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Abstract

The purpose of this article is to generalize the following results existing respectively in ([6], [8]) and in [7] to Riemannian foliations with dense leaves on a compact manifold:

i) If $\mathcal{F}$ is a Lie G-foliation with dense leaves on a compact manifold, then there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie(G)$ and $\mathcal{F}$ extensions,

ii) if $\mathcal{F}_{\mathcal{H}}$ is an extension of a Lie $\mathcal{G}$-foliation with dense leaves on a compact manifold corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ and $\ell(M, \mathcal{F}_{\mathcal{H}})$ the Lie algebra of $\mathcal{F}_{\mathcal{H}}-$foliated transverse vectors fields then we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \, [u, h] = 0 \ \text{ for every } h \in \mathcal{H}\}.$$

Let G be a Lie simply connected group and let $\mathcal{G} = Lie(G)$ be the structural Lie algebra of a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold.

This study show us that there exists a discreet group H_0 and a representation $\rho : H_0 \to Diff(V)$ where V is an open of G such as:

i) there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie(G)$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and extensions of $\mathcal{F}$,

ii) if $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ then we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\, (h, a, v) \in \mathcal{H} \times H_0 \times V, \ [u, h] = 0 \ \text{ and } Ad_{(\rho_a(v))^{-1}.v}(u) = u\}.$$

Keywords: *Lie foliation, Riemannian foliation, foliation with dense leaves, extension of a foliation.*

1 Introduction

The purpose of this article is to generalize the following results existing respectively in ([6], [8]) and in [7] to Riemannian foliations with dense leaves on a compact manifold:

i) If $\mathcal{F}$ is a Lie G-foliation with dense leaves on a compact manifold, so it exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = lie\,(G)$ and $\mathcal{F}$ extensions,

ii) if $\mathcal{F}_{\mathcal{H}}$ is an extension of a Lie $\mathcal{G}$-foliation with dense leaves on a compact manifold corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ and $\ell(M, \mathcal{F}_{\mathcal{H}})$ the Lie algebra of $\mathcal{F}_{\mathcal{H}}-$foliated transverse vectors fields then we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \, [u, h] = 0 \ \text{ for every } h \in \mathcal{H}\}.$$

To achieve there, we first established that the closure of a leaf $F^{\natural}$ of lifted foliation $\mathcal{F}^{\natural}$ on the orthonormal transverse frame bundle $M^{\natural}$ of a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold is a covering of the manifold M.

We note that the fact that $\overline{F^{\natural}}$ is a covering of the manifold M entails that the dimension of the Lie structural algebra of a Riemannian foliation $\mathcal{F}$ on a compact manifold which is less than or equal to the codimension of the foliation $\mathcal{F}$. And this allows us to say that we should limit the classification of Molino in codimension 1, 2 and 3 of Riemannian foliation on a compact manifold [12] to only a few cases of this classification.

We note by H_0 the structure group of covering $\phi : \overline{F^{\natural}} \to M$. And, H_0 will be called *the discreet goup of Riemannian foliation* $\mathcal{F}$ on a compact manifold with dense leaves.

That said, we show in this paper that if $\mathcal{G}$ is the Lie structural algebra of a Riemannian foliation with dense leaves $(M, \mathcal{F})$ on a compact manifold M there exists a representation $\rho : H_0 \to Diff\,(V)$ where V is an open of G such as:

i) there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie\,(G)$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and $\mathcal{F}$ extensions,

ii) an extension is a Lie foliation if the subalgebra corresponding is an ideal of $\mathcal{G}$,

iii) every extension $\mathcal{F}'$ of $\mathcal{F}$ is a Riemannian foliation and there exists a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$,

iv) if $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ then to isomorphism nearly of Lie algebras we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\,(h, a, v) \in \mathcal{H} \times H_0 \times V, \ [u, h] = 0 \ \text{ and } Ad_{(\rho_a(v))^{-1}.v}\,(u) = u \ \}.$$

In particular

$$\ell(M, \mathcal{F}) = \{u \in \mathcal{G} / \forall\,(a, v) \in H_0 \times V, \ Ad_{(\rho_a(v))^{-1}.v}\,(u) = u\}.$$

Our paper is divided into two parts:

-the first part is devoted to reminders on Riemannian foliations and on extensions of foliations,

- the second part is devoted to the establishment of the already stated primary outcome.

In all that follows, the manifolds considered are supposed connected and differentiability is C^∞.

2 Definitions and Reminders

In this section, in the direction that is helpful, we reformulate some existing definitions and theorems in ([4], [6], [7], [8], [10] [11], [12], [13]).

Definition 2.1 *Let M be a manifold.*

An extension of a codimension q foliation $(M,\mathcal{F})$ is a codimension q' foliation $(M,\mathcal{F}')$ such that $0 < q' < q$ and $(M,\mathcal{F}')$ leaves are $(M,\mathcal{F})$ leaves meetings (it is noted $\mathcal{F} \subset \mathcal{F}'$).

We show that if $(M,\mathcal{F}')$ is a simple extension of a simple foliation $(M,\mathcal{F})$ and if $(M,\mathcal{F})$ and $(M,\mathcal{F}')$ are defined respectively by submersions $\pi : M \to T$ and $\pi' : M \to T'$, then there exists a submersion $\theta : T \to T'$ such that $\pi' = \theta \circ \pi$.

We say that the submersion θ is a bond between the foliation $(M,\mathcal{F})$ and its extension foliation $(M,\mathcal{F}')$.

It is shown in [6] that if the foliation $(M,\mathcal{F})$ and its extension $(M,\mathcal{F}')$ are defined respectively by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ and $(U_i, f'_i, T', \gamma'_{ij})_{i\in I}$ then we have

$$f'_i = \theta_i \circ f_i \quad \text{and } \gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}$$

where θ_s is a bond between the foliation $(U_s,\mathcal{F})$ and its extension foliation $(U_s,\mathcal{F}')$.

Definition 2.2 *Let $\mathcal{F}_q$ be a codimension q foliation on a manifold M.*

A flag of extensions of $\mathcal{F}_q$ is a sequence $\mathcal{D}^k_{\mathcal{F}_q} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_k)$ of foliations on the manifold M such that $\mathcal{F}_q \subset \mathcal{F}_{q-1} \subset \mathcal{F}_{q-2} \subset ... \subset \mathcal{F}_k$ and each foliation $\mathcal{F}_s$ is a codimension s foliation.

For $k = 1$, the flag of extensions $\mathcal{D}^k_{\mathcal{F}_q}$ will be called complete and will be denoted $\mathcal{D}_{\mathcal{F}_q}$.

If each foliation $\mathcal{F}_s$ is a Riemannian foliation, the flag of extensions $\mathcal{D}^k_{\mathcal{F}_q}$ will be called flag of Riemannian extensions of $\mathcal{F}_q$.

The following theorem is the biunivocal correspondence theorem between Lie subalgebras of $\mathcal{G} = Lie(G)$ and the extensions of a Lie $G-$foliation with dense leaves existing in [8].

Theorem 2.3 [8] *Let $(M,\mathcal{F})$ be a Lie G-foliation with dense leaves on compact connected manifold and let $\mathcal{G}$ be the Lie algebra of G.*

Then:

1-There exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G}$ (or if you prefer between the connected Lie subgroups of G) and extensions of $\mathcal{F}$.

2- An extension of $\mathcal{F}$ is a Riemannian $\frac{\mathcal{G}}{\mathcal{H}}$-foliation having trivial normal bundle and defined by a 1-form with values in $\frac{\mathcal{G}}{\mathcal{H}}$.

3- An extension of $\mathcal{F}$ is transversely homogeneous (resp. Lie) if and only if the Lie subgroup of G corresponding is a closed subgroup (resp. Normal subgroup) in G.

In [7] was calculated $\ell(M, \mathcal{F}_{\mathcal{H}})$ where $\mathcal{F}_{\mathcal{H}}$ is the extension of a Lie $\mathcal{G}-$foliation with dense leaves on a compact manifold corresponding to a Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$. This calculation gave us the following result:

Theorem 2.4 [7] *Let $\mathcal{H}$ be a Lie subalgebra of Lie algebra $\mathcal{G}$ of a Lie foliation $\mathcal{F}$ with dense leaves on a compact manifold, let ω be a 1-form of Fedida defining $\mathcal{F}$ and let $\mathcal{F}_{\mathcal{H}}$ be the extension of $\mathcal{F}$ corresponding to $\mathcal{H}$.*

Then:

i)

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \underset{h\in\mathcal{H}}{\cap} \ell(M, \mathcal{F}_{<h>})$$

where $\mathcal{F}_{<h>}$ is the extension of $\mathcal{F}$ corresponding to the Lie subalgebra $< h >$ of $\mathcal{G}$ generated by h.

ii)

$$\omega\left(\ell(M, \mathcal{F}_{\mathcal{H}})\right) \subset \mathcal{H}^{\perp}$$

where $\mathcal{H}^{\perp}$ is the ortho-complementary of $\mathcal{H}$ in $\mathcal{G}$ by the transverse metric associated with $\mathcal{F}$.

iii) For $h \in \mathcal{G}$,

$$\omega\left(\ell(M, \mathcal{F}_{<h>})\right) = \{u \in < h >^{\perp} / \ [u, h] = 0 \ \}.$$

iv)

$$\omega\left(\ell(M, \mathcal{F}_{\mathcal{H}})\right) = \{u \in \mathcal{H}^{\perp} / \, [u, h] = 0 \quad \textit{for every } h \in \mathcal{H}\}.$$

The theorem and the three following proposals allow us to give a generalization of the previous two theorems for Riemannian foliations with dense leaves on a compact manifold .

Theorem 2.5 [7] *Let $\mathcal{F}$ be a Lie $G-$foliation with dense leaves on a compact manifold M, let λ be a metric on M which is bundle-like for $\mathcal{F}$ and which admits $\lambda_{_T}$ as its associated transverse metric and let X be a $\mathcal{F}-$transverse foliated vectors field.*

Then:

i) for every point $a \in M$ there exist an open $\mathcal{F}$-distinguished V_a of M containing the point a such that the restriction X_{V_a} of X at V_a is a local $\mathcal{F}$-transverse Killing vectors field,

ii) any vectors field left invariant of G is a Killing vectors field for the $\mathcal{F}$-transverse metric $\lambda_{_T}$ left invariant,

iii) in the case where G is connected, the right translation R_a associated with the element $a \in G$ is an isometry for the metric $\lambda_{_T}$ left invariant.

Proposition 2.6 [7] *Let $\mathcal{F}''$ be an extension of a Riemannian foliation $\mathcal{F}'$ on a manifold M.*

Then $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$.

Proposition 2.7 [12] *Let $\mathcal{F}$ be a codimension q Riemannian foliation on a compact connected manifold M, let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$, let ϕ:$M^\natural \to M$ be the projection which to a frame at x associates x.*

Them:

i) $\phi\left(F^\natural\right)$ is a leaf of $\mathcal{F}$ and $\phi\left(\overline{F^\natural}\right) = \overline{\phi\left(F^\natural\right)}$,

ii) the map $\phi : \overline{F^\natural} \to \phi\left(\overline{F^\natural}\right)$ is a locally trivial fibration.

Proposition 2.8 [12] *Let (M, λ) be a Riemannian connected manifold and let $\mathcal{K}(M, \lambda)$ be a Lie algebra of Killing vectors field on (M, λ).*

Then the orbits of $\mathcal{K}(M, \lambda)$ having maximal dimension form an open dense in M.

3 Riemannian foliation with dense leaves on a compact manifold

In what follows G is a Lie group of Lie algebra $\mathcal{G}$, X^r (resp. X^l) is the vectors field on G right (resp. Left) invariant obtained from $X \in \mathcal{G}$ and L_a (resp. R_a) is the left (resp. right) translation associated with $a \in G$.

Proposition 3.1 *Let $\mathcal{K}(M, \lambda)$ be a Lie algebra of Killing vectors fields on a Riemannian connected manifold (M, λ), let $i_x = \{X \in \mathcal{K}(M, \lambda) / X_x = 0\}$ be the isotropy of $\mathcal{K}(M, \lambda)$ at the point $x \in (M, \lambda)$, let $\mathcal{O}_x$ be the orbit at the point x of $\mathcal{K}(M, \lambda)$ and let $\mathcal{O}_{(x_0,x)}$ be the orbit at the point $x \in (M, \lambda)$ of the isotropy i_{x_0} where $x_0 \in (M, \lambda)$.*

If for $x \in (M, \lambda)$, all orbits $\mathcal{O}_x$ have the same dimension then $\mathcal{O}_{(x_0,x)} = \{x\}$ for every $(x_0, x) \in (M, \lambda)^2$.

Proof . Let $x \in (M, \lambda)$. We have $\dim i_x = \dim \mathcal{K}(M, \lambda) - \dim \mathcal{O}_x$.

As all orbits $\mathcal{O}_x$ have the same dimension then $\dim i_x = \dim i_y$ for every $y \in (M, \lambda)$.

We note before continuing that i_x is a Lie subalgebra of Killing vectors fields of $\mathcal{K}(M, \lambda)$.

Let $x_0 \in (M, \lambda)$ and let $i_{(x_0,x)}$ be the isotropy at the point x of i_{x_0}.

We know [12] there exists an open neighborhood U_{x_0} of x_0 such as if $x' \in U_{x_0}$ then $i_{(x_0,x')} = i_{x'}$ that is to say the isotropy at the point x' of i_{x_0} is equal to the isotropy at the point x' of $\mathcal{K}(M, \lambda)$ for every $x' \in U_{x_0}$. For proof we assume

that $(k_1, k_2, ..., k_r, k'_1, k'_2, ..., k'_s)$ is a base of $\mathcal{K}(M, \lambda)$ such as $(k_1, k_2, ..., k_r)$ is a base of i_{x_0}.

For every $j \in \{1, ..., s\}$, $k'_j(x_0) \neq 0$. Hence there exists an open U_{x_0} containing x_0 such as for every $x' \in U_{x_0}$ and for every $j \in \{1, ..., s\}$, $k'_j(x') \neq 0$.

Let $x \in U_{x_0}$ and let $Y \in i_x$. there exists two finite sequences $(y_j)_{1 \leq j \leq r}$ and $\left(y'_j\right)_{1 \leq j \leq s}$ of real numbers such that $Y = \sum_{j=1}^{r} y_j.k_j + \sum_{j=1}^{s} y'_j.k'_j$.

We have $\sum_{j=1}^{r} y_j.k_j(x) + \sum_{j=1}^{s} y'_j.k'_j(x) = 0$ because $Y \in i_x$.

Quits to reorder the base of vectors fields $(k_1, k_2, ..., k_r)$ of i_{x_0} we can assume that there exists $r' \in \{0, 1, ..., r\}$ such as $k_j(x) \neq 0$ for $j \leq r'$ and $k_j(x) = 0$ for $r' + 1 \leq j \leq r$.

Thus, the fact that $Y \in i_x$ implies that $\sum_{j=1}^{r'} y_j.k_j(x) + \sum_{j=1}^{s} y'_j.k'_j(x) = 0 \quad (*)$.

As $k_j(x) \neq 0$ for $j \leq r'$ and $k'_j(x) \neq 0$ and $(k_1, k_2, ..., k_{r'}, k'_1, k'_2, ..., k'_s)$ is a free system of vectors fields of $\mathcal{K}(M, \lambda)$ then $(k_1(x), ..., k_{r'}(x), k'_1(x), ..., k'_s(x))$ form a free system. Therefore the equality $(*)$ shows that $y_j = 0$ for $j \leq r'$ and $y'_j = 0$ for every j.

It follows from this that the equality $Y = \sum_{j=1}^{r} y_j.k_j + \sum_{j=1}^{s} y'_j.k'_j$ implies that $Y = \sum_{j=r'+1}^{r} y_j.k_j$. Therefore $Y \in i_{x_0}$ and $i_x \subset i_{x_0}$.

The constancy of the dimension of i_y for all $y \in (M, \lambda)$ implies that $i_x = i_{x_0}$.

Thus for any $x \in U_{x_0}$, $i_{(x_0,x)} = i_x = i_{x_0}$ and the orbit $\mathcal{O}_{(x_0,x)}$ of i_{x_0} at the point x is the point x.

We know that ([14], [12]) the orbits of Lie algebra of Killing vectors fields on Riemannian connected manifold V having maximal dimension form an open dense in V. From where, the fact that the maximal dimension of the orbits of i_{x_0} in the open U_{x_0} is zero implies that $\mathcal{O}_{(x_0,x)} = \{x\}$ for all $x \in (M, \lambda)$. ■

We note that this proposition means that when a Lie algebra of Killing vectors fields $\mathcal{K}(M, \lambda)$ on a connected Riemannian manifold (M, λ) has his orbits having the same dimension then the isotropy i_x of $\mathcal{K}(M, \lambda)$ at any point $x \in (M, \lambda)$ *induces* on (M, λ) a null Lie algebra of Killing vectors fields.

That said, the previous proposition allows us to establish the following result:

Proposition 3.2 *Let H be a closed Lie subgroup of a connected Lie group G, let λ be a metric on G left invariant, let $\mathcal{G}$ =Lie(G) and let $\mathcal{H}$ =Lie(H).*

If λ is invariant by right translations obtained from the elements of H then the Lie subalgebra $\mathcal{H}$ is an ideal of $\mathcal{G}$.

Proof . If the Lie group H is discreet, then $\mathcal{H} = \{0\}$. And in this case $\mathcal{H}$ is an ideal of $\mathcal{G}$.

In what follows we assume that H is not discreet.

The Lie subgroup H is closed in the Lie group G. From where $\pi : G \to \frac{G}{H}$ is a principal fibration having H for structure group. We note in passing that π is a Riemannian submersion because the metric λ left invariant on G is invariant by right translations obtained from the elements of H.

Let α be the left Maurer Cartan form of G, let $p_{\mathcal{H}} : \mathcal{G} \to \mathcal{H}$ be the orthogonal projection on $\mathcal{H}$ and let $\alpha_{\mathcal{H}} = p_{\mathcal{H}} \circ \alpha$.

It is easy to see that $\alpha_{\mathcal{H}}$ is a differential form on G with values in $\mathcal{H}$.

Let $a \in H$, let $X \in TG$ and let $p_{\mathcal{H}^\perp} : \mathcal{G} \to \mathcal{H}^\perp$ be the orthogonal projection on $\mathcal{H}^\perp$ where $\mathcal{H}^\perp$ is the ortho-complementary of $\mathcal{H}$ in $\mathcal{G}$.

We have

$$\begin{aligned}
\left(R_a^* \alpha_{\mathcal{H}}\right)(X) &= \alpha_{\mathcal{H}}\left(R_{a*}(X)\right) \\
&= p_{\mathcal{H}} \circ \alpha\left(R_{a*}(X)\right) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}}\left(\alpha(X)\right) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}}\left(p_{\mathcal{H}}\left(\alpha(X)\right) + p_{\mathcal{H}^\perp}\left(\alpha(X)\right)\right) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}}\left(\alpha(X)\right) + p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}^\perp}\left(\alpha(X)\right).
\end{aligned}$$

We signal that for all $a \in H$, $\mathcal{H}$ and $\mathcal{H}^\perp$ are invariant by $Ad_{a^{-1}}$ because the left invariant metric λ is right invariant by translations obtained from elements of H.

It follows from this that

$$p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}}\left(\alpha(X)\right) = Ad_{a^{-1}} \circ p_{\mathcal{H}}\left(\alpha(X)\right) \text{ and } p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}^\perp}\left(\alpha(X)\right) = 0.$$

Thus, we get that for all $a \in H$ and for all $X \in TG$,

$$\left(R_a^* \alpha_{\mathcal{H}}\right)(X) = Ad_{a^{-1}} \circ p_{\mathcal{H}}\left(\alpha(X)\right) = Ad_{a^{-1}} \circ \alpha_{\mathcal{H}}(X).$$

And, this means that $\alpha_{\mathcal{H}}$ is a connection on the principal bundle $\pi : G \to \frac{G}{H}$.

Let $\mathcal{F}_{G,H}$ be the foliation obtained by the left translations of H and let $\left(T\mathcal{F}_{G,H}\right)^\perp$ be the orthogonal bundle of $T\mathcal{F}_{G,H}$.

It is clear to see that $\left(T\mathcal{F}_{G,H}\right)^\perp \subset \ker \alpha_{\mathcal{H}}$. But $\dim \ker \alpha_{\mathcal{H}} = \dim \frac{G}{H} = \dim \left(T\mathcal{F}_{G,H}\right)^\perp$ so $\ker \alpha_{\mathcal{H}} = \left(T\mathcal{F}_{G,H}\right)^\perp$.

In what follows the transverse bundle $\mathcal{V}\left(\mathcal{F}_{G,H}\right)$ is identified to $\left(T\mathcal{F}_{G,H}\right)^\perp$.

Above all, we note that H is not discreet then foliation $\mathcal{F}_{G,H}$ is not a foliation by points.

Let $\mathcal{G}^r$ be the Lie algebra of right invariant vectors fields of G and let $\mathcal{H}^r$ be the Lie subalgebra of right invariant vectors fields obtained from the vectors of $\mathcal{H}$.

Any vectors field $u^r \in \mathcal{G}^r$ associated with the vector $u \in \mathcal{G}$ commutes with every vectors field left invariant. From where u^r is a $\mathcal{F}_{G,H}-$foliated vectors field.

As the submersion $\pi : G \to \frac{G}{H}$ defined the foliation $\mathcal{F}_{G,H}$ then u^r is projected by π on $\frac{G}{H}$ following a vectors field notes that $\underline{u}^r$.

Let $\mathcal{X}\left(\frac{G}{H}\right)$ be the Lie algebra of vectors fields tangent to $\frac{G}{H}$, let $\mathcal{X}(G)$ be the Lie algebra of vectors fields tangent to G, let $w \in \mathcal{X}\left(\frac{G}{H}\right)$, let $X \in \mathcal{X}(G)$,

let $\widetilde{w}$ be the horizontal lift of w, let X^h be the horizontal component of X and let X^v be the vertical component of X.

We have ([1], [13]) for $w_1 \in \mathcal{X}\left(\frac{G}{H}\right)$ and $w_2 \in \mathcal{X}\left(\frac{G}{H}\right)$,

$$\begin{aligned}\pi_*\left([\widetilde{w}_1, \widetilde{w}_2]\right) &= \pi_*\left([\widetilde{w}_1, \widetilde{w}_2]^h\right) \\ &= \pi_*\left(\widetilde{[w_1, w_2]}\right) \\ &= [w_1, w_2] \\ &= [\pi_*(\widetilde{w}_1), \pi_*(\widetilde{w}_2)].\end{aligned}$$

Furthermore, for all for all $(u_1, u_2) \in \mathcal{G}^2$ as u_1^r and u_2^r are $\mathcal{F}_{G,H}-$foliated vectors fields then $\left[(u_1^r)^h, (u_2^r)^v\right]$ is a section of $T\mathcal{F}_{G,H}$ because $(u_1^r)^v$ and $(u_2^r)^v$ are sections of $T\mathcal{F}_{G,H}$ and

$$\left[(u_1^r)^h, (u_2^r)^v\right] = [u_1^r, (u_2^r)^v] - [(u_1^r)^v, (u_2^r)^v].$$

It follows from the foregoing that $\pi_*([u_1^r, u_2^r]) = [\pi_*(u_1^r), \pi_*(u_2^r)]$ for all $(u_1, u_2) \in \mathcal{G}^2$ because

$$\begin{aligned}\pi_*([u_1^r, u_2^r]) &= \pi_*\left(\left[(u_1^r)^v + (u_1^r)^h, (u_2^r)^v + (u_2^r)^h\right]\right) \\ &= \pi_*([(u_1^r)^v, u_2^r]) + \pi_*\left(\left[(u_1^r)^h, (u_2^r)^v\right]\right) + \pi_*\left(\left[(u_1^r)^h, (u_2^r)^h\right]\right) \\ &= \pi_*\left(\left[(u_1^r)^h, (u_2^r)^h\right]\right) \\ &= \pi_*\left(\left[\widetilde{\pi_*(u_1^r)}, \widetilde{\pi_*(u_2^r)}\right]\right) \\ &= \left[\pi_*\left(\widetilde{\pi_*(u_1^r)}\right), \pi_*\left(\widetilde{\pi_*(u_2^r)}\right)\right] \\ &= [\pi_*(u_1^r), \pi_*(u_2^r)]\end{aligned}$$

The equality $\pi_*([u_1^r, u_2^r]) = [\pi_*(u_1^r), \pi_*(u_2^r)]$ for all $(u_1, u_2) \in \mathcal{G}^2$ show that $\underline{\mathcal{G}}^r = \pi_*(\mathcal{G}^r)$ is a Lie algebra because $\mathcal{G}^r$ is a Lie algebra.

As the right invariant vectors fields u^r for all $u \in \mathcal{G}$ are Killing fields for the metric λ, then the fact that π is a Riemannian submersion implies that the vectors fields $\underline{u}^r$ are also Killings vectors fields. Thus $\underline{\mathcal{G}}^r$ is a Lie algebra of Killing fields on $\frac{G}{H}$.

We note that $(u^r)^h$ is the $\mathcal{F}_{G,H}-$foliated transverse vectors field associated with u^r since $\ker \alpha_{\mathcal{H}} = (T\mathcal{F}_{G,H})^{\perp}$ and we have identified $(T\mathcal{F}_{G,H})^{\perp}$ and the transverse bundle $\mathcal{V}(\mathcal{F}_{G,H})$.

We note also that the horizontal lift $\widetilde{\underline{u}}^r$ of $\underline{u}^r = \pi_*(u^r)$ checks

$$\widetilde{\underline{u}}^r = (u^r)^h.$$

Let $(u_1, u_2 ... u_s, u_{s+1}, ..., u_q)$ be an orthonormal base of $\mathcal{G}$ such as $(u_{s+1}, ..., u_q)$ is a base of $\mathcal{H}$ and u_i^r the right invariant vectors field obtained from u_i.

It results from the equality $\underline{\widetilde{u}}_i^r = (u_i^r)^h$ that $(u_i^r)^h$ is invariant by right translations obtained from the elements of H. Therefore, these right translations being isometric, we have for all $i \leq s$ and for all $a \in H$, $\left((u_1^r)^h(a), (u_2^r)^h(a), ..., (u_s^r)^h(a)\right)$ is an orthonormal base of $(T_aH)^\perp$ and

$$\underline{\widetilde{u}}_i^r(a) = (u_i^r)^h(a) = u_i^r(a).$$

Thus the fact that π is a riemannian submersion implies that $\left(\underline{u}_1^r\left(\overset{\bullet}{e}\right), \underline{u}_2^r\left(\overset{\bullet}{e}\right), ..., \underline{u}_s^r\left(\overset{\bullet}{e}\right)\right)$ is an orthonormal base of $T_{\overset{\bullet}{e}}\frac{G}{H}$ where $\overset{\bullet}{e}$ is the class of the identity element e of G in $\frac{G}{H}$. And, this entails that $\underline{\mathcal{G}}^r$ is a Lie algebra of Killing fields on $\frac{G}{H}$ having $\dim\left(\frac{G}{H}\right)$ for maximal dimension of its orbits.

We know that ([14], [12]) the orbits of maximal dimension of a Lie algebra of Killing fields on a connected manifold V form an open dense in V. From where there is an open U_r of $\frac{G}{H}$ dense in $\frac{G}{H}$ and containing $\overset{\bullet}{e}$ on which the orbits of $\underline{\mathcal{G}}^r$ are the dimension of $\frac{G}{H}$.

We note that there exists an open $U_r' \subset U_r$ such as U_r' is connected and $\overset{\bullet}{e} \in U_r'$.

We note also that $\frac{G}{H}$ is connected because G is connected and the map $\pi : G \to \frac{G}{H}$ is continuous.

According to the Proposition 3.1 the dimension of any orbit $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)}$ of isotropy $i_{\overset{\bullet}{e}}$ of $\underline{\mathcal{G}}^r$ at every point $\overset{\bullet}{x}$ of U_r' is null. Now the orbits of maximal dimension of the Lie algebra of Killing fields $i_{\overset{\bullet}{e}}$ on the manifold $\frac{G}{H}$ form a dense open of $\frac{G}{H}$. From where the fact that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in U_r'$ implies that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in \frac{G}{H}$.

It is easy to see that $\underline{\mathcal{H}}^r = \pi_*(\mathcal{H}^r)$ is a Lie subalgebra of $i_{\overset{\bullet}{e}}$. Therefore the fact that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in \frac{G}{H}$ shows us that $\underline{\mathcal{H}}^r = \pi_*(\mathcal{H}^r)$ is null. This means that any vector field right invariant obtained from the vectors of $\mathcal{H}$ is tangent to the foliation $\mathcal{F}_{G,H}$.

Thus for every $a \in G$, we have $aH = Ha$. In other words $\mathcal{H}$ is an ideal of $\mathcal{G}$.

■

The following result is another consequence of the proposition 3.1. This result is the basis of the generalization that we do in this paper. It allows us to look Riemannian foliations with dense leaves on a compact manifold with a new look. With this result we can for example associated with a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold a finished group whose properties depend on the nature of $\mathcal{F}$.

Proposition 3.3 *Let $\mathcal{F}$ be a riemannian foliation with dense leaves on a compact connected manifold M and let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$.*

Then $\overline{F^\natural}$ is a compact covering of M.

Proof . Let $\mathcal{F}$ be a riemannian foliation with dense leaves on a compact connected manifold M, let $\phi : \overline{F^\natural} \to M$ be the projection which to a frame at x associates x, let $f : U \to \overline{U}$ be a Riemannian submersion defining Riemannian foliation $\mathcal{F}$ on a distinguished connected open U and let $f^\natural : \phi^{-1}(U) \to E^\natural\left(\overline{U}\right)$ be the projection of $\phi^{-1}(U)$ on the orthonormal frame bundle of the local $\mathcal{F}$-quotient manifold $\overline{U}$.

We know that [12] $\phi : \overline{F^\natural} \to M$ is a principal fibration and the submersion $f^\natural$ defined the lifted foliation $\mathcal{F}^\natural$ on $\phi^{-1}(U)$ and there exists a submersion $\overline{\phi} : E^\natural\left(\overline{U}\right) \to \overline{U}$ making the diagram

$$\begin{array}{ccc} \phi^{-1}(U) & \stackrel{f^\natural}{\to} & E^\natural\left(\overline{U}\right) \\ \phi \downarrow & & \downarrow \overline{\phi} \\ U & \stackrel{f}{\to} & \overline{U} \end{array}$$

commutative because ϕ sends the fibers of $f^\natural$ on the fibers of f. According to Molino [12] $\overline{\phi} : E^\natural\left(\overline{U}\right) \to \overline{U}$ is the orthonormal transverse frame bundle above the local $\mathcal{F}$-quotient manifold $\overline{U}$.

Let X be a Killing vector fields on $\overline{U}$.

We recall that if $\left(\varphi_t^X\right)_{|t|<\varepsilon}$ is the local 1 parameter group associated to X then the local 1 parameter group $\left(\varphi_t^X\right)_* \circ x^\natural$ where $x^\natural \in E^\natural\left(\overline{U}\right)$ defined a vectors field $X^\natural$ on $E^\natural\left(\overline{U}\right)$ that we call the lifted vectors field of X on $E^\natural\left(\overline{U}\right)$ [12]. This lifted vectors field commutes with the canonical parallelism [12] of $E^\natural\left(\overline{U}\right)$.

We also recall that [12] any vectors field $Y^\natural$ of $E^\natural\left(\overline{U}\right)$ coincides in a neighborhood of each point of $E^\natural\left(\overline{U}\right)$ with the lift of a local Killing vectors field on $\overline{U}$ if and only if $Y^\natural$ commutes with the canonical parallelism of $E^\natural\left(\overline{U}\right)$.

That said, as $\mathcal{F}$ is a Riemannian foliation with dense leaves then the Lie algebra $\mathcal{G}^r$ of right invariant vectors fields obtained from the structural Lie algebra $\mathcal{G}$ of $\mathcal{F}$ is a Lie algebra of Killing vectors fields operating transitively [12] respectively on each connected component of $E^\natural\left(\overline{U}\right)$ and on $\overline{U}$. To be specific it is the Lie algebra of Killing fields $\overline{\phi}_*\left(\mathcal{G}^r\right)$ isomorphic ([12], [11]) to $\mathcal{G}^r$ which operates transitively on $\overline{U}$ and the lifted Lie algebra of $\overline{\phi}_*\left(\mathcal{G}^r\right)$ on $E^\natural\left(\overline{U}\right)$ is $\mathcal{G}^r$.

Let $x_0 \in \overline{U}$. According to Proposition 3.1 the dimension of any orbit $\mathcal{O}_{(x_0,x)}$ of the isotropy i_{x_0} of $\overline{\phi}_*\left(\mathcal{G}^r\right)$ at every point x of $\overline{U}$ is null.

Thus, the isotropy i_x of $\overline{\phi}_*\left(\mathcal{G}^r\right)$ for every $x \in \overline{U}$ is null.

As

$$\dim i_x = \dim\left(\overline{\phi}_*\left(\mathcal{G}^r\right)\right) - \dim \mathcal{O}_x = \dim\left(\overline{\phi}_*\left(\mathcal{G}^r\right)\right) - \dim \overline{U}$$

where $\mathcal{O}_x$ is the orbit at the point $x \in \overline{U}$ of $\overline{\phi}_*\left(\mathcal{G}^r\right)$ then

$$\dim \mathcal{G} = \dim \mathcal{G}^r = \dim\left(\overline{\phi}_*\left(\mathcal{G}^r\right)\right) = \dim \overline{U} = co\dim \mathcal{F}.$$

Let H be the structure group of the principal bundle $\phi : \overline{F^\natural} \to M$.

Using the fact that the restriction $\mathcal{F}^\natural_{\overline{F^\natural}}$ of $\mathcal{F}^\natural$ at $\overline{F^\natural}$ is a Lie $\mathcal{G}$-foliation with dense leaves is obtained that:

$$\dim M + \dim H = \dim \overline{F^\natural} = \dim \mathcal{F} + \dim \mathcal{G} = \dim \mathcal{F} + co\dim \mathcal{F} = \dim M.$$

And, equality $\dim M + \dim H = \dim M$ show that $\dim H = 0$. Which means that H is discreet.

It follows from the foregoing that the principal bundle $\phi : \overline{F^{\natural}} \to M$ is a covering.

Before concluding, we signal that H is finished because it is a structure group of a compact covering. ■

We note that this proposition allows us to say that the dimension of the Lie structural algebra of a Riemannian foliation on a compact connected manifold is lower or equal to the codimension of this foliation.

Corollary 3.4 *All Riemannian foliation $\mathcal{F}$ with dense leaves on a compact connected simply connected manifold M is an abelian $\mathcal{G}$-foliation with dense leaves.*

Proof. Let $\mathcal{F}$ be a Riemannian foliation with dense leaves on a compact connected simply connected manifold M and let $\overline{F^{\natural}}$ be the closure of a leaf $F^{\natural}$ of lifted foliation $\mathcal{F}^{\natural}$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^{\natural}$.

According to the previous proposition $\overline{F^{\natural}}$ is a covering of M. As M is a simply connected manifold then M is diffeomorphic to $\overline{F^{\natural}}$. And, that implies that $\mathcal{F}$ is a Lie $\mathcal{G}$-foliation with dense leaves.

As the Lie structural algebra of a Riemannian foliation on a simply connected compact connected manifold is an abelian Lie algebra [12] then we obtain that $\mathcal{F}$ is an abelian Lie $\mathcal{G}$-foliation with dense leaves. ■

The following theorem is the main result of this article. This is a generalization of theorems 2.3 and 2.4 to Riemannian foliations with dense leaves on a compact manifold.

In what follows the structure group of covering $\phi : \overline{F^{\natural}} \to M$ will be noted H_0 and we will call it *the discreet goup of Riemannian foliation* $\mathcal{F}$ with dense leaves.

Theorem 3.5 *Let $\mathcal{G} = Lie(G)$ be the Lie structural algebra of a Riemannian foliation with dense leaves $(M, \mathcal{F})$ on a compact manifold M, let H_0 be the discreet group of $\mathcal{F}$ and let λ be a metric on M which is bundle-like for $\mathcal{F}$ and which admits λ_T as its associated transverse metric.*

Then there exists a representation $\rho : H_0 \to Diff(V)$ where V is an open of G such as:

i) there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie(G)$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and $\mathcal{F}$ extensions,

ii) an extension is a Lie foliation if the subalgebra corresponding is an ideal of $\mathcal{G}$,

iii) every extension $\mathcal{F}'$ of $\mathcal{F}$ is a Riemannian foliation and there exists a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$,

iv) if $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ then to Lie algebra isomorphism nearly we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\, (h, a, v) \in \mathcal{H} \times H_0 \times V,\ [u, h] = 0 \ \text{ and } Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

In particular

$$\ell(M,\mathcal{F}) = \{u \in \mathcal{G} / \forall\,(a,v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

Proof. Let $\mathcal{F}$ be a Riemannian foliation on a compact connected manifold M, let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$, let $\phi{:}\overline{F^\natural} \to M$ be the projection which to a frame at x associates x and let $\mathcal{F}^\natural_{\overline{F^\natural}}$ be the restriction of $\mathcal{F}^\natural$ at $\overline{F^\natural}$.

We now that $\phi : \overline{F^\natural} \to M$ is a covering having H_0 for structure goup (*cf. Proposition* 3.3).

In what follows $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ denotes a foliated cocycle defining the Riemannian foliation $\mathcal{F}$ such as open U_i are open of local trivialization of covering $\phi : \overline{F^\natural} \to M$ and $f_i^\natural : \phi^{-1}(U_i) \to E^\natural\left(\overline{U}_i\right)$ denotes the projection of $\phi^{-1}(U_i)$ on the orthonormal transverse frame bundle $\overline{\phi}_i : E^\natural\left(\overline{U}_i\right) \to \overline{U}_i$ above the local $\mathcal{F}$-quotient manifold $\overline{U}_i$ of U_i.

We note that the fact that $\phi : \overline{F^\natural} \to M$ is a covering of M implies that for every open U_i of foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ of $\mathcal{F}$ there are open $U^\natural_{ia}$ of $\overline{F^\natural}$ where $a \in H_0$ such that:

- $\phi^{-1}(U_i) = \underset{a\in H_0}{\cup} U^\natural_{ia}$ and $\phi : U^\natural_{ia} \to U_i$ is a local difféomorphism,

- for every $(a,b) \in H_0 \times H_0$, $R^\natural_b\left(U^\natural_{ia}\right) = U^\natural_{iab}$ where $R^\natural_b$ is the right translation on $\overline{F^\natural}$ associated to b. And, $\left(U^\natural_{ia}\right)_{(i,a)\in I\times H_0}$ is an open cover of $\overline{F^\natural}$.

According to Molino [12], each submersion $f_i^\natural : \phi^{-1}(U_i) \to E^\natural\left(\overline{U}_i\right)$ defined the Lie $G-$foliation $\mathcal{F}^\natural_{\overline{F^\natural}}$ on $\phi^{-1}(U_i)$. Thus $E^\natural\left(\overline{U}_i\right) \subset G$.

We note also that according to Molino [12] for all $a \in H_0$ and all $x^\natural \in \phi^{-1}(U_i)$ we have

$$f_i^\natural \circ R^\natural_a\left(x^\natural\right) = \rho_a \circ f_i^\natural\left(x^\natural\right)$$

where ρ_a is a diffeomorphism on the open $V = E^\natural\left(\overline{U}_i\right)$ of G induce by $R^\natural_a$ because $\mathcal{F}^\natural_{\overline{F^\natural}}$ is invariant by $R^\natural_a$.

We easily verify that $\rho : H_0 \to Diff(V)$ is a representation and $\rho(H_0) \backsimeq H_0$.

Let λ be a $\mathcal{F}-$bundle-like metric on M.

As $\phi : \overline{F^\natural} \to M$ is a covering and $\mathcal{F}^\natural_{\overline{F^\natural}} = \phi^*\mathcal{F}$ then the metric $\widetilde{\lambda} = \phi^*\lambda$ is a $\mathcal{F}^\natural_{\overline{F^\natural}}-$bundle-like metric on $\phi : \overline{F^\natural} \to M$ and the right translation $R^\natural_a$ on $\overline{F^\natural}$ associated to $a \in H_0$ is an isometry for the metric $\widetilde{\lambda}$.

The equality $f_i^\natural \circ R^\natural_a = \rho_a \circ f_i^\natural$ implies that ρ_a is an isometry because $R^\natural_a$ is an isometry and $f_i^\natural$ is a riemannian submersion defining $\mathcal{F}^\natural_{\overline{F^\natural}}$ on $\phi^{-1}(U_i)$.

Before finishing these remarks we signal that [12] for all $i \in I$ the diagram

$$\begin{array}{ccc} \phi^{-1}(U_i) & \overset{f_i^\natural}{\to} & E^\natural\left(\overline{U}_i\right) \\ \phi \downarrow & & \downarrow \overline{\phi}_i \\ U_i & \overset{f_i}{\to} & \overline{U}_i \end{array}$$

is commutative and the $f_i^\natural \circ R_a^\natural = \rho_a \circ f_i^\natural$ implies again that $\overline{\phi}_i : E^\natural\left(\overline{U}_i\right) \to \overline{U}_i$ is a covering having $\rho\left(H_0\right) \backsimeq H_0$ for structure group.

That said, as $\mathcal{F}$ is a Riemannian foliation with dense leaves then the Lie algebra $\mathcal{G}^r$ of right invariant vectors fields obtained from the structural Lie algebra $\mathcal{G}$ of $\mathcal{F}$ is a Lie algebra of Killing vectors fields operating transitively [12] on each connected component of $E^\natural\left(\overline{U}\right)$ and for each $X^r \in \mathcal{G}^r$, $\left(\overline{\phi}_i\right)_*\left(X^r\right)$ is a vector fields on $\overline{U}$ then for all $(v,a) \in E^\natural\left(\overline{U}_i\right) \times H_0$ and for all $X^r \in \mathcal{G}^r$,

$$\left(\rho_a\right)_*\left(X_v^r\right) = X_{\rho_a(v)}^r = R_{v^{-1}.\rho_a(v)}\left(X_v^r\right).$$

We note that the fact $\mathcal{G}^r$ operate transitively [12] on each connected component of $E^\natural\left(\overline{U}\right)$ implies that for all $(v, X_v) \in E^\natural\left(\overline{U}_i\right) \times T_v E^\natural\left(\overline{U}\right)$ there exists $X^r \in \mathcal{G}^r$ such as $X_v = X_v^r$.

Thus for all $\left(v, x^\natural, a\right) \in E^\natural\left(\overline{U}_i\right) \times \phi^{-1}\left(U_i\right) \times H_0$ and for all $\left(X_v, Y_{x^\natural}^\natural\right) \in T_v E^\natural\left(\overline{U}\right) \times T_{x^\natural}\phi^{-1}\left(U_i\right)$,

$$\left(\rho_a\right)_*\left(X_v\right) = R_{v^{-1}.\rho_a(v)}\left(X_v^r\right) = R_{v^{-1}.\rho_a(v)}\left(X_v\right)$$

and

$$\left(f_i^\natural \circ R_a^\natural\right)_*\left(Y_{x^\natural}^\natural\right) = \left(\rho_a \circ f_i^\natural\right)_*\left(Y_{x^\natural}^\natural\right) = R_{v^{-1}.\rho_a(v)}\left(\left(f_i^\natural\right)_*\left(Y_{x^\natural}^\natural\right)\right) \text{ for } v = f_i^\natural\left(x^\natural\right).$$

In what follows we will designated by $\omega^\natural$ the 1-form of Fedida of Lie $G-$foliation $\mathcal{F}^\natural_{\overline{F^\natural}}$ whose restriction $\omega_i^\natural$ at each open $U_i^\natural = \phi^{-1}\left(U_i\right)$ is such that

$$\omega_i^\natural = \left(f_i^\natural\right)^* \alpha_i$$

where α_i is the restriction of left Maurer–Cartan form α of G at $E^\natural\left(\overline{U}_i\right)$.

For all $\left(x^\natural, a\right) \in \phi^{-1}\left(U_i\right) \times H_0$ and for all $Y_{x^\natural}^\natural \in T_z\phi^{-1}\left(U_i\right)$ we have

$$\begin{aligned}
\left(R_a^\natural\right)^* \omega^\natural\left(Y_{x^\natural}^\natural\right) &= \omega_i^\natural \circ \left(R_a^\natural\right)_*\left(Y_{x^\natural}^\natural\right) \\
&= \alpha_i\left(\left(f_i^\natural \circ R_a^\natural\right)_*\left(Y_{x^\natural}^\natural\right)\right) \\
&= \alpha_i\left(\left(\rho_a\right)_*\left(\left(f_i^\natural\right)_*\left(Y_{x^\natural}^\natural\right)\right)\right) \\
&= \alpha_i\left(R_{v^{-1}.\rho_a(v)}\left(\left(f_i^\natural\right)_*\left(Y_{x^\natural}^\natural\right)\right)\right) \text{ for } v = f_i^\natural\left(x^\natural\right) \\
&= Ad_{(\rho_a(v))^{-1}.v} \circ \omega^\natural\left(Y_{x^\natural}^\natural\right).
\end{aligned}$$

i) Let $\mathcal{F}'$ be an extension of $\mathcal{F}$ and let $\phi^*\mathcal{F}'$ be the inverse image of $\mathcal{F}'$.

We easily verify that $\mathcal{F}^\natural_{\overline{F^\natural}} \subset \phi^*\mathcal{F}'$. This implies that (*cf. theoreme 2.3*) there exists a Lie subalgebra of $\mathcal{G} = Lie\left(G\right)$ corresponding to $\phi^*\mathcal{F}'$. It will be noted $\mathcal{G}_{\mathcal{F}'}$.

We know that [6] $\phi^{*}\mathcal{F}'$ is defined by the differential system $\mathcal{P}^{\natural}$ defined on $\overline{F^{\natural}}$ by

$$\mathcal{P}^{\natural}\left(x^{\natural}\right)=T_{x^{\natural}}\mathcal{F}^{\natural}_{\overline{F^{\natural}}}\oplus ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)$$

where $\mathcal{G}^{\natural}_{\mathcal{F}'}$ is the Lie algebra of $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}-$foliated transverse vectors fields associated to $\mathcal{G}_{\mathcal{F}'}$ and $ev_{x^{\natural}}\left(X^{\natural}\right)=X^{\natural}_{x^{\natural}}$ for every $X^{\natural}\in\mathcal{G}^{\natural}_{\mathcal{F}'}$.

Let $a\in H_0$. The foliation $\phi^{*}\mathcal{F}'$ is invariant by the right translation $R^{\natural}_{a}$ on $\overline{F^{\natural}}$ associated to a. Hence the differential system $x^{\natural}\mapsto\mathcal{P}^{\natural}\left(x^{\natural}\right)$ is invariant by $R^{\natural}_{a}$.

As $T\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$ and the differential system $x^{\natural}\mapsto\mathcal{P}^{\natural}\left(x^{\natural}\right)$ are invariant by the isometry $R^{\natural}_{a}$ and as the ortho-complementary of $T_{x^{\natural}}\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$ in $\mathcal{P}^{\natural}\left(x^{\natural}\right)$ is $ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)$ then for all $x^{\natural}\in\phi^{-1}\left(U_i\right)$,

$$\left(R^{\natural}_{a}\right)_{*}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)=ev_{R^{\natural}_{a}\left(x^{\natural}\right)}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right).$$

But [6] for all $x^{\natural}\in\phi^{-1}\left(U_i\right)$

$$\omega^{\natural}_{i}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)=\mathcal{G}_{\mathcal{F}'}$$

so for all $v=f^{\natural}_{i}\left(x^{\natural}\right)\in E^{\natural}\left(\overline{U}_{i}\right)$ we have

$$\begin{aligned}
\mathcal{G}_{\mathcal{F}'} &= \omega^{\natural}_{i}\left(ev_{R^{\natural}_{a}\left(x^{\natural}\right)}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\\
&= \omega^{\natural}_{i}\left(\left(R^{\natural}_{a}\right)_{*}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\right)\\
&= \left(f^{\natural}_{i}\right)^{*}\alpha_i\left(R^{\natural}_{a*}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\right)\\
&= \alpha\left(\left(f^{\natural}_{i}\right)_{*}\circ R^{\natural}_{a*}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\right)\\
&= \alpha\left(R_{v^{-1}.\rho_a(v)}\left(\left(f^{\natural}_{i}\right)_{*}\right)\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\right)\\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\alpha\left(\left(f^{\natural}_{i}\right)_{*}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\right)\\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\left(\left(f^{\natural}_{i}\right)^{*}\alpha_i\right)\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\omega^{\natural}_{i}\left(ev_{x^{\natural}}\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right)\right)\\
&= Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}_{\mathcal{F}'}\right)
\end{aligned}$$

Reciprocally, suppose there is a Lie subalgebra $\mathcal{G}'$ of $\mathcal{G}$ such as $Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right)=\mathcal{G}'$ for all $v=f^{\natural}_{i}\left(x^{\natural}\right)\in E^{\natural}\left(\overline{U}_{i}\right)$ and for all $a\in H_0$.

As $Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right)=\mathcal{G}'$ for all $a\in H_0$ and for all $v=f^{\natural}_{i}\left(x^{\natural}\right)\in E^{\natural}\left(\overline{U}_{i}\right)$ then the differential system $\mathcal{S}^{\natural}$ defined on $\overline{F^{\natural}}$ by $\mathcal{S}^{\natural}\left(x^{\natural}\right)=ev_{x^{\natural}}\left(\mathcal{G}'^{\natural}\right)$ where $\mathcal{G}'^{\natural}$ is the Lie algebra of $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}-$foliated transverse vectors fields associated to $\mathcal{G}'$, is invariant by $R^{\natural}_{a}$ for all $a\in H_0$.

Indeed for $a \in H_0$ we have:

$$\begin{aligned}
\omega_i^\natural\left(ev_{R_a^\natural(x^\natural)}\left(\mathcal{G}'^\natural\right)\right) &= \mathcal{G}' \\
&= Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right) \quad \text{for } v = f_i^\natural\left(x^\natural\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) \\
&= \left(R_a^\natural\right)^* \omega_i^\natural\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) \\
&= \omega_i^\natural\left(\left(R_a^\natural\right)_*\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right)\right).
\end{aligned}$$

The fact that $\omega_i^\natural : \left(T_{R_a^\natural(x^\natural)}\mathcal{F}^\natural_{\overline{F^\natural}}\right)^\perp \to \mathcal{G}$ is an isomorphism, because $\mathcal{F}^\natural_{\overline{F^\natural}}$ is a $\mathcal{G}-$Lie foliation with dense leaves, implies that

$$\left(R_a^\natural\right)_{*x^\natural}\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) = \left(ev_{R_a^\natural(x^\natural)}\left(\mathcal{G}'^\natural\right)\right)$$

for all $a \in H_0$ and $x^\natural \in \overline{F^\natural}$.

It follows from the foregoing that the extension $\mathcal{F}^\natural_{\mathcal{G}'}$ of $\mathcal{F}^\natural_{\overline{F^\natural}}$ corresponding to $\mathcal{G}'$ is invariant by the right translation of the elements of H_0 because ([6], [8])

$$T_{x^\natural}\mathcal{F}^\natural_{\mathcal{G}'} = T_{x^\natural}\mathcal{F}^\natural_{\overline{F^\natural}} \oplus ev_{x^\natural}\left(\mathcal{G}'^\natural\right)$$

for all $x^\natural \in \overline{F^\natural}$ and $T\mathcal{F}^\natural_{\overline{F^\natural}}$ is invariante by $R_a^\natural$ for all $a \in H_0$.

Thus $\mathcal{F}^\natural_{\mathcal{G}'}$ is projected by ϕ into an extension $\mathcal{F}'$ of $\mathcal{F}$.

ii) Let $\mathcal{F}'$ be an extension of Lie of Riemannian foliation $(M, \mathcal{F})$ with dense leaves on a compact manifold M, let $\phi^*\mathcal{F}'$ be the lifted foliation of $\mathcal{F}'$ on the covering $\phi : \overline{F^\natural} \to M$, let $\ell(M, \mathcal{F}')$ be the Lie algebra of $\mathcal{F}'-$foliated transverse vectors fields and let $\ell^\natural(M, \mathcal{F}')$ be the lifted of $\ell(M, \mathcal{F}')$ on the covering $\overline{F^\natural}$.

For all $X^\natural \in \ell^\natural(M, \mathcal{F}')$ and for all $Y^\natural \in \mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right)$ where $\mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right)$ is the Lie algebra of vectors fields tangent to $\phi^*\mathcal{F}'$, we have $\forall\,(i,a) \in I \times H_0$,

$$\phi_*\left(\left[X^\natural, Y^\natural\right]_{/U_{ia}^\natural}\right) = \left[\phi_*\left(X^\natural_{/U_{ia}^\natural}\right), \phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)\right] \quad (**).$$

We note that $\forall\,(i,a) \in I \times H_0$, $\left[\phi_*\left(X^\natural_{/U_{ia}^\natural}\right), \phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)\right]$ is tangent to $\mathcal{F}'$ because $\phi_*\left(X^\natural_{/U_{ia}^\natural}\right)$ is $\mathcal{F}'-$foliated and $\phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)$ is tangent to $\mathcal{F}'$. It follows from this that $\phi_*\left(\left[X^\natural, Y^\natural\right]_{/U_{ia}^\natural}\right)$ is tangent to $\mathcal{F}'$ for all $(i,a) \in I \times H_0$. Which causes that $\left[X^\natural, Y^\natural\right]$ is tangent to $\phi^*\mathcal{F}'$.

Thus all vectors fields of $\ell^\natural(M, \mathcal{F}')$ is $\phi^*\mathcal{F}'-$foliated.

Using the fact that for any $(i,a) \in I \times H_0$, $\phi : U_{ia}^\natural \to U_i$ is an isometry (relatively to the metrics λ and $\widetilde{\lambda}$), it is easily verified that all vectors field of $\ell^\natural(M, \mathcal{F}')$ is $\phi^*\mathcal{F}'-$tranverse since we identify the orthogonal bundle $(T\phi^*\mathcal{F}')^\perp$ of $T\phi^*\mathcal{F}'$ and the transverse bundle $\mathcal{V}\left(\phi^*\mathcal{F}'\right)$.

We note that $\ell^\natural(M,\mathcal{F}')$ is stable for the Lie bracket of two vectors fields.

Indeed for all $X^\natural \in \ell^\natural(M,\mathcal{F}')$ and $Y^\natural \in \ell^\natural(M,\mathcal{F}')$, there exists $X \in \ell(M,\mathcal{F}')$ and $Y \in \ell(M,\mathcal{F}')$ such as $\phi_*\left(X^\natural\right) = X$ and $\phi_*\left(Y^\natural\right) = Y$.

The lifted vectors fields $X^\natural$ and $Y^\natural$ are invariant by the right translations obtained from the elements of H_0. It follows from this that $\left[X^\natural, Y^\natural\right]$ is invariant by the right translations obtained from elements of H_0.

Thus $\left[X^\natural, Y^\natural\right]$ is projected by ϕ along a vectors field on M. And we have for all $(i,a) \in I \times H_0$,

$$\begin{aligned}
\left(\phi_*\left(\left[X^\natural, Y^\natural\right]\right)\right)_{/U_i} &= \phi_*\left(\left[X^\natural, Y^\natural\right]_{/U_{ia}^\natural}\right) \\
&= \left[\phi_*\left(X^\natural_{/U_{ia}^\natural}\right), \phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)\right] \\
&= \left[\left(\phi_*\left(X^\natural\right)\right)_{/U_i}, \left(\phi_*\left(Y^\natural\right)\right)_{/U_i}\right] \\
&= \left(\left[\left(\phi_*\left(X^\natural\right)\right), \left(\phi_*\left(Y^\natural\right)\right)\right]\right)_{/U_i} \\
&= [X,Y]_{/U_i}.
\end{aligned}$$

The above shows that $\phi_*\left(\left[X^\natural, Y^\natural\right]\right) = [X,Y]$.

As $[X,Y] \in \ell(M,\mathcal{F}')$ then $\left[X^\natural, Y^\natural\right] \in \ell^\natural(M,\mathcal{F}')$. Which means that $\ell^\natural(M,\mathcal{F}')$ is stable for the Lie bracket.

It follows from the above that $\ell^\natural(M,\mathcal{F}')$ is a Lie algebra of $\phi^*\mathcal{F}'-$foliated transverse vectors fields.

We note that $\ell^\natural(M,\mathcal{F}')$ has the same dimension as $\ell(M,\mathcal{F}')$.

The fact that $\phi^*\mathcal{F}'$ is an extension of the Lie foliation $\mathcal{F}^\natural_{\overline{F^\natural}}$ ensures by Proposition 2.6 that $\ell^\natural(M,\mathcal{F}')$ is a Lie algebra of vectors fields $\mathcal{F}^\natural_{\overline{F^\natural}}-$foliated transverse.

Let $\mathcal{G}^\natural_{\mathcal{F}'}$ be the Lie algebra of $\mathcal{F}^\natural_{\overline{F^\natural}}-$foliated transverse vectors field associated to the Lie subalgebra $\mathcal{G}_{\mathcal{F}'}$ corresponding of the extension $\phi^*\mathcal{F}'$ of $\mathcal{F}^\natural_{\overline{F^\natural}}$.

We signal that $\ell^\natural(M,\mathcal{F}')$ is ortho- complementary to $\mathcal{G}^\natural_{\mathcal{F}'}$ in $\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}})$.

Indeed $\dim\left(\ell^\natural(M,\mathcal{F}')\right) = \dim\left(\ell(M,\mathcal{F}')\right) = co\dim\left(\phi^*\mathcal{F}'\right)$ and [6]

$$\dim\left(\mathcal{G}^\natural_{\mathcal{F}'}\right) = co\dim\left(\mathcal{F}^\natural_{\overline{F^\natural}}\right) - co\dim\left(\phi^*\mathcal{F}'\right) = \dim\left(\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}})\right) - \dim\left(\ell^\natural(M,\mathcal{F}')\right)$$

and $\ell^\natural(M,\mathcal{F}')$ is orthogonal to $\mathcal{G}^\natural_{\mathcal{F}'}$ because $\ell^\natural(M,\mathcal{F}')$ is a Lie algebra of $\phi^*\mathcal{F}'-$foliated transverse vectors fields.

For $X^\natural \in \ell^\natural(M,\mathcal{F}')$ and $Y^\natural \in \mathcal{G}^\natural_{\mathcal{F}'}$ we have $\left[X^\natural, Y^\natural\right] \in \ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}})$ and the equality $(**)$ show us that $\left[X^\natural, Y^\natural\right]$ is tangent to $\phi^*\mathcal{F}'$ that is to say $\left[X^\natural, Y^\natural\right] \in \mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right)$.

Thus for all $X^\natural \in \ell^\natural(M,\mathcal{F}')$ and $Y^\natural \in \mathcal{G}^\natural_{\mathcal{F}'}$ we have $\left[X^\natural, Y^\natural\right] \in \mathcal{G}^\natural_{\mathcal{F}'}$ because [6]

$$\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}}) \cap \mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right) = \mathcal{G}^\natural_{\mathcal{F}'}.$$

It results from the above that $\mathcal{G}^{\natural}_{\mathcal{F}'}$ is an ideal of $\ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}})$. And this implies that $\mathcal{G}_{\mathcal{F}'}$ is an ideal of $\mathcal{G}$.

iii) Let $\mathcal{F}'$ be an extension of $\mathcal{F}$ and let λ be a $\mathcal{F}$-bundle-like metric on M.

Quits to reduce the size of opens U_i, we can assume distinguished both for $\mathcal{F}$ and $\mathcal{F}'$ each open U_i.

The local isometry $\phi : U^{\natural}_{ia} \to U_i$ sends the leaves of $\left(U^{\natural}_{ia}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}}\right)$ on leaves of $(U_i, \mathcal{F})$ and leaves of $\left(U^{\natural}_{ia}, \phi^*\mathcal{F}'\right)$ on leaves of $(U_i, \mathcal{F}')$ for all $(i, a) \in I \times H_0$.

As the metric $\widetilde{\lambda} = \phi^*\lambda$ is bundle-like for $\left(U^{\natural}_{ia}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}}\right)$ and $\left(U^{\natural}_{ia}, \phi^*\mathcal{F}'\right)$ then the metric λ is bundle-like for $(U_i, \mathcal{F})$ and $(U_i, \mathcal{F}')$. And, this implies that the metric λ is a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$.

iv) Let $\mathcal{F}_{\mathcal{H}}$ be the extension of $\mathcal{F}$ corresponding to a Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$, let $\mathcal{F}^{\natural}_{\mathcal{H}}$ be the extension of $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$ corresponding to $\mathcal{H}$, let $\ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\mathcal{H}})$ be the Lie algebra of $\mathcal{F}^{\natural}_{\mathcal{H}}-$foliated transverse vectors fields and let $\ell_{inv}(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\mathcal{H}})$ be the Lie subalgebra of $\ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\mathcal{H}})$ of invariant vectors fields by the right translations $R^{\natural}_a$ for all $a \in H_0$.

One checks easily through the equality

$$\mathcal{F}^{\natural}_{\mathcal{H}} = \phi^*\mathcal{F}_{\mathcal{H}}$$

and thanks to the fact that $\forall\,(i, a) \in I \times H_0$, $\phi : U^{\natural}_{ia} \to U_i$ is a local isometry that $\phi_* : \ell_{inv}(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\mathcal{H}}) \to \ell(M, \mathcal{F}_{\mathcal{H}})$ is an isomorphism of Lie algebras.

Using the theorem 2.4 and the fact that for all $a \in H_0$ and $Y^{\natural}_{x^{\natural}} \in T_{x^{\natural}}\phi^{-1}(U_i)$

$$\left(R^{\natural}_a\right)^* \omega^{\natural}\left(Y^{\natural}_{x^{\natural}}\right) = Ad_{(\rho_a(v))^{-1}.v} \circ \omega^{\natural}\left(Y^{\natural}_{x^{\natural}}\right) \text{ for } v = f^{\natural}_i\left(x^{\natural}\right)$$

one easily gets equality between $\omega^{\natural}\left(\ell_{inv}(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\mathcal{H}})\right)$ and

$$\{u \in \mathcal{H}^{\perp} /\ \forall h \in \mathcal{H},\ [u, h] = 0 \ \text{ and } \forall\,(a, v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

But $\omega^{\natural} : \ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}}) \to \mathcal{G}$ is an isomorphism of Lie algebras because the Lie foliation $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$ has leaves that are dense, so $\omega^{\natural} \circ (\phi_*)^{-1}$ is an isomorphism of Lie algebras between $\ell(M, \mathcal{F}_{\mathcal{H}})$ and

$$\{u \in \mathcal{H}^{\perp} /\ \forall h \in \mathcal{H},\ [u, h] = 0 \ \text{ and } \forall\,(a, v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

■

Corollary 3.6 *Let $\mathcal{G}$ be the structural Lie algebra of a Riemannian foliation with dense leaves $\mathcal{F}$ on a compact manifold M.*

Then $\mathcal{F}$ admits a complete flag of extensions if and only if $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$ foliation with dense leaves.

Proof. Let q be the codimension of $\mathcal{F}$ and let λ be a $\mathcal{F}$-bundle-like metric on M.

We suppose that $\mathcal{F}$ admits a complete flag of extensions
$\mathcal{D}_{\mathcal{F}} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$.

It is easily verified for $i \in \{1, 2, ..., q\ \}$ that the dimension of the bundle $(T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1}$ is 1. We denote by X_i the unified field that orients $(T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1}$.

We note in passing that the foliation $\mathcal{F}_0$ has one leaf and this leaf is the manifold M. We note also that

$$\begin{aligned} T\mathcal{F}_{i-1} &= T\mathcal{F}_i \oplus (T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1} \\ &= T\mathcal{F}_i \oplus < X_i > \end{aligned}$$

That said, we can say that X_i is a $\mathcal{F}_i-$foliated transverse vectors field.

Indeed $\mathcal{F}$ is a Riemannian foliation with dense leaves then according to the theorem 3.7, $\mathcal{D}_{\mathcal{F}}$ is a complete flag of Riemannian extensions and the metric λ is a common bundle-like metric for the foliations $\mathcal{F}_i$.

As X_i is tangent to $(T\mathcal{F}_i)^{\perp}$ then we have for all $Y \in \mathcal{X}(\mathcal{F}_i)$,

$$0 = Y\lambda(X_i, X_i) = 2\lambda([Y, X_i], X_i).$$

Therefore $[Y, X_i]$ is orthogonal to X_i. But $[Y, X_i] \in \mathcal{X}(\mathcal{F}_{i-1})$ and

$$T\mathcal{F}_{i-1} = T\mathcal{F}_i \oplus < X_i >$$

so $[Y, X_i]$ is tangent to $T\mathcal{F}_i$. Which implies that $X_i \in \ell(M, \mathcal{F}_i)$ because X_i is tangent to $(T\mathcal{F}_i)^{\perp}$.

In what follows we denote $\mathcal{F}$ by $\mathcal{F}_q$.

According to the proposition 2.6 for all i and j such as $1 \leq i < j \leq q$ we have $\ell(M, \mathcal{F}_i) \subset \ell(M, \mathcal{F}_j)$.

It follows that for all $i \leq j$ we have $X_i \in \ell(M, \mathcal{F}_j)$.

The fact that the unified vectors fields X_i are orthogonal two by two implies that $\mathcal{F}_j$ is a Lie G_j-foliation for all j and

$$\ell(M, \mathcal{F}_j) =< X_j, X_{j-1}, ...X_1 > .$$

Indeed for all j the codimension of $\mathcal{F}_j$ is j and the leaves of $\mathcal{F}_j$ are dense.

Let U be a $\mathcal{F}_q-$distinguished open, let $f : U \to \overline{U}$ be a Riemannian submersion defining $\mathcal{F}_q$ on U, let ω_q be a Fedida 1-forme of $\mathcal{F}_q$ such as $\omega_{q/U} = (f)^* \alpha_q$ where α_q is the left Maurer Cartan form of G_q, let $\omega_q(X_i) = \overline{X}_i$ and let $\mathcal{G}_q = Lie(G_q)$.

We note in passing that ω_q:$\ell(M, \mathcal{F}_q) \to \mathcal{G}_q$ is an isomorphism of Lie algebra because $\mathcal{F}_q$ is a Lie G_q-foliation with dense leaves.

It is easily verified using the equality $\omega_{q/U} = (f)^* \alpha_q$ and the fact that the unified vectors fields X_i are orthogonal two by two that $\left(\overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_1\right)$ is an orthonormal base for $\mathcal{G}_q$.

Let $\mathcal{H}_j$ be the Lie subalgebra of $\mathcal{G}_q$ corresponding to the extension $\mathcal{F}_j$ of $\mathcal{F}_q$.

We note [6] that $\dim(\mathcal{H}_j) = q - j$. And that implies that $\dim\left((\mathcal{H}_j)^{\perp}\right) = j$.

According to the Theorem 2.4 we have $\omega_q(\ell(M,\mathcal{F}_j)) \subset (\mathcal{H}_j)^{\perp}$.

Consequently the fact that $\dim\left((\mathcal{H}_j)^{\perp}\right) = j$ and

$$\omega_q(\ell(M,\mathcal{F}_j)) = \omega_q(< X_j, X_{j-1}, ...X_1 >) =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 >$$

implies that

$$(\mathcal{H}_j)^{\perp} =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 > .$$

Thus we have

$$\mathcal{H}_j =< \overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_{j+1} > .$$

As

$$\omega_q(\ell(M,\mathcal{F}_j)) =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 > \text{ and } \mathcal{H}_j =< \overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_{j+1} >$$

then according to the theorem 2.4 for all i and k such as $1 \leq i \leq j < k \leq q$ we have $\left[\overline{X}_i, \overline{X}_k\right] = 0$.

Is thus obtained for all i and k such that $i < k$, $\left[\overline{X}_i, \overline{X}_k\right] = 0$. And this implies that $\mathcal{G}_q$ is an abelian Lie algebra. Consequently $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$foliation because $\mathcal{G} = \mathcal{G}_q$.

Reciprocally we suppose that $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$foliation.

In this case the theorem 2.3 show us that $\mathcal{F}$ admits a complete flag of extensions. ■

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