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# NUMERICAL SEMIGROUPS WITH TWO COPRIME GENERATORS: JAMES J. SYLVESTER'S PROOF

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ABSTRACT. This paper examines in a new way some known facts about numerical semigroups especially when the number of minimal generators (that is the embedding dimension) is at most three and at least two minimal generators are coprime; for such semigroups an algorithm is reinvestigated to find the pseudo-Frobenius numbers, in particular we fully characterize those of  $\langle a, a + 1, a + p \rangle$  for any  $p \geq 2$ ,  $\langle ab + a + 1, bc + b + 1, ca + c + 1 \rangle$  and consider  $\langle n^2, (n + 1)^2, (n + 2)^2 \rangle$  semigroups; a particular family of  $n$  dimensional numerical semigroups of type at most  $n - 1$  is also given.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $a_1, \dots, a_n$  be  $n$  positive integers with  $\gcd(a_1, \dots, a_n) = 1$ . The set  $S := \left\{ \sum_{i=1}^s \lambda_i a_i \mid s \in \mathbb{N}, \lambda_i \geq 0, \text{ for all } i \right\}$  is called the numerical semigroup  $S$  and  $a_1, \dots, a_n$  are called the generators of  $S$ , they are minimal generators if we cannot take out a generator  $a_i$  without changing the set  $S$ , in this case we denote  $S$  by  $\langle a_1, \dots, a_n \rangle$ . For a numerical semigroup  $S \neq \mathbb{N}$  the number  $F(S) := \max\{n \in \mathbb{N} \mid n \notin S\}$  exists (see [11, Theorem 1.0.1] for proof). It is called the Frobenius number of  $S$ . We will suppose hereafter that  $a_1 < a_2$ . The interested reader can see [12] and the references therein (in particular [13]) for an extensive reading on numerical semigroups. The next definition and Lemma 1 follow Sylvester's terminology in [1] on p. 134; in order to give the number of solutions to  $px + py < pq$  where  $p$  and  $q$  are coprime positive integers Sylvester gave a second proof which we quote the referred paragraph as it led to this article:

*This result admits of a somewhat piquant verification. The number of integers less than  $pq$  and containing neither  $p$  nor  $q$  is  $(p - 1)(q - 1)$ , and if every two of these which are supplementary to one another (I mean whose sum is  $pq$ ) be made into a pair, it is easily demonstrable, but by no means an unimportant fact, that one of the pair will be a compound and the other a non-compound of  $p$  and  $q$ . Hence the total number of solutions of  $px + qy < pq$  will be the remainder when the above is subtracted from  $pq$ , i.e.  $\frac{pq + q + p - 1}{2}$  as previously determined.*

**Definition 1.** Let  $a_1$  and  $a_2$  be positive integers with  $\gcd(a_1, a_2) = 1$ . The set of positive integers  $x$  of the form  $x = \alpha a_i - \beta a_{3-i}$ ,  $i = 1$  or  $2$  with  $0 < \alpha < a_{3-i}$  and  $0 < \beta < a_i$  is called the non-compound set and is denoted by  $NC$ . The set of

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integers  $x$  of the form  $x = \alpha a_1 + \beta a_2$  with  $\alpha > 0$ ,  $\beta > 0$  and  $x < a_1 a_2$  is called the compound set and is denoted by  $C$ .

**Lemma 1.** [4] *Let  $S = \langle a_1, a_2 \rangle$  and let  $x$  be a positive integer. Then  $x \in NC$  if and only if  $0 < x < a_1 a_2$  and  $x \notin S$ .*

*Proof.* Suppose  $x \in S$  and in  $NC$ ,  $x < a_1 a_2$ . So  $x = \alpha a_1 + \beta a_2 = \alpha a_i - \beta a_{3-i}$  for some nonnegative numbers  $\alpha, \beta$ , but this contradicts the fact that  $\alpha < a_{3-i}$  and  $\beta < a_i$  (for example if  $i = 1$ ,  $(\alpha - \beta)a_1 = -\beta a_2$  and  $|\alpha - \beta| < a_2$ ). Take the numbers less than  $a_1 a_2$  having neither  $a_1$  nor  $a_2$  as a divisor, there are  $(a_1 - 1)(a_2 - 1)$  such numbers, arrange them as couples summing  $a_1 a_2$  if  $x \in NC$ ,  $a_1 a_2 - x = y \in S$ , and if  $y \in C$ ,  $(a_1 a_2 - y) = x \notin S$ , that means that  $x \in NC$ . Finally  $|NC| = \frac{(a_1 - 1)(a_2 - 1)}{2}$  and  $|NC| + |C| = (a_1 - 1)(a_2 - 1)$ .  $\square$

We have proven that  $|NC| = |C| = \frac{(a_1 - 1)(a_2 - 1)}{2}$ . The number  $F(S) = (a_1 - 1)(a_2 - 1) - 1 = a_1 a_2 - a_1 - a_2$  doesn't belong to  $S$  since  $a_1 a_2 - F(S)$  is compound. Any number  $y > a_1 a_2$  can be written as  $\alpha a_1 a_2 + r$  and it is easily seen that since  $r < a_1 a_2$ ,  $r$  is either in  $S$  or in  $NC$  so all  $n > F(S)$  do belong to  $S$ .

Now we can characterize all the numbers that aren't in  $S := \langle a_1, a_2 \rangle$ .

**Proposition 1.** *The set  $NC$  is the set  $\{F(S) - y > 0 | y \in S, y < F(S)\}$*

*Proof.* A direct application of previous result.  $\square$

**Remark 1.** *Notice at last that any integer  $x < a_1 a_2$  in  $S$  has a unique representation as  $x = \alpha a_1 + \beta a_2$  with  $(\alpha, \beta) \in \mathbb{N}^2$ ,  $\alpha < a_2$  and  $\beta < a_1$  because if not  $\alpha a_1 + \beta a_2 = \nu a_1 + \gamma a_2$  and  $(\alpha - \nu)a_1 = (\gamma - \beta)a_2$  and we can't have  $a_2$  divides  $(\alpha - \nu)$  or  $a_1$  divides  $(\gamma - \beta)$ . A number  $x$  in  $NC$  can be written as  $a_1 \cdot a_2 - w a_1 - r a_2 = (a_2 - w) \cdot a_1 - r \cdot a_2$  where  $1 \leq w < a_2$  and  $1 \leq r < a_1$  hence the uniqueness of  $(w, r) \in \mathbb{N}^2$  follows from the last argument.*

We end this expository section by the following definition

**Definition 2.** *For a numerical semigroup  $S$ ,  $T(S) := \{x \in \mathbb{N} | x \notin S, x + s \in S, \text{ for all } s \in S, s > 0\}$ . The cardinality of  $T(S)$  is called the type of  $S$  and a number in  $T(S)$  is called a pseudo-Frobenius number.*

## 2. ADDING MINIMAL GENERATOR

Here we consider numerical semigroups  $S$  minimally generated by  $a_1, a_2, a_3$ ;  $a_1$  and  $a_2$  are coprime with  $a_1 < a_2$ . The next remark is already well known and is very useful:

**Remark 2.** [6, 12, 13] *Let  $a_1, a_2, \dots, a_k$  be positive integers. If  $\gcd(a_2, \dots, a_n) = d$  and  $a_j = d \cdot a'_j$  for each  $j > 1$  then*

- $F(\langle a_1, a_2, \dots, a_n \rangle) = d \cdot F(\langle a_1, a'_2, \dots, a'_n \rangle) + a_1 \cdot (d - 1)$  (Johnson formula).
- The type of  $\langle a_1, a_2, \dots, a_n \rangle$  equals type of  $\langle a_1, a'_2, \dots, a'_n \rangle$ .
- The type of  $S = \langle a_1, a_2, a_3 \rangle$  is at most two.

**Remark 3.** *A number  $x$  in  $NC$  is written as  $a_1 \cdot a_2 - k \cdot a_2 - j \cdot a_1$  and we have  $0 < k < a_1$  if and only if  $0 < j < a_2$ . This is useful when  $i \cdot x \in NC$  and:  $0 < i \cdot k < a_1$  so  $0 < i \cdot j - (i - 1) \cdot a_2 < a_2$  or  $0 < i \cdot j < a_2$  so  $0 < i \cdot k - (i - 1) \cdot a_1 < a_1$ .*

From the unique representation of  $NC$  numbers we have:

**Lemma 2.** *Let  $S = \langle a_1, a_2, a_3 \rangle$ ,  $Q = \langle a_1, a_2 \rangle$ ,  $a_3 = a_1 \cdot a_2 - k \cdot a_2 - j \cdot a_1$  for some  $1 \leq k < a_1$ ;  $1 \leq j < a_2$ . If  $2 \cdot a_3 \in Q$  then the possible pseudo-Frobenius numbers are  $\{a_1 \cdot a_2 - (k+1) \cdot a_2 - a_1, a_1 \cdot a_2 - a_2 - (j+1) \cdot a_1\}$*

**Example 1.** *Take  $\langle 4, 9, 15 \rangle$  with  $a_3 = 15$ ,  $F(S) = 14 = 4 \cdot 9 - 2 \cdot 9 - 4$  and the other pseudo-Frobenius number is  $11 = 4 \cdot 9 - 9 - 4 \cdot 4$ .*

**Corollary 1.** *Let  $S := \langle a, ha + d, ha + 2d \rangle$  with  $d \geq 1$ ,  $\gcd(a, d) = 1$ ,  $h \geq 1$  and  $a$  is odd then we have:*

$$T(S) = \{F_1 := a \cdot (ha + 2d) - a - (\frac{a-1}{2} + 1) \cdot (ha + 2d), F_0 := a \cdot (ha + 2d) - (h \cdot \frac{a-1}{2} + d + 1) \cdot a - (ha + 2d)\}.$$

*Proof.* Take  $Q = \langle a, ha + 2d \rangle$  where  $a_1 = a$ ,  $a_2 = (ha + 2d)$  and apply Lemma 2, since  $F_0 + a_3 = (h-1) \cdot (a_1) + i \cdot (a_2) \in Q$  and  $F_1 + a_3 = a_1(d-1 + h(i+1)) \in Q$  we get the result.  $F_1 - F_0 = d$ .  $\square$

In Corollary 1 one can check that  $F_1 = F(S) = a \cdot (ha + 2d) - a - (\frac{a-1}{2} + 1) \cdot (ha + 2d)$  and equalize it with the formula of  $F(S)$  given in [11, Theorem 5.4.3] (or in [14]) for a more general family of numerical semigroups with an Arithmetic progression of generators, see also [15].

**Definition 3.** *Let  $S = \langle a_1, a_2, a_3 \rangle$ ,  $Q = \langle a_1, a_2 \rangle$ . If  $x_1 := a_1 \cdot a_2 - k \cdot a_2 - j \cdot a_1$ ,  $1 \leq k < a_1$  and  $1 \leq j < a_2$ ,  $k$  is called the level and  $j$  the under level of  $x_1$ . Let  $m$  be the least integer such that  $m \cdot a_3 \in Q$ ,  $x_1 = a_3$  and let  $x_i := ia_3 \notin Q$  for every  $i$ ,  $i \leq m-1$ . By Remark 3 we have either  $i \cdot k < a_1$  (so  $i \cdot j > a_2$ ) with  $i \cdot j - (i-1) \cdot a_2 < a_2$  (upgrade start level) or  $i \cdot j < a_2$  (so  $i \cdot k > a_1$ ) with  $i \cdot k - (i-1) \cdot a_1 < a_1$  (downgrade start level) where  $i \leq s_1 \leq m-1$  for a certain  $s_1$  (the  $s$  means the start level -see Example 2-).*

*If  $h$  denotes the highest level and  $l$  the highest under-level among  $x_i$ 's then our pseudo-Frobenius numbers candidates are of the form  $F_0 := a_1 \cdot a_2 - a_2 - (l+1) \cdot a_1 + w \cdot a_3$  for some  $w \geq 0$  and  $F_1 := a_1 \cdot a_2 - (h+1) \cdot a_2 - a_1 + r \cdot a_3$  for some  $r \geq 0$ ,  $w$  resp.  $r$  are the least integers such that  $F_1 + a_3 \in Q$  resp.  $F_0 + a_3 \in Q$ .*

It is worth mentioning that the two previous pseudo-Frobenius candidates are the precise formulas for pseudo-Frobenius numbers of  $S$ , this is proved in [13]-Theorem 11-Second proof- where similar characterizations of pseudo-Frobenius numbers for any  $S = \langle a_1, a_2, a_3 \rangle$  are given yielding that the type of  $\langle a_1, a_2, a_3 \rangle$  is at most two.

**Example 2.** *We illustrate the above definition here: Take  $\langle 11, 13, 10 \rangle$ ,  $a_3 = 10 = 11 \cdot 13 - 5 \cdot 11 - 6 \cdot 13$  and it can be verified that  $m = 5$ , ( $5 \cdot a_3 \in Q = \langle 11, 13 \rangle$ ), it has a downgrade start level,  $s_1 = 2$ ,  $h = 7$ ,  $l = 10$  and the pseudo-Frobenius numbers are  $11 \cdot 13 - 11 - 8 \cdot 13 + 10 = 38$  and  $11 \cdot 13 - 11 \cdot 11 - 13 + 20 = 29$ .*

Lemma 2 has the following extension:

**Theorem 1.** *Let  $S := \langle a_1, \dots, a_n \rangle$ ,  $Q = \langle a_1, a_2 \rangle$  where  $a_1$  and  $a_2$  are two coprime minimal generators. If for all  $i$  and  $j \geq 3$ ,  $a_i + a_j \in Q$  then the type of  $S$  is at most  $n-1$ .*

*Proof.* The idea is simple in the sense that we get at most  $n-1$  possible pseudo-Frobenius numbers  $F_i \notin S$  verifying both  $F_i + a_1 \in S$  and  $F_i + a_2 \in S$ . By Lemma 1 we can write  $a_i := a_1 a_2 - \alpha_i a_2 - \beta_i a_1$ ,  $i \geq 3$  for some  $1 \leq \alpha_i < a_1$  and

$1 \leq \beta_i < a_2$ . Arrange these generators so sequence  $\alpha_i$  is decreasing (strictly) and thus  $\beta_i$  is strictly increasing since otherwise the difference between two consecutive minimal generators is in  $Q$ . Set

$$F_1 = a_1 a_2 - a_2 - \max_{3 \leq i \leq n} (\beta_i + 1) a_1$$

$$F_2 = a_1 a_2 - \max_{3 \leq i \leq n} (\alpha_i + 1) a_2 - a_1$$

and for  $3 \leq i \leq n - 1$  set

$$F_i = a_1 a_2 - (\min(\alpha_i, \alpha_{i+1}) + 1) a_2 - (\min(\beta_i, \beta_{i+1}) + 1) a_1.$$

The proof is direct for possible pseudo-Frobenius numbers having their level or under level equals 1 ( $F_1$  and  $F_2$ ;  $F_1 \notin S$  and  $F_2 \notin S$ .  $F_i + a_j \in S$  for any  $i = 1, 2$  and  $j = 1, 2$ ).

Otherwise we have the result from:  $\{a_i + s | s \in Q, i \geq 3\} \cup Q = S$  and Remark 1. For example one can prove that if  $F_i = a_1 \cdot a_2 - k \cdot a_2 - j \cdot a_1$  is a pseudo-Frobenius number  $k > 1$  and  $j > 1$  then for some  $i$ ,  $\alpha_{i+1} < k < \alpha_i$  and  $\beta_i < j < \beta_{i+1}$ , to get the necessity of the given expressions.  $\square$

### 3. ALGORITHM'S APPLICATIONS

As the reader may have noticed computing the Frobenius number for  $S = \langle a_1, a_2, a_3 \rangle$  as presented may require to explicit all the integers in  $\mathbb{N} \setminus S$ . However for some particular cases the answer is known. The interesting particular case that was unsolved in the general case (for any  $p$ ) is  $S = \langle a, a + 1, a + p \rangle$ . It turns out from the following theorems that knowing this Frobenius number we can find the Frobenius number of any  $S = \langle a_1, a_2, a_3 \rangle$ . We only state the algorithms layout without going further in details and complexity.

**Theorem 2.** [2] *Let  $S = \langle a, b, c \rangle$ ,  $a < b < c$ ,  $d = \gcd(a, b)$ ,  $n := \frac{bv}{d} - 1$ ,  $p = n(\frac{cdv}{a} - 1)$ ,  $v \in \mathbb{N}$  such that  $bv \equiv d \pmod{a}$  with  $vd < a$  then*

$$F(\langle a, b, c \rangle) = \frac{ab(F(\langle n, n + 1, n + p \rangle) + 2n + 1)}{dn(n + 1)} + (d - 1)c - (a + b).$$

As stated in [2] this follows by applying Johnson formula (Remark 2) three times: It could be better seen if one starts considering  $b = db'$ ,  $a = da'$ , replacing the different variables  $n, p$  by the given values the computation should be applied two times to the semigroup  $\langle n, n + 1, n + p \rangle$ . Lastly the formula is applied for  $\langle a, b, c \rangle$  with  $\gcd(a, b) = d$ .

**Theorem 3.** [2] *Let  $S = \langle a, a + 1, a + p \rangle$ . If  $a \geq p(p - 4) + 2$  then*

$$F(S) = \left\lfloor \frac{a}{p} \right\rfloor (a + 1) + (p - 1) \left\lfloor \frac{a + 1}{p} \right\rfloor + (p - 3)a - 1.$$

**Lemma 3.** *Let  $S = \langle a, a + 1, ia + d \rangle$ ,  $a = dk + j$ ,  $0 < i < d < a$ ,  $1 \leq j < d$  and  $(k + 1) \cdot i + j + 1 = d$ . Then the type of  $S$  is one.*

*Proof.* The proof makes use of Remark 2 by finding a common factor  $f$  of  $a + 1$ ,  $ia + d$  and showing that  $a \in \langle \frac{a+1}{f}, \frac{ia+d}{f} \rangle$ . Notice that  $a = j \cdot (k + 1) + k \cdot (i(1 + k) + 1)$  from the definition of  $a$  and  $d$ ,  $a + 1 = (d - i) \cdot (k + 1)$  and  $ia + d = (d - i) \cdot (1 + i(1 + k))$ . The result follows taking  $f = d - i$ .  $\square$

**Theorem 4.** Let  $S = \langle a, a+1, ia+d \rangle$ ,  $Q = \langle a, a+1 \rangle$  where  $a = dk+j$ ,  $a_3 = ia+d$ ,  $0 < i < d < a$ ,  $1 \leq j < d$ , we set by definition  $wd := 3 + x$  and  $d \geq 3$ . Then  $T(S) = \{F_1, F_2\}$ ,

$$F_1 = a \cdot (a+1) - (k(d-i)+1) \cdot a - (a+1) + w \cdot a_3$$

$$F_2 = a \cdot (a+1) - a - (a-d+1) \cdot (a+1) + (k-1)a_3$$

where  $w = 0$  if  $(k+1)i+j \geq d$  and  $w = k$  otherwise.

*Proof.* Write  $ia+d = a \cdot (a+1) - (d-i) \cdot a - (a-d) \cdot (a+1)$ , we are in a downgrading stage with

$$na_3 = a \cdot (a+1) - n(d-i) \cdot a - (a-dn) \cdot (a+1)$$

for all  $n \leq k$  and  $(k+1) \cdot a_3 \in Q$ . If

$$F = a \cdot (a+1) - (k(d-i)+1) \cdot a - (a+1)$$

$F \notin S$  and  $F + w \cdot a_3 = (-2 + j - x + wi - 1 + ki) \cdot (dk+j) + (2+x) \cdot (dk+j+1)$  is in  $Q$  if and only if:

$$ki + j \geq w(d-i) \text{ and } 2 + x = wd - 1 \geq 0$$

$$\text{or } ki + j - w(d-i) < 0 \text{ and } wd - 1 \geq a,$$

we call  $w_0$  the smallest such integer  $\geq 1$ . But  $w_0$  exists and  $w_0 = 1$  if  $(k+1)i+j \geq d$ ,  $w_0 = k+1$  otherwise; so first

$$F_1 = a \cdot (a+1) - (k(d-i)+1) \cdot a - (a+1) + (w_0-1) \cdot (ia+d) \subset T(S).$$

Take  $G = a \cdot (a+1) - a - (a-d+1) \cdot (a+1) + k \cdot (ia+d)$  and set  $y = d-1-j$ ,  $G = a \cdot (ki+j) + y \cdot (a+1) \in Q$ . Now take  $F = a \cdot (d-2) + d-1 + (k-t) \cdot (ia+d)$  clearly  $F - G = -(td+ita)$  thus

$$\begin{aligned} F &= a \cdot (ki+j+td+ita) - (-d+1+j+td+ita) \cdot (a+1) \\ &= a \cdot (ki+j+td-it) - (1+j+(t-1)d) \cdot (a+1) \end{aligned}$$

and clearly  $F$  is not in  $Q$  if and only if  $ki+j+t(d-i) < a+1$  and  $1+j+(t-1)d > 0$ . We call  $t_0$  the smallest such integer  $\geq 1$ , but it can be verified that  $(k-1)i+j+d < a+1$  and so if  $r = k-t_0 = k-1$ , the second pseudo-Frobenius is  $F_2 := a \cdot (d-2) + d-1 + (k-1) \cdot (ia+d)$ .

$F_1 = (j+ki-w(d-i)) \cdot a + (wd-1) \cdot (a+1)$  and  $F_2 = a \cdot ((k-1)i+j+d) - (1+j) \cdot (a+1)$ ; if  $w = 0$  then  $F_1 \neq F_2$  since  $1 < 1+j$  and for  $w = k$  it is straightforward that  $F_1 \leq F_2$  with equality if and only if  $(k+1)i+j+1 = d$  which yields the result by Lemma 3.  $\square$

By Remark 2 the case  $d$  divides  $a$  is almost trivial. Note that Theorem 4 gives by Theorem 2 a new way to find the Frobenius number of any  $S = \langle a, b, c \rangle$ . Particular cases of Theorem 4 can be deduced from the following result:

**Theorem 5.** [3] Let  $S = \langle a, b, c \rangle$  with pairwise coprime generators and so  $c = bh - ad = a \cdot b - (a-h) \cdot b - d \cdot a = (b-d) \cdot a - (a-h)b$ . If  $\frac{b}{d} \geq \left\lceil \frac{a}{h} \right\rceil$  then

$$T(S) = \left\{ \left\lfloor \frac{a-1}{h} \right\rfloor c + da - b - a, (a-1)b - \left\lfloor \frac{a-1}{h} \right\rfloor da - a \right\}$$

**Example 3.** For  $S = \langle 34, 35, 46 \rangle$  we cannot apply Theorem 5 because the pairwise coprime condition is not satisfied, also Theorem 3 cannot be applied, applying Theorem 4 we have  $d = 12$ ,  $a = 34 = 2 \cdot 12 + 10$ ,  $i = 1$ ,  $k = 2$ ,  $j = 10$ , here  $w = 0$  and  $T(S) = \{373, 397\}$ .

One can verify that  $(n+1)(n+2)^2 = (n+1)n^2 + 4(n+1)^2$  and  $n(2n+1)^2 = n(n+1)^2 + (3n+2)n^2$  for any  $n$ . Some results for example Frobenius number formulas of three consecutive squares were done in [16, 17].

**Lemma 4.** *Let  $S = \langle (4s-1)^2, (4s)^2, (4s+1)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 4s-1$  odd ( $s > 1$ ) then  $T(S) = \{256s^3 - 144s^2 + 8s - 1, 272s^3 - 168s^2 + s - 2\}$ .*

*Proof.* First let us write

$$(n+2)^2 = n^2(n+1)^2 - (n^2 - 2n - 4)n^2 - (4n-4)(n+1)^2,$$

if  $n = 4s - 1$ ,  $s > 1$  then we verify that  $s(n+2)^2 \in Q$  and

$$(s-1)(n+2)^2 = (16s^2 - 15s - 1)n^2 - (n^2 - 4n + 3)(n+1)^2 \notin Q$$

by Definition 3 notations we have that  $S$  is an upgrade start level semigroup with  $s_1 = s - 1$ ,  $l = n^2 - 2n - 4$  and  $h = n^2 - 4n + 3$ . Finally we have:

$$\begin{aligned} & n^2(n+1)^2 - (n^2 - 2n - 3)n^2 - (n+1)^2 + (\alpha)(n+2)^2 \\ &= (4n+4 + \alpha(4n+5))n^2 - (1 + \alpha(4n-4))(n+1)^2 \end{aligned}$$

so we verify

$$n^2(n+1)^2 - (n^2 - 2n - 3)n^2 - (n+1)^2 + (s-1)(n+2)^2 = (s-1)n^2 + (16s-8)(n+1)^2$$

$$\text{and } n^2(n+1)^2 - n^2 - (n^2 - 4n + 4)(n+1)^2 + (n+2)^2 = 16sn^2. \quad \square$$

**Lemma 5.** *Let  $S = \langle (4s-3)^2, (4s-2)^2, (4s-1)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 4s-3$  odd ( $s > 1$ ) then  $T(S) = \{144s^3 - 296s^2 + 193s - 42, 160s^3 - 352s^2 + 234s - 51\}$ .*

*Proof.* If  $n = 4s - 3$  for some  $s > 1$  then  $(2s-1)(n+2)^2 \in Q$ , the highest under level is for

$$s(n+2)^2 = n^2(n+1)^2 - (16s^2 - 25s + 8)n^2 - (8s-9)(n+1)^2$$

so  $l = 16s^2 - 25s + 8$  and the highest level is for  $h = 16s^2 - 32s + 16$  with

$$(s-1)(n+2)^2 = n^2(n+1)^2 - (7s-3)n^2 - (16s^2 - 32s + 16)(n+1)^2$$

$$(2s-2)(n+2)^2 = n^2(n+1)^2 - (14s-6)n^2 - (16s^2 - 40s + 23)(n+1)^2.$$

Here  $s_1 = s - 1$  and we verify that

$$\begin{aligned} & n^2(n+1)^2 - n^2 - (16s^2 - 32s + 17)(n+1)^2 + \alpha(n+2)^2 \\ &= (\alpha(4n+5) - 1)n^2 - (8 - 8s + \alpha(4n-4))(n+1)^2 \end{aligned}$$

$$\begin{aligned} & n^2(n+1)^2 - (16s^2 - 25s + 9)n^2 - (n+1)^2 + \alpha(n+2)^2 \\ &= (9s-5 + \alpha(4n+5))n^2 - (1 + \alpha(4n-4))(n+1)^2 \end{aligned}$$

$$n^2(n+1)^2 - n^2 - (16s^2 - 32s + 17)(n+1)^2 + s(n+2)^2 = (9(s-1) + 4)n^2 + (n+1)^2$$

$$\text{and } n^2(n+1)^2 - (16s^2 - 25s + 9)n^2 - (n+1)^2 + (s-1)(n+2)^2 = (2(s-1))n^2 + (8(s-1))(n+1)^2. \quad \square$$

**Lemma 6.** *Let  $S = \langle (4s)^2, (4s+1)^2, (4s+2)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 4s$ ,  $s \geq 2$  then  $T(S) = \{112s^3 + 48s^2 + 8s - 1, 128s^3 - 20s - 5\}$ .*

*Proof.*  $(n+2)^2 = n^2(n+1)^2 - (n^2 - 2n - 4)n^2 - (4n - 4)(n+1)^2$ , here we need the following expressions:

$$\left\{ \begin{array}{l} s(n+2)^2 = n^2(n+1)^2 - (3s+1)n^2 - (16s^2 - 4s)(n+1)^2 \\ (s+1)(n+2)^2 = n^2(n+1)^2 - (16s^2 - 5s - 3)n^2 - (12s - 4)(n+1)^2 \\ 2s(n+2)^2 = n^2(n+1)^2 - (2 + 6s)n^2 - (16s^2 - 8s)(n+1)^2 \\ (2s+1)(n+2)^2 = n^2(n+1)^2 - (16s^2 - 2s - 2)n^2 - (8s - 4)(n+1)^2 \\ 3s(n+2)^2 = n^2(n+1)^2 - (9s+3)n^2 - (16s^2 - 12s)(n+1)^2 \\ (3s+1)(n+2)^2 = n^2(n+1)^2 - (16s^2 + s - 1)n^2 - (4s - 4)(n+1)^2 \\ n(n+2)^2 = n^2(n+1)^2 - (12s+4)n^2 - (16s^2 - 16s)(n+1)^2 \end{array} \right.$$

with  $(n+1)(n+2)^2 \in Q$ .  $s_1 = s$ ,  $h = 16s^2 - 4s$  and  $l = 16s^2 + s - 1$ . Finally we have

$$\begin{aligned} & n^2(n+1)^2 - n^2 - (16s^2 - 4s + 1)(n+1)^2 + \alpha(n+2)^2 \\ &= (\alpha(4n+5) - 1)n^2 - (1 - 4s + 16\alpha s - 4\alpha)(n+1)^2 \\ & n^2(n+1)^2 - (16s^2 + s)n^2 - (n+1)^2 + \alpha(n+2)^2 \\ &= (2n+1 - s + \alpha(4n+5))n^2 - (1 + \alpha(4n-4))(n+1)^2. \end{aligned}$$

$$n^2(n+1)^2 - n^2 - (16s^2 - 4s + 1)(n+1)^2 + (3s+1)(n+2)^2 = 3(n+1)^2 + (7s+1)n^2$$

and

$$n^2(n+1)^2 - (16s^2 + s)n^2 - (n+1)^2 + s(n+2)^2 = (4s)n^2 + (4s-1)(n+1)^2.$$

□

**Lemma 7.** Let  $S = \langle (4s-2)^2, (4s-1)^2, (4s)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 4s - 2$ ,  $(s > 1)$  then  $T(S) = \{80s^3 - 80s^2 + 28s - 5, 128s^3 - 160s^2 + 60s - 9\}$ .

*Proof.*  $(n+2)^2 = n^2(n+1)^2 - (n^2 - 2n - 4)n^2 - (4n - 4)(n+1)^2$ .

$$\left\{ \begin{array}{l} (s-1)(n+2)^2 = n^2(n+1)^2 - (11s-2)n^2 - (16s^2 - 28s + 12)(n+1)^2 \\ s(n+2)^2 = n^2(n+1)^2 - (16s^2 - 13s + 2)n^2 - (4s-4)(n+1)^2 \\ (2s-1)(n+2)^2 = n^2(n+1)^2 - (6s-1)n^2 - (16s^2 - 24s + 8)(n+1)^2 \\ (2s)(n+2)^2 = n^2(n+1)^2 - (16s^2 - 18s + 3)n^2 - (8s-8)(n+1)^2 \\ (3s-1)(n+2)^2 = n^2(n+1)^2 - (s)n^2 - (16s^2 - 20s + 4)(n+1)^2 \\ (3s)(n+2)^2 = n^2(n+1)^2 - (16s^2 - 23s + 4)n^2 - (12s-12)(n+1)^2 \\ n(n+2)^2 = n^2(n+1)^2 - (12s-2)n^2 - (16s^2 - 32s + 12)(n+1)^2 \end{array} \right.$$

with  $(n+1)(n+2)^2 \in Q$ .  $s_1 = s - 1$ ,  $h = 16s^2 - 20s + 4$  and  $l = 16s^2 - 13s + 2$ . Finally we have

$$\begin{aligned} & n^2(n+1)^2 - n^2 - (16s^2 - 20s + 5)(n+1)^2 + \alpha(n+2)^2 \\ &= (\alpha(4n+5) - 1)n^2 - (1 - 4s + 16\alpha s - 12\alpha)(n+1)^2 \\ & n^2(n+1)^2 - (16s^2 - 13s + 3)n^2 - (n+1)^2 + \alpha(n+2)^2 \\ &= (5s - 2 + \alpha(4n+5))n^2 - (1 + \alpha(4n-4))(n+1)^2. \end{aligned}$$



$$n^2(n+1)^2 - n^2 - (16s^2 - 20s + 5)(n+1)^2 + s(n+2)^2 = 3(n+1)^2 + (5s-2)n^2$$

and

$$n^2(n+1)^2 - (16s^2 - 13s + 3)n^2 - (n+1)^2 + (3s-1)(n+2)^2 = (n)n^2 + (n+1)(n+1)^2.$$

□

**Lemma 8.** Let  $S = \langle (2s)^2, (4s+1)^2, (2s+1)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 2s$ , ( $s > 1$ ) then  $T(S) = \{32s^3 - 12s^2 - 11s - 2, 28s^3 - 4s - 1\}$ .

*Proof.* First

$$(2n+1)^2 = n^2(n+1)^2 - (2n+1)n^2 - (n^2 - 2n - 1)(n+1)^2.$$

$n = 2s$ ,  $S$  is a downgrade semigroup with  $s_1 = s - 1$ ,  $l = n^2 - 3s - 1$ ,  $h = n^2 - 2n - 1$  by computation we have

$$\begin{aligned} n^2(n+1)^2 - n^2 - (n^2 - 2n)(n+1)^2 + \alpha(2n+1)^2 \\ = (2n + \alpha(2n+1))(n+1)^2 - (1 + \alpha(2n+1))n^2. \end{aligned}$$

$$n^2(n+1)^2 - n^2 - (n^2 - 2n)(n+1)^2 + (s-1)(2n+1)^2 = (s-1)(n+1)^2 + (7s+1)n^2$$

$$\text{and } n^2(n+1)^2 - (n^2 - 3s)n^2 - (n+1)^2 + (2n+1)^2 = 3sn^2 + 4s(n+1)^2. \quad \square$$

**Lemma 9.** Let  $S = \langle (2s+1)^2, (4s+3)^2, (2s+2)^2 \rangle$  and  $Q = \langle n^2, (n+1)^2 \rangle$  with  $n = 2s+1 > 1$ , then  $T(S) = \{20s^3 + 28s^2 + 9s - 1, 32s^3 + 44s^2 + 13s - 2\}$ .

*Proof.* First we got

$$(2n+1)^2 = n^2(n+1)^2 - (2n+1)n^2 - (n^2 - 2n - 1)(n+1)^2.$$

Also if  $n = 2s+1$ ,  $s(2n+1)^2 \notin Q$  and  $(s+1)(2n+1)^2 = (s+1)n^2 + (3s+2)(n+1)^2$  so  $s_1 = s$ ,  $l = n^2 - s - 1$ ,  $h = n^2 - 2n - 1$ .

$$\begin{aligned} \text{From computation: } n^2(n+1)^2 - n^2 - (n^2 - 2n)(n+1)^2 + \alpha(2n+1)^2 \\ = (2n + \alpha(2n+1))(n+1)^2 - (1 + \alpha(2n+1))n^2. \end{aligned}$$

By a direct verification

$$n^2(n+1)^2 - n^2 - (n^2 - 2n)(n+1)^2 + s(2n+1)^2 = (5s+3)n^2 + (3s+1)(n+1)^2$$

$$\text{and } n^2(n+1)^2 - (n^2 - s)n^2 - (n+1)^2 + (2n+1)^2 = sn^2 + (4s+2)(n+1)^2. \quad \square$$

The next lemma is the particular three dimensional case of what it seems a conjecture of Prof. Dao at <http://mathoverflow.net/questions/292507/a-formula-for-frobenius-number-of-certain-numerical-semigroups>

**Lemma 10.** [12, Proposition 10.16] Let  $a, b, c$  be three positive integers such that  $ab + b + 1$  and  $ca + c + 1$  are coprime. For  $S = \langle ab + a + 1, bc + b + 1, ca + c + 1 \rangle$ ,  $T(S) = \{abc - 1, 2(abc - 1)\}$ .

*Proof.* First it is clear from the cyclic expression that if any two of the generators share a common factor  $d$  the third one will be divisible by  $d$ , assuming the hypothesis we write:

$$bc + b + 1 = (ab + a + 1)(ca + c + 1) - c(ab + a + 1) - ((b+1)a - b)(ca + c + 1).$$

The generators are  $a_3 = bc + b + 1$ ,  $Q = \langle ab + a + 1, ca + c + 1 \rangle$ .

We get  $m = a + 1$ , and

$$\begin{aligned} (ab + a + 1)(ca + c + 1) - (ab + a + 1) - ((b + 1)a - b + 1)(ca + c + 1) + \alpha a_3 \\ = (b + \alpha(b + 1))(ca + c + 1) - (1 + \alpha c)(ab + a + 1). \end{aligned}$$

$$\begin{aligned} F_1 &= (ab + a + 1)(ca + c + 1) - (ab + a + 1) - ((b + 1)a - b + 1)(ca + c + 1) \\ &\quad + (a - 1)(bc + b + 1) = 2(abc - 1) \end{aligned}$$

$$F_2 = (ab + a + 1)(ca + c + 1) - (ca + 1)(ab + a + 1) - (ca + c + 1) = abc - 1.$$

□

The algorithm that we studied first appeared in [13] Theorem-11. The arguments are closely related to the results in [4]. There are different algorithms computing  $F(S)$  with  $S$  a three minimally generated semigroup, among them is [6] also [8], Rödseth's formula [5, 7]; and more recently [9, 10]. We refer to [11] for a further discussion on these and other generalized algorithms.

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