t)\partial_t + a(t, \mathrm{x}, \partial_{\mathrm{x}}),$$

where $r(t)$ is a multiplication operator and $a(t, \mathrm{x}, D_{\mathrm{x}})$ is a differential operator with principal symbol $k \cdot h_t^{-1}(\mathrm{x})k$.

To account for the decay properties that follow from assumption (aM), we introduce a pseudodifferential calculus $\Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R}; \mathbb{R}^d)$ $(m, \delta \in \mathbb{R})$ consisting of pseudodifferential operators with time-depending symbols $a(t, \mathrm{x}, k)$ satisfying

$$\partial_t^\gamma \partial_{\mathrm{x}}^\alpha \partial_k^\beta a(t, \mathrm{x}, k) \in O\big((\langle t \rangle + \langle \mathrm{x} \rangle)^{\delta - \gamma - |\alpha|} \langle k \rangle^{m - |\beta|}\big), \ \gamma \in \mathbb{N}, \ \alpha, \beta \in \mathbb{N}^d.$$

Thus, δ accounts for the asymptotic behaviour for large $|\mathrm{x}|$ and/or t. We also introduce its time-independent counterpart, denoted by $\Psi_{\mathrm{sd}}^{m,\delta}(\mathbb{R}^d)$. We then find that our hypothesis (aM) implies that one is reduced to the situation covered by the following list of assumptions (keeping in mind that $\delta > 1$ is as in (aM)):

$$r(t) \in \Psi_{\mathrm{std}}^{0,-1-\delta}(\mathbb{R}; \mathbb{R}^d),$$

(std) $\quad a(t, \mathrm{x}, D_{\mathrm{x}}) - a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) \in \Psi_{\mathrm{std}}^{2,-\delta}(\mathbb{R}; \mathbb{R}^d)$ on $\mathbb{R}^{\pm} \times \mathbb{R}^d$ where:

$$a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) \in \Psi_{\mathrm{sd}}^{2,0}(\mathbb{R}^d) \text{ is elliptic (in the usual } \Psi^2(\mathbb{R}^d) \text{ calculus)},$$

$$a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) = a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}})^* \geq C_\infty > 0.$$

In this setup, using standard arguments from Fredholm analysis and the well-posedness of the inhomogeneous Cauchy problem

$$\begin{cases} Pu = f, & f \in \mathcal{Y}^m \\ \varrho_t u = v, & v \in H^{m+1}(\mathbb{R}^d) \oplus H^m(\mathbb{R}^d), \end{cases}$$

we deduce that the Fredholm property of P acting on the spaces $\mathcal{X}_{\mathrm{F}}^m$, $\mathcal{Y}^m$ (strictly speaking their analogue in the reduced setting (std), so it is not necessary to use a diffeomorphism χ in the definition) is equivalent to the Fredholm property of the operator

$$W_{\mathrm{F}} := W_{\mathrm{out}} c_{\mathrm{free}}^{+,\mathrm{vac}} + W_{\mathrm{in}} c_{\mathrm{free}}^{-,\mathrm{vac}} \in B(H^{m+1}(\mathbb{R}^d) \oplus H^m(\mathbb{R}^d)).$$

We actually prove the stronger statement that the operators

$$(1.8) \qquad W_{\mathrm{F}}^{\dagger} W_{\mathrm{F}} - \mathbf{1}, \ \ W_{\mathrm{F}} W_{\mathrm{F}}^{\dagger} - \mathbf{1}$$

are *smoothing* and have *decay* properties that imply their compactness (here $\dagger$ denotes the adjoint with respect to the canonical non-positive charge inner product preserved by the evolution, so that $W_{\mathrm{out}}^{\dagger} W_{\mathrm{out}} = W_{\mathrm{in}}^{\dagger} W_{\mathrm{in}} = \mathbf{1}$). The proof is based on the method of approximate diagonalization of the Cauchy evolution, developed successively in the works [Ju, JS, GW1, GW2, GOW] and improved in the present paper to yield estimates on the decay (in both time and spatial directions) of various terms that account for the relation between the full dynamics and its asymptotic counterparts. The role of this approximate diagonalization, beside providing the basis for the construction of a parametrix, is to split the Cauchy evolution in two parts corresponding to propagation within $\mathcal{N}^{+}$ and $\mathcal{N}^{-}$, and then our estimates allow us to relate this splitting to the canonical one for the asymptotic dynamics.

The construction of the Feynman parametrix G_{F} is based on a formula that is a time-ordered expression[6], but involving only the 'diagonalizable' part of the evolution. The evolution being in fact diagonalizable in the sense considered here modulo terms that are smoothing and decaying, this produces a parametrix indeed, and the proof of the wave front set condition follows by standard arguments, detailed previously in [GOW].

It is worth mentioning that compactness of the remainder term in (1.8) was already studied in an analogous problem for the Dirac operator on Minkowski space with external potentials [Ma1, Ma2, BH], where index formulas have also been derived and the interpretation of the index in terms of particle creation was discussed (see also [BS1, BS2]). An interesting topic of further research would thus be to find a short-hand index formula in our setting.

1.4. **Plan of the paper.** The paper is organized as follows.

In Sect. 2 we introduce the time-dependent pseudodifferential operator classes $\Psi_{\mathrm{std}}^{m,\delta}$ and state some of their properties. We recall the method of approximate diagonalization of the Cauchy evolution from [GOW, GW4], and we then give a refinement in the setup of assumption (std) by showing decay of various remainder terms.

In Sect. 3 we set up a Fredholm problem for the Klein-Gordon operator, assuming hypothesis (std), and then we construct a Feynman parametrix and prove that the remainder terms are compact operators. An important role is played by the approximate diagonalization and the estimates from Sect. 2.

Finally, in Sect. 4 we consider asymptotically Minkowski spacetimes (aM). We show that in this case, using the procedure from [GOW, GW4] one is reduced to assumption (std). This allows us to adapt the results from Sect. 3 and to prove Thm. 1.2. As an aside, we show that the retarded and advanced propagators can be obtained as inverses of a bounded operator acting on Hilbert spaces; this gives another analogy to the setting of [GHV].

Several auxiliary proofs are collected in Appendix A.

2. MODEL KLEIN-GORDON OPERATOR

2.1. **Notation.** The space of differential operators on $\mathbb{R}^{k}$ of order m is denoted by $\mathrm{Diff}^{m}(\mathbb{R}^{k})$. The space of smooth functions with compact support is denoted $C_{\mathrm{c}}^{\infty}(\mathbb{R}^{k})$.

---

[6]We mean specifically that its integral kernel is of the form $\theta(t-t')\Lambda^{+}(x,x')+\theta(t'-t)\Lambda^{-}(x,x')$ with $\Lambda^{\pm} \geq 0$.

From now on, the operator of multiplication by a function f will be denoted by f, while the operators of partial differentiation will be denoted by $\overline{\partial}_i$, so that $[\overline{\partial}_i, f] = \partial_i f$.

2.2. Klein-Gordon operator.
In what follows we use the notation $x = (t, \mathrm{x})$ for points in $\mathbb{R}^{1+d}$, $d \geq 1$.

Before considering the Klein-Gordon equation on actual asymptotically Minkowski spacetimes, we first work with a simpler set of assumptions that allow us to write the Klein-Gordon operator in the form

$$(2.9) \qquad P = \overline{\partial}_t^2 + r(t, \mathrm{x})\overline{\partial}_t + a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}}) \in \mathrm{Diff}^2(\mathbb{R}^{1+d}),$$

where $r(t, \mathrm{x}) \in C^\infty(\mathbb{R}^{1+d})$, and $a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}}) \in \mathrm{Diff}^2(\mathbb{R}^d)$ depends smoothly on t. Specifically, if $h(t)$ is a smooth family of smooth Riemannian metrics on $\mathbb{R}^d$ and $V \in C^\infty(\mathbb{R}^{1+d})$ is real-valued, we consider the metric

$$g = -dt^2 + h_{ij}(t, \mathrm{x})d\mathrm{x}^i d\mathrm{x}^j$$

on $\mathbb{R}^{1+d}$. Then, the Klein-Gordon operator $\tilde{P} := -\Box_g + V$ equals

$$(2.10) \qquad P = |h|^{-\frac{1}{2}}\overline{\partial}_t |h|^{\frac{1}{2}}\overline{\partial}_t - |h|^{-\frac{1}{2}}\overline{\partial}_i h^{ij}|h|^{\frac{1}{2}}\overline{\partial}_j + V,$$

which is indeed of the form (2.9) with

$$a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}}) = -|h|^{-\frac{1}{2}}\overline{\partial}_i h^{ij}|h|^{\frac{1}{2}}\overline{\partial}_j + V(t, \mathrm{x})$$

and

$$r(t, \mathrm{x}) = |h|^{-\frac{1}{2}}\partial_t(|h|^{\frac{1}{2}})(t, \mathrm{x}).$$

In the sense of formal adjoints, the Klein-Gordon operator satisfies $P^* = P$ with respect to the $L^2(\mathbb{R}^{1,d}, |h|^{\frac{1}{2}}dtd\mathrm{x})$ scalar product, and similarly $a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}})^* = a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}})$ with respect to the t-dependent $L^2(\mathbb{R}^d, |h|^{\frac{1}{2}}d\mathrm{x})$ scalar product.

We will often abbreviate $a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}})$ by $a(t)$ or simply by a.

In what follows we introduce the terminology needed to formulate additional assumptions on a and r. These will correspond to a scattering situation, i.e. to the case of a metric g (resp. potential V) converging to some asymptotic static metrics $g_{\mathrm{out/in}} = -dt^2 + h_{\mathrm{out/in,ij}}(\mathrm{x})d\mathrm{x}^i d\mathrm{x}^j$ (resp. to some time-independent potentials $V_{\mathrm{out/in}}$) as $t \to \pm\infty$.

2.3. Scattering pseudodifferential calculus.
We start by introducing a time-dependent pseudodifferential calculus on $\mathbb{R}^d$ that allows to control both Sobolev regularity in the usual sense and decay at space-time infinity, and which is therefore the natural calculus on asymptotically Minkowski spacetimes.

For $m, \delta \in \mathbb{R}$ we denote by $S_{\mathrm{std}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)$ the space of smooth functions $a(t, \mathrm{x}, k)$ such that

$$\partial_t^\gamma \partial_\mathrm{x}^\alpha \partial_k^\beta a(t, \mathrm{x}, k) \in O((\langle t\rangle + \langle \mathrm{x}\rangle)^{\delta - \gamma - |\alpha|}\langle k\rangle^{m-|\beta|}), \ \gamma \in \mathbb{N}, \ \alpha, \beta \in \mathbb{N}^d.$$

The subscript std refers to the space-time decay properties of the symbols in (t, x). The subspace of symbols which are poly-homogeneous in k will be denoted by $S_{\mathrm{std,ph}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)$. We denote by $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$ the space of operator-valued functions $a(t)$ such that

$$\|(D_\mathrm{x}^2 + \mathrm{x}^2 + 1)^m \partial_t^n a(t)(D_\mathrm{x}^2 + \mathrm{x}^2 + 1)^m\|_{B(L^2(\mathbb{R}^d))} \in O(\langle t\rangle^{-p}), \ \forall m, n, p \in \mathbb{N}.$$

These will play the role of error terms in the calculus. Correspondingly, we set:

$$\Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R}; \mathbb{R}^d) := \mathrm{Op}^{\mathrm{w}}(S_{\mathrm{std,ph}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)) + \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d),$$

where Op^{w} is the well-known Weyl quantization, defined by

$$\mathrm{Op}^{\mathrm{w}}(a)u(\mathrm{x}) = (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(\mathrm{x}-\mathrm{y})\cdot k}a\big(t, \tfrac{\mathrm{x}+\mathrm{y}}{2}, k\big)u(\mathrm{y})d\mathrm{y}dk.$$

It is easy to see that

$$S_{\mathrm{std,ph}}^{m,\delta}(\mathbb{R};T^*\mathbb{R}^d) = (\langle \mathrm{x} \rangle + \langle t \rangle)^\delta S_{\mathrm{std,ph}}^{m,0}(\mathbb{R};T^*\mathbb{R}^d),$$

$$\Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R};\mathbb{R}^d) = (\langle \mathrm{x} \rangle + \langle t \rangle)^\delta \Psi_{\mathrm{std}}^{m,0}(\mathbb{R};\mathbb{R}^d),$$

which allows us to reduce ourselves to the case $\delta = 0$.

Omitting the variable t in the above definitions, we also obtain classes of (time-independent) symbols and pseudodifferential operators on $\mathbb{R}^d$, which will be denoted respectively by $S_{\mathrm{sd}}^{m,\delta}(T^*\mathbb{R}^d)$, $\Psi_{\mathrm{sd}}^{m,\delta}(\mathbb{R}^d)$ and $\mathcal{W}_{\mathrm{sd}}^{-\infty}(\mathbb{R}^d)$, where the subscript sd refers to space decay properties of the symbols or operators.

The classes $\Psi_{\mathrm{sd}}^{m,\delta}(\mathbb{R}^d)$ are the well-known 'scattering pseudodifferential operators', see e.g. [Co, Pa, Sh1]. The error terms $\mathcal{W}_{\mathrm{sd}}^{-\infty}(\mathbb{R}^d)$ can be described as

(2.11) $$\mathcal{W}_{\mathrm{sd}}^{-\infty}(\mathbb{R}^d) = \bigcap_{m\in\mathbb{R}} B\big(\langle \mathrm{x} \rangle^m H^{-m}(\mathbb{R}^d), \langle \mathrm{x} \rangle^{-m} H^m(\mathbb{R}^d)\big),$$

where $H^m(\mathbb{R}^d)$ is the standard Sobolev space of order m. Using (2.11) we equip $\mathcal{W}_{\mathrm{sd}}^{-\infty}(\mathbb{R}^d)$ with its canonical Fréchet space topology; this is needed to define in the obvious way spaces such as $C^\infty(\mathbb{R}^2; \mathcal{W}_{\mathrm{sd}}^{-\infty}(\mathbb{R}^d))$. We will also occasionally need the broader class of smoothing operators $\mathcal{W}^{-\infty}(\mathbb{R}^d)$, defined as in (2.11) but with the $\langle \mathrm{x} \rangle^{\pm m}$ weights ommitted.

We will need an appropriate notion of *ellipticity* and principal symbol for the $\Psi_{\mathrm{sd}}^{m,0}(\mathbb{R}^d)$ and $\Psi_{\mathrm{std}}^{m,0}(\mathbb{R};\mathbb{R}^d)$ classes. For our purposes, elliptic operators in $\Psi_{\mathrm{sd}}^{m,0}(\mathbb{R}^d)$ will be simply those which are elliptic in the usual sense[7] in $\Psi^m(\mathbb{R}^d)$, i.e. $a \in \Psi^m(\mathbb{R}^d)$ is elliptic if there exists $C > 0$ such that

(2.12) $$|\sigma_{\mathrm{pr}}(a)| \geq C|k|^m, \quad |k| \geq 1,$$

where $\sigma_{\mathrm{pr}}(a)$ is the principal symbol of a defined in the usual way, see e.g. [Sh2]. Furthermore, $a \in \Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R};\mathbb{R}^d)$ is said to be elliptic if $a(t)$ is elliptic for all $t \in \mathbb{R}$ and the constant C in (2.12) is uniform in t.

Let us remark that the $\Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R};\mathbb{R}^d)$ pseudodifferential calculus has structural properties very analogous to the $\Psi_{\mathrm{td}}^{m,\delta}(\mathbb{R};\mathbb{R}^d)$ calculus introduced in [GW4], which is defined by omitting $\langle x \rangle$ in the relevant definitions (so that only decay in time is taken into account). As a consequence, many results from [GW4] can be adapted to the present setup. In particular, a variant of Seeley's theorem is valid for the $\Psi_{\mathrm{std}}^{m,0}(\mathbb{R};\mathbb{R}^d)$ classes; it is proved in complete analogy to [GW4, Thm. 3.7] by reduction to the general framework of [ALNV], see also the arguments in [GOW, Subsect. 5.3].

Theorem 2.1. *Let $a \in \Psi_{\mathrm{std}}^{m,0}(\mathbb{R};\mathbb{R}^d)$ be elliptic, selfadjoint with $a(t) \geq c_0\mathbf{1}$ for some $c_0 > 0$. Then for any $\alpha \in \mathbb{R}$, $a^\alpha \in \Psi_{\mathrm{std}}^{\alpha m,0}(\mathbb{R};\mathbb{R}^d)$ and $\sigma_{\mathrm{pr}}(a^\alpha)(t) = \sigma_{\mathrm{pr}}(a(t))^\alpha$.*

Proof. We will simply outline the main steps of the proof, which consists in verifying the abstract conditions in [ALNV]. We use the notations in [GOW, Subsect. 5.3].

We choose as Hilbert space $\mathcal{H} = L^2(\mathbb{R}_t \times \mathbb{R}_{\mathrm{x}}^d)$. As injective operator on $\mathcal{H}$ belonging to $\mathcal{W}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d)$ we choose $R = \mathrm{e}^{-(D_{\mathrm{x}}^2+\mathrm{x}^2+t^2+1)}$. It is easy to see that if $a \in S_{\mathrm{std,ph}}^{m,0}(\mathbb{R};T^*\mathbb{R}^d)$ and $\mathrm{Op}(a) \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R};\mathbb{R}^d)$ then $a \in S_{\mathrm{std,ph}}^{-\infty,0}(\mathbb{R};T^*\mathbb{R}^d)$. It remains to check the spectral invariance property of $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R};\mathbb{R}^d)$, ie to prove that if $R \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R};\mathbb{R}^d)$ and $\mathbf{1} - R$ is boundedly invertible on $\mathcal{H}$, then $(\mathbf{1} - R)^{-1} \in \mathbf{1} + \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R};\mathbb{R}^d)$. We write $\mathcal{H}$ as $L^2(\mathbb{R}_t; L^2(\mathbb{R}^d))$ and $R = \int_{\mathbb{R}}^{\oplus} R(t)dt$, hence

---

[7]Thus, we do *not* consider here ellipticity in the sense of the scattering pseudodifferential calculus [Co, Pa, Sh1].

$(\mathbf{1} - R)^{-1} = \int_{\mathbb{R}}^{\oplus} (\mathbf{1} - R(t))^{-1} dt$. As in [GOW, Lem. 5.5], $\mathbf{1} - R(t)$ is boundedly invertible on $L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ and we have

$$(\mathbf{1} - R(t))^{-1} = \mathbf{1} + R(t) + R(t)(\mathbf{1} - R(t))^{-1} R(t) := \mathbf{1} + R_1(t).$$

Using that $\partial_t(\mathbf{1} - R(t))^{-1} = (\mathbf{1} - R(t))^{-1}\partial_t R(t)(\mathbf{1} - R(t))^{-1}$ and Leibniz rule we obtain that $R_1(t) \in \mathcal{W}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d)$. $\square$

Later on we will need the following auxiliary result on fractional powers of elliptic operators, the proof of which is deferred to Subsect. A.1.

Proposition 2.2. *Let* $a_i \in \Psi_{\mathrm{std}}^{2,0}(\mathbb{R}; \mathbb{R}^d)$ *(i = 1, 2) be elliptic, with* $a_i = a_i^*$ *and* $a_i(t) \geq c_0 \mathbf{1}$ *for some* $c_0 > 0$. *Suppose that* $a_1 - a_2 \in \Psi_{\mathrm{std}}^{2,-\delta}(\mathbb{R}; \mathbb{R}^d)$ *with* $\delta > 0$. *Then for each* $\alpha \in \mathbb{R}$ *one has:*

$$a_1^\alpha - a_2^\alpha \in \Psi_{\mathrm{std}}^{2\alpha,-\delta}(\mathbb{R}; \mathbb{R}^d).$$

Let us now consider the ressummation of symbols. We denote

$$\Psi_{\mathrm{std}}^{-\infty,-\delta}(\mathbb{R}; \mathbb{R}^d) := \bigcap_{m \in \mathbb{R}} \Psi_{\mathrm{std}}^{m,-\delta}(\mathbb{R}; \mathbb{R}^d),$$

which is a subclass of (but is not equal to) $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$. The lemma below is obtained by exactly the same arguments as in [GW4, Lem. 3.11].

Lemma 2.3. *Let* $\delta \in \mathbb{R}$ *and let* (m_j) *be a real sequence decreasing to* $-\infty$. *Then if* $a_j \in \Psi_{\mathrm{std}}^{m_j,-\delta}(\mathbb{R}; \mathbb{R}^d)$ *there exists* $a \in \Psi_{\mathrm{std}}^{m_0,-\delta}(\mathbb{R}; \mathbb{R}^d)$, *unique modulo* $\Psi_{\mathrm{std}}^{-\infty,-\delta}(\mathbb{R}; \mathbb{R}^d)$, *such that*

$$a \sim \sum_{j=0}^\infty a_j, \ i.e. \ \forall N \in \mathbb{N}, \ a - \sum_{j=0}^N a_j \in \Psi_{\mathrm{std}}^{m_{N+1},-\delta}(\mathbb{R}; \mathbb{R}^d).$$

Thus when performing the ressummation of (a_j) we have control of the decay of the error terms (as they belong to $\Psi_{\mathrm{std}}^{-\infty,-\delta}(\mathbb{R}; \mathbb{R}^d)$).

2.4. **Assumptions of the model.** We are now ready to state the precise assumptions on the model Klein-Gordon operator

$$P = \overline{\partial}_t^2 + r(t, \mathrm{x})\overline{\partial}_t + a(t, \mathrm{x}, \overline{\partial}_{\mathrm{x}}).$$

Namely, we assume that there exist $\delta > 0$, $a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) \in \Psi_{\mathrm{sd}}^{2,0}(\mathbb{R}^d)$ and a constant C_∞ such that:

(std)
$$a(t, \mathrm{x}, D_{\mathrm{x}}) = a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) + \Psi_{\mathrm{std}}^{2,-\delta}(\mathbb{R}; \mathbb{R}^d) \text{ on } \mathbb{R}^\pm \times \mathbb{R}^d,$$
$$r(t) \in \Psi_{\mathrm{std}}^{0,-1-\delta}(\mathbb{R}; \mathbb{R}^d),$$
$$a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) \in \Psi_{\mathrm{sd}}^{2,0}(\mathbb{R}^d) \text{ is elliptic,}$$
$$a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}}) = a_{\mathrm{out/in}}(\mathrm{x}, D_{\mathrm{x}})^* \geq C_\infty > 0.$$

Note that this gives a setup which is a particular case of that considered recently in [GW4]. Here we assume decay of various terms both in time and in the spatial variables, while in [GW4] only time decay is required (and more general manifolds are allowed in the place of $\mathbb{R}^d$).

2.5. **Approximate diagonalization of the Cauchy evolution.** For $t \in \mathbb{R}$ let us denote by ϱ_t the Cauchy data map

$$\varrho_t : C^\infty(\mathbb{R}; \mathscr{D}'(\mathbb{R}^d)) \to \mathscr{D}'(\mathbb{R}^d) \oplus \mathscr{D}'(\mathbb{R}^d)$$
$$u \mapsto (u(t), \mathrm{i}^{-1}\partial_t u(t)).$$

Furthermore, for $t, s \in \mathbb{R}$ let us denote by $\mathscr{U}(t,s)$ the *Cauchy evolution propagator* for P, i.e., $\mathscr{U}(t,s)v$ is by definition the Cauchy data $\varrho_t u$ of the unique solution of the Cauchy problem

$$\begin{cases} Pu = 0, \\ \varrho_s u = v. \end{cases}$$

The main ingredient in our approach is a refined version of the approximate diagonalization of $\mathscr{U}(t,s)$ developed in [GOW, GW4] in the context of Hadamard states. In what follows we recall its outcome and discuss how it can be improved in the setup of assumption (std).

The first step consists in performing an approximate factorization of the Klein-Gordon operator $P = \overline{\partial}_t^2 + r\overline{\partial}_t + a$. This means specifically that one constructs a family of elliptic pseudodifferential operators $b^+(t)$, smoothly depending on t, and such that

$$(2.1) \qquad P = (\overline{\partial}_t + \mathrm{i}b^+(t) + r(t)) \circ (\overline{\partial}_t - \mathrm{i}b^+(t)) + r_\infty^+(t),$$

where $r_\infty^+(t)$ is a smoothing error. Once this is done, one actually gets immediately a second solution to this problem by setting

$$(2.2) \qquad b^- = -(b^+)^*, \quad r_\infty^- = (r_\infty^+)^*.$$

Indeed, by taking the adjoint of both sides of (2.1) with respect to the t-dependent inner product $L^2(\mathbb{R}^d, |h|^{\frac{1}{2}}dx)$, and using that

$$(\partial_t b^+)^* = \partial_t((b^+)^*) + r(b^+)^* - (b^+)^*r$$
$$= -\partial_t b^- - rb^- + b^-r,$$

one obtains that

$$P = (\overline{\partial}_t + \mathrm{i}b^-(t) + r(t)) \circ (\overline{\partial}_t - \mathrm{i}b^-(t)) + r_\infty^-(t).$$

The pair of identities

$$(2.3) \qquad P = (\overline{\partial}_t + \mathrm{i}b^\pm(t) + r(t)) \circ (\overline{\partial}_t - \mathrm{i}b^\pm(t)) + r_\infty^\pm(t).$$

is then used to re-express the Cauchy evolution of P in terms of the Cauchy evolution of $\overline{\partial}_t - \mathrm{i}b^+(t)$ and $\overline{\partial}_t - \mathrm{i}b^-(t)$.

In Prop. 2.4 below, we state our result on the existence of $b^\pm(t)$ as above and satisfying in addition decay estimates that are crucial for the proofs in the rest of the paper. To simplify notation we write $b_1(t) = b_2(t) + \Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R}^\pm; \mathbb{R}^d)$ if $b_1(t)$, $b_2(t)$ are two t-dependent operators such that $b_1(t) - b_2(t) \in \Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R}; \mathbb{R}^d)$ on $\mathbb{R}^\pm \times \mathbb{R}^d$. By an argument from [GW4] there exists $\varphi \in C_c^\infty(\mathbb{R})$ such that $a(t) + \varphi(a(t)) > 0$ and $a(t) + \varphi(a(t)) \sim a_{\mathrm{out/in}}$ on $\mathbb{R}^\pm$ in the sense that

$$\exists\, c > 0 \text{ s.t. } c^{-1}a_{\mathrm{out/in}} \leq a(t) + \varphi(a(t)) \leq c\, a_{\mathrm{out/in}} \text{ on } \mathbb{R}^\pm.$$

Moreover, $\varphi(a(t)) \in C_c^\infty(\mathbb{R}; \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}^d))$. We then set $\epsilon(t) := \big(a(t) + \varphi(a(t))\big)^{\frac{1}{2}}$ and $\epsilon_{\mathrm{out/in}} := a_{\mathrm{out/in}}^{\frac{1}{2}}$.

Proposition 2.4. *There exist* $b^\pm(t) = \pm\epsilon(t) + \Psi_{\mathrm{std}}^{0,-1-\delta}(\mathbb{R}; \mathbb{R}^d)$ *that solve (2.2) and (2.3) with* $r_{-\infty}^\pm \in \Psi_{\mathrm{std}}^{-\infty,-1-\delta}(\mathbb{R}; \mathbb{R}^d)$, *and that satisfy*

$$(2.4) \qquad b^\pm(t) = \pm\epsilon_{\mathrm{out/in}} + \Psi_{\mathrm{std}}^{1,-\delta}(\mathbb{R}^\pm; \mathbb{R}^d).$$

Moreover, $b^\pm(t)$ *can be chosen in such way that*

$$(2.5) \qquad (b^+(t) - b^-(t))^{-1} \geq C(t)\epsilon(t)^{-1}$$

holds for some $C(t) > 0$.

The proof is completely analogous to [GW4], using Prop. 2.2 instead of [GW4, Prop. 3.10].

The approximate diagonalization of $\{\mathscr{U}(t,s)\}_{t,s\in\mathbb{R}}$ proceeds now as follows. We set

$$
\begin{aligned}
(2.6) \qquad T(t) &:= \mathrm{i}^{-1} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ b^+ & -b^- \end{pmatrix} (b^+ - b^-)^{-\frac{1}{2}}, \\
T^{-1}(t) &= \mathrm{i}(b^+ - b^-)^{-\frac{1}{2}} \begin{pmatrix} -b^- & \mathbf{1} \\ -b^+ & \mathbf{1} \end{pmatrix},
\end{aligned}
$$

which is well defined by (2.5). We then define a new evolution group $\{\mathscr{U}^{\mathrm{ad}}(t,s)\}_{t,s\in\mathbb{R}}$ by

$$
(2.7) \qquad \mathscr{U}(t,s) =: T(t)\mathscr{U}^{\mathrm{ad}}(t,s)T(s)^{-1}.
$$

We denote by $H(t)$ the generator of $\{\mathscr{U}(t,s)\}_{t,s\in\mathbb{R}}$, defined by

$$
(2.8) \qquad \frac{\partial}{\partial t}\mathscr{U}(t,s) =: \mathrm{i}H(t)\mathscr{U}(t,s),
$$

and similarly we define $H^{\mathrm{ad}}(t)$, the generator of $\{\mathscr{U}^{\mathrm{ad}}(t,s)\}_{t,s\in\mathbb{R}}$. Explicitly, the former equals

$$
H(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & \mathrm{i}r(t) \end{pmatrix}.
$$

By a direct computation one finds that the latter is of the form

$$
(2.9) \qquad H^{\mathrm{ad}}(t) = H^{\mathrm{d}}(t) - V^{\mathrm{ad}}_{-\infty}(t),
$$

where $H^{\mathrm{d}}(t)$ is a diagonal matrix of pseudodifferential operators (smoothly depending on t) and $V^{\mathrm{ad}}_{-\infty}(t)$ is a smoothing, decaying remainder. More precisely,

$$
(2.10) \qquad H^{\mathrm{d}}(t) = \begin{pmatrix} \epsilon^+ & 0 \\ 0 & \epsilon^- \end{pmatrix},
$$

where the components $\epsilon^{\pm}(t)$ are

$$
\begin{aligned}
(2.11) \qquad \epsilon^{\pm} &= -b^{\mp} + \mathrm{i}r + [(b^+ - b^-)^{-\frac{1}{2}}, b^{\mp}] - \mathrm{i}\partial_t(b^+ - b^-)^{-\frac{1}{2}}(b^+ - b^-)^{\frac{1}{2}} \\
&\quad - (b^+ - b^-)^{-\frac{1}{2}}r^{\mp}_{-\infty}(b^+ - b^-)^{-\frac{1}{2}},
\end{aligned}
$$

and furthermore,

$$
\begin{aligned}
(2.12) \qquad V^{\mathrm{ad}}_{-\infty}(t) &= (b^+ - b^-)^{-\frac{1}{2}} \begin{pmatrix} r^-_{-\infty} & -r^-_{-\infty} \\ r^+_{-\infty} & -r^+_{-\infty} \end{pmatrix} (b^+ - b^-)^{-\frac{1}{2}} \\
&\quad \in \Psi^{-\infty, -1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d) \otimes B(\mathbb{C}^2),
\end{aligned}
$$

where $r^{\pm}_{-\infty} \in \Psi^{-\infty, -1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d)$ are the remainder terms from (2.3). This way, the evolution $\mathscr{U}^{\mathrm{d}}(t,s)$ generated by $H^{\mathrm{d}}(t)$ is diagonal, and moreover, as shown in [GOW]:

$$
\begin{aligned}
(2.13) \qquad \mathscr{U}(t,s) &= T(t)\mathscr{U}^{\mathrm{ad}}(t,s)T(s)^{-1} \\
&= T(t)\mathscr{U}^{\mathrm{d}}(t,s)T(s)^{-1} + C^{\infty}(\mathbb{R}^2;\mathcal{W}^{-\infty}(\mathbb{R}^d)).
\end{aligned}
$$

Proposition 2.5. *Assume* (std) *and let $\epsilon^{\pm}(t)$ be as defined in* (2.11). *Then*

$$
\begin{aligned}
(2.14) \qquad \epsilon^{\pm}(t) + b^{\mp}(t) &\in \Psi^{0, -1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d), \\
\epsilon^{\pm}(t) \mp \epsilon(t) &\in \Psi^{0, -1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d),
\end{aligned}
$$

and consequently,

$$
(2.15) \qquad H^{\mathrm{ad}}(t) = \begin{pmatrix} \epsilon(t) & 0 \\ 0 & -\epsilon(t) \end{pmatrix} + \Psi^{0, -1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d) \otimes B(\mathbb{C}^2).
$$

Proof. From the definition of $\epsilon^\pm$ and the fact that $r_{-\infty}^\pm \in \Psi_{\mathrm{std}}^{-\infty,-1-\delta}(\mathbb{R};\mathbb{R}^d)$ one obtains that $\epsilon^\pm = -b^\mp + r_b^\mp$, where

$$r_b^\pm = \mathrm{i} r + [(b^+ - b^-)^{-\frac{1}{2}}, b^\pm] - \mathrm{i}\partial_t (b^+ - b^-)^{-\frac{1}{2}}(b^+ - b^-)^{\frac{1}{2}}.$$

Thus, to get the first part of (2.14) it suffices to prove that $r_b^\pm \in \Psi_{\mathrm{std}}^{0,-1-\delta}(\mathbb{R};\mathbb{R}^d)$. This can be performed in exact analogy to [GW4] using Seeley's theorem and composition properties of the $\Psi_{\mathrm{std}}^{m,\delta}(\mathbb{R};\mathbb{R}^d)$ calculus.

The second part of (2.14) then follows using Prop. 2.4. The last statement is a direct consequence of (2.14) and (2.9)–(2.11).□

2.6. **Symplectic properties of the approximate diagonalization.** It is well known that there is a symplectic form preserved by the Cauchy evolution of P. In our setup this can be written as the identity

(2.16) $$\mathscr{U}(t,s)^* q \mathscr{U}(t,s) = q, \text{ where } q := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

for all $t, s \in \mathbb{R}$. The operators $T(t)$ are defined in such way that

$$T^*(t) q T(t) = q^{\mathrm{ad}}, \text{ where } q^{\mathrm{ad}} := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

and consequently we have for the approximately diagonalized evolution an analogue of (2.16), namely

(2.17) $$\mathscr{U}^{\mathrm{ad}}(t,s)^* q^{\mathrm{ad}} \mathscr{U}^{\mathrm{ad}}(t,s) = q^{\mathrm{ad}}.$$

On the level of the respective generators, this translates to

(2.18) $$H(t)^* q = q H(t), \quad H^{\mathrm{ad}}(t)^* q^{\mathrm{ad}} = q^{\mathrm{ad}} H^{\mathrm{ad}}(t).$$

Using the latter we can deduce that $H^{\mathrm{d}}(t) = H^{\mathrm{d}*}(t)$, and hence $\epsilon^\pm(t)^* = \epsilon^\pm(t)$.

It is convenient to denote by $A^\dagger$ the adjoint with respect to the inner product defined by either q or q^{ad} whenever it is clear from the context which of the two is meant. For instance, this way (2.18) can be rewritten as $H(t)^\dagger = H(t)$ and $H^{\mathrm{ad}}(t)^\dagger = H^{\mathrm{ad}}(t)$, though one has to keep in mind that the former refers to q and the latter to q^{ad}.

3. FEYNMAN INVERSES FROM SCATTERING DATA IN THE MODEL CASE

3.1. **Setup.** In this section we consider again the model Klein-Gordon operator studied in Subsect. 2.5:

(3.1) $$P = \overline{\partial}_t^2 + r(t,\mathrm{x})\overline{\partial}_t + a(t,\mathrm{x},\overline{\partial}_\mathrm{x}),$$

and denote by $P_{\mathrm{out/in}}$ the asymptotic Klein-Gordon operators

$$P_{\mathrm{out/in}} := \overline{\partial}_t^2 + a_{\mathrm{out/in}}(\mathrm{x},\overline{\partial}_\mathrm{x}).$$

We will assume conditions (std) with $\delta > 1$, which corresponds to a *short-range* situation.

By a parametrix for P we will mean an operator G_I such that $PG_I - \mathbf{1}$ and $G_I P - \mathbf{1}$ have smooth Schwartz kernel. Duistermaat and Hörmander proved the existence of a Feynman parametrix G_{F}, or parametrix with *Feynman type wave front set*, i.e.

$$\mathrm{WF}'(G_{\mathrm{F}}) = (\mathrm{diag}_{T^*M}) \cup \bigcup_{s \le 0}(\Phi_s(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\mathcal{N}).$$

This means that up to singularities on the full diagonal diag_{T^*M} of $T^*M \times T^*M$, $\mathrm{WF}'(G_{\mathrm{F}})$ is contained in the backward flowout of diag_{T^*M} by the bicharacteristic flow (here acting on the left component of $T^*M \times T^*M$, accordingly π is the projection to that component).

Our primary goal will be to prove that for suitably chosen Hilbert spaces of distributions $\mathcal{X}_I^m, \mathcal{Y}^m$, the operator $P : \mathcal{X}_I^m \to \mathcal{Y}^m$ is Fredholm, i.e. its kernel and cokernel are of finite dimension. This guarantees the existence of pseudo-inverses, i.e. operators $G_I : \mathcal{Y}^m \to \mathcal{X}_I^m$ such that $PG_I - \mathbf{1}$ and $G_I P - \mathbf{1}$ are compact.

We will be interested in constructing a pseudo-inverse that is at the same time a Feynman parametrix. This will be based on the reduction to the almost diagonalized dynamics $\mathscr{U}^{\mathrm{ad}}(t,s)$ introduced in Subsect. 2.5.

3.2. **Notation.** First, we introduce some further notation needed to relate in an efficient way various objects related to the different dynamics. As a rule, all objects related to the almost diagonalized situation are decorated with a superscript ad. We recall that $L^2(\mathbb{R}^{1+d})$ is equipped with the scalar product

$$(u|v) := \int \overline{u} v |h_t|^{\frac{1}{2}} dt d\mathrm{x}.$$

3.2.1. *Operators.* Let us recall that the operators $H(t)$, $H^{\mathrm{ad}}(t)$, $T(t)$ are defined respectively in (2.8), (2.9), (2.6).

- We set for $u \in C^\infty(\mathbb{R}; \mathscr{D}'(\mathbb{R}^d))$, $u^{\mathrm{ad}} \in C^\infty(\mathbb{R}; \mathscr{D}'(\mathbb{R}^d) \oplus \mathscr{D}'(\mathbb{R}^d))$:

$$\varrho_t u = (u(t), \mathrm{i}^{-1}\partial_t u(t)), \quad \varrho_t^{\mathrm{ad}} u^{\mathrm{ad}} := u^{\mathrm{ad}}(t),$$
$$(Tu^{\mathrm{ad}})(t) := T(t)u^{\mathrm{ad}}(t), \quad (\varrho u)(t) := \varrho_t u(t).$$

Setting also $\pi_i(u_0, u_1) = u_i$ and using the standard notation $D_t = \mathrm{i}^{-1}\overline{\partial}_t$, we have:

(3.2) $$P = -\pi_1(D_t - H(t))\varrho.$$

- We set:

$$P^{\mathrm{ad}} := D_t - H^{\mathrm{ad}}(t),$$

and an easy computation shows that:

(3.3) $$T P^{\mathrm{ad}} T^{-1} = D_t - H(t), \text{ hence } P = -\pi_1 T P^{\mathrm{ad}} T^{-1} \varrho.$$

- We denote by $H_{\mathrm{out/in}}, H_{\mathrm{out/in}}^{\mathrm{ad}}, T_{\mathrm{out/in}}$, the analogues of $H(t), H^{\mathrm{ad}}(t), T(t)$ with $a(t), r(t)$ replaced by $a_{\mathrm{out/in}}, 0$.

- The Cauchy evolutions generated by $H(t)$, $H_{\mathrm{out/in}}$, $H^{\mathrm{ad}}(t)$, $H_{\mathrm{out/in}}^{\mathrm{ad}}$ are denoted by $\mathscr{U}(t,s)$, $\mathscr{U}_{\mathrm{out/in}}(t,s)$, $\mathscr{U}^{\mathrm{ad}}(t,s)$, $\mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(t,s)$. We recall that:

(3.4) $$\mathscr{U}(t,s) = T(t)\mathscr{U}^{\mathrm{ad}}(t,s)T^{-1}(s), \quad \mathscr{U}_{\mathrm{out/in}}(t,s) = T_{\mathrm{out/in}}\mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(t,s)T_{\mathrm{out/in}}^{-1}.$$

We also recall that $\mathscr{U}_{(\mathrm{out/in})}^{(\mathrm{ad})}(t,s)$, $\mathscr{U}_{(\mathrm{out/in})}^{(\mathrm{ad})}(t,s)$ are symplectic for $q^{(\mathrm{ad})}$.

- Operators of the form $a \otimes \mathbf{1}_{\mathbb{C}^2}$ will often be abbreviated by a for simplicity.

3.2.2. *Function spaces.* We will abbreviate by H^m the Sobolev spaces $H^m(\mathbb{R}^d)$.

- Furthermore, we set:

$$\mathcal{E}^m := H^{m+1} \oplus H^m, \quad \mathcal{H}^m := H^m \oplus H^m, \ m \in \mathbb{R}.$$

As usual we define $\mathcal{E}^\infty := \bigcap_{m \in \mathbb{R}} \mathcal{E}^m$, $\mathcal{E}^{-\infty} := \bigcup_{m \in \mathbb{R}} \mathcal{E}^m$ and similarly for $\mathcal{H}^\infty$, $\mathcal{H}^{-\infty}$, equipped with their canonical topologies.

- If $\mathcal{E}$ is a Banach space, $k \in \mathbb{N}$, we denote by $C^k(\mathbb{R}; \mathcal{E})$ the Banach space of $\mathcal{E}-$valued functions with norm

$$\|u\|_{C^k(\mathbb{R};\mathcal{E})} = \sum_{0 \leq l \leq k} \sup_{t \in \mathbb{R}} \|\partial_t^l u(t)\|_\mathcal{E}.$$

We will frequently use the fact that $T(t) : \mathcal{H}^{m+\frac{1}{2}} \to \mathcal{E}^m$ is boundedly invertible with $\|T(t)\|, \|T^{-1}(t)\|$ uniformly bounded in t. From [GW4, Prop. 5.6] we know that:

$$(3.5) \qquad \sup_{t,s \in \mathbb{R}} \|\mathscr{U}^{\mathrm{ad}}(t,s)\|_{B(\mathcal{H}^m)} < \infty, \quad \sup_{t,s \in \mathbb{R}} \|\mathscr{U}(t,s)\|_{B(\mathcal{E}^m)} < \infty.$$

3.3. Møller (wave) operators. We now introduce the relevant objects from scattering theory. We will consider $t = 0$ as our fixed reference time. It is a standard fact, derived using (2.15), (2.14) and the so-called Cook argument (see e.g. [DG1]), that the Møller operators

$$(3.6) \qquad W^{\mathrm{ad}}_{\mathrm{out/in}} := \lim_{t \to \pm\infty} \mathscr{U}^{\mathrm{ad}}(0,t) \mathscr{U}^{\mathrm{ad}}_{\mathrm{out/in}}(t,0) \in B(\mathcal{H}^m)$$

exist and are invertible with inverses given by

$$(3.7) \qquad (W^{\mathrm{ad}}_{\mathrm{out/in}})^{-1} = (W^{\mathrm{ad}}_{\mathrm{out/in}})^\dagger = \lim_{t \to \pm\infty} \mathscr{U}^{\mathrm{ad}}_{\mathrm{out/in}}(0,t) \mathscr{U}^{\mathrm{ad}}(t,0) \in B(\mathcal{H}^m).$$

Using then (3.4) and the fact that $T^{-1}(t) T_{\mathrm{out/in}} - \mathbf{1}$ tends to 0 in $B(\mathcal{H}^m)$ when $t \to \pm\infty$, we obtain the existence of

$$(3.8) \qquad W_{\mathrm{out/in}} := \lim_{t \to \pm\infty} \mathscr{U}(0,t) \mathscr{U}_{\mathrm{out/in}}(t,0) \in B(\mathcal{E}^m),$$

with inverses

$$(3.9) \qquad (W_{\mathrm{out/in}})^{-1} = (W_{\mathrm{out/in}})^\dagger = \lim_{t \to \pm\infty} \mathscr{U}_{\mathrm{out/in}}(0,t) \mathscr{U}(t,0) \in B(\mathcal{E}^m),$$

and satisfying the identities

$$(3.10) \qquad W_{\mathrm{out/in}} = T(0) W^{\mathrm{ad}}_{\mathrm{out/in}} T^{-1}_{\mathrm{out/in}}.$$

Remark 3.1. *Strictly speaking* $W^{(\mathrm{ad})}_{\mathrm{out/in}}$ *acting on* $\mathcal{E}^m$ *or* $\mathcal{H}^m$ *should be denoted by, e.g.,* $W^{(\mathrm{ad}),m}_{\mathrm{out/in}}$ *to indicate its dependence on* m. *However since* $W^{(\mathrm{ad}),m}_{\mathrm{out/in}}$ *is the closure of* $W^{(\mathrm{ad}),m'}_{\mathrm{out/in}}$ *for any* $m' > m$, *we will often dispense with the exponent* m *in the sequel. The same remark applies to* $(W^{(\mathrm{ad})}_{\mathrm{out/in}})^{-1}$.

3.4. Compactness properties of $W^{\mathrm{ad}}_{\mathrm{out/in}}$. Let us denote by $\pi^\pm$ the projections

$$(3.11) \qquad \pi^+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

The decay properties implied by assumption (std) have the following important consequence.

Proposition 3.2. *Assume condition* (std) *with* $\delta > 1$ *and let* $\alpha < \delta/2$. *Then*

$$W^{\mathrm{ad}}_{\mathrm{out/in}} \pi^+ (W^{\mathrm{ad}}_{\mathrm{out/in}})^{-1} - \pi^+ \in \langle \mathrm{x} \rangle^{-\alpha} \mathcal{W}^{-\infty}(\mathbb{R}^d) \langle \mathrm{x} \rangle^{-\alpha} \otimes B(\mathbb{C}^2).$$

It follows that $[W^{\mathrm{ad}}_{\mathrm{out/in}}, \pi^+]$ *is a compact operator on* $\mathcal{H}^m$ *for* $m \in \mathbb{R}$.

To prove Prop. 3.2, we will need the following lemma, the proof of which is given in Appendix A.2.

Lemma 3.3. *Assume conditions* (std) *for* $\delta > 0$. *Then for all* $m, k \in \mathbb{R}^+$:

$$\sup_{t \geq 0} \|\langle D_{\mathrm{x}} \rangle^m \langle \mathrm{x} \rangle^k \mathscr{U}^{\mathrm{ad}}(0,t) (\langle \mathrm{x} \rangle + \langle t \rangle)^{-k} \langle D_{\mathrm{x}} \rangle^{-m}\|_{B(\mathcal{H}^0)} < \infty.$$

Proof of Prop. 3.2. Let us set $W^{\mathrm{ad}}_{\mathrm{out/in}}(t) = \mathscr{U}^{\mathrm{ad}}(0,t)\mathscr{U}^{\mathrm{ad}}_{\mathrm{out/in}}(t,0)$. We have for $m \in \mathbb{N}$, $\alpha > 0$:

$$
\begin{aligned}
&\langle D_{\mathrm{x}}\rangle^m \langle \mathrm{x}\rangle^\alpha \partial_t (W^{\mathrm{ad}}_{\mathrm{out/in}}(t)\pi^+ W^{\mathrm{ad}}_{\mathrm{out/in}}(t)^{-1}) \langle \mathrm{x}\rangle^\alpha \langle D_{\mathrm{x}}\rangle^m \\
&= \langle D_{\mathrm{x}}\rangle^m \langle \mathrm{x}\rangle^\alpha \mathscr{U}^{\mathrm{ad}}(0,t)[R_{-\infty}(t), \pi^+]\mathscr{U}^{\mathrm{ad}}(t,0)\langle \mathrm{x}\rangle^\alpha \langle D_{\mathrm{x}}\rangle^m \\
&= \langle D_{\mathrm{x}}\rangle^m \langle \mathrm{x}\rangle^\alpha \mathscr{U}^{\mathrm{ad}}(0,t)(\langle \mathrm{x}\rangle + \langle t\rangle)^{-\alpha}\langle D_{\mathrm{x}}\rangle^{-m} \\
&\quad \times \langle D_{\mathrm{x}}\rangle^m (\langle \mathrm{x}\rangle + \langle t\rangle)^\alpha [R_{-\infty}(t), \pi^+](\langle \mathrm{x}\rangle + \langle t\rangle)^\alpha \langle D_{\mathrm{x}}\rangle^m \\
&\quad \times \langle D_{\mathrm{x}}\rangle^{-m}(\langle \mathrm{x}\rangle + \langle t\rangle)^{-\alpha}\mathscr{U}^{\mathrm{ad}}(t,0)\langle \mathrm{x}\rangle^\alpha \langle D_{\mathrm{x}}\rangle^m \\
&=: R_{m,\alpha}(t) \times M_{m,\alpha}(t) \times R_{m,\alpha}(t)^\dagger.
\end{aligned}
$$ (3.12)

Since $R_{-\infty}(t) \in \Psi^{-\infty,-1-\delta}_{\mathrm{std}}(\mathbb{R}; \mathbb{R}^d) \otimes B(\mathbb{C}^2)$ we know that

$$\|M_{m,\alpha}(t)\|_{B(\mathcal{H}^0)} \in O(\langle t\rangle^{-1-\delta+2\alpha}).$$

From Lemma 3.3 we know that $\|R_{m,\alpha}(t)\|_{B(\mathcal{H}^0)} \in O(1)$, which implies the same bound for $R_{m,\alpha}(t)^\dagger$. Thus from (3.12) we obtain that

$$\langle D_{\mathrm{x}}\rangle^m \langle \mathrm{x}\rangle^\alpha \partial_t (W^{\mathrm{ad}}_{\mathrm{out/in}}(t)\pi^+ W^{\mathrm{ad}}_{\mathrm{out/in}}(t)^{-1})\langle \mathrm{x}\rangle^\alpha \langle D_{\mathrm{x}}\rangle^m \in O(\langle t\rangle^{-1-\delta+2\alpha})$$

in $B(\mathcal{H}^0)$. This is integrable for $\alpha < \delta/2$. By integrating from $t=0$ to $t = \pm\infty$, since m is arbitrary this implies that:

$$\lim_{t\to\pm\infty} W^{\mathrm{ad}}_{\mathrm{out/in}}(t)\pi^+ W^{\mathrm{ad}}_{\mathrm{out/in}}(t)^{-1} - \pi^+ \in \langle \mathrm{x}\rangle^{-\alpha}\mathcal{W}^{-\infty}(\mathbb{R}^d)\langle \mathrm{x}\rangle^{-\alpha}.$$

Since $W^{\mathrm{ad}}_{\mathrm{out/in}} = \lim_{t\to\pm\infty} W^{\mathrm{ad}}_{\mathrm{out/in}}(t)$ this proves the proposition. $\square$

3.5. Inhomogeneous Cauchy problem.

Fixing γ with $\frac{1}{2} < \gamma < \frac{1}{2} + \delta$, we set:

$$\mathcal{Y}^m := \langle t\rangle^{-\gamma}L^2(\mathbb{R}; H^m), \quad \mathcal{Y}^{\mathrm{ad},m} := \langle t\rangle^{-\gamma}L^2(\mathbb{R}; \mathcal{H}^m).$$

The exponent γ is chosen so that $\mathcal{Y}^m \subset L^1(\mathbb{R}; \mathcal{H}^m)$, $\mathcal{Y}^{\mathrm{ad},m} \subset L^1(\mathbb{R}; \mathcal{H}^m)$. The benefit of working with $\mathcal{Y}^{(\mathrm{ad}),m}$ rather than with L^1 spaces is that the former are Hilbert spaces; this will be needed in Subsect. 4.5.

Definition 3.4. We denote by $\mathcal{X}^m$ the space of $u \in C^1(\mathbb{R}; H^{m+1}) \cap C^0(\mathbb{R}; H^m)$ such that $Pu \in \mathcal{Y}^m$, and similarly by $\mathcal{X}^{\mathrm{ad},m}$ the space of $u^{\mathrm{ad}} \in C^0(\mathbb{R}; \mathcal{H}^m)$ such that $P^{\mathrm{ad}}u^{\mathrm{ad}} \in \mathcal{Y}^{\mathrm{ad},m}$. We equip $\mathcal{X}^{(\mathrm{ad}),m}$ with the Hilbert norms:

$$
\begin{aligned}
\|u^{\mathrm{ad}}\|_m^2 &:= \|\varrho_0^{\mathrm{ad}}u^{\mathrm{ad}}\|_{\mathcal{H}^m}^2 + \|P^{\mathrm{ad}}u^{\mathrm{ad}}\|_{\mathcal{Y}^{\mathrm{ad},m}}^2, \\
\|u\|_m^2 &:= \|\varrho_0 u\|_{\mathcal{E}^m}^2 + \|Pu\|_{\mathcal{Y}^m}^2.
\end{aligned}
$$ (3.13)

The existence and uniqueness of the inhomogeneous Cauchy problem for P and P^{ad} implies that $\mathcal{X}^{(\mathrm{ad}),m}$ are Hilbert spaces, as stated implicitly in the following lemma.

Lemma 3.5. The map

$$
\begin{aligned}
\varrho_0 \oplus P : \mathcal{X}^m &\to \mathcal{E}^m \oplus \mathcal{Y}^m \\
u &\mapsto (\varrho_0 u, Pu)
\end{aligned}
$$ (3.14)

is boundedly invertible with inverse given by:

$$(\varrho_0 \oplus P)^{-1}(v, f) = \pi_0 \mathscr{U}(t,0)v - \mathrm{i}\pi_0 \int_0^t \mathscr{U}(t,s)\pi_1^* f(s)ds.$$ (3.15)

Similarly, the map

$$
\begin{aligned}
\varrho_0^{\mathrm{ad}} \oplus P^{\mathrm{ad}} : \mathcal{X}^{\mathrm{ad},m} &\to \mathcal{H}^m \oplus \mathcal{Y}^{\mathrm{ad},m} \\
u^{\mathrm{ad}} &\mapsto (\varrho_0^{\mathrm{ad}}u^{\mathrm{ad}}, P^{\mathrm{ad}}u^{\mathrm{ad}})
\end{aligned}
$$ (3.16)

is boundedly invertible with inverse given by:

$$(3.17) \qquad (\varrho_0^{\mathrm{ad}} \oplus P^{\mathrm{ad}})^{-1}(v^{\mathrm{ad}}, f^{\mathrm{ad}}) = \mathscr{U}^{\mathrm{ad}}(t,0)v^{\mathrm{ad}} + \mathrm{i}\int_0^t \mathscr{U}^{\mathrm{ad}}(t,s)f^{\mathrm{ad}}(s)ds.$$

It follows that

$$(3.18) \qquad \begin{aligned} \mathcal{X}^m &\hookrightarrow C^k(\mathbb{R}; H^{m+1-k}), \\ \mathcal{X}^{\mathrm{ad},m} &\hookrightarrow C^k(\mathbb{R}; \mathcal{H}^{m-k}), \end{aligned}$$

continuously for $m \in \mathbb{R}, \ k \in \mathbb{N}$.

The following facts are the result of easy computations that make use of (3.3):

$$(3.19) \qquad \begin{aligned} T^{-1}\varrho &\in B(\mathcal{X}^m, \mathcal{X}^{\mathrm{ad},m+\frac{1}{2}}), \quad -T^{-1}\pi_1^* \in B(\mathcal{Y}^m, \mathcal{Y}^{\mathrm{ad},m+\frac{1}{2}}), \\ \pi_0 T &\in B(\mathcal{X}^{\mathrm{ad},m+\frac{1}{2}}, \mathcal{X}^m), \quad -\pi_1 T \in B(\mathcal{Y}^{\mathrm{ad},m+\frac{1}{2}}, \mathcal{Y}^m). \end{aligned}$$

In the sequel we will also need the auxiliary identities

$$(3.20) \qquad \begin{aligned} \operatorname{Ran} T^{-1}\varrho &= \operatorname{Ker}(\varrho\pi_0 - \mathbf{1})T, \\ (T^{-1}\varrho)^{-1} &= \pi_0 T \ \text{ on } \ \operatorname{Ran} T^{-1}\varrho, \\ \operatorname{Ran} T^{-1}\pi_1^* &= \operatorname{Ker} \pi_0 T, \\ (T^{-1}\pi_1^*)^{-1} &= \pi_1 T \ \text{ on } \ \operatorname{Ran} T^{-1}\pi_1^*. \end{aligned}$$

3.6. Retarded and advanced propagators.

The retarded/advanced propagators for P^{ad} are defined as follows:

$$(3.21)$$
$$(G_+^{\mathrm{ad}} f^{\mathrm{ad}})(t) := \mathrm{i}\int_{-\infty}^t \mathscr{U}^{\mathrm{ad}}(t,s)f^{\mathrm{ad}}(s)ds, \quad (G_-^{\mathrm{ad}} f)(t) := -\mathrm{i}\int_t^{+\infty} \mathscr{U}^{\mathrm{ad}}(t,s)f^{\mathrm{ad}}(s)ds,$$

for $f^{\mathrm{ad}} \in L^1(\mathbb{R}; \mathcal{H}^m)$. Using (3.5) one obtains:

$$\begin{aligned} G_\pm^{\mathrm{ad}} &\in B(L^1(\mathbb{R}; \mathcal{H}^m), C^0(\mathbb{R}; \mathcal{H}^m)), \\ (G_\pm^{\mathrm{ad}})^\dagger &= G_\mp^{\mathrm{ad}} \text{ on } L^1(\mathbb{R}; \mathcal{H}^m), \quad P^{\mathrm{ad}}G_\pm^{\mathrm{ad}} = \mathbf{1} \text{ on } L^1(\mathbb{R}; \mathcal{H}^m). \end{aligned}$$

The analogous propagators for P are:

$$(3.22)$$
$$(G_+ f)(t) = -\mathrm{i}\pi_0 \int_{-\infty}^t \mathscr{U}(t,s)\pi_1^* f ds, \quad (G_- f)(t) = \mathrm{i}\pi_0 \int_t^{+\infty} \mathscr{U}(t,s)\pi_1^* f(s)ds,$$

for $f \in L^1(\mathbb{R}; H^m)$. One has:

$$\begin{aligned} G_\pm &\in B(L^1(\mathbb{R}; H^m), C^0(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m)), \\ G_\pm^* &= G_\mp \text{ on } L^1(\mathbb{R}; H^m), \quad PG_\pm = \mathbf{1} \text{ on } L^1(\mathbb{R}; H^m). \end{aligned}$$

Using (3.4) the relation between the propagators for P and P^{ad} is:

$$(3.23) \qquad G_\pm = -\pi_0 T G_\pm^{\mathrm{ad}} T^{-1}\pi_1^*.$$

3.7. Fredholm problems from scattering data.

We now want to define the maps that assign to an element of $\mathcal{X}^{(\mathrm{ad}),m}$ its scattering data in the standard sense, as well as its Feynman and anti-Feynman data. By *Feynman data* we mean positive-frequency data of a solution at $+\infty$ and negative-frequency data at $-\infty$, and by *anti-Feynman* the reverse.

Proposition 3.6. *The strong operator limits*

$$\mathrm{s}-\lim_{t\to\pm\infty} \mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(0,t)\varrho_t^{\mathrm{ad}}, \quad \text{resp. } \mathrm{s}-\lim_{t\to\pm\infty} \mathscr{U}_{\mathrm{out/in}}(0,t)\varrho_t,$$

exist in $B(\mathcal{X}^{\mathrm{ad},m}, \mathcal{H}^m)$, *resp. in* $B(\mathcal{X}^m, \mathcal{E}^m)$, *and equal* $(W_{\mathrm{out/in}}^{\mathrm{ad}})^{-1}$ *on* $\operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}}$, *resp.* $(W_{\mathrm{out/in}})^{-1}$ *on* $\operatorname{Ker} P|_{\mathcal{X}^m}$.

Proof. Let $u^{\mathrm{ad}} \in \mathcal{X}^{\mathrm{ad},m}$. By Lemma 3.5 we have

$$\mathscr{U}_{\mathrm{out}}^{\mathrm{ad}}(0,t)\varrho_t^{\mathrm{ad}} u^{\mathrm{ad}} = \mathscr{U}_{\mathrm{out}}^{\mathrm{ad}}(0,t)\mathscr{U}^{\mathrm{ad}}(t,0)v^{\mathrm{ad}}$$

$$+ \mathrm{i}\int_0^t \mathscr{U}_{\mathrm{out}}^{\mathrm{ad}}(0,t)\mathscr{U}^{\mathrm{ad}}(t,0)\mathscr{U}^{\mathrm{ad}}(0,s)f^{\mathrm{ad}}(s)ds,$$

which by dominated convergence tends to $(W_{\mathrm{out}}^{\mathrm{ad}})^{-1}(v^{\mathrm{ad}} - \varrho_0^{\mathrm{ad}}G_-^{\mathrm{ad}}f^{\mathrm{ad}})$ as $t \to +\infty$. Similarly we obtain that $\mathscr{U}_{\mathrm{in}}^{\mathrm{ad}}(0,t)\varrho_t^{\mathrm{ad}} u^{\mathrm{ad}}$ converges to $(W_{\mathrm{in}}^{\mathrm{ad}})^{-1}(v^{\mathrm{ad}} - \varrho_0^{\mathrm{ad}}G_+^{\mathrm{ad}}f^{\mathrm{ad}})$ as $t \to -\infty$. The proof in the scalar case is similar. $\square$

We can now introduce four scattering data maps $\varrho_I^{(\mathrm{ad})} : \mathcal{X}^{(\mathrm{ad}),m} \to \mathcal{H}^m$. Note the presence of the operators $T_{\mathrm{out}/\mathrm{in}}^{-1}$ below; this simplifies some considerations later on.

Definition 3.7. *We set:*

$$\varrho_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}} := \mathrm{s}-\lim_{t\to\pm\infty} \mathscr{U}_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}}(0,t)\varrho_t^{\mathrm{ad}},$$

$$\varrho_{\mathrm{out}/\mathrm{in}} := \mathrm{s}-\lim_{t\to\pm\infty} T_{\mathrm{out}/\mathrm{in}}^{-1}\mathscr{U}_{\mathrm{out}/\mathrm{in}}(0,t)\varrho_t,$$

$$\varrho_{\mathrm{F}}^{(\mathrm{ad})} := \pi^+ \varrho_{\mathrm{out}}^{(\mathrm{ad})} + \pi^- \varrho_{\mathrm{in}}^{(\mathrm{ad})},$$

$$\varrho_{\overline{\mathrm{F}}}^{(\mathrm{ad})} := \pi^- \varrho_{\mathrm{out}}^{(\mathrm{ad})} + \pi^+ \varrho_{\mathrm{in}}^{(\mathrm{ad})}.$$

Lemma 3.8. *For* $I \in \{\mathrm{in}, \mathrm{out}, \mathrm{F}, \overline{\mathrm{F}}\}$ *we have:*

$$(3.24) \qquad\qquad\qquad \varrho_I = \varrho_I^{\mathrm{ad}} T^{-1} \varrho,$$

$$\varrho_I^{\mathrm{ad}} = W_I^{\mathrm{ad}\dagger} \circ \varrho_0^{\mathrm{ad}} \quad on \ \mathrm{Ker}\, P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}},$$

$$(3.25)$$

$$\varrho_I = W_I^{\mathrm{ad}\dagger} T^{-1}(0)\varrho_0 \quad on \ \mathrm{Ker}\, P|_{\mathcal{X}^m},$$

for $I \in \{\mathrm{in}, \mathrm{out}, \mathrm{F}, \overline{\mathrm{F}}\}$, *where*

$$(3.26) \qquad W_{\mathrm{F}}^{\mathrm{ad}\dagger} := \pi^+ W_{\mathrm{out}}^{\mathrm{ad}\dagger} + \pi^- W_{\mathrm{in}}^{\mathrm{ad}\dagger}, \quad W_{\overline{\mathrm{F}}}^{\mathrm{ad}\dagger} := \pi^- W_{\mathrm{out}}^{\mathrm{ad}\dagger} + \pi^+ W_{\mathrm{in}}^{\mathrm{ad}\dagger}.$$

Proof. To prove (3.24) we write:

$$\varrho_{\mathrm{out}/\mathrm{in}} = T_{\mathrm{out}/\mathrm{in}}^{-1}\mathscr{U}_{\mathrm{out}/\mathrm{in}}(0,t)\varrho_t + o(1) = \mathscr{U}_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}}(0,t)T_{\mathrm{out}/\mathrm{in}}^{-1}\varrho_t + o(1)$$

$$= \mathscr{U}_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}}(0,t)T^{-1}(t)\varrho_t + o(1) = \mathscr{U}_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}}(0,t)\varrho_t^{\mathrm{ad}} T^{-1}\varrho + o(1).$$

This implies (3.24) for $I = \mathrm{out}/\mathrm{in}$ and then for $I = \mathrm{F}/\overline{\mathrm{F}}$. The first statement of (3.25) follows then from the fact that $\varrho_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}} = W_{\mathrm{out}/\mathrm{in}}^{\mathrm{ad}\dagger}$ on $\mathrm{Ker}\, P^{\mathrm{ad}}$, the second from (3.24) and the fact that $T^{-1}\varrho : \mathrm{Ker}\, P \to \mathrm{Ker}\, P^{\mathrm{ad}}$. $\square$

Lemma 3.9. *Let* $I \in \{\mathrm{F}, \overline{\mathrm{F}}\}$. *Then* $W_I^{\mathrm{ad}} W_I^{\mathrm{ad}\dagger} - \mathbf{1}$ *and* $W_I^{\mathrm{ad}\dagger} W_I^{\mathrm{ad}} - \mathbf{1}$ *are compact on* $\mathcal{H}^m$ *and hence* $W_I^{\mathrm{ad}}, W_I^{\mathrm{ad}\dagger}$ *are Fredholm. Moreover:*

$$\mathrm{Ker}\, W_I^{\mathrm{ad}(\dagger)}|_{\mathcal{H}^m} = \mathrm{Ker}\, W_I^{\mathrm{ad}(\dagger)}|_{\mathcal{E}^{\mathrm{ad},\infty}}, \quad \mathrm{coKer}\, W_I^{\mathrm{ad}(\dagger)}|_{\mathcal{H}^m} = \mathrm{coKer}\, W_I^{\mathrm{ad}(\dagger)}|_{\mathcal{E}^{\mathrm{ad},-\infty}},$$

and hence $\mathrm{ind}(W_I^{\mathrm{ad}(\dagger)})|_{\mathcal{H}^m}$ *is independent on* m.

Proof. We consider only the F case. We have

$$W_{\mathrm{F}}^{\mathrm{ad}} W_{\mathrm{F}}^{\mathrm{ad}\dagger} = \mathbf{1} + K_1, \quad K_1 = W_{\mathrm{out}}^{\mathrm{ad}}\pi^+(W_{\mathrm{out}}^{\mathrm{ad}})^{-1} - \pi^+ + W_{\mathrm{in}}^{\mathrm{ad}}\pi^-(W_{\mathrm{in}}^{\mathrm{ad}})^{-1} - \pi^-,$$

$$W_{\mathrm{F}}^{\mathrm{ad}\dagger} W_{\mathrm{F}}^{\mathrm{ad}} = \mathbf{1} + K_2, \quad K_2 = \pi^+(W_{\mathrm{out}}^{\mathrm{ad}})^{-1}W_{\mathrm{in}}^{\mathrm{ad}}\pi^- + \pi^-(W_{\mathrm{in}}^{\mathrm{ad}})^{-1}W_{\mathrm{out}}^{\mathrm{ad}}\pi^+.$$

By Prop. 3.2 we see that K_1, K_2 are compact on $\mathcal{H}^m$ and moreover map $\mathcal{H}^m$ to $\mathcal{H}^\infty$. Therefore $\mathrm{Ker}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^m} \subset \mathrm{Ker}(\mathbf{1} + K_1)|_{\mathcal{H}^m} \subset \mathcal{H}^\infty$ hence $\mathrm{Ker}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^m} = \mathrm{Ker}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^\infty}$. Similarly, identifying $(\mathcal{H}^m)^*$ with $\mathcal{H}^{-m}$ and $\mathrm{coKer}\, A$ with $\mathrm{Ker}\, A^*$ we have $\mathrm{coKer}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^m} \subset \mathrm{Ker}(\mathbf{1} + K_2^*)|_{\mathcal{H}^m} \subset \mathcal{H}^\infty$, hence $\mathrm{coKer}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^m} = \mathrm{coKer}\, W_{\mathrm{F}}^{\mathrm{ad}\dagger}|_{\mathcal{H}^{-\infty}}$. $\square$

We will need the following lemma, see [BB, Prop. A.1] for its proof. The next few results are obtained as simple applications of it, following the strategy in [BS1] (originally applied to the case of the Dirac equation on compact space-times with space-like boundary).

Lemma 3.10. *Let $\mathcal{K}$ be a Hilbert space and $\mathcal{E}$, $\mathcal{F}$ Banach spaces. Let $K : \mathcal{K} \to \mathcal{E}$, $Q : \mathcal{K} \to \mathcal{F}$ be bounded and assume that Q is surjective. Then $K : \operatorname{Ker} Q \to \mathcal{E}$ is Fredholm (of index l) iff $K \oplus Q : \mathcal{K} \to \mathcal{E} \oplus \mathcal{F}$ is Fredholm (of index l).*

Lemma 3.11. *For $I \in \{\mathrm{in}, \mathrm{out}, \mathrm{F}, \overline{\mathrm{F}}\}$, the operator*
$$\varrho_I^{(\mathrm{ad})} : \ \{u^{(\mathrm{ad})} \in \mathcal{X}^{(\mathrm{ad}),m} : \ P^{(\mathrm{ad})} u^{(\mathrm{ad})} = 0\} \to \mathcal{H}^m$$
is Fredholm of index equal $\operatorname{ind} W_I^{\mathrm{ad}\dagger}$ and is invertible for $I \in \{\mathrm{in}, \mathrm{out}\}$.

Proof. We apply (3.25) and the fact that $\varrho_0^{\mathrm{ad}} : \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}} \to \mathcal{H}^m$ and $T^{-1}(0)\varrho_0 : \operatorname{Ker} P|_{\mathcal{X}^m} \to \mathcal{H}^{m+\frac{1}{2}}$ are bijections, by Lemma 3.5. $\square$

Lemma 3.12. *The maps*
$$\varrho_I^{\mathrm{ad}} \oplus P^{\mathrm{ad}} : \mathcal{X}^{\mathrm{ad},m} \to \mathcal{H}^m \oplus \mathcal{Y}^{\mathrm{ad},m},$$
$$\varrho_I \oplus P : \mathcal{X}^m \to \mathcal{H}^{m+\frac{1}{2}} \oplus \mathcal{Y}^m,$$
are Fredholm of index $\operatorname{ind} W_I^{\mathrm{ad}\dagger}$.

Proof. We use Lemma 3.10 with $\mathcal{K} = \mathcal{X}^{\mathrm{ad},m}$ resp. $\mathcal{X}^m$, $\mathcal{E} = \mathcal{H}^m$ resp. $\mathcal{H}^{m+\frac{1}{2}}$, $\mathcal{F} = \mathcal{Y}^{\mathrm{ad},m}$ resp. $\mathcal{Y}^m$, $K = \varrho_I^{\mathrm{ad}}$ resp. ϱ_I, $Q = P^{\mathrm{ad}}$ resp. P. The assumptions of Lemma 3.10 are satisfied in view of Lemma 3.11 and Lemma 3.5 which gives surjectivity of P^{ad} resp. P. $\square$

Let us introduce the following notation: if $I = \mathrm{in/out}$ then $I^{\mathrm{c}} := \mathrm{out/in}$ and if $I = \mathrm{F}/\overline{\mathrm{F}}$ then $I^{\mathrm{c}} := \overline{\mathrm{F}}/\mathrm{F}$.

Theorem 3.13. *Assume* (std) *with $\delta > 1$. Let*
$$\mathcal{X}_I^{(\mathrm{ad}),m} := \{u \in \mathcal{X}^{(\mathrm{ad}),m} : \varrho_{I^{\mathrm{c}}}^{(\mathrm{ad})} u = 0\},$$
equipped with the topology of $\mathcal{X}^{(\mathrm{ad}),m}$. Then
$$P^{\mathrm{ad}} : \mathcal{X}_I^{\mathrm{ad},m} \to \mathcal{Y}^{\mathrm{ad},m},$$
$$P : \mathcal{X}_I^m \to \mathcal{Y}^m$$
are Fredholm of index $\operatorname{ind} W_{I^{\mathrm{c}}}^{\mathrm{ad}\dagger}$.

Proof. It suffices to check the assumptions of Lemma 3.10 for $\mathcal{K} = \mathcal{X}^{\mathrm{ad},m}$ resp. $\mathcal{X}^m$, $\mathcal{E} = \mathcal{Y}^{\mathrm{ad},m}$ resp. $\mathcal{Y}^m$, $\mathcal{F} = \mathcal{H}^m$ resp. $\mathcal{H}^{m+\frac{1}{2}}$, $K = P^{\mathrm{ad}}$ resp. P, and $Q = \varrho_{I^{\mathrm{c}}}^{\mathrm{ad}}$ resp. $\varrho_{I^{\mathrm{c}}}$. The Fredholm property of $K \oplus Q$ follows from Lemma 3.12, so it remains to check that $\varrho_{I^{\mathrm{c}}}^{\mathrm{ad}} : \mathcal{X}^{\mathrm{ad},m} \to \mathcal{H}^m$ and $\varrho_{I^{\mathrm{c}}} : \mathcal{X}^m \to \mathcal{H}^{m+\frac{1}{2}}$ are surjective. This is obvious if $I = \mathrm{out/in}$ using (3.25) and Lemma 3.5. Let us now consider the case $I = \mathrm{F}$.

Let $\eta_{\mathrm{out/in}} \in C^\infty(\mathbb{R})$ with $\eta_{\mathrm{in}}(t) + \eta_{\mathrm{out}}(t) = 1$ and $\eta_{\mathrm{out/in}}(t) = 1$ for large $\pm t$. Then
$$\varrho_{\mathrm{out/in}}^{(\mathrm{ad})} \circ \eta_{\mathrm{out/in}} = \varrho_{\mathrm{out/in}}^{(\mathrm{ad})}, \ \ \varrho_{\mathrm{in/out}}^{(\mathrm{ad})} \circ \eta_{\mathrm{out/in}} = 0.$$
Furthermore $\eta_{\mathrm{out/in}} \operatorname{Ker} P^{(\mathrm{ad})}|_{\mathcal{X}^{(\mathrm{ad}),m}} \subset \mathcal{X}^{(\mathrm{ad}),m}$. It follows that
$$\varrho_{\mathrm{F}}^{\mathrm{ad}} \mathcal{X}^{\mathrm{ad},m} \supset \varrho_{\mathrm{F}}^{\mathrm{ad}} (\eta_{\mathrm{in}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}} + \eta_{\mathrm{out}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}})$$
$$= (\pi^+ \varrho_{\mathrm{out}}^{\mathrm{ad}} + \pi^- \varrho_{\mathrm{in}}^{\mathrm{ad}})(\eta_{\mathrm{in}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}} + \eta_{\mathrm{out}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}})$$
$$= \pi^+ \varrho_{\mathrm{out}}^{\mathrm{ad}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}} + \pi^- \varrho_{\mathrm{in}}^{\mathrm{ad}} \operatorname{Ker} P^{\mathrm{ad}}|_{\mathcal{X}^{\mathrm{ad},m}}$$
$$= \pi^+ \mathcal{H}^m + \pi^- \mathcal{H}^m = \mathcal{H}^m.$$

This proves $\varrho_{\mathrm{F}}^{\mathrm{ad}} : \mathcal{X}^{\mathrm{ad},m} \to \mathcal{H}^m$ is surjective. The same argument shows that $\varrho_{\mathrm{F}} : \mathcal{X}^m \to \mathcal{H}^{m+\frac{1}{2}}$ is surjective. In the analogous way we obtain surjectivity of $\varrho_{\overline{\mathrm{F}}}^{\mathrm{ad}}$ and $\varrho_{\overline{\mathrm{F}}}$. $\square$

3.8. Retarded/advanced propagators.
We now show that as anticipated, the retarded/advanced propagators $G_{\pm}^{\mathrm{ad}}$ are the inverses of $P^{\mathrm{ad}} : \mathcal{X}_{\mathrm{out/in}}^{\mathrm{ad},m} \to \mathcal{Y}^{\mathrm{ad},m}$, and a similar statement holds true in the scalar case.

Proposition 3.14. $P^{(\mathrm{ad})} : \mathcal{X}_{\mathrm{out/in}}^{(\mathrm{ad}),m} \to \mathcal{Y}^{(\mathrm{ad}),m}$ *are boundedly invertible with inverse equal to* $G_{\pm}^{(\mathrm{ad})}$.

Proof. We only treat the case of $G_{+}^{(\mathrm{ad})}$, the other one being analogous. We have seen in Subsect. 3.6 that

$$G_{+}^{\mathrm{ad}} \in B(L^1(\mathbb{R}; \mathcal{H}^m), C^0(\mathbb{R}; \mathcal{H}^m))$$

and $P^{\mathrm{ad}} G_{+}^{\mathrm{ad}} = \mathbf{1}$ on $L^1(\mathbb{R}; \mathcal{H}^m)$, hence $G_{+}^{\mathrm{ad}} \in B(\mathcal{Y}^{\mathrm{ad},m}, \mathcal{X}^{\mathrm{ad},m})$ and $P^{\mathrm{ad}} G_{+}^{\mathrm{ad}} = \mathbf{1}$ on $\mathcal{Y}^{\mathrm{ad},m}$. Since $\lim_{t \to -\infty} \varrho_t^{\mathrm{ad}} G_{+}^{\mathrm{ad}} f^{\mathrm{ad}} = 0$, we have $G_{+}^{\mathrm{ad}} \mathcal{Y}^{\mathrm{ad},m} \subset \mathcal{X}_{\mathrm{out}}^{\mathrm{ad},m}$. It remains to show that $G_{+}^{\mathrm{ad}} P^{\mathrm{ad}} = \mathbf{1}$ on $\mathcal{X}_{\mathrm{out}}^{\mathrm{ad},m}$. If $u^{\mathrm{ad}} \in \mathcal{X}_{\mathrm{out}}^{\mathrm{ad},m}$ we have:

$$
\begin{aligned}
(G_{+}^{\mathrm{ad}} P^{\mathrm{ad}} u^{\mathrm{ad}})(t) &= \int_{-\infty}^{t} \mathscr{U}^{\mathrm{ad}}(t,s)(\overline{\partial}_s - iH^{\mathrm{ad}}(s)) u^{\mathrm{ad}} ds \\
&= \lim_{t_+ \to -\infty} \int_{-\infty}^{t} \mathscr{U}^{\mathrm{ad}}(t,s)(\overline{\partial}_s - iH^{\mathrm{ad}}(s)) u^{\mathrm{ad}} ds \\
&= \lim_{t_+ \to -\infty} \left[\mathscr{U}^{\mathrm{ad}}(t,s) u(s) \right]_{t_+}^{t} \\
&\quad - \lim_{t_+ \to -\infty} \int_{t_+}^{t} (-\overline{\partial}_s + iH^{\mathrm{ad}}(s)) \mathscr{U}^{\mathrm{ad}}(t,s) u^{\mathrm{ad}}(s) ds \\
&= u^{\mathrm{ad}}(t),
\end{aligned}
$$

since $\lim_{t_+ \to -\infty} u^{\mathrm{ad}}(t_+) = 0$ in view of $u^{\mathrm{ad}} \in \mathcal{X}_{\mathrm{out}}^{\mathrm{ad},m}$. In the scalar case we obtain from (3.3) that $(D_t - H(t)) T G_{+}^{\mathrm{ad}} T^{-1} = \mathbf{1}$ hence $(\varrho \pi_0 - \mathbf{1}) T G_{+}^{\mathrm{ad}} T^{-1} \pi_1^* = 0$ which implies that $P G_+ = \mathbf{1}$ on $\mathcal{Y}^m$. Conversely, by (3.19), (3.24) we know that $T^{-1} \varrho : \mathcal{X}_{\mathrm{out}}^m \to \mathcal{X}_{\mathrm{out}}^{\mathrm{ad},m+\frac{1}{2}}$. Since $G_{+}^{\mathrm{ad}} P^{\mathrm{ad}} = \mathbf{1}$ on $\mathcal{X}^{\mathrm{ad},m+\frac{1}{2}}$ this yields

$$T G_{+}^{\mathrm{ad}} T^{-1} (D_t - H(t)) \varrho = T G_{+}^{\mathrm{ad}} P^{\mathrm{ad}} T^{-1} \varrho = \varrho \text{ on } \mathcal{X}_{\mathrm{out}}^m,$$

hence $G_+ P = \mathbf{1}$ on $\mathcal{X}_{\mathrm{out}}^m$ using $(D_t - H(t)) \varrho = \pi_1^* \pi_1 (D_t - H(t))$. This completes the proof. $\square$

3.9. The Fredholm inverses for $P^{(\mathrm{ad})}$ on $\mathcal{X}_{\mathrm{F}}^{(\mathrm{ad}),m}$.
From Thm. 3.13 we know that $P^{(\mathrm{ad})} : \mathcal{X}_{\mathrm{F}}^{(\mathrm{ad}),m} \to \mathcal{Y}^{(\mathrm{ad}),m}$ are Fredholm. We will now construct explicit approximate inverses $G_{\mathrm{F}}^{(\mathrm{ad})}$ of $P^{(\mathrm{ad})} : \mathcal{X}_{\mathrm{F}}^{(\mathrm{ad}),m} \to \mathcal{Y}^m$, which requires some special care because of the requirement $\varrho_{\overline{\mathrm{F}}}^{(\mathrm{ad})} \circ G_{\mathrm{F}}^{(\mathrm{ad})} = 0$ that follows from the definition of $\mathcal{X}_{\mathrm{F}}^{(\mathrm{ad}),m}$ (in fact, for instance the time-ordered Feynman propagators associated to the *in* or *out* state, see [Is, GW4], fail in general to satisfy this condition). We will then show that G_{F} has the required wave front set.

3.9.1. *Auxiliary diagonal Hamiltonian.* Recall that

$$(3.27) \qquad\qquad H^{\mathrm{ad}}(t) = H^{\mathrm{d}}(t) - V^{\mathrm{ad}}_{-\infty}(t),$$

Where $V^{\mathrm{ad}}_{-\infty}(t) \in \Psi^{-\infty,-1-\delta}_{\mathrm{std}}(\mathbb{R};\mathbb{R}^d) \otimes \mathbb{C}^2$ and $H^{\mathrm{d}}(t)$ is the diagonal matrix with on-diagonal components $\epsilon^{\pm}(t)$.

Let $\mathscr{U}^{\mathrm{d}}(t,s)$ be the evolution generated by $H^{\mathrm{d}}(t)$. Using (3.27) we see that $\mathscr{U}^{\mathrm{d}}(t,s)$ is well defined and moreover,

$$\sup_{t,s\in\mathbb{R}} \|\mathscr{U}^{\mathrm{d}}(t,s)\|_{B(\mathcal{H}^m)} < \infty,$$

using the same argument as for $\mathscr{U}(t,s)$, see [GW4]. Since $H^{\mathrm{d}}(t) = H^{\mathrm{d}\dagger}(t)$ we also have

$$(3.28) \qquad\qquad \mathscr{U}^{\mathrm{d}}(t,s)^{\dagger} = \mathscr{U}^{\mathrm{d}}(s,t).$$

We set correspondingly

$$P^{\mathrm{d}} := D_t - H^{\mathrm{d}}(t) = P^{\mathrm{ad}} - V^{\mathrm{ad}}_{-\infty}(t).$$

Note that since $\|V^{\mathrm{ad}}_{-\infty}(t)\|_{B(\mathcal{H}^m)} = O(\langle t\rangle^{-1-\delta})$ and we have assumed that $\gamma < \frac{1}{2}+\delta$, we see that for $u \in C^0(\mathbb{R};\mathcal{H}^m)$ we have $P^{\mathrm{ad}}u \in \mathcal{Y}^{\mathrm{ad},m}$ if and only if $P^{\mathrm{d}}u \in \mathcal{Y}^{\mathrm{ad},m}$ and the two norms in (3.13) on $\mathcal{X}^{\mathrm{ad},m}$ defined with P^{ad} and P^{d} are equivalent.

3.9.2. *Fredholm inverse for P^{ad} on $\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}$.*

Definition 3.15. *We set for $f^{\mathrm{ad}} \in \mathcal{Y}^{\mathrm{ad},m}$:*

$$G^{\mathrm{ad}}_{\mathrm{F}} f^{\mathrm{ad}}(t) := \mathrm{i} \int_{-\infty}^{t} \mathscr{U}^{\mathrm{d}}(t,0)\pi^{+}\mathscr{U}^{\mathrm{d}}(0,s)f^{\mathrm{ad}}(s)ds$$

$$- \mathrm{i} \int_{t}^{+\infty} \mathscr{U}^{\mathrm{d}}(t,0)\pi^{-}\mathscr{U}^{\mathrm{d}}(0,s)f^{\mathrm{ad}}(s)ds.$$

Using the 'time-kernel notation' $A(t,s) := \varrho^{\mathrm{ad}}_t \circ A \circ (\varrho^{\mathrm{ad}}_s)^*$ we can write:

$$G^{\mathrm{ad}}_{\mathrm{F}}(t,s) = \mathrm{i}\theta(t-s)\mathscr{U}^{\mathrm{d}}(t,0)\pi^{+}\mathscr{U}^{\mathrm{d}}(0,s) - \mathrm{i}\theta(s-t)\mathscr{U}^{\mathrm{d}}(t,0)\pi^{-}\mathscr{U}^{\mathrm{d}}(0,s)$$

$$= \mathrm{i}\mathscr{U}^{\mathrm{d}}(t,0)\pi^{+}\mathscr{U}^{\mathrm{d}}(0,s) - \mathrm{i}\theta(s-t)\mathscr{U}^{\mathrm{d}}(t,s),$$

where θ is the Heaviside step function. Let us also observe that since $[\mathscr{U}^{\mathrm{d}}(t,s),\pi^{+}] = 0$, we have

$$(3.29) \qquad\qquad G^{\mathrm{ad}}_{\mathrm{F}} = G^{\mathrm{d}}_{+}\pi^{+} + G^{\mathrm{d}}_{-}\pi^{-},$$

where $G^{\mathrm{d}}_{\pm}$ are the retarded/advanced propagators for $H^{\mathrm{d}}(t)$, defined in analogy to $G^{\mathrm{ad}}_{\pm}$.

Theorem 3.16. *Let $m \in \mathbb{R}$. We have:*

i) $G^{\mathrm{ad}}_{\mathrm{F}} \in B(\mathcal{Y}^{\mathrm{ad},m}, \mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}})$, $P^{\mathrm{ad}}G^{\mathrm{ad}}_{\mathrm{F}} = \mathbf{1}_{\mathcal{Y}^{\mathrm{ad},m}} + K_{\mathcal{Y}^{\mathrm{ad},m}}$,
 where $K_{\mathcal{Y}^{\mathrm{ad},m}}$ is compact on $\mathcal{Y}^{\mathrm{ad},m}$

ii) $G^{\mathrm{ad}}_{\mathrm{F}}P^{\mathrm{ad}} = \mathbf{1}_{\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}} + K_{\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}}$, *where $K_{\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}}$ is compact on $\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}$,*

iii) $\mathrm{i}^{-1}q^{\mathrm{ad}}(G^{\mathrm{ad}}_{\mathrm{F}} - (G^{\mathrm{ad}}_{\mathrm{F}})^{\dagger}) \geq 0$ *on $\mathcal{Y}^{\mathrm{ad},m}$, for $m \geq 0$.*

To prove Thm. 3.16 we will need the following lemma.

Lemma 3.17. $V^{\mathrm{ad}}_{-\infty} : \mathcal{X}^{\mathrm{ad},m} \to \mathcal{Y}^{\mathrm{ad},m}$ *is compact.*

Proof. From (3.18) we first obtain that the injection $\mathcal{X}^{\mathrm{ad},m} \hookrightarrow C^{k+1}(\mathbb{R}; \mathcal{H}^{m-k-1})$ is bounded for any $k \in \mathbb{N}$, $m \in \mathbb{R}$. We pick $\varepsilon > 0$ such that $\gamma < \frac{1}{2} + \delta - 2\varepsilon$ and write $V^{\mathrm{ad}}_{-\infty}(t)$ as $\langle t \rangle^{-1-\delta+\varepsilon} \langle \mathrm{x} \rangle^{-\varepsilon} Y^{\mathrm{ad}}(t)$, where $Y^{\mathrm{ad}}(t) \in C^{\infty}(\mathbb{R}; \mathcal{W}^{-\infty}(\mathbb{R}^d) \otimes B(\mathbb{C}^2))$. It follows that $Y^{\mathrm{ad}} : C^k(\mathbb{R}; \mathcal{H}^m) \to C^k(\mathbb{R}; \mathcal{H}^{m'})$ is bounded for any m, m', hence

$$V^{\mathrm{ad}}_{-\infty} : \mathcal{X}^m \to \langle t \rangle^{-1-\delta+\varepsilon} C^{k+1}(\mathbb{R}; \langle \mathrm{x} \rangle^{-\varepsilon} \mathcal{H}^{m'+1}) \text{ is bounded for any } k \in \mathbb{N}, \ m, m' \in \mathbb{R}.$$

This implies that
(3.30)
$$V^{\mathrm{ad}}_{-\infty} : \mathcal{X}^m \to \langle t \rangle^{-1-\delta+2\varepsilon} C^k(\mathbb{R}; \mathcal{H}^{m'}) \text{ is compact for any } k \in \mathbb{N}, \ m, m' \in \mathbb{R}.$$

We use (3.30) for $k = 0, m' = m$, and the fact that the injection $\langle t \rangle^{-1-\delta+2\varepsilon} C^0(\mathbb{R}; \mathcal{H}^m) \hookrightarrow \langle t \rangle^{-\gamma} L^2(\mathbb{R}; \mathcal{H}^m) = \mathcal{Y}^{\mathrm{ad},m}$ is bounded since $\gamma < \frac{1}{2} + \delta - 2\varepsilon$. It follows that $V^{\mathrm{ad}}_{-\infty} : \mathcal{X}^{\mathrm{ad},m} \to \mathcal{Y}^{\mathrm{ad},m}$ is compact. $\square$

Proof of Thm. 3.16. *Proof of i)*: note first that $G^{\mathrm{ad}}_{\mathrm{F}} = G^{\mathrm{d}}_{+} \pi^{+} + G^{\mathrm{d}}_{-} \pi^{-} \in B(\mathcal{Y}^{\mathrm{ad},m}, \mathcal{X}^{\mathrm{ad},m})$ since $G^{\mathrm{d}}_{\pm} \in B(\mathcal{Y}^{\mathrm{ad},m}, \mathcal{X}^{\mathrm{ad},m})$. We then have:

$$P^{\mathrm{ad}} G^{\mathrm{ad}}_{\mathrm{F}} = P^{\mathrm{d}} G^{\mathrm{ad}}_{\mathrm{F}} + V^{\mathrm{ad}}_{-\infty} G^{\mathrm{ad}}_{\mathrm{F}}$$

$$= P^{\mathrm{d}} G^{\mathrm{d}}_{+} \pi^{+} + P^{\mathrm{d}} G^{\mathrm{d}}_{-} \pi^{-} + V^{\mathrm{ad}}_{-\infty} G^{\mathrm{ad}}_{\mathrm{F}}$$

$$= \mathbf{1}_{\mathcal{Y}^{\mathrm{ad},m}} + V^{\mathrm{ad}}_{-\infty} G^{\mathrm{ad}}_{\mathrm{F}},$$

by Prop. 3.14 applied to P^{d}. By Lemma 3.17, $V^{\mathrm{ad}}_{-\infty} G^{\mathrm{ad}}_{\mathrm{F}}$ is compact on $\mathcal{Y}^{\mathrm{ad},m}$.

It remains to check that $G^{\mathrm{ad}}_{\mathrm{F}} : \mathcal{Y}^{\mathrm{ad},m} \to \mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}$, i.e. $\pi^{+} \varrho^{\mathrm{ad}}_{\mathrm{in}} G^{\mathrm{ad}}_{\mathrm{F}} = \pi^{-} \varrho^{\mathrm{ad}}_{\mathrm{out}} G^{\mathrm{ad}}_{\mathrm{F}} = 0$. We have:

$$\pi^{+} \varrho^{\mathrm{ad}}_{\mathrm{in}} G^{\mathrm{ad}}_{\mathrm{F}} = \varrho^{\mathrm{ad}}_{\mathrm{in}} \pi^{+} (G^{\mathrm{d}}_{+} \pi^{+} + G^{\mathrm{d}}_{-} \pi^{-}) = \varrho^{\mathrm{ad}}_{\mathrm{in}} G^{\mathrm{d}}_{+} \pi^{+} = 0,$$

since $[G^{\mathrm{d}}_{\pm}, \pi^{\pm}] = 0$ and $\varrho^{\mathrm{ad}}_{\mathrm{in}} G^{\mathrm{d}}_{+} = 0$. Similarly we obtain that $\pi^{-} \varrho^{\mathrm{ad}}_{\mathrm{out}} G^{\mathrm{ad}}_{\mathrm{F}} = 0$, which completes the proof of *i)*.

Proof of ii): we have by (3.29):

$$G^{\mathrm{ad}}_{\mathrm{F}} P^{\mathrm{ad}} = G^{\mathrm{ad}}_{\mathrm{F}} P^{\mathrm{d}} + G^{\mathrm{ad}}_{\mathrm{F}} V^{\mathrm{ad}}_{-\infty} = G^{\mathrm{d}}_{+} P^{\mathrm{d}} \pi^{+} + G^{\mathrm{d}}_{-} P^{\mathrm{d}} \pi^{-} + G^{\mathrm{ad}}_{\mathrm{F}} V^{\mathrm{ad}}_{-\infty}.$$

If $u \in \mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}$ we have $\pi^{+} \varrho^{\mathrm{ad}}_{\mathrm{in}} u = \varrho^{\mathrm{ad}}_{\mathrm{in}} \pi^{+} u = 0$ and $\pi^{-} \varrho^{\mathrm{ad}}_{\mathrm{out}} u = \varrho^{\mathrm{ad}}_{\mathrm{out}} \pi^{-} u = 0$. By Prop. 3.14 applied to P^{d} we have $G^{\mathrm{d}}_{+} P^{\mathrm{d}} \pi^{+} u = \pi^{+} u$, $G^{\mathrm{d}}_{-} P^{\mathrm{d}} \pi^{-} u = \pi^{-} u$, hence

$$G^{\mathrm{ad}}_{\mathrm{F}} P^{\mathrm{ad}} = \mathbf{1}_{\mathcal{X}^{\mathrm{ad},m}_{\mathrm{F}}} + G^{\mathrm{ad}}_{\mathrm{F}} V^{\mathrm{ad}}_{-\infty}.$$

Again, by Lemma 3.17 $G^{\mathrm{ad}}_{\mathrm{F}} V^{\mathrm{ad}}_{-\infty}$ is compact on $\mathcal{X}^{\mathrm{ad},m}$.

Proof of iii): using the time-kernel notation we have by (3.28)

$$(G^{\mathrm{ad}}_{\mathrm{F}})^{\dagger}(t, s) = G^{\mathrm{ad}}_{\mathrm{F}}(s, t)^{\dagger}$$

$$= \mathrm{i}\theta(t - s) \mathscr{U}^{\mathrm{d}}(t, 0) \pi^{-} \mathscr{U}^{\mathrm{d}}(0, s) - \mathrm{i}\theta(s - t) \mathscr{U}^{\mathrm{d}}(t, 0) \pi^{+} \mathscr{U}^{\mathrm{d}}(0, s).$$

Hence, defining $\mathscr{U}^{\mathrm{d}} \in B(\mathcal{H}^m, \mathcal{X}^{\mathrm{ad},m})$ by

(3.31) $$\mathscr{U}^{\mathrm{d}} v^{\mathrm{ad}}(t) := \mathscr{U}^{\mathrm{d}}(t, 0) v^{\mathrm{ad}}, \quad v^{\mathrm{ad}} \in \mathcal{H}^m,$$

we have

$$\mathrm{i}^{-1}(G^{\mathrm{ad}}_{\mathrm{F}} - (G^{\mathrm{ad}}_{\mathrm{F}})^{\dagger})(t, s) = \mathscr{U}^{\mathrm{d}}(t, 0)(\pi^{+} - \pi^{-}) \mathscr{U}^{\mathrm{d}}(0, s) = \mathscr{U}^{\mathrm{d}}(t, 0) q^{\mathrm{ad}} \mathscr{U}^{\mathrm{d}}(0, s).$$

It follows that

$$\mathrm{i}^{-1}(f^{\mathrm{ad}} | q^{\mathrm{ad}}(G^{\mathrm{ad}}_{\mathrm{F}} - (G^{\mathrm{ad}}_{\mathrm{F}})^{\dagger}) f^{\mathrm{ad}})_{\mathcal{H}^0} = ((\mathscr{U}^{\mathrm{d}})^{\dagger} f^{\mathrm{ad}} | (q^{\mathrm{ad}})^2 (\mathscr{U}^{\mathrm{d}})^{\dagger} f^{\mathrm{ad}})_{\mathcal{H}^0} \geq 0,$$

hence $\mathrm{i}^{-1} q^{\mathrm{ad}}(G^{\mathrm{ad}}_{\mathrm{F}} - (G^{\mathrm{ad}}_{\mathrm{F}})^{\dagger}) \geq 0$ on $\mathcal{H}^0$ hence on $\mathcal{H}^m$ for $m \geq 0$. $\square$

3.9.3. *Fredholm inverse for P on $\mathcal{X}_{\mathrm{F}}^m$.*

Theorem 3.18. *Assume* (std) *with $\delta > 1$. Let*

$$(3.32) \qquad G_{\mathrm{F}} := -\pi_0 T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^*.$$

We have:

    i) $G_{\mathrm{F}} \in B(\mathcal{Y}^m, \mathcal{X}_{\mathrm{F}}^m)$, $PG_{\mathrm{F}} = \mathbf{1}_{\mathcal{Y}^m} + K_{\mathcal{Y}^m}$, *where $K_{\mathcal{Y}^m}$ is compact on $\mathcal{Y}^m$,*

    ii) $G_{\mathrm{F}} P = \mathbf{1}_{\mathcal{X}_{\mathrm{F}}^m} + K_{\mathcal{X}_{\mathrm{F}}^m}$, *where $K_{\mathcal{X}_{\mathrm{F}}^m}$ is compact on $\mathcal{X}_{\mathrm{F}}^m$,*

    iii) $\mathrm{i}(G_{\mathrm{F}} - G_{\mathrm{F}}^*) \geq 0$ *on $\mathcal{Y}^m$, for $m \geq 0$,*

    iv) $PG_{\mathrm{F}} - \mathbf{1}$, $G_{\mathrm{F}} P - \mathbf{1}$ *are smoothing operators,*

    v) $\mathrm{WF}(G_{\mathrm{F}})' = (\mathrm{diag}_{T^*M}) \cup \bigcup_{t \leq 0}(\Phi_t(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\mathcal{N})$.

In particular G_{F} is a Feynman parametrix of P in the sense of Def. 1.1.

Proof. *Proof of i):* from (3.19) and Thm. 3.16 we see that $G_{\mathrm{F}} \in B(\mathcal{Y}^m, \mathcal{X}^m)$. Let us show that G_{F} maps $\mathcal{Y}^m$ into $\mathcal{X}_{\mathrm{F}}^m$. For $V_{-\infty}^{\mathrm{ad}}$ the operator introduced in (3.27) we have:

$$P^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} = \mathbf{1} + V_{-\infty}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} \ \Rightarrow\ T P^{\mathrm{ad}} T^{-1} T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} = \mathbf{1} + T V_{-\infty}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1}.$$

Using (3.3) this implies that:

$$(3.33) \qquad (D_t - H(t)) T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* = \pi_1^* + T V_{-\infty}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* =: \pi_1^* + R_1,$$

where $R_1 \in B(\mathcal{Y}^m, \langle t \rangle^{-1-\delta} C^0(\mathbb{R}; \mathcal{E}^m))$, using that $V_{-\infty}^{\mathrm{ad}} \in \Psi_{\mathrm{std}}^{-\infty,-1-\delta}(\mathbb{R}; \mathbb{R}^d) \otimes B(\mathbb{C}^2)$. This implies that

$$\varrho \pi_0 T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* = T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* + R_2,$$

where $R_2 \in B(\mathcal{Y}^m, \langle t \rangle^{-1-\delta} C^0(\mathbb{R}; \mathcal{E}^m))$. We now have:

$$\mathscr{U}_{\mathrm{out/in}}(0,t) \varrho_t G_{\mathrm{F}} f = -\mathscr{U}_{\mathrm{out/in}}(0,t) \varrho_t \pi_0 T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* f$$

$$= -\mathscr{U}_{\mathrm{out/in}}(0,t) T(t) \varrho_t^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* f + o(1)$$

$$= -T_{\mathrm{out/in}} \mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(0,t) T_{\mathrm{out/in}}^{-1} T(t) \varrho_t^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* f + o(1)$$

$$= -T_{\mathrm{out/in}} \mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(0,t) \varrho_t^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* f + o(1)$$

$$= -T_{\mathrm{out/in}} \varrho_{\mathrm{out/in}}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^* f + o(1),$$

hence

$$(3.34) \qquad \varrho_{\mathrm{out/in}} G_{\mathrm{F}} = -\varrho_{\mathrm{out/in}}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} \pi_1^*.$$

By Thm. 3.16 we have $\varrho_{\overline{\mathrm{F}}}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} = 0$, i.e. $\pi^- \varrho_{\mathrm{out}}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} = \pi^+ \varrho_{\mathrm{in}}^{\mathrm{ad}} G_{\mathrm{F}}^{\mathrm{ad}} = 0$, which by (3.34) gives $\varrho_{\overline{\mathrm{F}}} G_{\mathrm{F}} = 0$. It follows that G_{F} maps $\mathcal{Y}^m$ to $\mathcal{X}_{\mathrm{F}}^m$ as claimed.

From (3.33), we obtain by an easy computation:

$$(3.35) \qquad PG_{\mathrm{F}} = \mathbf{1} - \pi_1 R_1 - D_t \pi_0 R_1 + \mathrm{i} r \pi_0 R_1.$$

Using (3.30) we obtain that $R_1 : \mathcal{Y}^m \to \langle t \rangle^{-1-\delta+2\varepsilon} C^k(\mathbb{R}; \mathcal{E}^{m'})$ is compact for any m, m', k, hence $PG_{\mathrm{F}} - \mathbf{1}$ is compact on $\mathcal{Y}^m$.

Proof of ii): by Thm. 3.16 and (3.3) we know that:

$$G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} (D_t - H(t)) T = G_{\mathrm{F}}^{\mathrm{ad}} P^{\mathrm{ad}} = \mathbf{1} + G_{\mathrm{F}}^{\mathrm{ad}} V_{-\infty}^{\mathrm{ad}} \text{ on } \mathcal{X}_{\mathrm{F}}^{\mathrm{ad}, m+\frac{1}{2}},$$

hence

$$T G_{\mathrm{F}}^{\mathrm{ad}} T^{-1} (D_t - H(t)) T = T + T G_{\mathrm{F}}^{\mathrm{ad}} V_{-\infty}^{\mathrm{ad}} \text{ on } \mathcal{X}_{\mathrm{F}}^{\mathrm{ad}, m+\frac{1}{2}}.$$

By (3.19), (3.24) we know that $T^{-1}\varrho : \mathcal{X}_{\mathrm{F}}^m \to \mathcal{X}_{\mathrm{F}}^{\mathrm{ad},m+\frac{1}{2}}$. It follows that

$$TG_{\mathrm{F}}^{\mathrm{ad}}T^{-1}(D_t - H(t))\varrho = \varrho + TG_{\mathrm{F}}^{\mathrm{ad}}V_{-\infty}^{\mathrm{ad}}T^{-1}\varrho, \text{ on } \mathcal{X}_{\mathrm{F}}^m.$$

Since $(D_t - H(t))\varrho = \pi_1^*\pi_1(D_t - H(t))$, we obtain that

$$G_{\mathrm{F}}P = \pi_0 T G_{\mathrm{F}}^{\mathrm{ad}}T^{-1}\pi_1^*\pi_1(D_t - H(t))\varrho$$

$$(3.36) \qquad\qquad = \pi_0\varrho + \pi_0 T G_{\mathrm{F}}^{\mathrm{ad}}V_{-\infty}^{\mathrm{ad}}T^{-1}\varrho$$

$$\qquad\qquad = \mathbf{1} + \pi_0 T G_{\mathrm{F}}^{\mathrm{ad}}V_{-\infty}^{\mathrm{ad}}T^{-1}\varrho \text{ on } \mathcal{X}_{\mathrm{F}}^m.$$

Using (3.19) and Lemma 3.17 we obtain that $G_{\mathrm{F}}P - \mathbf{1}$ is compact on $\mathcal{X}_{\mathrm{F}}^m$, which proves $ii)$.

Proof of iii): we note that for any operator A^{ad} one has

$$\pi_0 A^{\mathrm{ad}}\pi_1^* = \pi_1 q A^{\mathrm{ad}}\pi_1^*, \quad \text{hence } (\pi_0 A^{\mathrm{ad}}\pi_1^*)^* = \pi_1 q A^{\mathrm{ad}\dagger}\pi_1^*.$$

This gives

$$\mathrm{i}(G_{\mathrm{F}} - G_{\mathrm{F}}^*) = \mathrm{i}^{-1}\pi_1 q T(G_{\mathrm{F}}^{\mathrm{ad}} - G_{\mathrm{F}}^{\mathrm{ad}\dagger})T^{-1}\pi_1^* = \mathrm{i}^{-1}(\pi_1 T)q^{\mathrm{ad}}(G_{\mathrm{F}}^{\mathrm{ad}} - G_{\mathrm{F}}^{\mathrm{ad}\dagger})(\pi_1 T)^* \geq 0,$$

by Thm. 3.16 iii).

Proof of iv): we first see using $\mathscr{U}^{\mathrm{d}}(t,s)(D_s - H^{\mathrm{d}}(s)) = 0$ and integration by parts that $G_{\mathrm{F}}^{\mathrm{ad}}$ maps compactly supported elements of $H^{-p}(\mathbb{R}; \mathcal{H}^{-k})$ into $H^p(\mathbb{R}; \mathcal{H}^{-k-2p})$ for $k, p \in \mathbb{N}$, hence $V_{-\infty}^{\mathrm{ad}}G_{\mathrm{F}}^{\mathrm{ad}}$ maps compactly supported elements of $H^{-p}(\mathbb{R}; \mathcal{H}^k)$ into $H^p(\mathbb{R}, \mathcal{H}^k)$. The same argument shows that $G_{\mathrm{F}}^{\mathrm{ad}}$ maps also compactly supported elements $H^{-p}(\mathbb{R}; \mathcal{H}^k)$ into $H^p(\mathbb{R}, \mathcal{H}^k)$. By (3.33), (3.35), (3.36) this implies iv).

Proof of v): Let $G_{\mathrm{F,ref}}$ be given by the analog of (3.32) with $\mathscr{U}^{\mathrm{d}}$ replaced by $\mathscr{U}^{\mathrm{ad}}$ in the definition of $G_{\mathrm{F}}^{\mathrm{ad}}$. By [GOW, Thm. 7.10, Prop. 7.11], $G_{\mathrm{F,ref}}$ is a Feynman parametrix. From (2.13) we obtain that $\mathscr{U}^{\mathrm{d}}(t,s) - \mathscr{U}^{\mathrm{ad}}(t,s) \in C^{\infty}(\mathbb{R}^2; \mathcal{W}^{-\infty}(\mathbb{R}^d))$, which implies that $G_{\mathrm{F}} - G_{\mathrm{F,ref}}$ is smoothing and completes the proof of of v). $\square$

4. Asymptotically Minkowski spacetimes

4.1. Assumptions. In this final section we consider asymptotically Minkowski spacetimes and prove analogues of the results from Sect. 3 by a reduction procedure.

We work on $M = \mathbb{R}^{1+d}$ and use the notation $y = (t, \mathrm{y})$ for its elements.

For $\delta \in \mathbb{R}$ we denote by $S_{\mathrm{std}}^{\delta}(\mathbb{R}^{1+d})$, resp. $S_{\mathrm{std}}^{\delta}(\mathbb{R}^{\pm} \times \mathbb{R}^d)$, the class of smooth functions such that

$$\partial_y^{\alpha} f \in O(\langle y \rangle^{\delta - |\alpha|}), \ \alpha \in \mathbb{N}^{1+d},$$

holds on $\mathbb{R}^{1+d}$, resp. $\mathbb{R}^{\pm} \times \mathbb{R}^d$.

The analogous spaces on $\mathbb{R}^d$ will be denoted by $S_{\mathrm{sd}}^{\delta}(\mathbb{R}^d)$.

We denote by $\eta_{\mu\nu}$ the Minkowski metric on $\mathbb{R}^{1+d}$, fix a Lorentzian metric g on $\mathbb{R}^{1+d}$ and consider the Klein-Gordon operator

$$(4.1) \qquad\qquad P = -\Box_g + V(y),$$

where V is again a smooth real function on $\mathbb{R}^{1+d}$. We assume that (M, g) is asymptotically Minkowski and V is asymptotically constant in the following sense:

(aM)
- $i)$ $g_{\mu\nu}(y) - \eta_{\mu\nu} \in S_{\mathrm{std}}^{-\delta}(\mathbb{R}^{1+d})$, $\delta > 1$,
- $ii)$ $V(y) - m^2 \in S_{\mathrm{std}}^{-\delta}(\mathbb{R}^{1+d})$, $m > 0$, $\delta > 1$,
- $iii)$ $(\mathbb{R}^{1+d}, g)$ is globally hyperbolic,
- $iv)$ $(\mathbb{R}^{1+d}, g)$ has a time function $\tilde{t}$ with $\tilde{t} - t \in S_{\mathrm{std}}^{1-\epsilon}(\mathbb{R}^{1+d})$ for $\epsilon > 0$.

We recall that a smooth function $\tilde{t}$ is called a *time function* if $\nabla\tilde{t}$ is a timelike vector field.

Remark 4.1. *We conjecture that* (aM) *iv*) *follows from* (aM) *i*), *iii*).

4.2. **Global hyperbolicity and non-trapping condition.** In what follows we discuss the relation of our hypothesis (aM) with non-trapping assumptions for null geodesics.

The null geodesics of g coincide modulo reparametrization with the projections on the base of null bicharacteristics of $m(x,\xi) = |\xi|^{-1}\xi \cdot g^{-1}(x)\xi$. We recall that Φ_s is the flow of the Hamiltonian vector field H_p, $p(y,\xi) = \xi \cdot g^{-1}(y)\xi$, which acts naturally on $S^*\mathbb{R}^{1+d} = T^*\mathbb{R}^{1+d} \cap \{|\xi| = 1\}$. Null bicharacteristics stay in one of the two connected components $\mathcal{N}^\pm$ of $\mathcal{N}$. We set

$$\Gamma^\pm_{\mathrm{in/out}} = \{X \in S^\pm : \ \phi_s(X) \not\to \infty \text{ as } s \to \pm\infty\}, \ S^\pm = \mathcal{N}^\pm \cap \{|\xi| = 1\}.$$

The familiar *non-trapping condition* is:

(nt) there are no trapped null geodesics of g, i.e. $\Gamma^\pm = \Gamma^\pm_{\mathrm{in}} \cap \Gamma^\pm_{\mathrm{out}} = \emptyset$.

By a well-known argument, this actually implies that $\Gamma^\pm_{\mathrm{in/out}} = \emptyset$, see Lemma 4.2 below, hence any null geodesic escapes to infinity both when the affine parameter s tends to $+\infty$ *and* to $-\infty$.

Lemma 4.2. *If* (nt) *holds then* $\Gamma^\pm_{\mathrm{in}} = \Gamma^\pm_{\mathrm{out}} = \emptyset$.

Proof. We drop the $\pm$ superscript. We claim that $\Gamma_{\mathrm{in}} \neq \emptyset$ or $\Gamma_{\mathrm{out}} \neq \emptyset$ implies $\Gamma \neq \emptyset$. In fact Let $X_0 \in \Gamma_{\mathrm{in}}$, K_1 a compact set such that $\{\Phi_s(X_0) : \ s \geq 0\} \subset K_1$. Let $s_j \to +\infty$ a sequence such that $X_j = \Phi_{s_j}(X_0) \to X_\infty \in K_1$. Clearly $\Phi_s(X_j) = \Phi_{s+s_j}(X_0) \to \Phi_s(X_\infty)$ for any $s \in \mathbb{R}$. For j large enough we have $\Phi_{s+s_j}(X_0) \in K_1$ hence $\Phi_s(X_\infty) \in K_1$ for any $s \in \mathbb{R}$, which means that $X_\infty \in \Gamma$. $\square$

Proposition 4.3. *Assume* (aM) *i*). *Then*
(1) $(\mathbb{R}^{1+d}, g)$ *is globally hyperbolic iff* (nt) *holds,*
(2) *if* (aM), *iii*) *and iv*) *hold then there exists a Cauchy time function $\tilde{t}$ such that $\tilde{t} - t \in C^\infty_c(\mathbb{R}^{1+d})$.*

In the sequel we will work with the Cauchy time function $\tilde{t}$ obtained in (2) of Prop. 4.3 .

Proof. First let us prove (1). By (aM) *i*) we have

$$\{p, t\} = \partial_\tau(|\xi|^{-1}(\tau^2 - k^2)) + O(\langle x\rangle^{-\delta}|\xi|^{-1}) \geq \tau|\xi|^{-1} + O(\langle x\rangle^{-\delta}|\xi|^{-1}).$$

It follows that there exist $c_0 > 0$ and compact sets $K^\pm \subset \mathcal{N}^\pm$ such that

(4.2) $\pm\{p, t\} \geq c_0$ on $S^\pm \setminus K^\pm$

This implies that if $X \in S^+$ and $\phi_s(X) \to \infty$ when $s \to \pm\infty$ then $t \circ \phi_s(X) \to \pm\infty$ when $s \to \pm\infty$. Of course a similar statement is true for $X \in S^-$ with the reversed sign.

Let us set $\Sigma_s = t^{-1}(s)$. Using (4.2) we obtain that there exists $T_0 > 0$ such that any null geodesic intersects $\Sigma_{\pm T}$ transversally for $T \geq T_0$ and hence enters $I^\pm(\Sigma_{\pm T})$. Moreover $\Sigma_{\pm T}$ is achronal for T large enough, since ∂_t is a future directed time-like vector field in $\{\pm t \geq \pm T\}$ for T large enough. We can apply then the Geroch-Sánchez theorem (see for instance [Wa2, Thm. 8.3.7] for its basic version), which implies that $\Sigma_{\pm T}$ are Cauchy hypersurfaces for T large enough, which completes the proof of (1), $\Leftarrow$.

Assume now that $(\mathbb{R}^{1+d}, g)$ is globally hyperbolic and (nt) is violated. Let $\gamma = \{x(s) : \ s \in \mathbb{R}\}$ be an (affine parametrized) null geodesic which is past and future trapped, ie $\gamma \subset K$ for some compact set K. Since $(\mathbb{R}^{1+d}, g)$ is strongly causal,

for each $x \in K$ there exists an open neighborhood $U(x)$ of x such that γ enters $U(x)$ only once, ie $\{s \in \mathbb{R} : x(s) \in U(x)\} := I(x)$ is a bounded open interval. By compactness of K we have $x(s) \notin K$ for $s \notin \cup_{i=1}^{n} I(x_i)$, which is a contradiction. This completes the proof of (1), $\Rightarrow$.

Now let us prove (2). Let $\tilde{t}$ be the time function in (aM) iii). First of all, we note that it follows from (aM) that $-C^{-1} \leq d\tilde{t} \cdot g^{-1}d\tilde{t} \leq -C$ for some $C > 0$. We fix a cutoff function $\chi \in C_c^{\infty}(\mathbb{R}^{1+d})$ with $0 \leq \chi \leq 1$, $\chi = 1$ near 0 and set $\chi_R(y) = \chi(R^{-1}y)$, $\hat{t}_R = \chi_R \tilde{t} + (1 - \chi_R)t$. We have:

$$d\hat{t}_R = \chi_R d\tilde{t} + (1 - \chi_R)dt + (\tilde{t} - t)d\chi_R.$$

The covector $\alpha_R = \chi_R d\tilde{t} + (1 - \chi_R)dt$ is a convex combination of future directed timelike covectors, which using (aM) i) implies that there exists $C > 0$ such that $-C^{-1} \leq \alpha_R \cdot g^{-1}\alpha_R \leq -C$, uniformly for $R \geq 1$. The error term $(\tilde{t} - t)d\chi_R$ is of norm $O(R^{-\epsilon})$, which shows that $\hat{t}_R$ is a time function for R large enough. Let us fix such an R and denote $\hat{t}_R$ by $\hat{t}$. Clearly $\hat{t} - t \in C_c^{\infty}(\mathbb{R}^{1+d})$. It remains to check that $\hat{t}$ is a Cauchy time function. First using that $\hat{t} = t + C_c^{\infty}(\mathbb{R}^{1+d})$ we obtain that

$$(4.3) \qquad \lim_{T \to +\infty} \sup_{\Sigma_{-T}} \hat{t} = -\infty, \quad \lim_{T \to +\infty} \inf_{\Sigma_T} \hat{t} = +\infty.$$

Let now γ be an inextendible future directed continuous causal curve and $s \in \mathbb{R}$. Since $\hat{t}$ is a time function, γ intersects $\hat{t}^{-1}(s)$ at most once. By global hyperbolicity, γ intersects the Cauchy hypersurfaces $\Sigma_{\pm T}$ for T large enough. By (4.3) this implies choosing T very large that γ intersects $\hat{t}^{-1}(s^{\pm})$ for some $s_- < s < s_+$ hence also $\hat{t}^{-1}(s)$. Therefore $\hat{t}^{-1}(s)$ is a Cauchy hypersurface for each s and $\hat{t}$ is a Cauchy time function. $\square$

4.3. Reduction to the model case.
We now perform the reduction to the model case following the method explained in [GW4], here however we need to take into account the space-time decay of g and V.

After possibly redefining $\tilde{t}$ by adding a constant, we can assume that $\Sigma := \tilde{t}^{-1}(\{0\}) = \{0\} \times \mathbb{R}^d$, so that Σ is a Cauchy hypersurface both for g and η.

We set $v = \dfrac{g^{-1}d\tilde{t}}{d\tilde{t} \cdot g^{-1}d\tilde{t}}$, so that $v = \partial_t$ outside a compact set. If ϕ_t is the flow of v, we set:

$$\chi : \mathbb{R} \times \Sigma \in (t, \mathrm{x}) \mapsto \phi_t(0, \mathrm{x}) \in \mathbb{R}^{1+d},$$

so that $\tilde{t}(\chi(t, \mathrm{x})) = t$. Thanks to the space-time decay properties (aM), the diffeomorphism χ has the following proprieties.

Lemma 4.4. *Assume* (aM). *Then*

$$\chi^* g = -\hat{c}^2(t, \mathrm{x})dt^2 + \hat{h}(t, \mathrm{x})d\mathrm{x}^2, \quad \chi^* V = \hat{V},$$

where:

$$\hat{h}, \hat{h}^{-1}, \hat{c}, \hat{c}^{-1}, \hat{V} \in S_{\mathrm{std}}^0(\mathbb{R}^{1+d}).$$

Moreover there exist diffeomorphisms $\mathrm{y}_{\mathrm{out/in}}$ *of* Σ *with*

$$\mathrm{y}_{\mathrm{out/in}}(\mathrm{x}) - \mathrm{x} \in S_{\mathrm{sd}}^{1-\delta}(\mathbb{R}^d),$$

such that if

$$\hat{h}_{\mathrm{out/in}} := \mathrm{y}_{\mathrm{out/in}}^* \delta,$$

where δ *is the flat Riemannian metric on* $\mathbb{R}^d$, *we have:*

$$\hat{h} - \hat{h}_{\mathrm{out/in}}, \hat{c} - 1, \hat{V} - m^2 \in S_{\mathrm{std}}^{-\delta}(\mathbb{R}^{\pm} \times \mathbb{R}^d).$$

Proof. We have $v = \partial_t + S_{\text{std}}^{-\delta}$, which also implies that

$$(4.4) \qquad \langle \phi_s(0, \text{x}) \rangle \geq C(\langle s \rangle + \langle \text{x} \rangle), \quad C > 0.$$

Setting $w := \pi_{\text{y}} v$, we have $\pi_{\text{y}} \chi(t, \text{x}) = \pi_{\text{y}} \text{x} + \int_0^t w(\phi_s(\text{x})) ds$. Using that $w \in S_{\text{std}}^{-\delta}(\mathbb{R}^{1+d})$, we obtain that

$$\text{y}_{\text{out/in}}(\text{x}) := \lim_{t \to \pm \infty} \pi_{\text{y}} \chi(t, \text{x})$$

exist and:

$$(4.5) \qquad \pi_{\text{y}} \chi(t, \text{x}) - \text{y}_{\text{out/in}}(\text{x}) \in S_{\text{std}}^{1-\delta}(\mathbb{R}^{\pm} \times \mathbb{R}^d), \quad \text{y}_{\text{out/in}}(\text{x}) - \text{x} \in S_{\text{sd}}^{1-\delta}(\mathbb{R}^d).$$

Since $\tilde{t} = t$ outside a compact set, we also have $\chi(t, \text{x}) = (t, \pi_{\text{y}} \chi(t, \text{x}))$ for $|t| + |\text{x}| \geq C$, hence

$$D\chi(t, \text{x}) = \begin{pmatrix} 1 & 0 \\ 0 & D\text{y}_{\text{out/in}}(\text{x}) \end{pmatrix} + S_{\text{std}}^{-\delta}(\mathbb{R}^{\pm} \times \mathbb{R}^d).$$

This estimate and (4.4) imply the assertion. $\square$

If $P \in \text{Diff}(\mathbb{R}^{1+d})$ we denote by $\chi^* P$ the pullback of P by χ defined by $(\chi^* P)u \circ \chi = (Pu) \circ \chi$.

We set $\hat{P} = \chi^* P$ and $\tilde{P} = \hat{c}^{1-n/2} \hat{P} \hat{c}^{1+n/2}$. By a direct computation, one finds

$$(4.6) \qquad \tilde{P} = \overline{\partial}_t^2 + r(t, \text{x}) \overline{\partial}_t + a(t, \text{x}, \overline{\partial}_\text{x}),$$

where by Lemma 4.4, r, a satisfy (std) for $\delta > 1$, with

$$(4.7) \qquad a_{\text{out/in}}(\text{x}, \overline{\partial}_\text{x}) := -\Delta_{\hat{h}_{\text{out/in}}} + m^2.$$

Thus, one is reduced to the setting from previous sections, with the mere notational difference that the model Klein-Gordon operator from Sect. 2 is now denoted $\tilde{P}$.

Note that

$$a_{\text{out/in}}(\text{x}, \overline{\partial}_\text{x}) = \chi_{\text{out/in}}^*(-\Delta_\text{x} + m^2), \quad \text{where } \chi_{\text{out/in}}(t, \text{x}) := (t, \text{y}_{\text{out/in}}(\text{x})).$$

There are several inconveniences related to the possibility that $a_{\text{out}} \neq a_{\text{in}}$. It turns out, however, that they can be circumvented by considering the dynamics associated to the free Laplace operator $-\Delta_\text{x} + m^2$ instead of the asymptotic dynamics $\mathscr{U}_{\text{out/in}}(t, s)$ associated to $a_{\text{out/in}}(\text{x}, \overline{\partial}_\text{x})$.

4.4. Wave operators.

Let $\Sigma_s := \tilde{t}^{-1}(\{s\})$ for $m \in \mathbb{R}$. Using the diffeomorphism χ we identify Σ_s with $\mathbb{R}^d$ to define the Sobolev spaces $H^m(\Sigma_s)$. We introduce the energy spaces:

$$\mathcal{E}^m(s) := H^{1+m}(\Sigma_s) \oplus H^m(\Sigma_s), \quad m \in \mathbb{R}.$$

Of course, under χ all spaces $\mathcal{E}^m(s)$ equal $\mathcal{E}^m(\mathbb{R}^d)$ with uniformly equivalent norms. We denote by $\mathscr{U}(t, s) : \mathcal{E}^m(s) \to \mathcal{E}^m(t)$ the Cauchy evolution associated to P.

The approximate diagonalization of $\mathscr{U}(t, s)$ can be performed after reduction to the model Klein-Gordon operator $\tilde{P}$, thus one needs to take into account the diffeomorphism χ and the conformal transformation involving appropriate powers of $\hat{c}$. This is done in the lemma below, which is proved by straightforward computations detailed in [GW4].

Lemma 4.5. *Let $\chi_t(\text{x}) = \text{y}(t, 0, \text{x})$. Let $Z(t) : \mathcal{E}^m(t) \to \mathcal{H}^{m+\frac{1}{2}}$ be defined by*

$$Z(t) := (\chi_t^*)^{-1} \hat{c}^{n/2-1} \begin{pmatrix} 1 & 0 \\ -\text{i}(n/2 - 1)\partial_t \ln(\hat{c}) & 1 \end{pmatrix} \circ T(t),$$

and similarly let us define $Z_{\text{out/in}} := (\chi_{\text{out/in}}^*)^{-1} T_{\text{out/in}}$. Then

$$(4.8) \qquad \mathscr{U}(t, s) = Z(t) \mathscr{U}^{\text{ad}}(t, s) Z^{-1}(s),$$

where $\mathscr{U}^{\mathrm{ad}}(t,s)$ is the approximately diagonalized Cauchy evolution of $\tilde{P}$ defined in (2.7). Furthermore,

$$(4.9) \qquad \lim_{t\to\pm\infty} Z(t)^{-1} Z_{\mathrm{out/in}} - \mathbf{1} = 0 \ \mathit{in} \ B(\mathcal{H}^m).$$

We set $P_{\mathrm{free}} := \overline{\partial}_t^2 - \Delta_{\mathrm{x}} + m^2$. Denoting by $\mathscr{U}_{\mathrm{free}}(t,s)$ the usual Cauchy evolution for P_{free}, we have by (4.7):

$$\mathscr{U}_{\mathrm{free}}(t,s) = Z_{\mathrm{out/in}} \mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(t,s) Z_{\mathrm{out/in}}^{-1}.$$

Proposition 4.6. *The limits*

$$(4.10) \qquad W_{\mathrm{out/in}} := \lim_{t\to\pm\infty} \mathscr{U}(0,t) \mathscr{U}_{\mathrm{free}}(t,0)$$

exist in $B(\mathcal{E}^m(0))$ *with inverses*

$$(4.11) \qquad W_{\mathrm{out/in}}^{-1} = W_{\mathrm{out/in}}^{\dagger} = \lim_{t\to\pm\infty} \mathscr{U}_{\mathrm{free}}(0,t) \mathscr{U}(t,0).$$

Moreover one has:

$$(4.12) \qquad W_{\mathrm{out/in}} = Z(0) W_{\mathrm{out/in}}^{\mathrm{ad}} Z_{\mathrm{out/in}}^{-1},$$

where we recall that $W_{\mathrm{out/in}}^{\mathrm{ad}} = \lim_{t\to\pm\infty} \mathscr{U}^{\mathrm{ad}}(0,t) \mathscr{U}_{\mathrm{out/in}}^{\mathrm{ad}}(t,0).$

Proof. The existence of the limits (4.10), (4.11) follows from the Cook argument, using the short range condition $\delta > 1$. The identity (4.12) follows from (4.9). $\square$

4.5. Fredholm problems and Feynman pseudo-inverse.

Following the notation in (4.6), the objects introduced in Sect. 3 will be denoted with tildes, like $\tilde{\mathcal{X}}^m, \tilde{\mathcal{Y}}^m$, etc. We define the spaces
$$(4.13)$$
$$\mathcal{Y}^m := (\chi^{-1})^* \tilde{\mathcal{Y}}^m = (\chi^{-1})^* \big(\langle t \rangle^{-\gamma} L^2(\mathbb{R}; H^m)\big),$$
$$\mathcal{X}^m := (\chi^{-1})^* \tilde{\mathcal{X}}^m = (\chi^{-1})^* \big\{ \tilde{u} \in \big(C^0(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m)\big) : \ \tilde{P}\tilde{u} \in \tilde{\mathcal{Y}}^m \big\}.$$

In particular $\mathcal{X}^m$ is the space of $u \in \mathscr{D}'(\mathbb{R}^{1+d})$ such that $u \circ \chi \in C^0(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m)$ and $Pu \in \mathcal{Y}^m$. We equip $\mathcal{Y}^m$ and $\mathcal{X}^m$ with the norms obtained from $\tilde{\mathcal{Y}}^m$ and $\tilde{\mathcal{X}}^m$.

We now introduce the projections
$$c_{\mathrm{free}}^{\pm,\mathrm{vac}} = \frac{1}{2} \begin{pmatrix} 1 & \pm\sqrt{-\Delta_{\mathrm{x}} + m^2} \\ \pm\sqrt{-\Delta_{\mathrm{x}} + m^2} & 1 \end{pmatrix}.$$

An easy computation shows that

$$(4.14) \qquad c_{\mathrm{free}}^{\pm,\mathrm{vac}} = Z_{\mathrm{out}} \pi^{\pm} Z_{\mathrm{out}}^{-1} = Z_{\mathrm{in}} \pi^{\pm} Z_{\mathrm{in}}^{-1}.$$

Definition 4.7. *We set* $\varrho_{\mathrm{out/in}} := \mathrm{s-}\lim_{t\to\pm\infty} \mathscr{U}_{\mathrm{free}}(0,t) \varrho_t$ *and*

$$\varrho_{\mathrm{F}} := c_{\mathrm{free}}^{+,\mathrm{vac}} \varrho_{\mathrm{out}} + c_{\mathrm{free}}^{-,\mathrm{vac}} \varrho_{\mathrm{in}}, \quad W_{\mathrm{F}}^{\dagger} := c_{\mathrm{free}}^{+,\mathrm{vac}} W_{\mathrm{out}}^{\dagger} + c_{\mathrm{free}}^{-,\mathrm{vac}} W_{\mathrm{in}}^{\dagger},$$

$$\varrho_{\overline{\mathrm{F}}} := c_{\mathrm{free}}^{-,\mathrm{vac}} \varrho_{\mathrm{out}} + c_{\mathrm{free}}^{+,\mathrm{vac}} \varrho_{\mathrm{in}}, \quad W_{\overline{\mathrm{F}}}^{\dagger} := c_{\mathrm{free}}^{-,\mathrm{vac}} W_{\mathrm{out}}^{\dagger} + c_{\mathrm{free}}^{+,\mathrm{vac}} W_{\mathrm{in}}^{\dagger},$$

and for $I \in \{\mathrm{in}, \mathrm{out}, \mathrm{F}, \overline{\mathrm{F}}\}$:

$$\mathcal{X}_I^m := \{u \in \mathcal{X}^m : \ \varrho_{I^c} u = 0\}.$$

Theorem 4.8. *Assume* (aM) *and let* P, $\mathcal{X}_I^m$ *be as defined in* (4.1) *and Def. 4.7 for* $m \in \mathbb{R}$ *and* $I \in \{\mathrm{in}, \mathrm{out}, \mathrm{F}, \overline{\mathrm{F}}\}$. *Then:*

(1) $P : \mathcal{X}_I^m \to \mathcal{Y}^m$ *is Fredholm of index* $\mathrm{ind}\, W_{I^c}^{\dagger}$, *and invertible with inverse* $G_{\pm}$ *if* $I = \mathrm{out/in}$. *Furthermore,* $\mathrm{Ker}\, P|_{\mathcal{X}_I^m} \subset C^{\infty}(M)$ *and the index does not depend on the Sobolev order* m.

(2) *Let*

$$G_{\mathrm{F}} := (\chi^{-1})^*(\hat{c}^{1+n/2}\tilde{G}_{\mathrm{F}}\hat{c}^{1-n/2}),$$

where $\tilde{G}_{\mathrm{F}}$ is the operator defined in (3.32) and $\hat{c}$, χ are defined in Subsect. 4.3. Then:

i) $G_{\mathrm{F}} \in B(\mathcal{Y}^m, \mathcal{X}_{\mathrm{F}}^m)$, $PG_{\mathrm{F}} = \mathbf{1}_{\mathcal{Y}^m} + K_{\mathcal{Y}^m}$, *where $K_{\mathcal{Y}^m}$ is compact on $\mathcal{Y}^m$,*

ii) $G_{\mathrm{F}}P = \mathbf{1}_{\mathcal{X}_{\mathrm{F}}^m} + K_{\mathcal{X}_{\mathrm{F}}^m}$, *where $K_{\mathcal{X}_{\mathrm{F}}^m}$ is compact on $\mathcal{X}_{\mathrm{F}}^m$,*

iii) $\mathrm{i}(G_{\mathrm{F}} - G_{\mathrm{F}}^*) \geq 0$ *on $\mathcal{Y}^m$, for $m \geq 0$,*

iv) $PG_{\mathrm{F}} - \mathbf{1}, G_{\mathrm{F}}P - \mathbf{1}$ *are smoothing operators,*

v) $\mathrm{WF}(G_{\mathrm{F}})' = (\mathrm{diag}_{T^*M}) \cup \bigcup_{t \leq 0}(\Phi_t(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\mathcal{N})$.

In particular, G_{F} is a Feynman parametrix of P in the sense of Def. 1.1.

Proof. The maps

$$\mathcal{Y}^m \ni f \mapsto \tilde{f} := \hat{c}^{1-n/2}f \circ \chi \in \tilde{\mathcal{Y}}^m,$$

$$\mathcal{X}^m \ni u \mapsto \tilde{u} := \hat{c}^{1+n/2}u \circ \chi \in \tilde{\mathcal{Y}}^m,$$

are boundedly invertible and furthermore, $Pu = f$ iff $\tilde{P}\tilde{u} = \tilde{f}$. Moreover, by a direct computation we obtain that $Z^{-1}\varrho u = T^{-1}\tilde{\varrho}\tilde{u}$ and hence $u \in \mathcal{X}_I^m$ iff $\tilde{u} \in \tilde{\mathcal{X}}_I^m$. The theorem follows hence from Thm. 3.18 provided we check that

(4.15) $$\mathrm{ind}\, W_I^\dagger = \mathrm{ind}\, W_I^{\mathrm{ad}\dagger}.$$

This is obvious for $I = \mathrm{out/in}$ since the operators are then bijective. Let us check (4.15) for $I = \overline{\mathrm{F}}$ for example. We denote by Z_{free} the analog of $Z_{\mathrm{out/in}}$ with $\epsilon_{\mathrm{out/in}}$ replaced by $\epsilon_{\mathrm{free}} = (-\Delta_{\mathrm{x}} + m^2)^{\frac{1}{2}}$ and $\chi_{\mathrm{out/in}}$ replaced by $\mathbf{1}$. Using (4.12) and (4.14) we obtain that

$$Z_{\mathrm{free}}^{-1}W_{\overline{\mathrm{F}}}^\dagger = (Z_{\mathrm{free}}^{-1}Z_{\mathrm{out}}\pi^- W_{\mathrm{out}}^{\mathrm{ad}\dagger} + Z_{\mathrm{free}}^{-1}Z_{\mathrm{in}}\pi^+ W_{\mathrm{in}}^{\mathrm{ad}\dagger})Z(0)^{-1} = S \circ W_{\overline{\mathrm{F}}}^{\mathrm{ad}\dagger} \circ Z(0)^{-1},$$

for $S = Z_{\mathrm{free}}^{-1}Z_{\mathrm{out}}\pi^- + Z_{\mathrm{free}}^{-1}Z_{\mathrm{in}}\pi^+$. But since $c_{\mathrm{free}}^{\pm,\mathrm{vac}} = Z_{\mathrm{free}}\pi^{\pm}Z_{\mathrm{free}}^{-1}$, $Z_{\mathrm{free}}^{-1}Z_{\mathrm{out/in}}$ commutes with π^+ and π^-, using again (4.14). Therefore $S = \pi^- Z_{\mathrm{free}}^{-1}Z_{\mathrm{out}}\pi^- + \pi^+ Z_{\mathrm{free}}^{-1}Z_{\mathrm{in}}\pi^+$ is invertible and hence $\mathrm{ind}\, W_{\overline{\mathrm{F}}}^\dagger = \mathrm{ind}\, W_{\overline{\mathrm{F}}}^{\mathrm{ad}\dagger}$. $\square$

APPENDIX A

A.1. **Proof of Prop. 2.2.** To prove Prop. 2.2 we first need an auxiliary lemma about parameter-dependent pseudodifferential calculus.

We start by introducing parameter dependent versions of the $\Psi_{\mathrm{std}}^{m,0}(\mathbb{R}; \mathbb{R}^d)$ calculus. Namely, we define $\widetilde{S}_{\mathrm{std}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)$ to be the space of functions $c(t, \mathrm{x}, k, \lambda)$ such that:

$$\partial_t^n \partial_\lambda^p \partial_{\mathrm{x}}^\alpha \partial_k^\beta c(t, \mathrm{x}, k, \lambda) \in O\big((\langle x \rangle + \langle t \rangle)^{\delta - n - |\alpha|}(\langle k \rangle + \langle \lambda \rangle)^{m - |\beta| - p}\big), \; \alpha, \beta \in \mathbb{N}^d, \; p, n \in \mathbb{N}.$$

The typical element of $\widetilde{S}_{\mathrm{std}}^{m,0}(\mathbb{R}; T^*\mathbb{R}^d)$ is $c(t, \mathrm{x}, k, \lambda) = a(t, \mathrm{x}, k) + \lambda^m$ for $a \in S_{\mathrm{std}}^{m,0}(\mathbb{R}; T^*\mathbb{R}^d)$.

We denote by $\widetilde{S}_{\mathrm{std,ph}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)$ the subspace of symbols wich are polyhomogeneous in (k, λ).

Furthermore, we define $\widetilde{\mathcal{W}}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$ as the set of maps $\mathbb{R} \ni \lambda \mapsto a(t, \lambda) \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$ such that:

$$\|(D_{\mathrm{x}}^2 + \mathrm{x}^2 + \lambda^2 + 1)^m \partial_t^n \partial_\lambda^p a(t, \lambda)(D_{\mathrm{x}}^2 + \mathrm{x}^2 + \lambda^2 + 1)^m\|_{B(L^2(\mathbb{R}^d))} \in O(\langle t \rangle^{-m}\langle \lambda \rangle^{-m}),$$

for all $m, n, p \in \mathbb{N}$. We set

$$\widetilde{\Psi}_{\mathrm{std}}^{m,\delta}(\mathbb{R}; \mathbb{R}^d) = \mathrm{Op}\big(\widetilde{S}_{\mathrm{std,ph}}^{m,\delta}(\mathbb{R}; T^*\mathbb{R}^d)\big) + \widetilde{\mathcal{W}}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d).$$

Lemma A.1. *Suppose that* $a(t) \in \Psi_{\mathrm{std}}^{2,0}(\mathbb{R}; \mathbb{R}^d)$ *and that* $a(t)$ *is elliptic, selfadjoint on* $L^2(\mathbb{R}^d)$ *and satisfies* $a(t) \geq c_0 \mathbf{1}$, $c_0 > 0$. *Then* $(a(t) + \lambda^2)^{-1} \in \widetilde{\Psi}_{\mathrm{std}}^{-2,0}(\mathbb{R}; \mathbb{R}^d)$.

Proof. Let us consider the operator $A(t) = a(t) + D_l^2$ acting on $L^2(\mathbb{R}^d \times \mathbb{R})$, where l is the dual variable to λ. It is selfadjoint on $H^2(\mathbb{R}^d \times \mathbb{R})$ and satisfies $A(t) \geq c_0 \mathbf{1}$. The idea of the proof is to construct a time-dependent pseudodifferential calculus on $\mathbb{R}_{\mathrm{x}}^d \times \mathbb{R}_l$ in which $A(t)$ is elliptic, and such that Seeley's theorem holds true.

Step 1. In step 1 we construct a convenient pseudodifferential calculus acting on $\mathbb{R}^{d+1} = \mathbb{R}_{\mathrm{x}}^d \times \mathbb{R}_l$. Specifically, the symbol classes $S_{\mathrm{std}}^m(\mathbb{R}; T^*\mathbb{R}^{d+1})$ are defined by the conditions

$$\partial_t^n \partial_{\mathrm{x}}^\alpha \partial_k^\beta \partial_l^p \partial_\lambda^m a(t, \mathrm{x}, k, l, \lambda) \in O\big((\langle \mathrm{x} \rangle + \langle t \rangle)^{-n-|\alpha|}(\langle k \rangle + \langle \lambda \rangle)^{m-|\beta|-m}\big),$$

for all $\alpha, \beta \in \mathbb{N}^d$, $m, n, p \in \mathbb{N}$. We denote by $S_{\mathrm{std,ph}}^m(\mathbb{R}; T^*\mathbb{R}^{d+1})$ the subspace of symbols which are polyhomogeneous in (k, λ).

The class of $t-$dependent pseudodifferential operators $\widetilde{\Psi}_{\mathrm{std}}^m(\mathbb{R}; \mathbb{R}^{d+1})$ consists by definition of sums of $\widetilde{\mathrm{Op}}(S_{\mathrm{std,ph}}^m(\mathbb{R}; T^*\mathbb{R}^{d+1}))$, where $\widetilde{\mathrm{Op}}$ denotes the Weyl quantization in (x, l), and of elements of the ideal $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$ of operator-valued functions $\mathbb{R} \ni t \mapsto a(t)$ such that

$$(\mathrm{A.1}) \quad \|(D_l^2 + D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^n \partial_t^m \mathrm{ad}_l^p a(t)(D_l^2 + D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^n\|_{B(L^2(\mathbb{R}_{\mathrm{x}}^d \times \mathbb{R}_l))} \in O(\langle t \rangle^{-n}),$$

for all $m, n, p \in \mathbb{N}$, where we use the notation $\mathrm{ad}_A B = [A, B]$.

Let us check that Seeley's theorem is still valid for this class of operators, by verifying once again the abstract conditions in [ALNV]. As in the proof of Thm. 2.1 we use the notations in [GOW, Subsect. 5.3]. We take as Hilbert space $\mathcal{H} = L^2(\mathbb{R}_t \times \mathbb{R}^{d+1})$, as injective operator in $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$ the operator $\mathrm{e}^{-D_{\mathrm{x}}^2 + D_l^2 + \mathrm{x}^2 + t^2 + 1}$. The fact that $a \in S_{\mathrm{std,ph}}^m(\mathbb{R}; T^*\mathbb{R}^{d+1})$ and $\widetilde{\mathrm{Op}}(a) \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$ implies $a \in S_{\mathrm{std,ph}}^{-\infty,0}(\mathbb{R}; T^*\mathbb{R}^{d+1})$ is easy to check.

The proof of the spectral invariance of $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$ is done as in Thm. 2.1, using that the operator ad_l satisfies Leibniz rule and the identity $\mathrm{ad}_l(\mathbf{1} - R)^{-1} = (\mathbf{1} - R)^{-1} \mathrm{ad}_l R (\mathbf{1} - R)^{-1}$.

Step 2. In step 2 we describe the relationship between the classes $\Psi_{\mathrm{std}}^m(\mathbb{R}; \mathbb{R}^{1+d})$ and $\widetilde{\Psi}_{\mathrm{std}}^m(\mathbb{R}; \mathbb{R}^d)$. Denoting by $\mathscr{F}$ the Fourier transform in l and by T_l the group of translations in l, we see that

$$c \in S_{\mathrm{std,ph}}^m(\mathbb{R}; T^*\mathbb{R}^{1+d}), \ [T_l, \widetilde{\mathrm{Op}}(c)] = 0$$
$$\Leftrightarrow \quad \mathscr{F}\mathrm{Op}(c)\mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} \mathrm{Op}(c(\lambda))d\lambda, \ \text{for } c(\lambda) \in \widetilde{S}_{\mathrm{std,ph}}^m(\mathbb{R}; T^*\mathbb{R}^d).$$

Let now $w \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$ with $[w, T_l] = 0$. We have:

$$(\mathrm{A.2}) \qquad\qquad \mathscr{F}w\mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} w(t, \lambda)d\lambda.$$

From (A.1) we obtain that:

$$\int_{\mathbb{R}} \langle \lambda \rangle^n \|(D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^n \partial_t^m \partial_\lambda^p w(t, \lambda)u(\lambda)\|_{L^2(\mathbb{R}^d)}^2 d\lambda$$

$$\leq C_{n,p} \langle t \rangle^{-n} \int_{\mathbb{R}} \|(D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^{-n}u(\lambda)\|_{L^2(\mathbb{R}^d)}^2 d\lambda, \quad \forall m, n, p \in \mathbb{N},$$

or equivalently

$$\int_{\mathbb{R}}^{\oplus} \langle \lambda \rangle^n (D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^{n/2} \partial_t^m \partial_\lambda^p w(\lambda)(D_{\mathrm{x}}^2 + \langle \mathrm{x} \rangle^2)^{n/2} d\lambda$$

is of norm $O(\langle t \rangle^{-n})$ in $B(L^2(\mathbb{R}^{d+1}))$. By Sobolev's embedding theorem this implies that

$$\|(D_x^2 + \langle x \rangle^2)^{n/2} \partial_t^m \partial_\lambda^p w(\lambda)(D_x^2 + \langle x \rangle^2)^{n/2}\|_{B(L^2(\mathbb{R}^d))} \in O(\langle t \rangle^n \langle \lambda \rangle^{-n}) \ \forall m, n, p \in \mathbb{N},$$

hence $w(t, \lambda) \in \widetilde{\mathcal{W}}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$. Conversely, if $w(t, \lambda) \in \widetilde{\mathcal{W}}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$ it is immediate that w defined by (A.2) belongs to $\mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^{d+1})$. Hence we have shown

$$w \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R} : \mathbb{R}^{d+1}), \ \ [w, T_l] = 0$$

(A.3)
$$\Leftrightarrow \mathscr{F} w \mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} w(\lambda) d\lambda, \ \text{for } w(\lambda) \in \widetilde{\mathcal{W}}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d).$$

Step 3. We can now conclude the proof of the lemma. By assumption $A(t) = a(t) + D_l^2 \in \Psi_{\mathrm{std}}^2(\mathbb{R}; \mathbb{R}^{d+1})$ and is uniformly elliptic in that class, hence by Seeley's theorem $A(t)^{-1} \in \Psi_{\mathrm{std}}^{-2}(\mathbb{R}; \mathbb{R}^{d+1})$. We have

$$\mathscr{F} A(t)^{-1} \mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} (a(t) + \lambda^2)^{-1} d\lambda,$$

where by Step 2 $(a(t) + \lambda^2)^{-1} \in \widetilde{\Psi}_{\mathrm{std}}^{-2}(\mathbb{R}; \mathbb{R}^d)$. This completes the proof of the lemma. $\square$

Proof of Prop. 2.2 In view of the identity

$$a_1^{1+\alpha} - a_2^{1+\alpha} = (a_1 - a_2)a_1^\alpha + a_2(a_1^\alpha - a_2^\alpha),$$

we see that it suffices to prove the proposition for $0 < \alpha < 1$. We will use the following formula, valid for example if a is a selfadjoint operator on a Hilbert space $\mathcal{H}$ with $a \geq c\mathbf{1}$, $c > 0$:

(A.4)
$$a^\alpha = C_\alpha \int_0^{+\infty} (a + s)^{-1} s^\alpha ds = C_\alpha \int_{\mathbb{R}} (a + \lambda^2)^{-1} \lambda^{2\alpha+1} d\lambda, \ \alpha \in \mathbb{R},$$

where the integrals are norm convergent in say, $B(\mathrm{Dom}\, a^m, \mathcal{H})$ for m large enough.

We have for $r(t) = a_1(t) - a_2(t)$:

$$\begin{aligned}
(a_1(t) + \lambda^2)^{-1} &= (a_2(t) + \lambda^2)^{-1}(\mathbf{1} + r(t)(a_1(t) + \lambda^2)^{-1}) \\
&= (a_2(t) + \lambda^2)^{-1} + (a_2(t) + \lambda^2)^{-2}(a_2(t) + \lambda^2)r(t)(a_1(t) + \lambda^2)^{-1} \\
&= (a_2(t) + \lambda^2)^{-1} + (a_2(t) + \lambda^2)^{-2} a_2(t) c_1(t, \lambda) \\
&= (a_2(t) + \lambda^2)^{-1} + a_2(t) c_2(t, \lambda),
\end{aligned}$$

where using Lemma A.1, $c_1(t, \lambda) \in \widetilde{\Psi}^{0, -\delta}(\mathbb{R}; \mathbb{R}^d))$ and $c_2(t, \lambda) \in \widetilde{\Psi}^{-4, -\delta}(\mathbb{R}; \mathbb{R}^d)$. From (A.4) we obtain that:

(A.5)
$$a_1^\alpha(t) - a_2^\alpha(t) = C_\alpha a_2(t) \int_{\mathbb{R}} c_2(t, \lambda) \lambda^{2\alpha+1} d\lambda.$$

We now write $c_2(t, \lambda)$ as $\mathrm{Op}(d_2(t, \lambda)) + w_2(t, \lambda)$, for $d_2 \in \widetilde{S}_{\mathrm{std,ph}}^{-4, -\delta}(\mathbb{R}; T^*\mathrm{r}^d)$ and $w_2(t, \lambda) \in \widetilde{\mathcal{W}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$. Using that

$$\int_{\mathbb{R}} (\langle \xi \rangle + \langle \lambda \rangle)^{-4-k} \lambda^{2\alpha+1} d\lambda \sim \langle \xi \rangle^{2\alpha-2-k},$$

we first obtain that

$$\int_{\mathbb{R}} d_2(t, \lambda) \lambda^{2\alpha+1} d\lambda \in S_{\mathrm{std,ph}}^{2\alpha-2, -\delta}(\mathbb{R}; \mathbb{R}^d).$$

Similarly we obtain that $\int_{\mathbb{R}} w_2(t, \lambda) \lambda^{2\alpha+1} d\lambda \in \mathcal{W}_{\mathrm{std}}^{-\infty}(\mathbb{R}; \mathbb{R}^d)$. Using (A.5) this implies that $a_1^\alpha(t) - a_2^\alpha(t) \in \Psi^{2\alpha, -\delta}(\mathbb{R}; \mathbb{R}^d)$, as claimed. $\square$

A.2. **Proof of Lemma 3.3.** By interpolation, it suffices to prove the lemma for $m, k \in \mathbb{N}$. Let us set

$$T_{m,k}(t) = \langle \epsilon(0) \rangle^m \langle \mathrm{x} \rangle^k \mathscr{U}^{\mathrm{ad}}(0,t)(\langle \mathrm{x} \rangle + \langle t \rangle)^{-k} \langle \epsilon(0) \rangle^{-m},$$

$$R_{m,k}(t,s) = \mathscr{U}^{\mathrm{ad}}(t,s) \langle \epsilon(s) \rangle^m \langle \mathrm{x} \rangle^k \mathscr{U}^{\mathrm{ad}}(s,t)(\langle \mathrm{x} \rangle + \langle t \rangle)^{-k} \langle \epsilon(0) \rangle^{-m}.$$

Using the uniform ellipticity of $\epsilon(t)$ it suffices to prove that

(A.6)
$$\sup_{t \geq 0} \|T_{m,k}(t)\|_{B(\mathcal{H}^0)} < \infty.$$

We claim that

(A.7)
$$\sup_{0 \leq s \leq t} \|R_{m,k}(t,s)\|_{B(\mathcal{H}^0)} < \infty,$$

This of course implies (A.6) by taking $s = 0$ and using that $\mathscr{U}^{\mathrm{ad}}(t,s)$ is uniformly bounded in $B(\mathcal{H}^0)$, see [GW4]. To prove (A.7) we compute
(A.8)
$$\partial_s R_{m,k}(t,s)$$
$$= \mathscr{U}^{\mathrm{ad}}(t,s) \left(\partial_s \langle \epsilon(s) \rangle^m + [H^{\mathrm{ad}}(s), -\mathrm{i}\langle \epsilon(s) \rangle^m] \right) \langle \mathrm{x} \rangle^k \mathscr{U}^{\mathrm{ad}}(s,t)(\langle \mathrm{x} \rangle + \langle t \rangle)^{-k} \langle \epsilon(0) \rangle^{-m}$$
$$+ \mathscr{U}^{\mathrm{ad}}(t,s) \langle \epsilon(s) \rangle^m [H^{\mathrm{ad}}(s), -\mathrm{i}\langle \mathrm{x} \rangle^k] \mathscr{U}^{\mathrm{ad}}(s,t)(\langle \mathrm{x} \rangle + \langle t \rangle)^{-k} \langle \epsilon(0) \rangle^m.$$

Recall that

$$H^{\mathrm{ad}}(t) = \begin{pmatrix} \epsilon(t) & 0 \\ 0 & -\epsilon(t) \end{pmatrix} + \Psi_{\mathrm{std}}^{0,-1-\delta}(\mathbb{R}; \mathbb{R}^d) \otimes B(\mathbb{C}^2)$$

by Prop. 2.5. Hence:

$$\left(\partial_s \langle \epsilon(s) \rangle^m + [H^{\mathrm{ad}}(s), -\mathrm{i}\langle \epsilon(s) \rangle^m] \right) \in \Psi_{\mathrm{std}}^{m,-1-\delta}(\mathbb{R}; \mathbb{R}^d) \otimes B(\mathbb{C}^2),$$

and we can write:

(A.9)
$$\left(\partial_s \langle \epsilon(s) \rangle^m + [H^{\mathrm{ad}}(s), -\mathrm{i}\langle \epsilon(s) \rangle^m] \right) = A_m(s) \langle \epsilon(s) \rangle^m \langle \mathrm{x} \rangle^{-1},$$

where

(A.10)
$$\|A_m(s)\|_{B(\mathcal{H}^0)} \in O(1),$$

since $(\langle \mathrm{x} \rangle + \langle t \rangle)^{-1-\delta} \leq \langle \mathrm{x} \rangle^{-1}$. Similarly, we have

(A.11)
$$\langle \epsilon(s) \rangle^m [H^{\mathrm{ad}}(s), \mathrm{i}\langle \mathrm{x} \rangle^k] = C_{m,k}(s) \langle \epsilon(s) \rangle^m \langle \mathrm{x} \rangle^{k-1},$$

where

(A.12)
$$\|C_{m,k}(s)\|_{B(\mathcal{H}^0)} \in O(1).$$

We also set

$$B_{m,k}(t) = (\langle \mathrm{x} \rangle + \langle t \rangle)^{-k+1} \langle \epsilon(0) \rangle^{-m} (\langle \mathrm{x} \rangle + \langle t \rangle)^{k-1} \langle \epsilon(0) \rangle^m,$$

and we have by pseudodifferential calculus

(A.13)
$$\|B_{m,k}(t)\|_{B(\mathcal{H}^0)} \in O(1).$$

Hence, we can rewrite (A.8) as

(A.14)
$$\begin{aligned} &\partial_s R_{m,k}(t,s) \\ &= \mathscr{U}^{\mathrm{ad}}(t,s) D_{m,k}(s) \mathscr{U}^{\mathrm{ad}}(s,t) \times R_{m,k-1}(t,s) \times B_{m,k}(t) \times (\langle \mathrm{x} \rangle + \langle t \rangle)^{-1}, \end{aligned}$$

where

(A.15)
$$D_{m,k}(s) = A_m(s) + C_{m,k}(s), \quad \|D_{m,k}(s)\|_{B(\mathcal{H}^0)} \in O(1).$$

We can prove now (A.7) by induction for k. First, note that by 1) of [GW4, Prop. 5.6], (A.7) holds for $k = 0$. Assume that (A.7) holds for $k - 1$. Integrating (A.14) from t to s we obtain:

$$\|R_{m,k}(t,s) - R_{m,k}(t,t)\| \leq \int_0^t \|R_{m,k-1}(t,\sigma)\|\langle t\rangle^{-1}dt \in O(1), \text{for } 0 \leq s \leq t$$

by the induction hypothesis. We conclude the proof of (A.7) using that

$$\|R_{m,k}(t,t)\| = \|\langle\epsilon(t)\rangle^m\langle\mathrm{x}\rangle^k(\langle\mathrm{x}\rangle + \langle t\rangle)^{-k}\langle\epsilon(0)\rangle^m\| \in O(1).$$

This completes the proof of the lemma. $\square$

References

[ALNV] B. Ammann, R. Lauter, V. Nistor, A. Vasy, *Complex powers and non-compact manifolds*, Comm. in PDE **29** (2004), 671–705.

[APS1] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. I.* Math. Proc. Cambridge Philos. Soc., 77, (1975), 43–69.

[APS2] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. III.* Math. Proc. Cambridge Philos. Soc., 79(1), (1976), 71–99.

[BB] C. Bär, W. Ballmann, *Boundary value problems for elliptic differential operators of first order*, Surveys in differential geometry, vol. 17, Int. Press, Boston MA, (2012), 1–78.

[BGP] C. Bär, N. Ginoux, F. Pfäffle, *Wave equation on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, EMS (2007).

[BF1] R. Brunetti, K. Fredenhagen, *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds*, Comm. Math. Phys. **208** (2000), 623-661.

[BF2] C. Bär (ed.), K. Fredenhagen (ed.), *Quantum Field Theory on Curved Spacetimes*, Lect. Notes Phys. 786 (2009).

[BF3] M. Brum, K. Fredenhagen, *'Vacuum-like' Hadamard states for quantum fields on curved spacetimes*, Class. Quantum Grav. 31 , no. 2, (2014), 025024.

[BH] U. Bunke, T. Hirschmann, *The index of the scattering operator on the positive spectral subspace*, Comm. Math. Phys. 148 (1992), 487–502.

[BS1] C. Bär, A. Strohmaier, *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, preprint `arXiv:1506.00959`, (2015).

[BS2] C. Bär, A. Strohmaier, *A rigorous geometric derivation of the chiral anomaly in curved backgrounds*, preprint `arXiv:1508.05345`, (2015).

[BVW] D. Baskin, A. Vasy, J. Wunsch, *Asymptotics of radiation fields in asymptotically Minkowski space*, to appear in Am. J. Math., `arXiv:1212.5141` (2012).

[CC] Y. Choquet-Bruhat, Y. Cotsakis, *Global hyperbolicity and completeness*, J. Geom. and Phys. **43**, (2002), 345–350.

[Co] H.O. Cordes, *A global parametrix for pseudo-differential operators over $\mathbb{R}^n$, with applications*, preprint no. 90, SFB 72, Bonn

[Da] N.V. Dang, *Renormalization of quantum field theory on curved spacetimes, a causal approach*, Ph.D. thesis, Paris Diderot University, (2013).

[DG1] J. Dereziński, C. Gérard, *Scattering Theory of Classical and Quantum N-particle Systems*, Texts and Monographs in Physics, Springer, 1997.

[DG2] J. Dereziński, C. Gérard, *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs in Mathematical Physics, Cambridge University Press, 2012.

[DH] J.J. Duistermaat J.J., L. Hörmander, *Fourier integral operators II*, Acta Math. **128** (1972), 183–269.

[DMP1] C. Dappiaggi, V. Moretti, N. Pinamonti, *Distinguished quantum states in a class of cosmological spacetimes and their Hadamard property*, J. Math. Phys. 50 (2009) 062304.

[DMP2] C. Dappiaggi, V. Moretti, N. Pinamonti, *Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime*, Adv. Theor. Math. Phys. 15 (2011) 355.

[FMR]   F. Finster, S. Murro, and C. Röken, *The fermionic projector in a time-dependent external potential: Mass oscillation property and Hadamard states*, preprint `arXiv:1501.05522`, (2015).

[FNW]   S.A. Fulling, F.J. Narcowich, R.M. Wald, *Singularity structure of the two-point function in quantum field theory in curved space-time, II*, Annals of Physics, **136** (1981), 243-272.

[FV1]   C.J. Fewster, R. Verch, *On a recent construction of "vacuum-like" quantum field states in curved spacetime*, Class. Quant. Grav. 29(20), (2012), 205017.

[FV2]   C.J. Fewster, R. Verch, *Algebraic quantum field theory in curved spacetimes*, in: Advances in Algebraic Quantum Field Theory, Springer (2015).

[Gi1]   G.W. Gibbons, *Cosmological fermion-number non-conservation*, Phys. Lett. B, 84(4), (1979), 431–434.

[Gi2]   G.W. Gibbons, *Spectral asymmetry and quantum field theory in curved spacetime*, Ann. of Phys., 125(1), (1980), 98–116.

[GHV]   J. Gell-Redman, N. Haber, A. Vasy, *The Feynman propagator on perturbations of Minkowski space*, Comm. Math. Phys., 342, 1, (2016), 333–384.

[GOW]   C. Gérard, O. Oulghazi, M. Wrochna, *Hadamard states for the Klein-Gordon equation on Lorentzian manifolds of bounded geometry*, preprint `arXiv:1602.00930` (2016).

[GR]   G.W. Gibbons and J. Richer, *Gravitational creation of odd numbers of fermions*, Phys. Lett. B, 89(3), (1980), 338-340.

[GW1]   C. Gérard, M. Wrochna, *Construction of Hadamard states by pseudo-differential calculus*, Comm. Math. Phys. **325** (2) (2014), 713-755.

[GW2]   C. Gérard, M. Wrochna, *Hadamard states for the linearized Yang-Mills equation on curved spacetime*, Comm. Math. Phys. 337 (1) (2015), 253–320.

[GW3]   C. Gérard, M. Wrochna, *Construction of Hadamard states by characteristic Cauchy problem*, Anal. PDE 9 (1), 111-149 (2016).

[GW4]   C. Gérard, M. Wrochna, *Hadamard states from scattering data for the Klein-Gordon equation on asymptotically static spacetimes*, preprint `arXiv:1603.07465` (2016).

[Hö]   L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis*, Springer (1985).

[HV1]   N. Haber, A. Vasy, *Propagation of singularities around a Lagrangian submanifold of radial points*, Microlocal Methods in Mathematical Physics & Global Analysis, Springer, (2013).

[HV2]   P. Hintz, A. Vasy, *Semilinear wave equations on asymptotically de Sitter, Kerr-de Sitter and Minkowski spacetimes*, to appear in Anal. PDE, `arXiv:1306.4705` (2013).

[HW1]   S. Hollands, R.M. Wald, *Existence of local covariant time ordered products of quantum fields in curved spacetime*, Comm. Math. Phys. 231, no. 2 (2002), 309–345.

[HW2]   S. Hollands, R.M. Wald, *Conservation of the stress tensor in perturbative interacting quantum field theory in curved spacetimes*, Rev. Math. Phys. 17.3 (2005), 277-311.

[HW3]   S. Hollands, R.M. Wald, *Quantum fields in curved spacetime*, in: General Relativity and Gravitation: A Centennial Perspective, Cambridge University Press (2015).

[Is]   H. Isozaki, *QFT for scalar particles in external fields on Riemannian manifolds*, Rev. Math. Phys. 13, No. 6 (2001), 767–798.

[Ju]   W. Junker, *Adiabatic Vacua and Hadamard States for Scalar Quantum Fields on Curved Spacetime*, PhD thesis, University of Hamburg 1995.

[JS]   W. Junker, E. Schrohe, *Adiabatic Vacuum States on General Space-time Manifolds: Definition, Construction, and Physical Properties*, Ann. Henri Poincaré, **3** (2002), 1113-1181.

[KM]   I. Khavkine, V. Moretti, *Algebraic QFT in curved spacetime and quasifree Hadamard states: an introduction*, in: Advances in Algebraic Quantum Field Theory, Springer (2015).

[Ma1]   T. Matsui, *The index of scattering operators of Dirac equations*, Comm. Math. Phys. 110, (1987), 553–571.

[Ma2]   T. Matsui, *The index of scattering operators of Dirac equations, II*, J. Funct. Anal. 94, (1990), 93–109.

[Pa]   C. Parenti, *Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni*, Annali Mat. Pura ed Appl. 93, 359–389.

[Ra]   M. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Phys. **179** (1996), 529–553.

[RS]   M. Reed, B. Simon, *Methods of modern mathematical physics II : Fourier analysis, self-adjointness*, Academic Press, 1975.

[Sh1]   M.A. Shubin, *Pseudodifferential operators in $\mathbb{R}^n$*, Dokl. Akad. Nauk SSSR 196, N2 (1971), 316–319.

[Sh2]   M.A. Shubin, *Pseudo-differential operators and spectral theory*, Springer (2001).

[Va1]   A. Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, MSRI Publications, vol. 60, Cambridge University Press, (2012).

[Va2] A. Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces, (With an appendix by S. Dyatlov)*, Inventiones Math., (2013), 194.2: 381–513.

[Va3] A. Vasy, *On the positivity of propagator differences*, `arXiv:1411.7242` (2014).

[VW] A. Vasy, M. Wrochna, *Quantum fields from global propagators on asymptotically Minkowski and extended de Sitter spacetimes*, preprint `arXiv:1512.08052`, (2015).

[Wa2] R.M. Wald, *General Relativity*, University of Chicago Press (1984).

Université Paris-Sud XI, Département de Mathématiques, 91405 Orsay Cedex, France

E-mail address: `christian.gerard@math.u-psud.fr`

Université Grenoble Alpes, Institut Fourier, UMR 5582 CNRS, CS 40700, 38058 Grenoble Cedex 09, France

E-mail address: `michal.wrochna@ujf-grenoble.fr`