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On the diameter of cut polytopes

José Neto

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Abstract Given an undirected graph G with node set V, the cut polytope is defined as the convex hull of the incidence vectors of all the cuts in G. And for a given integer $k \in \{1, 2, \ldots, \lfloor \frac{|V|}{2} \rfloor\}$, the uniform cut polytope $CUT_{=k}(G)$ is defined as the convex hull of the cuts which correspond to a bipartition of the node set into sets with cardinalities k and $|V| - k$.

In this paper, we study the diameter of these two families of polytopes. With respect to the cut polytope, we show a linear upper bound on its diameter (improving on one stemming from the reference: [F. Barahona and A.R. Mahjoub: *On the cut polytope*, Mathematical Programming 36, 157–173 (1986)]), give its value for trees and complete bipartite graphs. Then concerning uniform cut polytopes, we establish bounds on their diameter for different graph families, we provide some connections with other partition polytopes in the literature, and introduce sufficient and necessary conditions for adjacency on their 1-skeleton.

Keywords (uniform) cut polytope · diameter · graph partitioning

Mathematics Subject Classification (2000) 05C12 · 90C05 · 90C57

1 Introduction

Given a nonempty polytope P, the *1-skeleton* of P is defined as the set of vertices and edges of P. Given two vertices v_1, v_2 of P, let $d(v_1, v_2)$ denote the length (w.r.t. the number of edges) of a shortest path in the graph corresponding to the 1-skeleton of P between the nodes representing v_1 and v_2. The *diameter* of P, which is denoted by $diameter(P)$ is the maximum

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length of a shortest path between any two vertices of P, i.e. $\max\{d(v_1, v_2) : v_1, v_2$ are vertices of $P\}$.

The notion of diameter of a polyhedron presents, among others, some connections with linear programming and linear optimization methods that are based on the fact that there exists an optimal solution that is a vertex of the feasible region whenever the latter is nonemtpy and bounded. For example, consider the simplex algorithm applied to a linear program: starting from a particular vertex v_0 of the feasible region F (a polyhedron), this method consists in iteratively moving from a vertex (*the current basic feasible solution*) of F to a neighbor (with better or equal objective value), until an optimal vertex solution v^* is found. Given a nonempty polytope P, let $V(P)$ denote the set of vertices of P and $I(P)$ denote the set of all the instances of linear programming problems having P as feasible region. Given an instance $I \in I(P)$ and $v_0 \in V(P)$, let $s(I, v_0)$ denote the minimum number of iterations the simplex method may apply before reaching an optimal solution, starting with the vertex s_0. Then the diameter provides a lower bound on the quantity:

$$\max_{(I,v_0):I\in I(P),v_0\in V(P)} \{s(I, v_0)\}$$

that is, on the number of iterations the procedure applies at least, in a 'worst-case' (i.e. when the diameter coincides with the distance on the 1-skeleton of P between the starting point v_0 and the optimal solution v^*).

For further information on polytopes and related notions, the reader may consult, e.g., the textbooks by Grünbaum [6] and Ziegler [11]. In the present paper we study the diameter of cut polytopes. To be more precise, the polytopes we consider correspond to the convex hull of the incidence vectors of all the cuts in some given graph or to the convex hull of the incidence vectors of the cuts that satisfy an additional constraint with respect to the cardinalities of their shores. Our objective is to develop new insights and improve our knowledge on the polyhedral structure of these notorious and related polytopes (for further information on properties and applications of these families of polytopes the reader may consult, e.g., [4]). For, in the rest of this section, we introduce some notation and preliminaries.

Notation and preliminaries

Let G stand for an undirected graph with node set V and edge set E. Given a node set $S \subseteq V$, let $\delta_G(S)$ (or $\delta(S)$ when G is clear from the context) stand for the cut that is defined by S, i.e., the set of edges in E with exactly one endpoint in S: $\delta(S) = \{e \in E : |e \cap S| = 1\}$. The node sets S and $V \setminus S$ are called the *shores* of the cut $\delta(S)$.

Given a set F of edges in G, its incidence vector $\chi^F \in \mathbb{R}^{|E|}$ is defined by $\chi^F_e = 1$ if $e \in F$ and 0 otherwise. Given a node $v \in V$, $N(v)$ denotes the set of nodes in $V \setminus \{v\}$ that are adjacent to v in the graph G, and $deg(v)$ stands for the degree of this node, i.e. $deg(v) = |N(v)|$. The dimension of a polyhedron

P is the maximum number of affinely independent points in P minus 1. It is denoted by $\dim(P)$.

Given $S \subseteq V$, $G[S]$ denotes the subgraph of G that is induced by the node set S.

The *cut polytope* $CUT(G)$ is defined as the convex hull of the incidence vectors of all the cuts in the graph G, i.e. $CUT(G) = \text{conv}\{\chi^{\delta_G(S)}: \ S \subseteq V\}$.

Given some integer k satisfying $k \leq \lfloor \frac{n}{2} \rfloor$, $CUT_{=k}(G) = \text{conv}\{\chi^{\delta_G(S)}: \ S \subseteq V \text{ and } |S| = k\}$ denotes the convex hull of the incidence vectors of the cuts in G having one shore with size k. Polytopes of the latter form are called *uniform cut polytopes*. In the particular case when the graph G is complete, we shall simply write: $CUT^n_{=k}$.

A d-dimensional polytope P with f facets is said to satisfy the Hirsch property if its diameter verifies the inequality: $diameter(P) \leq f - d$.

To the author's knowledge, the first study of the 1-skeleton of the cut polytope appears in the paper by Barahona and Mahjoub [1]. There, the authors namely provide the following characterization of the adjacency of incidence vectors of cuts on the cut polytope (Theorem 4.1 in [1]).

Theorem 1 *Let $G = (V, E)$ be a connected graph. Let x^I, x^J be extreme points of $CUT(G)$; let I, J be the corresponding cuts. Let $F = E \setminus (I \Delta J)$. Then x^I and x^J are adjacent in $CUT(G)$ iff $H_{I\Delta J} = (V, F)$ has two connected components.*

In the same reference [1] the authors also establish that the Hirsch property holds for $CUT(G)$. In fact, Naddef [10] proved later that this property holds for all $(0, 1)$-polytopes, and thus also for $CUT(G)$ and the uniform cut polytopes. What also follows from Theorem 1 and its proof is that $CUT(G)$ is a so-called *combinatorial polyhedron* [8,9], a family of polyhedra for which Matsui and Tamura also established further different properties related to the Hirsch property [8].

The present paper is organized as follows. In Section 2, we study the diameter of the cut polytope depending on the graph properties, give its exact value for some cases and provide upper bounds on some other classes. In Section 3 we report investigations on the diameter of uniform cut polytopes: after we discuss some particular cases (Subsection 3.1), we provide (Subsection 3.2) sufficient and necessary conditions for the incidence vectors of two cuts to be adjacent on these polytopes.

2 On the diameter of the cut polytope

For the particular case of complete graphs Barahona and Mahjoub [1] showed that the cut polytope has diameter 1: this is a direct consequence of Theorem 1. For general graphs they provided the following upper bound on the diameter that is given by the next theorem (Theorem 4.3 in [1]). Before we formulate it we give some additional notation. Let G be a connected graph and C be a cut

in G. Let $T(C)$ denote the graph obtained from G by contracting edges not in C and replacing multiple edges by single edges. Let $m(C)$ be the number of edges of $T(C)$. Given two cuts I, J of G, let $d(I, J)$ stand for the distance in the 1-skeleton of $CUT(G)$ between the vertices corresponding to the incidence vectors of I and J.

Theorem 2 *If I and J are two cuts in G, then $d(I, J) \leq m(I \Delta J)$.*

We now give a generally better bound on the distance between the incidence vectors of two cuts in the 1-skeleton of $CUT(G)$. Given a subset of edges I of the graph G, let H_I denote the graph obtained from G by removing all the edges in I. Given a graph G, let $\kappa(G)$ denote the number of connected components in G.

Proposition 1 *If I and J are two cuts in a connected graph G, then $d(I, J) \leq \kappa(H_{I \Delta J}) - 1$.*

Proof First notice that the proposition holds for the two basic cases when $H_{I \Delta J}$ has one or two components. The case when $\kappa(H_{I \Delta J}) = 1$ corresponds to $I = J$, and so $d(I, I) = 0 = \kappa(H_{I \Delta J}) - 1 = \kappa(G) - 1$. By Theorem 1 we have $d(I, J) = 1$ if and only if $\kappa(H_{I \Delta J}) = 2$.

So assume now $\kappa(H_{I \Delta J}) \geq 3$. Let $P = \delta(S)$, $\emptyset \neq S \subsetneq V$ denote a minimal (w.r.t. inclusion) proper cut in G which is contained in $I \Delta J$. Define $L := I \Delta P$. Notice that since P is minimal it follows from Theorem 1 that the cuts I and L have their respective incidence vectors adjacent on the cut polytope $CUT(G)$.

Also notice that L is a cut which differs from I only in the set $\delta(S)$. More precisely we have $L = (I \setminus \delta(S)) \cup (J \cap \delta(S))$. And as the graph G is assumed to be connected we have $(I \Delta J) \cap \delta(S) \neq \emptyset$. This implies that the graph $H_{L \Delta J}$ (which differs from $H_{I \Delta J}$ only in the set of edges $\delta(S)$) will contain at least one edge in $\delta(S)$, and thus $\kappa(H_{L \Delta J}) \leq \kappa(H_{I \Delta J}) - 1$.

We may then iterate this process (i.e. look for a minimal cut P' in G that is contained in $L \Delta J$, then define $L' = L \Delta P' \ldots$) until the graph H which is built at the current iteration (i.e. as $H_{I \Delta J}$ above) has only two connected components (which, by Theorem 1, corresponds to the case when a cut, whose incidence vector is adjacent to the one of J on the cut polytope, has been reached).

As the number of connected components of the graph H decreases by at least one at each iteration and each operation of the form $I \Delta P$ (as described above) corresponds to moving from one vertex to an adjacent one on the cut polytope, it follows that $d(I, J) \leq \kappa(H_{I \Delta J}) - 1$.

Remark. To be more accurate, one can give the following upper bound on $d(I, J)$. The underlying idea is to use a minimal cut that is contained in $I \Delta J$ and induces the maximum reduction in the number of connected components in the auxiliary graph H that will be used in the next iteration. We have: $d(I, J) \leq \kappa(H_{I \Delta J}) - \eta_{I,J}$ with
$\eta_{I,J} = \max_{C=I,J}\{cn(S, C) \colon \delta(S)$ minimal cut of G contained in $I \Delta J\}$. Given a node set $S \subseteq V$ and a cut C in $G = (V, E)$, $cn(S, C)$ denotes the number of

different connected components $(C_i)_{i=1}^r$ in $H_{I\Delta J}$, none of which is contained in S and such that for each component C_j, $j \in \{1, \ldots, r\}$ there is an edge in C having one endnode in S and the other in C_j.

For any connected graph G, Proposition 1 provides the upper bound $n-1$ on the diameter of the cut polytope $CUT(G)$. This bound can also be shown to hold and improved for disconnected graphs. But before we give an improved bound, we shall first mention some results with respect to the diameter and the adjacency relation between vertices of cartesian products of polyhedra. The proofs are simple and given here for completeness and since the author is not currently aware of a reference in the literature stating them explicitly.

Given a set of $q \in \mathbb{N}$, $q \geq 2$, polyhedra $(P_i)_{i=1}^q$ with $P_i \subseteq \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, for all $i \in \{1, 2, \ldots, q\}$, their cartesian product is the set $P = \times_{i=1,\ldots,q} P_i = \{(x_1, x_2, \ldots, x_q) \in \times_{i=1,\ldots,q} \mathbb{R}^{n_i} : x_i \in P_i, i = 1, \ldots, q\}$.

Proposition 2 *The vector $(x_1, x_2, \ldots, x_q) \in P(= \times_{i=1,\ldots,q} P_i)$ is an extreme point of P if and only if x_i is an extreme point of P_i, for all $i \in \{1, 2, \ldots, q\}$.*

Proof [$\Rightarrow$] Assume the vector $x = (x_1, x_2, \ldots, x_q)$ is an extreme point of the polyhedron P and that some vector $x_k, k \in \{1, 2, \ldots, q\}$, is not an extreme point of P_k. Then, there exist different vectors $y_k^1, y_k^2 \in P_k, \alpha \in]0, 1[$ such that $x_k = \alpha y_k^1 + (1-\alpha) y_k^2$. Let $z^i = (x_1, \ldots, x_{k-1}, y_k^i, x_{k+1}, \ldots, x_q)$, for $i = 1, 2$. Since both vectors z^1, z^2, belong to the polyhedron P and $x = \alpha z^1 + (1-\alpha) z^2$, we get a contradiction with x being an extreme point of P.
[$\Leftarrow$] Assuming that all the vectors x_i for $i = 1, \ldots, q$ are extreme points of the polyhedra $(P_i)_{i=1}^q$ implies that the vector $x = (x_1, x_2, \ldots, x_n)$ belongs to P and cannot be expressed as a convex combination of other points in P. Thus, x is an extreme point of P.

Proposition 3 *Let $x = (x_1, x_2, \ldots, x_q)$ and $y = (y_1, y_2, \ldots, y_q)$ denote two different extreme points of the polyhedron P. Then, x and y are adjacent on the 1-skeleton of P if and only if*

i) $|\{i \in \{1, 2, \ldots, q\} : x_i \neq y_i\}| = 1$, *and*
ii) x_k *and* y_k *are adjacent extreme points on the 1-skeleton of* P_k, *with* $k \in \{1, 2, \ldots, q\}$ *and such that* $x_k \neq y_k$.

Proof [$\Rightarrow$] *Necessity of i).* Assume that the vectors x and y are such that $x_r \neq y_r$ and $x_s \neq y_s$ for at least two different indexes $r, s \in \{1, 2, \ldots, q\}$. Let $w = (x_1, \ldots, x_{r-1}, y_r, x_{r+1}, \ldots, x_q)$, and $z = (y_1, \ldots, y_{r-1}, x_r, y_{r+1}, \ldots, y_q)$. Since w and z both differ from the vectors x, y and are two different points of P such that $x + y = w + z$, we get a contradiction with x, y being adjacent extreme points of P.
Necessity of ii). We make use of the property that two extreme points x^1, x^2, of a polyhedron $Q \subseteq \mathbb{R}^n$ are adjacent if and only if there exists some vector $c \in \mathbb{R}^n$ such that x^1 and x^2 are the only two extreme points of Q which maximize $c^t x$ over $x \in Q$.

Let $\overline{c} = (\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_q) \in \times_{i=1,\ldots,q} \mathbb{R}^{n_i}$ be a vector such that the optimum of the linear program $\max_{x \in P} c^t x$, is attained by x and y but no other extreme

point of P. Then, necessarily the vector $\overline{c}_k \in \mathbb{R}^{n_k}$ is such that the linear program $\max_{x \in P_k} c^t x$ is attained by x_k and y_k but no other extreme point of P_k.

[$\Leftarrow$] Let $x = (x_1, x_2, \ldots, x_q)$ and $y = (y_1, y_2, \ldots, y_q)$ denote two different extreme points of P such that properties $i)$ and $ii)$ hold. For each $j \in \{1, 2, \ldots, q\} \setminus \{k\}$, let $\overline{c}_j \in \mathbb{R}^{n_j}$ denote a vector such that x_j is the only extreme point of P_j which is optimal for the linear program $\max_{x \in P_j} \overline{c}_j^t x$. And let $\overline{c}_k \in \mathbb{R}^{n_k}$ be such that the linear program $\max_{x \in P_k} c^t x$ is attained by x_k and y_k but no other extreme point of P_k. Considering now the vector $\overline{c} = (\overline{c}_1, \overline{c}_2 \ldots, \overline{c}_q)$ it is obvious that x and y are the only two extreme points of P attaining the optimum of the linear program $\max_{x \in P} \overline{c}^t x$.

Proposition 4 *The following equation holds*

$$diameter(P) = \sum_{i=1}^{q} diameter(P_i).$$

Proof Let the sequence of extreme points of the polyhedron P: $(x^k)_{k=0}^L$, $L \in \mathbb{N}$, denote a shortest path τ in the 1-skeleton of P (w.r.t. the number of edges) joining two of his extreme points: $x = (x_1, x_2, \ldots, x_q)$ and $y = (y_1, y_2, \ldots, y_q)$. Let $\tau = (x^0 = x, x^1, \ldots, x^L = y)$, with $x^j = (x_1^j, x_2^j, \ldots, x_q^j)$, $j = 0, \ldots, L$. From Proposition 3 and removing redundancies, the sequence $(x_i^k)_{k=0}^L$ with $i \in \{1, 2, \ldots, q\}$, corresponds to a shortest path τ_i in the 1-skeleton of P_i joining x_i and y_i, and the length $l(\tau)$ (i.e. the number of edges) of the path τ corresponds to the sum of these paths τ_i, $i = 1, \ldots, q$: $l(\tau) = \sum_{i=1}^q l(\tau_i)$. It follows that $diameter(P) \leq \sum_{i=1}^q diameter(P_i)$.

Now, for each $i \in \{1, 2, \ldots, q\}$, let $(\overline{x}^i, \overline{y}^i)$ denote a pair of extreme points of the polyhedron P_i such that their distance in the 1-skeleton of P_i equals $diameter(P_i)$. Then, considering the length of a shortest path τ in the 1-skeleton of P joining its extreme points $\overline{x} = (\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^q)$ and $\overline{y} = (\overline{y}^1, \overline{y}^2, \ldots, \overline{y}^q)$, from Proposition 3, we deduce $diameter(P) \geq l(\tau) \geq \sum_{i=1}^q diameter(P_i)$.

A corollary of Proposition 4 is the following result relating the diameter of $CUT(G)$ to that of the cut polytopes corresponding to the connected components of the graph G.

Corollary 1 *For a graph $G = (V, E)$ with connected components corresponding to the subgraphs $(G_i = (V_i, E_i))_{i=1}^r$, $r \geq 2$, the following equation holds : $diameter(CUT(G)) = \sum_{i=1}^r diameter(CUT(G_i))$.*

Proof The result follows from Proposition 4 and the relation $CUT(G) = \times_{i=1}^r CUT(G_i)$.

Corollary 2 *For any graph $G = (V, E)$ we have $diameter(CUT(G)) \leq |V| - \kappa(G)$.*

Proof Applying Corollary 1 and using the upper bound $|V_i| - 1$ on the diameter of $CUT(G_i)$ for each connected subgraph $G_i = (V_i, E_i)$ of G (which follows from Proposition 1), leads to the result.

Remark. Notice that for the case of trees, for example if we consider a tree with n nodes, then the cut polytope has dimension $n-1$ and $2(n-1)$ facets (each one corresponds to a trivial inequality), so that the diameter (by Proposition 6) coincides here with the number of facets minus the dimension of the polytope. So, in some sense, this class of graphs is *tight* for the Hirsch property.

The upper bounds given by Theorem 2 and Corollary 2 on the diameter of the cut polytope are both tight for trees. However, for general graphs, both may strongly differ. For, we mention hereafter a case where the bound from Theorem 2 may be quadratic in the number of vertices of the graph whereas the one from Corollary 2 is linear. Consider the case of a complete bipartite graph K_{k_1,k_2} with node bipartition (V_1, V_2) with $|V_i| = k_i$ for $i = 1, 2$. The upper bound on the diameter of $CUT(K_{k_1,k_2})$ provided by Theorem 2 is k_1k_2 whereas that from Corollary 2 is $n-1 = k_1 + k_2 - 1$.

Note that the value k_1k_2 from Theorem 2 corresponds to $C = I\Delta J$ with $I = \emptyset$ and $J = \delta(V_1)$ (or vice-versa). Assume without loss of generality $k_1 \leq k_2$. Proceeding as in the proof of Proposition 1 we can show $d(\delta(\emptyset), \delta(V_1)) \leq k_1$. Although better, we will see next that the bound from Corollary 2 is not tight in general, e.g., for the case of complete bipartite graphs by showing that the value of the diameter of the cut polytope for a family of such graphs is 2 (see Proposition 7).

Proposition 5 *Let $G^2 = (V, E')$ be a connected subgraph of $G^1 = (V, E)$ and let S_1, S_2 denote two subsets of the node set V. Then if the incidence vectors of the cuts $\delta(S_1)$ and $\delta(S_2)$ are adjacent on $CUT(G^2)$, they are also adjacent on $CUT(G^1)$.*

Proof Assuming $\delta_{G^2}(S_1)$ and $\delta_{G^2}(S_2)$ are adjacent on $CUT(G^2)$, then from Theorem 1, the graph $H^2 = H_{\delta_{G^2}(S_1)\Delta\delta_{G^2}(S_2)}$, has two connected components. Since $E' \subseteq E$, each connected component of H^2 is contained in one of $H^1 = H_{\delta_{G^1}(S_1)\Delta\delta_{G^1}(S_2)}$ (defined similarly as H^2 with respect to the graph G^1). So $\kappa(H^1) \leq 2$ and since $\delta_{G^1}(S_1) \neq \delta_{G^1}(S_2)$, necessarily $\kappa(H^1) \geq 2$. So $\kappa(H^1) = 2$ and by Theorem 1, the incidence vectors of the cuts $\delta_{G^1}(S_1)$ and $\delta_{G^1}(S_2)$ are adjacent on $CUT(G^1)$.

Corollary 3 *Let $G^2 = (V, E')$ be a connected subgraph of $G^1 = (V, E)$. Then $diameter(CUT(G^2)) \geq diameter(CUT(G^1))$.*

Proof Let v^i_S denote the vertex in the 1-skeleton of G^i corresponding to the cut $\delta_{G^i}(S)$, for $i = 1, 2$, $S \subseteq V$. Let $S_1, S_2 \subseteq V$, $S_1 \neq S_2$ and consider a shortest path (w.r.t. the number of edges) $(v_0 = v^2_{S_1}, v_1, \ldots, v_l = v^2_{S_2})$ joining $v^2_{S_1}$ and $v^2_{S_2}$ in the 1-skeleton of $CUT(G^2)$. From Proposition 5 the sequence of cuts corresponding to this path also corresponds to a path in the 1-skeleton of $CUT(G^1)$ (joining $v^1_{S_1}$ and $v^1_{S_2}$).

From Corollary 3, it follows that among the connected graphs, the trees constitute a class with largest diameter with respect to the cut polytope. Since

the cut polytope of a tree with $n+1$ nodes corresponds to the hypercube $[0,1]^n$, we deduce the following proposition.

Proposition 6 *If $G = (V, E)$ is a tree, then diameter$(CUT(G)) = |V| - 1$.*

Proposition 7 *For any two integers $k_1, k_2 \geq 2$, diameter$(CUT(K_{k_1,k_2})) = 2$.*

Proof Let (V_1, V_2) denote the node bipartition associated with the graph K_{k_1,k_2}. (So we have $|V_i| = k_i$ for $i = 1, 2$). We start defining four types of cuts in $G = K_{k_1,k_2}$:

- *Type 0*: $\delta(\emptyset) = \delta(V) = \emptyset$.
- *Type 1*: $\delta(V_1) = \delta(V_2)$.
- *Type 2*: $\delta(W) = \delta(V \setminus W)$ with W such that $V_i \setminus W \neq \emptyset$ and $W \cap V_i \neq \emptyset$ for $i = 1, 2$.
- *Type 3*: $\delta(W) = \delta(V \setminus W)$ with $W \neq \emptyset$ such that either $W \subsetneq V_1$ or $W \subsetneq V_2$.

From Theorem 1, it follows that each cut of type 0 or 1 is adjacent to all the cuts of type 2. This implies that the distance in the 1-skeleton of $CUT(K_{k_1,k_2})$ between two cuts having their types in the set $\{0, 1, 2\}$ is at most 2.

The following three claims imply that the distance between a cut of type 3 and another one of type 0, 1 or 3 is at most 2.

Claim 1. If $\emptyset \neq W^i \subsetneq V_i$ for $i = 1, 2$, then the incidence vectors of the cuts $\delta(W^i)$ for $i = 1, 2$ are adjacent on $CUT(G)$.
Proof of Claim 1. Let $\delta_i = \delta(W^i)$ for $i = 1, 2$. Then the claim follows by Theorem 1, observing that the graph $H_{\delta_1 \Delta \delta_2}$ has two connected components: they correspond to the node sets $W^1 \cup W^2$ and its complement in G. $\diamond$

Note that in the following claim the node sets W^1 or W^2 may correspond to the empty set $\emptyset$. The case $W^1, W^2 \subseteq V_2$ is symmetric.
Claim 2. Assume $W^1, W^2 \subseteq V_1$. Then the incidence vector of the cut $\delta(W^1 \cup W^2 \cup \{z\})$ with $z \in V_2$ is adjacent on $CUT(G)$ to the incidence vectors of the cuts $\delta(W^i)$ for $i = 1, 2$.
Proof of Claim 2. Let $W' := W^1 \cup W^2 \cup \{z\}$. The claim follows by Theorem 1, observing that the graphs $H_{\delta_i \Delta \delta'}$ with $\delta_i = \delta(W^i)$ and $\delta' = \delta(W')$ have two connected components: they correspond to the node sets $W^i \cup (V_1 \setminus W') \cup (V_2 \setminus \{z\})$ and its complement in G, for $i = 1, 2$. $\diamond$

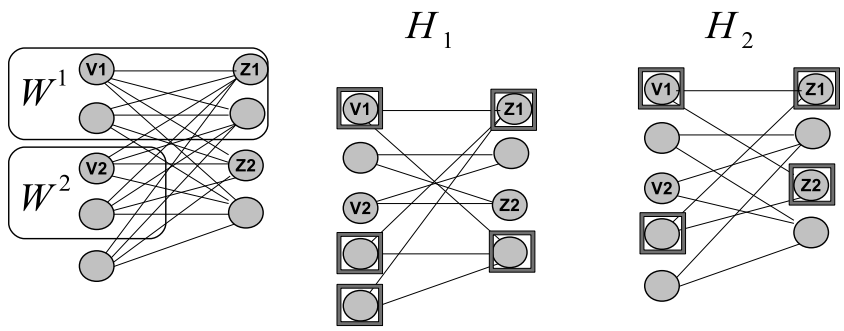
It remains to consider the distance between two cuts, one of which is of type 2 and the other of type 3. So assume $\delta(W^1)$ and $\delta(W^2)$ are two cuts of type 2 and 3 respectively, with $W^1 \subset V$ and $W^2 \subsetneq V_1$ (the case $W^2 \subsetneq V_2$ is symmetric). Consider now a set U which consists of four nodes in G: $U = \{v_1, v_2, z_1, z_2\}$ and satisfying the following requirements:

- Properties verified by the nodes v_1, v_2. We distinguish between two cases. First, if $(W^1 \cap V_1) \setminus W^2 \neq \emptyset$ and $W^2 \setminus W^1 \neq \emptyset$, then $v_1 \in (W^1 \cap V_1) \setminus W^2$ and $v_2 \in W^2 \setminus W^1$. Otherwise (i.e. $W^2 \subseteq (W^1 \cap V_1)$ or $(W^1 \cap V_1) \subseteq W^2$), then $v_1 \in W^1 \cap W^2$ and $v_2 \in V_1 \setminus (W^1 \cup W^2)$.

– Properties verified by the nodes z_1, z_2: $z^1 \in W^1 \cap V_2$ and $z_2 \in V_2 \setminus W^1$.

Claim 3. The incidence vector of a cut $\delta(U)$ in G satisfying the two properties mentioned above is adjacent to the incidence vectors of the cuts $\delta(W^1)$ and $\delta(W^2)$ on $CUT(G)$.
Proof of Claim 3. The claim follows from Proposition 1. For the two cases mentioned above concerning the properties verified by the nodes v_1 and v_2 in U, we mention explicitly the node set corresponding to one of the two connected component arising in the graphs $H_{\delta(W^i)\Delta\delta(U)}$ for $i = 1, 2$. To ease the exposition, an illustration is provided for each case by the Figures 1 and 2.

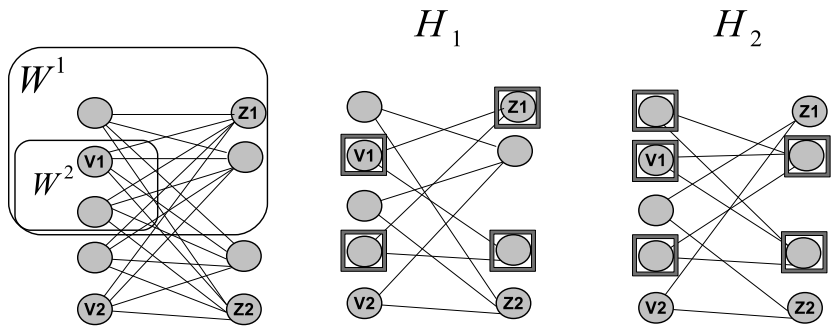
Fig. 1 Illustration for the proof of Proposition 7 on the graph $K_{5,4}$: case $(W^1 \cap V_1) \setminus W^2 \neq \emptyset$ and $W^2 \setminus (W^1 \cap V_1) \neq \emptyset$. The first graph gives the sets W^1 and W^2. The second and third graphs correspond to $H_i = H_{\delta(W^i)\Delta\delta(U)}$ for $i = 1, 2$, respectively, showing by Proposition 1 that the incidence vectors of the cuts $\delta(W^i)$ and $\delta(U)$ are adjacent on $CUT(G)$. The set of nodes that are squared in H_1 and H_2 correspond to one of the two connected components in those graphs.



– Case $(W^1 \cap V_1) \setminus W^2 \neq \emptyset$ and $W^2 \setminus W^1 \neq \emptyset$. One of the two connected components of the graph $H_{\delta(W^i)\Delta\delta(U)}$ corresponds to the node set $\{v_1\} \cup (V_1 \setminus (W^1 \cup \{v_2\})) \cup \{z_1\} \cup (V_2 \setminus (W^1 \cup \{z_2\}))$ for $i = 1$ and to the node set $\{v_1, z_1, z_2\} \cup W^2 \setminus \{v_2\}$ for $i = 2$.
– Case $W^2 \subseteq (W^1 \cap V_1)$ or $(W^1 \cap V_1) \subseteq W^2$. One of the two connected components of the graph $H_{\delta(W^i)\Delta\delta(U)}$ corresponds to the node set $\{v_1, z_1\} \cup (V_1 \setminus (W^1 \cup \{v_2\})) \cup (V_2 \setminus (W^1 \cup \{z_2\}))$ for $i = 1$ and to the node set $\{v_1\} \cup (V_1 \setminus (W^2 \cup \{v_2\})) \cup (V_2 \setminus \{z_1, z_2\})$ for $i = 2$.

◇

Fig. 2 Illustration for the proof of Proposition 7 on the graph $K_{5,4}$: case $W^2 \subseteq (W^1 \cap V_1)$ or $(W^1 \cap V_1) \subseteq W^2$.



3 On the diameter of uniform cut polytopes

In this section we start dealing with some simple cases illustrating when the diameters of the cut and uniform cut polytopes may coincide or differ. We then express necessary and sufficient conditions for the incidence vectors of two cuts to be adjacent on a uniform cut polytope.

3.1 Particular cases

3.1.1 Complete graphs

A consequence of Theorem 1 is that any two different cuts in a complete graph have incidence vectors which are adjacent on $CUT(K_n)$, i.e. the cut polytope $CUT(K_n)$ has diameter one. In particular, the incidence vectors of any two different uniform cuts are adjacent, so that the latter property is 'inherited' by uniform cut polytopes.

Corollary 4 *For all integers $n \geq 1$ and $k \leq \lfloor \frac{n}{2} \rfloor$, the uniform cut polytopes $CUT_{=k}(K_n)$ have diameter 1.*

3.1.2 Uniform cut polytopes with diameter 1

For a general graph G, both the general and uniform cut polytopes may have a diameter strictly larger than one. Consider for example the case when G consists of an elementary path with four nodes: (v_1, v_2, v_3, v_4). Then the incidence vectors of the cuts $\delta(v_2)$ and $\delta(v_4)$ are not adjacent on $CUT_{=1}(G)$. (This can be seen from the relation $\chi^{\delta(v_2)} + \chi^{\delta(v_4)} = \chi^{\delta(v_1)} + \chi^{\delta(v_3)}$). In fact, by Theorem 1 we have $diameter(G) > 1$ for any connected graph $G = (V, E)$ with at least three nodes and that is not complete. (For, one may consider for

example the incidence vectors of the cuts $\delta(v)$ and $\delta(w)$ for two nonadjacent vertices $v, w \in V$).

The following proposition gives a sufficient condition on G for the uniform cut polytope $CUT_{=1}(G)$ to have diameter 1.

Proposition 8 *If $G = (V, E)$ is a connected graph satisfying the following condition then the diameter of $CUT_{=1}(G)$ is 1. For each pair of nodes $(i, j) \in V^2$ either:*

- *$ij \in E$, or*
- *there exists an elementary path in G joining i and j with length (= number of edges) at least 3, or*
- *all the elementary paths joining the nodes i and j have length 2 and either*
  - *$\min(deg(i), deg(j)) \geq 2$ and $\max(deg(i), deg(j)) \geq 3$, or*
  - *all the nodes in $N(i) \cap N(j)$ have degree at least 3.*

Proof We prove the proposition by exhibiting an edge-weight function w for each pair of nodes $(i, j) \in V^2$ such that the incidence vectors of the cuts $\delta(i)$ and $\delta(j)$ are the only two optimal solutions of the problem: $\max_{v \in V} w(\delta(v))$.

Case 1: $ij \in E$. Set $w_{ij} := 1$ and $w_e := 0, \forall e \in E \setminus \{ij\}$.

Case 2: There exists a path from i to j denoted $(v_0 = i, v_1, \ldots, v_{l-1}, v_l = j)$ with length $l \geq 3$. Set $w_{v_0 v_1} := w_{v_{l-1} v_l} := 1$, $w_{v_1 v_2} := w_{v_{l-2} v_{l-1}} := -2$, and a weight of value 0 for all the other edges.

Case 3: All the elementary paths in G joining the nodes i and j have length (= number of edges) 2 and either $\min(deg(i), deg(j)) \geq 2$ and $\max(deg(i), deg(j)) \geq 3$, or all the nodes in $N(i) \cap N(j)$ have degree at least 3. Set $w_e := \frac{1}{deg(k)}$ if $e \in \delta(k)$ for $k \in \{i, j\}$ and $w_e := -2$ otherwise.

So as Theorem 1, Proposition 8 implies that for the case of a complete graph G the diameter of $CUT_{=1}(G)$ is 1. In addition, among others, Proposition 8 provides us with other family of graphs, namely the stars (or e.g. cycles) with at least 4 nodes, for which the diameter of the cut polytope (recall from Proposition 6 that the diameter of $CUT(K_{1,p})$ with $p \geq 4$ is p) is strictly larger than that of the uniform cut polytope $CUT_{=1}(K_{1,p}), p \geq 4$ (with value 1).

3.1.3 3-*vertex connected graphs and connections with partition polytopes*

In this subsection we establish a simple upper bound on the diameter of uniform cut polytopes of 3-vertex connected graphs and discuss some connections between the structures of uniform cut polytopes and partition polytopes for this graph family.

Recall that a graph $G = (V, E)$ is said to be k-vertex connected if, for any node subset $Q \subseteq V$ with cardinality $|Q| < k$, the node induced subgraph $G[V \setminus Q]$ is connected. The next proposition is a consequence of Proposition 12 that we prove later.

Corollary 5 *If the graph G is 3-vertex connected then the incidence vectors of the cuts $\delta(S \cup \{v\})$ and $\delta(S \cup \{w\})$ are adjacent on $CUT_{=k}(G)$, $\forall S \subseteq V$ with $|S| = k-1$, and $\forall v, w \in V \setminus S$.*

Proof Setting $C_1 := \{v\}$, $C_2 := \{w\}$ and using the 3-vertex connectivity of the graph G, the assumptions of Proposition 12 are satisfied.

It directly leads to the following upper bound on the diameter of uniform cut polytopes for 3-vertex connected graphs.

Proposition 9 *If the graph G is 3-vertex connected, then*

$$diameter(CUT_{=k}(G)) \leq k.$$

Remark. We may establish some connections between the 1-skeletons of uniform cut polytopes and bipartition polytopes. The bipartition polytope $BP(k,n)$, with $k, n \in \mathbb{N}$, $k \leq \lfloor \frac{n}{2} \rfloor$, (a special case of the partition polytope [3,7]) is formally defined as the set of vectors $y = (y^1, y^2) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the following constraints

$$\sum_{j=1}^{n} y_j^1 = k \tag{1}$$

$$\sum_{j=1}^{n} y_j^2 = n - k \tag{2}$$

$$y_j^1 + y_j^2 = 1, \forall j \in \{1, \ldots, n\} \tag{3}$$

$$y \in \mathbb{R}_+^{2n}. \tag{4}$$

Bipartition polytopes arise in problems where the aim is to divide a set of n items into two sets of prescribed sizes: k and $n-k$. So, differently from the uniform cut polytope there is no underlying graph structure that is associated with the items and it is (a $(n-1)$-dimensional polytope) expressed in $\mathbb{R}^{2n}$ (while in $\mathbb{R}^{|E|}$ for $CUT_{=k}(G)$ with $G = (V, E)$). Also as the matrix corresponding to the constraints (1)-(4) is totally unimodular, it corresponds to an explicit description of the convex hull of the incidence vectors of bipartitions having one part with cardinality k (and the other with cardinality $n-k$), whereas such an explicit description is unknown for general uniform cut polytopes.

The following proposition (see, e.g., Lemma 5 in [2] which corresponds to an extension of this result for more general partition polytopes) gives a characterization of the adjacency relation between vertices of the bipartition polytope.

Proposition 10 *The incidence vectors of the two bipartitions $(S_1, V \setminus S_1)$ and $(S_2, V \setminus S_2)$ with $S_1, S_2 \subseteq V$, $|S_1| = |S_2| = k \leq \frac{|V|}{2}$ (if $k = \frac{|V|}{2}$ then the set S_2 defining the second partition is chosen such that $|S_1 \cap S_2| \geq |S_1 \cap (V \setminus S_2)|$) are adjacent on the 1-skeleton of the bipartition polytope $BP(k,n)$ if and only if the sets S_1 and S_2 differ in exactly one item, i.e. there exists $(v_1, v_2) \in S_1 \times (V \setminus S_1)$ such that $S_2 = S_1 \Delta \{v_1, v_2\}$.*

Now let $G^P = (V^P, E^P)$ (resp. $G^C = (V^C, E^C)$) denote the graph corresponding to the 1-skeleton of the bipartition polytope $BP(k,n)$ (resp. of the uniform cut polytope $CUT_{=k}(G)$), with $n = |V|$, $G = (V, E)$ 3-vertex connected.

Consider the application $\Psi : V^P \to V^C$, which to any vertex $v_1 \in V^P$ corresponding to the bipartition $(X_1, V \setminus X_1)$ with $|X_1| = k$, associates the vertex $\Psi(v_1) \in V^C$ corresponding to the cut $\delta(X_1)$.

By Corollary 5 and Proposition 10 it follows that if $(v_1, v_2) \in E^P$ then either $\Psi(v_1) = \Psi(v_2)$ or $(\Psi(v_1), \Psi(v_2)) \in E^C$. And as each vertex in V^C is the image of at least one vertex of V^P by Ψ, this namely implies that the graph G^C may be obtained from G^P by merging nodes and adding edges. Thus the following inequality holds

$$diameter(CUT_{=k}(G)) \leq diameter(BP(k,n)) = k \tag{5}$$

where G stands for any 3-vertex connected graph with order n (see, e.g. Corollary 6 in [2] for the last equality and Theorem 5 in the same reference for further results on the diameter of more general partition polytopes). $\diamond$

3.2 Necessary and sufficient conditions for adjacency on uniform cut polytopes

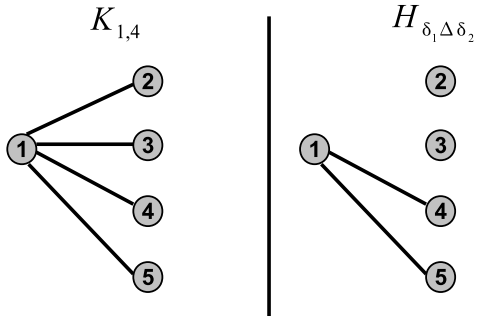
In this section we formulate necessary and sufficient conditions for the incidence vectors of two cuts to be adjacent on a uniform cut polytope $CUT_{=k}(G)$ since in general, the characterization of adjacency between the incidence vectors of cuts as is given by Theorem 1 for the general cut polytope, does not hold for uniform cut polytopes. For, we may start giving an elementary example to illustrate this by considering a star on five nodes, see Figure 3. The two cuts we consider are $\delta_1 = \delta(\{1,2\})$ and $\delta_2 = \delta(\{1,3\})$. Both graphs G and $H_{\delta_1 \Delta \delta_2}$ (as defined in Theorem 1) are depicted. Since $H_{\delta_1 \Delta \delta_2}$ contains three connected components the incidence vectors of these two cuts are not adjacent on the cut polytope $CUT(G)$. Consider now the following cost function defined on the edges of G: $w_{12} = w_{13} = 1$ and $w_{14} = w_{15} = 2$. Then we can easily check that the only cuts solving the following optimization problem: $\max_{S \subseteq V(G) : |S|=2} w(\delta(S))$ are δ_1 and δ_2, and thus they are adjacent on $CUT_{=2}(G)$.

3.2.1 A necessary condition for adjacency on uniform cut polytopes

We now formulate a necessary condition for two incidence vectors of cuts in $CUT_{=k}(G)$ to be adjacent. For, we start introducing some notation and terminology. Let $C_i \subseteq V$ and $\delta_i = \delta(C_i)$, with $|C_i| = k$, for $i = 1, 2$. Given a node set $W \subseteq V$, the cuts δ_1 and δ_2 will be said to *coincide* (resp. *be opposed*) on W if $C_1 \cap W = C_2 \cap W$ (resp. $C_1 \cap W = W \setminus C_2$).

Let I denote an index set on the connected components of the graph $H_{\delta_1 \Delta \delta_2}$. To each connected component of $H_{\delta_1 \Delta \delta_2}$ corresponding to a set of

Fig. 3 An illustration for the case of two cuts with incident vectors that are adjacent on the uniform cut polytope $CUT_{=2}(G)$ but not on $CUT(G)$



vertices $A^i \subseteq V$, $i \in I$, we can associate a bipartition (V_1^i, V_2^i) of A^i with $V_1^i = C_1 \cap A^i$. For each $i \in I$, we have the property that the cuts δ_1 and δ_2 either coincide or are opposed on A^i (this follows from the definition of $H_{\delta_1 \Delta \delta_2}$).

Let $\overline{I} \subseteq I$ denote the index set on the connected components of $H_{\delta_1 \Delta \delta_2}$ on which δ_1 and δ_2 are opposed. We can then express a necessary condition for δ_1 and δ_2 to be adjacent on a uniform cut polytope as follows.

Proposition 11 *With the sets $\delta_1, \delta_2, V_1^i, V_2^i, I, \overline{I}$ as defined above, if the incidence vectors of the cuts δ_1 and δ_2 are adjacent on $CUT_{=k}(G)$ then there does not exist any set $I' \subseteq I$ with $I \neq I' \neq \overline{I}$ such that $\sum_{i \in I'} |V_1^i| = \sum_{i \in I'} |V_2^i|$ and $\sum_{i \in I' \cap \overline{I}} |V_2^i| + \sum_{i \in I' \setminus \overline{I}} |V_1^i| = \sum_{i \in I' \cap \overline{I}} |V_1^i| + \sum_{I' \setminus \overline{I}} |V_2^i|$.*

Proof By contradiction. For, assume there exists $I' \neq \overline{I}$ such that $\sum_{i \in I'} |V_1^i| = \sum_{i \in I'} |V_2^i|$ and $\sum_{i \in I' \cap \overline{I}} |V_2^i| + \sum_{i \in I' \setminus \overline{I}} |V_1^i| = \sum_{i \in I' \cap \overline{I}} |V_1^i| + \sum_{I' \setminus \overline{I}} |V_2^i|$. Then consider the two cuts $\delta(C_i')$ with $C_i' \subseteq V$, $i = 1, 2$, defined as follows:

- $C_1' := (\cup_{i \in I \setminus I'} V_1^i) \cup (\cup_{i \in I'} V_2^i)$,
- $C_2' := (\cup_{i \in I \setminus (\overline{I} \Delta I')} V_1^i) \cup (\cup_{i \in \overline{I} \Delta I'} V_2^i)$.

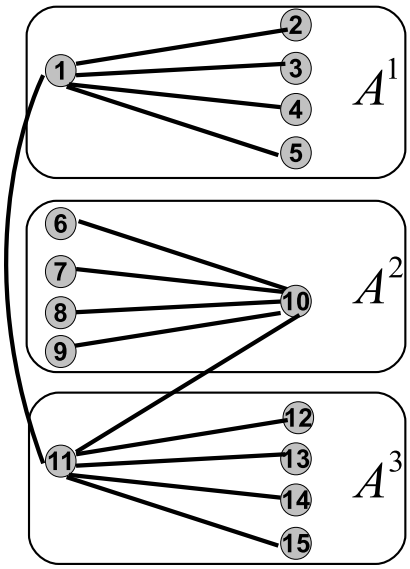
By the definition of the sets C_j, $(V_j^i)_{i \in I}$, for $j = 1, 2$ and the assumption made on the set I', we have $|C_1'| = |C_2'| = k$. Then observe that for an edge

- $e \in \delta_1 \cap \delta_2$, we have $e \in \delta(C_1') \cap \delta(C_2')$,
- $e \notin \delta_1 \cup \delta_2$, we have $e \notin \delta(C_1') \cup \delta(C_2')$,
- $e \in \delta_1 \setminus \delta_2$ we can write $e = v_i v_j \in A^i \times A^j$ with $(i, j) \in (I \setminus \overline{I}) \times \overline{I}$, $i \neq j$, and either $j \in \overline{I} \setminus I'$, $e \in \delta(C_1')$ or $j \in \overline{I} \cap I'$, $e \in \delta(C_2')$,
- $e \in \delta_2 \setminus \delta_1$, we can proceed as in the case before, showing it belongs either to $\delta(C_1')$ or $\delta(C_2')$.

It follows that $\chi^{\delta_1} + \chi^{\delta_2} = \chi^{\delta(C_1')} + \chi^{\delta(C_2')}$, implying that χ^{δ_1} and χ^{δ_2} are not adjacent on $CUT_{=k}(G)$.

Roughly speaking, the two conditions in Proposition 11 express the fact that it is not possible to permute a subset of the vertices in one shore of the cut $\delta_1 = \delta(C_1)$ (or $\delta_2 = \delta(C_2)$) with a subset of vertices in the other shore, such that the cardinality of the resulting shores is unchanged and the set of vertices permuted corresponds to the union of connected components in $H_{\delta_1 \Delta \delta_2}$. We provide a simple example in Figure 4 illustrating the fact that the condition expressed in Proposition 11 (and in particular the second equation) is necessary for the incidence vectors of the cuts δ_1 and δ_2 to be adjacent on $CUT_{=k}(G)$. In this example we consider cuts having one shore with cardinality $k = 6$. Let $C_1 = \{1, 6, 7, 8, 9, 11\}$, and $C_2 = \{2, 3, 4, 5, 10, 11\}$ define the cuts $\delta_i = \delta(C_i)$ for $i = 1, 2$. The figure represents the graph G and indicates the sets of vertices $(A^i)_{i=1}^3$ corresponding to the connected components of $H_{\delta_1 \Delta \delta_2}$. We have $\overline{I} = \{1, 2\}$ and $\delta_1 \Delta \delta_2 = \{(1, 11), (10, 11)\}$. Considering the following edge cost function: $c_e = 0$ if $e \in \delta_1 \Delta \delta_2$ and $c_e = 1$ otherwise, we can check that there exist 3 cuts of maximum value in $CUT_{=6}(G)$ w.r.t. this cost function: δ_1, δ_2 and $\delta_3 = \delta(\{1, 10, 12, 13, 14, 15\})$. So the incidence vectors of these cuts define a face of dimension 2 of this polyhedron (since they are affinely independent) and they are all adjacent on $CUT_{=6}(G)$. Considering the index set $I' = \{2, 3\}$, this illustrates the need for the satisfaction of the two conditions formulated in Proposition 11 (in this example I' violates the second equation).

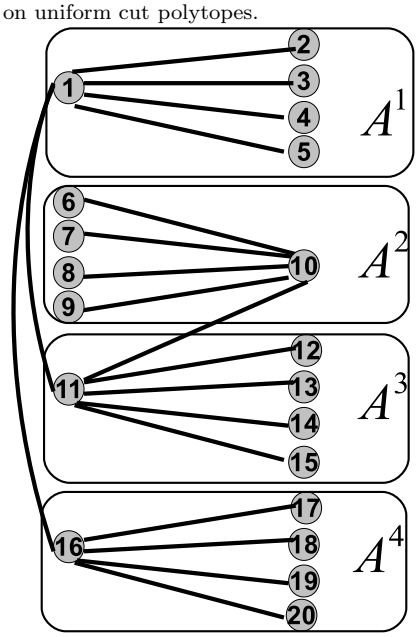
Fig. 4 Illustration for Proposition 11.



We terminate our discussion on Proposition 11 by giving an example showing that the given necessary condition is generally not sufficient for two cuts to

have their incidence vectors adjacent on a uniform cut polytope. The graph G is displayed in Figure 5, and we consider the uniform cut polytope $CUT_{=7}(G)$. The figure also represents the node sets $(A^i)_{i=1}^4$ which correspond to the connected components of the graph $H_{\delta_1 \Delta \delta_2}$ with $\delta_1 = \delta(\{1,6,7,8,9,11,16\})$ and $\delta_2 = \delta(\{2,3,4,5,10,11,16\})$. So in this example we have $\overline{I} = \{1,2\}$ and the only two possibilities for a set $I' \neq \overline{I}$, $I' \subseteq I$ such that $\sum_{i \in I'} |V_1^i| = \sum_{i \in I'} |V_2^i|$ are $\{2,3\}$ and $\{2,4\}$ and none verifies $\sum_{i \in I' \cap \overline{I}} |V_2^i| + \sum_{i \in I' \setminus \overline{I}} |V_1^i| = \sum_{i \in I' \cap \overline{I}} |V_1^i| + \sum_{I' \setminus \overline{I}} |V_2^i|$. So the necessary condition expressed by Proposition 11 is satisfied. But considering now the cuts $\delta_3 = \delta(\{1,10,12,13,14,15,16\})$ and $\delta_4 = \delta(\{1,10,11,17,18,19,20\})$, we have $\chi^{\delta_1} + \chi^{\delta_2} = \chi^{\delta_3} + \chi^{\delta_4}$, implying that χ^{δ_1} and χ^{δ_2} are not adjacent on $CUT_{=7}(G)$.

Fig. 5 Illustration for the fact that Proposition 11 does not provide a characterization for the adjacency relation on uniform cut polytopes.



3.2.2 Sufficient conditions for adjacency on uniform cut polytopes

The following proposition, which straightforwardly follows from Theorem 1, formulates a simple sufficient condition for the incidence vectors of two cuts to be adjacent on a uniform cut polytope. It corresponds to the adjacency relations on the uniform cut polytopes which are "inherited" from those on the cut polytope $CUT(G)$.

Corollary 6 *Let $G = (V, E)$ be a connected graph and let (C_1, C_2, S, W) denote a partition of V such that $|C_1| = |C_2| \neq 0$, $|C_1| + |S| = k$, $G[C_1 \cup C_2]$ and $G[S \cup W]$ are connected. Then the incidence vectors of the cuts $\delta(S \cup C_1)$ and $\delta(S \cup C_2)$ are adjacent on $CUT_{=k}(G)$.*

We formulate hereafter further sufficient conditions for the incidence vectors of two cuts to be adjacent on a uniform cut polytope.

The next one may be applied to the example illustrated in Figure 3 to prove adjacency of the two cuts mentioned there (whereas Corollary 6 cannot be applied for that case). It can be proved in a similar manner. We give the proof for completeness.

Proposition 12 *Let $G = (V, E)$ be a connected graph and let (C_1, C_2, S, W) denote a partition of V such that $|C_1| = |C_2| \neq 0$, $|C_1| + |S| = k$, $G[C_1]$, $G[C_2]$ and $G[S \cup W]$ are connected. Then the incidence vectors of the cuts $\delta(S \cup C_1)$ and $\delta(S \cup C_2)$ are adjacent on $CUT_{=k}(G)$.*

Proof The case when $G[C_1 \cup C_2]$ is connected follows from Corollary 6, and so we may assume that $G[C_1 \cup C_2]$ is not connected whereas $G[C_1]$ and $G[C_2]$ are.

Let T denote a tree on the graph G and consider the following edge cost function

$$c_e = \begin{cases} 1 \text{ if } e \in T \cap \delta(S \cup C_1) \cap \delta(S \cup C_2) \\ -1 \text{ if } e \in E \setminus (\delta(S \cup C_1) \cup \delta(S \cup C_2)) \\ 0 \text{ otherwise.} \end{cases}$$

Then the only integer solutions for the optimization problem $\max\{c^t x : x \in CUT_{=k}(G)\}$ are the incidence vectors of the cuts $\delta(S \cup C_1)$, $\delta(S \cup C_2)$ and possibly $\delta(W)$ (the latter case may occur only if $|W| = k$), with objective value $|T \cap \delta(S \cup C_1) \cap \delta(S \cup C_2)|$. Thus the incidence vectors of the two cuts $\delta(S \cup C_1)$ and $\delta(S \cup C_2)$ are adjacent on $CUT_{=k}(G)$.

A similar reasoning (based on connectivity properties of the graph $H_{\delta_1 \Delta \delta_2}$) leads to the next two propositions giving other sufficient conditions for adjacency on the 1-skeleton of uniform cut polytopes. Their proofs is similar to that of Proposition 12 and hence omitted.

Proposition 13 *Let $G = (V, E)$ be a connected graph and let (C_1, C_2, W) denote a partition of V such that $|C_1| = |C_2| = k$, $G[C_1]$ is connected and all connected components of $G[W]$ have cardinality at least $k + 1$ (or possibly, there is a single connected component with cardinality k and all the others with a strictly larger cardinality). Then the incidence vectors of the cuts $\delta(C_1)$ and $\delta(C_2)$ are adjacent on $CUT_{=k}(G)$.*

Proposition 14 *Let $G = (V, E)$ be a connected graph. If the graph $H_{\delta_1 \Delta \delta_2}$ with $\delta_i = \delta(C_i)$, $C_i \subseteq V$ and $|C_i| = k$ for $i = 1, 2$ has at most one connected component with cardinality lower than or equal to $2k$, then the incidence vectors of the cuts δ_1 and δ_2 are adjacent on $CUT_{=k}(G)$.*

We end this section with another formulation of a sufficient condition for the adjacency on uniform cut polytopes. As Proposition 12 it may be applied to the example that is illustrated by Figure 3. We first introduce some notation. Given two incidence vectors of cuts in $CUT_{=k}(G)$: $\delta_j = \delta(C_j)$, with $|C_j| = k$ ($k \leq \lfloor \frac{n}{2} \rfloor$) for $j = 1, 2$, we denote by $(A^i)_{i=1}^m$ the sets of vertices corresponding to the connected components of the graph $H_{\delta_1 \Delta \delta_2}$ and we set $i^* := Argmax_{i=1,\ldots,m}(\min_{j=1,2} |A^i \setminus C_j|)$.

Proposition 15 *With the notation above, if the graph G is connected and the cuts δ_1 and δ_2, $\delta_1 \neq \delta_2$ are such that*

i) $H_{\delta_1 \Delta \delta_2}$ *has three connected components,*
ii) $\min_{j=1,2} |A^{i^*} \setminus C_j| \geq k$
iii) $\exists i \in \{1, 2, 3\} \setminus \{i^*\}$ *such that* $|A^i \cap C_1| \neq |A^i \cap C_2|$

then the incidence vectors of the cuts δ_1 and δ_2 are adjacent on the polytope $CUT_{=k}(G)$.

Proof For simplicity in the presentation and without loss of generality assume that the node sets $(A^i)_{i=1}^3$ corresponding to the three connected components of $H_{\delta_1 \Delta \delta_2}$ are such that $i^* = 3$ and $|A^1 \cap C_1| \neq |A^1 \cap C_2|$. (For the example mentioned in Figure 3, one may take $A^1 = \{2\}, A^2 = \{3\}, A^3 = \{1, 4, 5\}$).
Claim 1. We have: $C_k \cap (A^1 \cup A^2) \neq \emptyset, \forall k \in \{1, 2\}$.
Proof of Claim 1. We do the proof for $k = 1$ (the case $k = 2$ is symmetric). The proof we give is by contradiction. Assume that $C_1 \cap (A^1 \cup A^2) = \emptyset$. Then $C_1 \subseteq A^3$. Also note that for each connected component A^i in $H_{\delta_1 \Delta \delta_2}$, the nodes in A^i can be partitioned into two node sets (V_1^i, V_2^i) such that each set V_k^i (for $k = 1, 2$) is contained in one of the shores of the cuts δ_1 and δ_2. With this observation it follows that necessarily $C_2 = A^3 \setminus C_1$ (a consequence of property (ii), the definition of the graph $H_{\delta_1 \Delta \delta_2}$ and the assumption that $\delta_1 \neq \delta_2 \in CUT_{=k}(G)$). But then we get a contradiction with $iii)$. $\diamond$

Note that a consequence of the claim to be used later is that $C_1 \cap A^3 = C_2 \cap A^3$.

Now, let T denote a tree on the graph G and consider the following edge cost function:

$$c_e = \begin{cases} 1 \text{ if } e \in T \cap \delta_1 \cap \delta_2 \\ -1 \text{ if } e \in E \setminus (\delta_1 \cup \delta_2) \\ 0 \text{ otherwise.} \end{cases}$$

Then the incidence vectors of the cuts δ_1 and δ_2 are optimal solutions for the problem $\max\{c^t x : x \in CUT_{=k}(G)\}$ with objective value $|T \cap \delta_1 \cap \delta_2|$. And notice that any other different optimal solution corresponding to the incidence vector of a cut $\delta(C)$, $|C| = k$ must verify either $C \cap A^3 = C_1 \cap A^3$ or $C \cap A^3 = A^3 \setminus C_1$. The latter case can occur only if $|A^3 \setminus C_1| = k$ (using assumption (ii)) and in that case there exists a unique optimal cut $\delta(C)$, $|C| = k$ in $CUT_{=k}(G)$ that is optimal w.r.t. the edge cost function c and such that $C \cap A^3 \neq C_1 \cap A^3$: it corresponds to $C = A^3 \setminus C_1$.

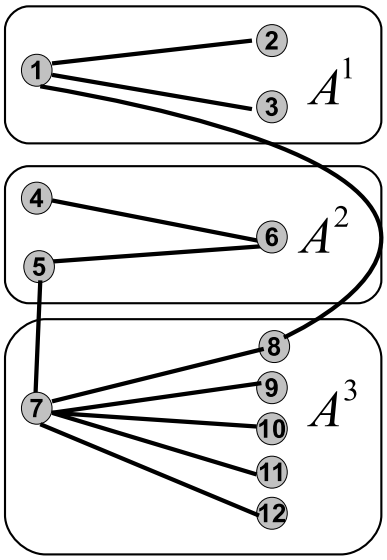
Consider now a cut $\delta(C)$, $|C| = k$ that is optimal w.r.t. the edge cost function c and such that $C \cap A^3 = C_1 \cap A^3$.

Then, from the observation reported above and *iii*) it cannot differ from both δ_1 and δ_2.

It follows that under the conditions formulated in the proposition there exist at most three extreme points of $CUT_{=k}(G)$ which correspond to optimal solutions w.r.t. the edge cost function c, among which the incidence vectors of the cuts δ_1 and δ_2. It then follows that the latter are adjacent on $CUT_{=k}(G)$.

We give in Figure 6 an example where Proposition 15 can be used to show that two cuts are adjacent on a uniform cut polytope. The figure displays the graph G and the connected components $(A^i)_{i=1}^3$ of the graph $H_{\delta_1 \Delta \delta_2}$ for the cuts $\delta_1 = \delta(\{1,4,5,7\})$ and $\delta_2 = \delta(\{2,3,6,7\})$ of $CUT_{=4}(G)$. Notice that Corollary 6 and Propositions 12, 13 and 14 do not apply here.

Fig. 6 Illustration for an example to which Proposition 15 may be applied to prove the adjacency of the incidence vectors of two cuts on a uniform cut polytope.



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