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On stationary solutions of two-dimensional Euler Equation

Nikolai Nadirashvili*

Abstract. We study the geometry of streamlines and stability properties for steady state solutions of the Euler equations for ideal fluid.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Let $v(x, t) = (v_1, v_2)$, $x \in \Omega$, $t \in \mathbb{R}$ be a solution of the Euler equation for an ideal fluid:

$$\begin{cases} \partial v / \partial t + v \nabla v = -\nabla p, & \text{in } \Omega \times \mathbb{R} \\ \operatorname{div} v = 0 & \text{in } \Omega \times \mathbb{R} \end{cases} \quad (1)$$

Together with the boundary condition,

$$(n, v) = 0 \quad \text{on } \partial\Omega, \quad (2)$$

the equation (1), (2) defines in spaces $C^{k,a}$, $k = 1, 2, \dots, 0 < a < 1$ an evolution operator, $e^t : v(x, 0) \rightarrow v(x, t)$, i.e., for any initial data $v_0 \in C^{k,a}(\Omega)$,

$$v(x, 0) = v_0(x)$$

there exists a unique solutions $v(t, x)$ of (1) defined for all $t \in \mathbb{R}$ such that for all $t \in \mathbb{R}$ $v \in C^{k,a}(\Omega)$, see [L]. For analytical v_0 the solution v remains analytic for all times t , [BBZ], [AM].

Vector field $v(x, t)$ defines a flow $g(x, t)$ on Ω ,

$$g : \Omega \rightarrow \Omega,$$

g is one-parametric group of area preserving diffeomorphisms of Ω .

Let $x_0 \in \Omega$. The curve $\gamma(t) \in \Omega$,

$$\gamma : t \in \mathbb{R} \rightarrow g(x_0, t) \in \Omega$$

called the streamline of a material particle x_0 of the fluid.

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Let $\omega = \text{curl } v$ be the vorticity of v . Then the equation (1) is equivalent to Euler-Helmholtz equation for the vortex, [AK],

$$\omega_t + \omega_v = 0.$$

The stationary (or steady) solutions are independent of t . Therefore the stationary Euler equation is

$$\begin{cases} v \nabla v + \nabla p = 0, & \text{in } \Omega \times \mathbb{R} \\ \text{div } v = 0 & \text{in } \Omega \times \mathbb{R} \end{cases} \quad (3)$$

In this paper we are concerned with the structure of steady solutions and the behavior of the flow e^t in a neighborhood of it. Let v be a solution of (3). Streamlines of v are the trajectories of the corresponding fluid motion, i.e., the integral curves of the vector field v .

Theorem 1.1. *Let $v \in C^1(\Omega)$ be a steady solution of the Euler equation, $c > |v| > c^{-1} > 0$. Then the streamlines of v are smooth (C^∞) curves. Moreover for any streamline γ , C^k -norms of γ at $x \in \Omega$ depend on the L_1 -norm of vorticity ω , on the constant c and the distance of x to the boundary of Ω .*

If additionally $v \in C^{3,a}$, $a > 0$, then the streamlines are real-analytic.

Theorem 1.1 gives a bound to the acceleration of individual material particles of the flow. In a sense, it explains a visible boundness of curvature of streamlines, which one can observe in a lot of experimental pictures of the flow.

Of course, in general v is not a smooth vector field on Ω . The phenomenon of a higher regularity of streamlines than the regularity of the solution itself has attracted a lot of attention. The first underlying ideas to it were suggested by Lichtenstein [Li]. Another approach to the problem is connected with the observation of Arnold, [A1]: flows generated by solutions of the Euler equation (1), (2) can be regarded as geodesics on the group of area preserving diffeomorphisms of Ω . More general, let M be a smooth n -dimensional Riemannian manifold with a smooth boundary ∂M . Denote by $S \text{Diff}(M)$ the Lie group of volume preserving diffeomorphisms of M . The tangent space TM are divergent free vector fields on M tangent to ∂M . The scalar product on TM defines a weak right-invariant metric g on $S \text{Diff}(M)$, [A1].

The geometry of volume preserving diffeomorphisms of a finite smoothness was studied by Ebin and Marsden, [EM]. Denote by $D^{1,a}$ the group of volume preserving diffeomorphisms of M which are in $C^{1,a}$, $a > 0$. Notice, the metric g is not complete on $D^{1,a}$. By the observation of Ebin and Marsden, [EM], metric g defines a *smooth* connection on $D^{1,a}$, and the geodesic exponential map on $(D^{1,a}, g)$, where it is defined, is smooth, [EM], Theorem 9.1. This does not imply that individual streamlines of the Euler equation are smoothly immersed curves into M , but rather the smoothness of the flow in an average sense. We notice that by the result of Milnor there are no analytical structures on $D^{1,a}$, [M], Lemma 9.1, and hence one can not directly generalize the results of [EM] into an analytic setting.

Smoothness of the individual streamlines of the Euler equation in \mathbb{R}^n was proved by Chemin, [C], for the initial data $v_0 \in C^{1,a}$, $a > 0$, so that C^k -norms of

the streamlines depend on $C^{1,a}$ -norm of v_0 . In Chemin's result the smoothness of the initial data v_0 can not be taken lower than $C^{1,a}$, $a > 0$, hence the result provides no bounds for the acceleration of flow's particles, or for the curvature of streamlines.

Theorem 1.1 has a local character, we do not assume any boundary condition on $\partial\Omega$. That also distinguish Theorem 1.1 from the previous results. The proof of the theorem is based on a detail analysis of the elliptic equation for the streamfunction of the flow.

As a corollary of Theorem 1.1 we show in Section 2 that the continuity (boundness) of the vorticity implies the continuity (correspondingly, boundness) of the first derivatives of the flow v . By Yudovich's theorem, [Y], the dynamics e^t of (1), (2) is well defined on the space of divergence free vector fields v with $\omega \in L_\infty$. Thus the last remark means that the Yudovich's space for the steady flows coincide with the space of Lipschitz, divergent free vector fields.

From Theorem 1.1 it follows that any individual streamline of a steady flow is defined uniquely by its any small segment. One can see easily that there is no unique continuation property for the continuation of v from subdomain of Ω on the whole domain.

Consider the steady state Euler-Helmholtz equation,

$$\omega_v = 0. \tag{4}$$

If we write equation (4) in the form of first order system we see that the characteristics of (4) coincide with the streamlines of the flow v . We show that the uniqueness of non-characteristically Cauchy problem for (4) requires very low smoothness of the solutions.

Theorem 1.2. *Let $v_1, v_2 \in C^1(\bar{\Omega})$ be steady solutions of the Euler equation. Assume that $\gamma \in C^1$ be an arc on $\partial\Omega$, and v_i flow inside Ω over γ , i.e., $(v_i, n) > 0$, $i = 1, 2$, where n be the inner normal to $\partial\Omega$. Assume $v_1 = v_2$, $\nabla v_1 = \nabla v_2$ on γ . Then streamlines of the flows v_1 and v_2 starting from same points of γ coincide. Moreover, the flows coincide on the union of these streamlines.*

As a complement to Theorem 1.1 in the following theorem we study the structure of the steady flow in a neighborhood of stagnation points (critical set) of the flow.

Theorem 1.3. *Let $v \in C^{1,a}$ be a steady flow defined in Ω . Assume that $0 \in \Omega$ is an isolated critical point of v , $v(0) = 0$. Then in a suitable orthonormal coordinates x_1, x_2 in a neighborhood of 0, v has one of the following expansions*

- (i) $v = (ax_2, bx_1) + o(|x|)$, $a, b \neq 0$,
- (ii) $v = (\Im(az^n), \Re(az^n)) + o(|z|^n)$, $z = x_1 + ix_2 \in \mathbb{C}$, $n \in \mathbb{N}$, $n \geq 2$, $a \neq 0$.
- (iii) $v = (ax_2 + o(x_2), \Re(\alpha z^n) + o(|x|^n))$, $\alpha, z = x_1 + ix_2 \in \mathbb{C}$, $n \geq 2$, $a, \alpha \neq 0$.

Let $G \subset \subset \Omega$, be a domain such that $v = 0$ on G and in $\Omega \setminus \bar{G}$ there are no stagnation points of v . Then there is a neighborhood of G consisting of closed streamlines encircling G .

One can easily see that in general the stagnation set of a steady flow is not necessarily discrete. For instance, for rotationally symmetric steady flows the

stagnation set can be a disk. As one can immediately see for the rotationally symmetric flow the domain of stagnation is always encircled by closed streamlines. The existence and the structure of Cantor type stagnation sets remains unclear.

Let v be a steady state solution of (2). Denote by u the stream function of the flow $v: v = (\partial u/\partial y, -\partial u/\partial x)$. Then we have

$$\Delta u = \omega,$$

and hence the gradients ∇u and $\nabla \Delta u$ are parallel. A steady solution $v \in C^{1,a}$ called stable in the sense of Arnold, if the stream function u satisfies the inequalities:

$$c < \nabla u / \nabla \Delta u < C, \tag{5}$$

where c, C are positive constants.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let v be a stable in the sense of Arnold steady state solution of (2), (3). Then v has a single critical point.*

We guess that the streamlines in Theorem 1.4 are convex curves, however, for general stationary solutions of (2), (3) in a convex domain there is no convex property for streamlines, [HNY].

By Arnold's theorem, [A2], [AK], if $v \in C^{1,a}$ is a steady flow stable in the sense of Arnold then v is a (Lyapunov) stable solution of (1) with respect to the norm $W^{1,2}(\Omega)$, i.e., for any $\delta > 0$ there is an $\varepsilon > 0$ such that any solution of (1) in $C^{1,a}$ which is at $t = 0$ in the ε -neighborhood of v in the space $W^{1,2}(\Omega)$ will never leave the δ -neighborhood of v . As a consequence v is also stable with respect to the norm $W^{1,p}(\Omega)$ for any $2 \leq p < \infty$. Unfortunately in the spaces $W^{1,p}(\Omega)$ dynamics of the Euler equation is unknown. It is interesting to understand the stability properties of steady solutions of the Euler equation in different functional spaces, especially in the spaces $C^{k,a}$ where dynamics of (1), (2) is defined. The following theorem shows that Arnold's stable solutions are extremely unstable in $C^{2,a}$.

Theorem 1.5. *Let $\tilde{v} \in C^{2,a}(A)$ be a steady radially symmetric solution of the Euler equation defined in the annulus $A = \{1 < r < 2\}$, $r = |x|$, such that the vorticity $\tilde{\omega}$ satisfies: $\tilde{\omega} > 0$, $\tilde{\omega}_r > 0$ (and therefore \tilde{v} stable in the sense of Arnold). Then there exists a neighborhood $G \subset C^{2,a}(A)$ of \tilde{v} such that for any $v_0 \in G$ the trajectory $v(\cdot, t)$ in $C^{2,a}(A)$ defined by (1) is either a stationary solution of the Euler equation, or there is $t_0 > 0$ such that $v(\cdot, t_0)$ is not in G .*

In particular from Theorem 1.5 it follows that in a neighborhood of \tilde{v} there are no periodic or quasi-periodic solutions of the Euler equation.

In [N] we proved that in any $C^{1,a}$ -neighborhood of \tilde{v} there is v_0 such that v is a wandering trajectory in $C^{1,a}(A)$. Shnirelman proved, [S], that a typical trajectory of the Euler equation is wandering. Koch shown, [K], that stable in $C^{1,a}$ steady state solution of the Euler equation generates a periodic flow. These results make natural the following conjecture:

Conjecture. *There are no stable in $C^{1,\alpha}$ stationary solutions of the Euler equation.*

Let v be a steady flow (2), (3) in Ω . Then v is an extremal of the kinetic energy of v ,

$$E(v) = \frac{1}{2} \int_{\Omega} v^2 dx,$$

under certain constraints.

Denote by $SDiff(\Omega)$ the group of area preserving diffeomorphisms of the domain Ω . We say that two divergence free vector fields v_1 and v_2 are isovorticed if their vorticity functions are the same up to $SDiff(\Omega)$ changes of variable x , i.e., if we denote by $S(\omega)$ the orbit of ω under the action of $SDiff(\Omega)$, we say v_1 and v_2 are isovorticed if $\omega_1 \in S(\omega_2)$.

Kelvin noticed that the steady flow v is an extremal of the kinetic energy $E(v)$ over the set of isovorticed with v divergence free vector fields.

Thus it is natural to consider the following variational problem (K): for a given function h on Ω , $h = \text{const}$ on $\partial\Omega$, find minimizers of the kinetic energy over divergence free vector fields v with vorticities in $S(h)$.

There are certain obstructions for the existence of a smooth minimizer v even if h is smooth, see Section 5. Thus, one can try to look for minimizers in the strong closure $\bar{S}(h)$, which coincide with the set of rearrangements of the function h , see, e.g., [AP]. Burton proved, [B2], that for any positive $h \in L_p(\Omega)$, $1 < p < \infty$, there exists a minimizer (and also a maximizer) of the variational problem (K), $\omega \in \bar{S}(h)$, and v , $curl v = \omega$, is a steady state solution of (2), (3). Thus $v \in W^{1,p}$. Questions remain for smooth h . First, for a general smooth h one can expect a better than $v \in W^{1,p}$ smoothness solutions of (K). Secondary, under some natural assumptions on h can one expect the existences of a smooth minimizer, i.e., a minimizer in $S(h)$. Regarding the second question we prove the following theorem.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain Let h be a smooth function on Ω and $h = \text{const}$ on $\partial\Omega$. Assume that $h < 0$ and has a single critical point in Ω . Then the global minimizer of the variational problem (K) is a smooth in $\Omega \setminus \{s\}$ steady flow v , where $\{s\}$ is a single critical point of v in Ω . Moreover v is an Arnold stable steady state flow in Ω .*

Remark. Notice that if h has a degenerate critical point then v might be not smooth (C^∞) in Ω .

It is easy to show that without assumptions on convexity of Ω or on single critical point of h the minimizer v might be not smooth. However, we guess that even without these assumptions the minimizer v will be in $C^{1,\alpha}$.

We show that Arnold steady state solution has vorticity of a constant sign. As a consequence we prove that in a convex domain the minimizers of variational problem (K) are exactly Arnold's stable steady state solutions.

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2 Level sets of semilinear elliptic equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in W^{2,p}$, $p > n$, be a solution of the equation

$$\Delta u = f(u) \text{ in } \Omega \quad (6)$$

where $f \in L_p$.

In this section we study the regularity of level sets of the solution u .

Theorem 2.1. *Let $x_0 \in \Omega$. Assume that $|u| < C$, $|\nabla u| > c > 0$ in Ω . Then the level set $\Gamma = \{x : u(x) = u(x_0)\}$, $x_0 \in \Omega$, is a smooth, C^∞ , surface and the normal derivative, $\partial u / \partial \nu$, is a smooth function on Γ . The C^k -norms of Γ and of $\partial u / \partial \nu$ on Γ at x_0 are bounded by C -norm of u in Ω , constant c , L_1 -norm of f and the distant of x_0 to the boundary $\partial\Omega$.*

If $\partial\Omega \in C^{k,a}$, $x_0 \in \partial\Omega$ and in a neighborhood of x_0 $u = 0$ on $\partial\Omega$ then the level sets of u in the neighborhood of x_0 have uniformly bounded $C^{k,a}$ -norm.

Theorem 2.1 shows a higher smoothness of level sets of (6) than it follows from the Schauder estimates. It is interesting to compare the theorem with the result of [HON] where we proved an additional regularity for nodal sets of solutions of a linear Schrödinger equation.

Corollary 2.2. *In the assumption of Theorem 2.1 let $f \in C(\mathbb{R})$ ($f \in L_\infty$) Then $u \in C^2$ (correspondingly, $D^2u \in L_\infty$) in a neighborhood of x_0 .*

Theorem 2.3. *Assume that $n = 2$ and $f \in C^{2,a}$ -function. Let $x_0 \in \Omega$ and let $\nabla u(x_0) \neq 0$ in Ω . Then the level set $\Gamma = \{x : u(x) = u(x_0)\}$ is a real-analytic curve and the normal derivative $\partial u / \partial \nu$ is a real analytic function on Γ .*

The similar result with a similar proof holds in dimension n .

Let u_1, u_2 be two solutions of (6) defined in Ω . We prove unique continuation results for the difference $u_1 - u_2$. For sufficiently regular function f the results are well known.

Theorem 2.4. *Assume that $n = 2$, $\partial\Omega \in C^1$ and $f \in C(\bar{\Omega})$, γ be an arc of $\partial\Omega$. Let u_1, u_2 be two solution of (6) and $u = u_1 - u_2$. Assume that $u = \nabla u = 0$ on γ and ∇u_1 is not vanishing on γ . Then $u_1 \equiv u_2$ in Ω .*

Let $H(x)$ be the fundamental solution of Laplace's equation in \mathbb{R}^n .

Proposition 2.5. *Let $\Gamma \subset \mathbb{R}^n$ be a bounded surface $\Gamma \in C^{k,a}$, let $\mu \in C^{k-1,a}(\Gamma)$. Let $l = l_\Gamma$ be the single layer potential of Γ with the density μ :*

$$l[\Gamma, \mu](x) = \int_{y \in \Gamma} \mu(y) H(x - y) ds.$$

Then the $C^{k+1,a}$ norms of l are finite in $\mathbb{R}^n \setminus \Gamma$ and bounded by the norms of Γ and μ in $C^{k,a}$ and $C^{k-1,a}$ correspondingly.

Proposition 2.5 is common knowledge. In such generality one can find a proof of the proposition in [W]. For $C^{1,a}$ -surfaces see, e.g., [G], [Mi]. Compare also with general results for the heat kernel in [Ka].

The single layer potential has a jump of normal derivative over the surface Γ . Thus on the whole space the single layer potential has only regularity

$$l \in C^{0,1}(\mathbb{R}^n).$$

Proof of Theorem 2.1. Let $x_0 \in \Omega$ and $\nabla u(x_0) \neq 0$. Let $u(x_0) = t_0$. Denote by G_t the level surface

$$G_t = \{x \in \Omega : u(x) = t\}.$$

We will prove that in the neighborhood of the point (x_0, t_0) the surfaces G_t are smooth. From the equation (6) it follows that $u \in C^{1,a}$ and hence outside the critical points of the function u the level surfaces of u are in $C^{1,a}$.

We will denote by \cong equality between functions up to a smooth function. Thus if $G(x, y)$ be the Green's function of the Dirichlet problem in Ω then

$$G(x, y) \cong P(x - y)$$

for x in a neighborhood of the point x_0 . The consideration in the proof below are local, in the neighborhood of the point x .

We will prove by induction over $k = 1, 2, \dots$ that

$$G_t \in C^{k,a} \tag{7}$$

and the normal derivatives

$$\partial u / \partial \nu \in C^{k-1,a}(G_t). \tag{8}$$

We know (7) and (8) hold for $k = 1$. Assume (7) and (8) hold for $k \in \mathbb{N}$. We prove the implications for $k + 1$.

Let t_1 be sufficiently close to t_0 . Without loss we will assume that $u > 0$ in Ω . Then we have

$$u \cong \int_0^\infty f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt. \tag{9}$$

We break the last integral into the sum of three integrals:

$$u \cong \int_0^{t_1-\varepsilon} f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt + \int_{t_1-\varepsilon}^{t_1+\varepsilon} f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt + \int_{t_1+\varepsilon}^\infty f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt.$$

Denote

$$u_1 = \int_0^{t_1-\varepsilon} f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt + \int_{t_1+\varepsilon}^\infty f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt,$$

$$u_2 = \int_{t_1-\varepsilon}^{t_1+\varepsilon} f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt.$$

Define vector functions $h_\varepsilon, q_\varepsilon$ on G_{t_1} taking the restrictions:

$$h_\varepsilon = \nabla u_1 \quad \text{on } G_{t_1},$$

$$q_\varepsilon = \nabla u_2 \quad \text{on } G_{t_1}.$$

The norm $\|h_\varepsilon\|_{C^{k,a}}$ is uniformly bounded for all small $\varepsilon > 0$ by Proposition 2.6 and by (7), (8). On the other hand we have the inequality

$$\|q_\varepsilon\| < C\varepsilon,$$

with the constant C independent of ε . Thus we can pass to the limit as ε goes to 0 and get

$$\nabla u|_{G_{t_1}} \in C^{k,a}(G_{t_1}).$$

Let $x \in G_t$. Denote by T the tangent plane to G_t at x . Let Δ_T the Laplace operator on the plane T . Since $u \in C^{1,a}(\Omega)$ we can compute $\Delta_T u(x)$ taking the derivative of ∇u along G_t and we get for $x \in G_t$,

$$\Delta_T u(x) \in C^{k-1,a}(G_t).$$

Denote by $M(x)$ the mean curvature of the surface G_t at the point x . Let T be the tangent to G_t plane at x , Then

$$\Delta_T u(x) = M(x) \frac{\partial u}{\partial \nu}(x).$$

Therefore

$$M(x) \in C^{k-1,a}(G_t).$$

Thus if we apply the Schauder estimates for the mean curvature type equation, see [GT], we get

$$G_t \in C^{k,a}.$$

The induction step is proved and hence the theorem proved for $x_0 \in \Omega$.

Now assume that $x_0 \in \partial\Omega$ and the restriction of u on $\partial\Omega$ is in $C^{k,a}$. In this case we need to introduce in (9) a correction term. Denote by g the restriction on $\partial\Omega$ of the following function:

$$\int_0^\infty f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt.$$

Let w be a solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0, & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases} \quad (10)$$

Then

$$u = w + \int_0^\infty f(t) l[G_t(\partial u / \partial \nu)^{-1}] dt.$$

Since by Schauder estimates $w \in C^{k,a}(\bar{\Omega})$, we can do the estimates of u as above and we get the desirable estimates near the boundary point. The theorem is proved.

Proof of Corollary 2.2. Let $0 \in \Omega$. Choose an orthonormal coordinate system x_1, \dots, x_n such that x_1 has a normal direction to the level surface of u at 0. By Theorem 2.1 all second derivatives of u at 0 except $\partial^2 u / \partial x_1^2$ are bounded and continuously dependent on point 0 on the level set. Since the derivative $\partial^2 u / \partial x_1^2$ is uniquely determined by the equation (6) and the rest second derivatives are in $C^2(L_\infty)$, the theorem follows.

Before going to the proof of Theorem 2.2 we prove some estimates for solutions of complex wave equation.

Proposition 2.6 Let $u \in C^2$, $u = u^1 + iu^2$, be a complex valued solution of an equation

$$\square u(t, x) + g^{00}(t, x)u_{tt} + g^{10}(t, x)u_{xt} + g^{11}(t, x)u_{xx} = f$$

$x \in \mathbb{R}$, $0 \leq t \leq T$, g be a complex valued C^1 -function and assume that $u = 0$ for large x . If

$$|g| = \sum |g^{ij}| < 1/2,$$

it follows that

$$\|u'(T, \cdot)\| \leq 2(\|u'(0, \cdot)\| + \int_0^T \|f(t, \cdot)\| dt) \exp\left(\int_0^T 2|g'(t)| dt\right), \quad (11)$$

where $\|\cdot\|$ are the L_2 norms with respect to x u' is the gradient of u with respect to x and t and

$$|g'(t)| = \sum \sup(|g_x^{ij}(t, \cdot)| + |g_t^{ij}(t, \cdot)|).$$

Proposition 2.6 is Proposition 6.3.2 from [H2] written for complex valued function. It easily follows from the integration of the identity

$$2\Re \bar{u}_t \square u = |u_t|_t^2 + |u_x|_t^2 - 2\Re(\bar{u}_t u_x)_x,$$

see the proof of Proposition 6.3.2 in [H2].

Proposition 2.7 Let $u \in C^2$, $u = u^1 + iu^2$, be a complex valued solution be a solution of the quasilinear wave equation

$$\square u(t, x) + g^{00}(u')u_{tt} + g^{10}(u')u_{xt} + g^{11}(u')u_{xx} = f(x) \quad (12)$$

$x \in \mathbb{R}$, $0 \leq t \leq T$, g, f be a complex valued C^2 -function and assume that $u = 0$ for large x . Assume that $\|u^1(0, \cdot)\|_{C^3} < C$, $\|u_t^1(0, \cdot)\|_{C^2} < C$, $u^2(0, \cdot) = 0$, $\|u_t^2(0, \cdot)\|_{C^2} < C$, $\sum |g^{ij}(0)| < 1/4$. Then there exists a constant $T_0 > 0$, $T_0 < T$ depending on g, f and C such that for any $0 < t < T_0$ $|u''(t, \cdot)| < 2C$.

Proof. There exists a constant $\delta > 0$ depending on g such that if $|u'(t, x) - u'(0)| < \delta$ then $\sum |g^{ij}(t, x)| < 1/2$. Applying inequality (11) to the second derivatives of equation (12) we get

$$\begin{aligned} \|u'''(T, \cdot)\| &\leq 2(\|u'''(0, \cdot)\| \\ &+ \int_0^T (|g'(t)| \|u'''\| + |g''(t)| \|u''\| + \|f''(t, \cdot)\|) dt) \exp\left(\int_0^T 2|g'(t)| dt\right), \end{aligned}$$

provided that $\sum |g^{ij}(t, x)| < 1/2$. Since $\|u''\|_C \leq \|u'''\|$ it follows from Gronwall's lemma that there exists a constant T_1 such that $|u''(t, \cdot)| < 2C$ for $0 < t < T_1$. Set $T_0 = \min\{\delta/2C, T_1\}$. Then for $0 < t < T_0$ $\sum |g^{ij}(t, x)| < 1/2$. Thus for $0 < t < T_0$ the proposition holds.

Consider the Cauchy problem

$$\begin{cases} \square u(t, x) = p(u), \\ u(0, x) = u_0, u_t(0, x) = u_1 \end{cases} \quad (13)$$

where $-1 < x < 1, 0 < t < 1, u \in \mathbb{C}, u_0 \in C^2, u_1 \in C^1, p$ be a polynomial $p \in \mathbb{C}$. By [HKM] Cauchy problem (13) locally has a classical solution.

Let $K \subset \mathbb{R}$ be a triangle with the vertices $(-1, 0), (1, 0), (0, 1 - \delta)$, where $\delta > 0$.

Proposition 2.8 *Let u be a classical solution of the Cauchy problem (13) defined in K . Assume u, u'' are uniformly bounded in K . Then solution u can be extended as a classical solution of the equation (13) in a neighborhood of the point $(0, 1 - \delta)$.*

Proof. By Proposition 2.7 u'' are uniformly bounded in K . Taking as initial data $u(1 - \delta - \varepsilon), u_t(1 - \delta - \varepsilon), \varepsilon > 0$ be sufficiently small, then by the result of [HKM] we get the existence of the solution of Cauchy problem for $1 - \delta - \varepsilon < t < 1 - \delta + \varepsilon$.

As a consequence of the last proposition we have.

Proposition 2.9 *Let $G \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary. Assume $u \in C^3(G) \cap L_\infty(\bar{G})$ is a solution of the wave equation (13) in G . Let $z \in \partial G$ and ∂G is not characteristic at z . Then u has an extension in a neighborhood of z as a classical solution of the equation (13).*

Proposition 2.10 *Let $u(x, y)$ be a solution of the equation (6) in $\Omega \subset \mathbb{R}^2$. Assume that function f is a polynomial. Let $G \subset \mathbb{C}^2$ be a bounded domain with a smooth boundary. Assume that u has a holomorphic extension on G as a function of complex variables $z_1 = x + ix', z_2 = y + iy'$ and D^2u is bounded in \bar{G} . Let $z' \in \partial G$ and T be the tangent plane at z' . Denote by L_1 2-dimensional plane $\{x, y'\}$ and by L_2 plane $\{x', y\}$. Let $l_1, l'_1 \subset L_1$ be the lines $x = y'$ and $x = -y'$. Let $l_2, l'_2 \subset L_2$ be the lines $x' = y$ and $x' = -y$. Assume that either $T \cap L_1$ is not l_1, l'_1 , or $T \cap L_2$ is not l_2, l'_2 . Then u has a holomorphic extension in a neighborhood of z' .*

Proof. Assume that $T \cap L_1$ is not l_1, l'_1 . Function u restricted on the planes parallel to L_1 satisfies nonlinear hyperbolic equation (13). Let point $z \in \partial G$ be sufficiently close to z' . Denote by L_z plane parallel to $L_1, z \in L_z$. Then by Proposition 2.10 solution of equation (13) can be defined in a neighborhood of z . Considering solutions of (13) for points $z \in \partial G$ in a neighborhood of z' we get an extension of the function u in a neighborhood of z' in \mathbb{C}^2 . Define smooth in G functions $\psi_j, j = 1, 2$, by

$$\psi_j = \frac{\partial y}{\partial \bar{z}_j}.$$

Taking the derivative of the equation (12) with respect to \bar{z}_j we get that the functions ψ_j are solutions of the hyperbolic equations on $L_z \cap G$. Since ψ_j vanishes on G then by the uniqueness of the solution of the Cauchy problem it follows that $\psi^j = 0$ in G . Thus u satisfies the Cauchy-Riemann equations in G and hence y is a holomorphic function in G .

Proof of Theorem 2.2. We assume first that f is a polynomial. For the polynomial f by the classical results of S. Bernshtein, [B], and H. Levy, [Le], the solution u is a real analytic function, having holomorphic extension in a domain $G, \Omega \subset G \subset \mathbb{C}^2$. Indeed the domain G depends on f . The complexification of real variable x, y we will denote by the same letters. Thus we will consider $u(x, y)$ as a holomorphic function of $x, y \in \mathbb{C}^2$.

We are going to prove that in the case of a polynomial f the radius of analyticity of the curve Γ and the estimates of the complex analytic extension depends only on the third derivative of f . Hence the proof of the theorem will follow after suitable approximations of f by polynomials in C^3 -norm.

For the proof we consider the complexification of u , that allows us to regard u as a solution of nonlinear wave equation. This method was developed by H. Levy, [Le], and I. Petrovsky, [P].

Choose orthonormal coordinates x, y in \mathbb{R}^2 such that $x_0 = \{0\}$ and coordinate y directed along $\nabla u(x_0)$. Then in a neighborhood of $\{0\}$ we can represent the graph of u as a function $y(x, u)$. Then $y \in C^{3,a}$ in a neighborhood G of $\{0\}$. Clearly the equation (6) for the function y takes the form of a quasilinear elliptic equation,

$$L(y) = \sum a_{ij}(y')y_{ij} - f(u) = 0,$$

where y' be the derivatives of y .

One can calculate the operator L directly,

$$L(y) = \frac{y_u^2 y_{xx} - y_x y_u y_{xu} + y_x^2 y_{uu}}{(y_x^2 + y_u^2)^{3/2} (1 + (y_x/y_u)^2)} - \frac{y_{uu}}{y_u^3} - f(u) = 0. \quad (14)$$

In a neighborhood of zero $y(x, u)$ is a holomorphic function of $x, u \in \mathbb{C}$. If $x_0, u_0 \in \mathbb{C}$ and $y_u(x_0, u_0) \neq 0$ then function u is holomorphic in a neighborhood of $(x_0, y(x_0, u_0))$. Denote by X a holomorphic map $X(x, u) = (x, y(x, u))$. Assume y is holomorphic in a domain $G' \subset \mathbb{C}^2$. Then y satisfies (14) in G' .

Let $z \in G', e \in \mathbb{C}^2$. Denote by $e(z)$ the maximal interval $z + te, 0 < t < T$ such that the function u has a holomorphic extension on $e(z)$.

We define a complexification of solutions y . Let $D_R \in \mathbb{R}^2$ be the disk, $|z| < R$. Assume that function y is defined on the disk D_{2R} on the real plane. Let $r \in \mathbb{R}^2$. Denote, $P(0) \in \mathbb{C}^2$ the plane $((0, ir_1), (r_2, 0))$, if $z \in \mathbb{C}^2$ by $P(z) \in \mathbb{C}^2$ denote the plane $P(0) + z$.

The equation (14) on $P(z)$ where y is defined has the form

$$\square y(r) + g^{kl}(y')y_{kl}(r) = f(r_2 + z). \quad (15)$$

where z is a parameter.

Define a 3-dimensional set $H \subset \mathbb{C}^2$. Let $q \in D_\varepsilon, -R < a < R, H = \{((q_1, 0), (a, q_2))\}$. We choose $\varepsilon > 0$ such small that y is holomorphic on H and for any $q \in D_\varepsilon$ and $z = ((q_1), (iq_2)), \|f\|_{C^2(D_R+z)} \leq 2\|f\|_{C^2(D_R)}$. Set $e = (i, 0) \in \mathbb{C}^2$. Define

$$Z = \cup_{z \in H} e(z).$$

Function y is defined and satisfies equation (14) on Z . Define function h on H setting $h = (\varepsilon^2 - q_1^2 - q_2^2)(R^2 - a^2)$. Denote

$$U_t = \cup_{z \in H} th(z)e,$$

where $t > 0$. For sufficiently small $t > 0$ we have $U_t \subset Z$.

From Cauchy-Riemann equations we have

$$y_{z_2}(\cdot, 0) = iy_{z_1}(\cdot, 0)$$

By scaling we may assume without loss that $y_u(0) = -1$. Then $g^{kl}(y'(0)) = 0$. Hence, since the restriction of y on the real plane is in C^3 it follows from Proposition 2.7 that for any $\delta > 0$ there exists $c > 0$ such for any $0 < t < c$, $-c < r_2 < c$,

$$|y'(0, \cdot) - y'(t, \cdot)| < \delta, \quad (16)$$

provided by $|g^{kl}y'| < 1/4$ on $(-c, c) \times (0, c)$ and $y \in Z$. Since a small $\delta > 0$ implies the last inequality it follows the existence of sufficiently small $c > 0$ such that inequality (16) holds. Constant c depends on C^3 -norm of y and on C^2 -norm of f on the real segment $[u_0 - 1, u_0 + 1]$ and is independent on the norms of f in \mathbb{C} . From (16) and Proposition 2.7 it follows that for $0 < t < c$, $-c < r_2 < c$, $|y''| < C$.

We are going to show that $y(t, \cdot)$ is defined for $0 < t < c$. Assume not. Let t_0 be the maximal t for which $G_t \subset Z$. Let $z_0 \in \partial Z \cap \partial U_{t_0}$

If $c > 0$ is sufficiently small we have $\Re y_{u_1}(z_0) < -1/2$ and hence if \tilde{y} be the restriction of y on ∂U_{t_0} then $\Re \tilde{y}_{u_1}(z_0) < -1/4$. Hence $X(\partial U_{t_0})$ is a smooth surface in a neighborhood of $X(z_0)$. Choosing constant $c > 0$ sufficiently small we have the inequality $|\Im \tilde{y}_{u_1}(z_0)| < 1/8$ and hence the tangent plane at $X(z_0)$ to $X(\Gamma_{s_0})$ satisfies assumptions of Proposition 2.10. Thus by Proposition 2.10 function u has a holomorphic extension in a neighborhood of the point $X(z_0)$ and since $u_y(X(z_0)) \neq 0$ function y has a holomorphic extension in a neighborhood of z_0 .

Thus we proved that function $y(x, u)$ has a holomorphic extension in the disk $(x_1 + ix_2, 0)$, $x \in D_c$ and bounded in this disk by a constant depending only on C^2 -norm of the function f . Thus the theorem is proved.

Proof of Theorem 2.4. Let $0 \in \gamma$. Choose orthonormal coordinates x, y in \mathbb{R}^2 such that coordinate y directed along $\nabla u(0)$. Then in a neighborhood of $\{0\}$ we can represent the graphs of u_i as a function $y^i(x, u)$ which satisfy the elliptic equation (6). Let Γ be the curve $(y^1, u_1(x, y))$, where $(x, y) \in \gamma$. Set $y = y^1 - y^2$. Then y is a solution of a linear elliptic equation of the form

$$\sum a_{ij}(x)y_{ij} = 0, \quad (17)$$

where $a_{ij} \in C^1$. Uniqueness of the Cauchy problem for the equation (17) is well known, see e.g., [H2], and since $y = \nabla y = 0$ on Γ it follows that $y \equiv 0$. Theorem 2.3 is proved.

3 Geometry of streamlines

In this section we prove Theorem 1.3.

Lemma 3.1 *Let $f, y, z : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = y(0) = z(0) = 0$. Let $y \in C^{k,a}$, $z \in C^{n-1,a}$, $k, n \in \mathbb{N}$, $a > 0$ and $y'(0) > 0$. Assume that f, y, z satisfy the functional equation*

$$z(x) = f(y(x)). \quad (18)$$

Then $f \in C^{m,a}$, where $m = \min(k, n - 1)$.

Proof. Since $y'(0) > 0$ it follows that $y^{(-1)} \in C^{k,a}$. Then $z(y^{(-1)}(x)) = f(y(y^{(-1)}(x))) = f(x)$ and the lemma follows.

Theorem 1.1 follows from Theorem 2.3 and Lemma 3.1. Remarks after Theorem 1.1 follows from Theorem 2.1. Theorem 1.2 is a consequence of Theorem 2.4.

Let v be a steady flow in Ω and $v(0) \neq 0$. The equation (4) implies that that the stream function u of v satisfies the equation

$$\Delta u = f(u) \quad (19)$$

in a neighborhood of 0. If $v(0) = 0$ the equation (19) might be not satisfied. However, for an isolated critical point velocity function v satisfies an elliptic equation.

Proposition 3.2. *Let $v \in C^2(\Omega)$ be a steady flow and $0 \in \Omega$ be an isolated critical point of v , $v(0) = 0$. Then in a neighborhood of 0 v satisfies the equation*

$$\Delta v = c(x)v \quad (20)$$

where $c \in L_\infty$ with the norm depending on C^2 -norm of v .

Proof. By our assumption there is a disk D , $0 \in D$ such that 0 is a single critical point of the stream function u in D . Hence it follows, that for any $x_0 \in D$ the connected component l of the level curve $\{u(x) = u(x_0)\}$ which contains the point x_0 has limit points on ∂D . Let $x_1 \in l \cap \partial D$. Then $\Delta u_i(x_0) = \Delta u_i(x_1)$. By Lemma 3.1 $|\Delta u_i(x_0)| \leq C|u_i|$. Therefore the proposition follows.

Proof of Theorem 1.3. Let u be the stream function of the flow v .

If 0 is a Morse's critical point of u , i.e., Hessian of u at 0 is not degenerate, then the singularity of v at 0 is obviously of the type (i).

Assume now that $D^2u(0) = 0$. In a neighborhood of any noncritical point of the stream function u it satisfy the equation $\Delta u = f(u)$ such that $f_u(u(x)) = \nabla \Delta u(x) / \nabla u(x) = c(x)$, where c is a coefficient of equation (20). By our assumption $\Delta u(0) = 0$, and hence if l is a level set $u(x) = u(0)$ then $\Delta u|_l = 0$. Set $w = u(x) - u(0)$. Since c is uniformly bounded we obtain that

$$\Delta w = d(x)w, \quad (21)$$

where d is a bounded function in a neighborhood of 0.

By the result of [HO] from equation (21) follows that

$$w = p_k + o(|x|^k),$$

where p_k is a homogeneous harmonic polynomial of order k , $k > 2$. Hence from the equation (21) we get

$$\nabla w = \nabla p_k + o(|x|^{k-1}).$$

Therefore, the singularity of v is of type (ii).

Assume finally that $D^2u(0)$ is degenerate and not equal to zero. Then after rotation axes the quadratic part of u will be ax_2^2 . Applying Proposition 3.2 to the first derivative of u we get that the singularity of v at 0 is of the type (iii).

Consider now the case of stagnating domain $G \subset \Omega$. Since the flow is area preserving any streamline in $\Omega \setminus G$ is either closed or has limit points on $\partial\Omega$. Denote by Q the union of non-closed streamlines. If Q has no limit points on ∂G then the theorem follows. If Q has a limit point on ∂G then by topological reason there are no closed streamlines enclosing G , and hence $\partial G \subset \partial Q$. Since each streamline from Q has limit points on $\partial\Omega$ it follows as in the proof of Proposition 3.2 that on Q v satisfies equation (20). Since $v \equiv 0$ on G and $c \in L_\infty$ then $v \equiv 0$ in Ω by the unique continuation theorem if ∂G is the limit set for Q . The theorem is proved.

4 Stable instability of unstable Arnold flows

In this section we prove Theorem 1.5. The main idea of the proof is similar to the approach we used for the proof of the existence of the wandering trajectories to the solution of the Euler equation, [N].

Let r, θ be the polar coordinates in A . Let $\tilde{v} \in C^{2,\alpha}(A)$ be a radial symmetric Arnold stable steady flow, $\tilde{v} = \tilde{v}(r)$. Since the flow \tilde{v} satisfies inequalities (5) it follows that $\tilde{\omega}_r > 0$ and we may assume without loss that $\tilde{\omega}_r > 1$ in A . Hence, if \tilde{v}^1, \tilde{v}^2 be the components of \tilde{v} in the coordinates r, θ then

$$\begin{aligned} \partial \tilde{v}^2 / \partial r &> 1. \\ \partial \tilde{\omega} / \partial r &> 1. \end{aligned} \tag{22}$$

Let $h \in C^1(A)$ and $\partial h / \partial r > 0$.

Set

$$\begin{aligned} h^+ &= \sup_{x \in A} \frac{\partial h(x) / \partial \theta}{|\nabla h(x)|}, \\ h^- &= \inf_{x \in A} \frac{\partial h(x) / \partial \theta}{|\nabla h(x)|}, \\ h^* &= h^+ - h^-. \end{aligned}$$

Thus if $h^* = 0$ then h is a function of radius.

Lemma 4.1. *There is a $\delta > 0$ such that if $\|v(x, t) - \tilde{v}(x)\|_{C^{2,\alpha}(A)} < \delta$ then the inequality $\omega^+ > -\omega^-$ yields,*

$$\frac{\partial \omega^+}{\partial t}(0) > c_0 \omega^+,$$

and the inequality $\omega^+ < -\omega^-$ yields,

$$\frac{\partial \omega^-}{\partial t}(0) > -c_0 \omega^-,$$

where c_0 be a positive constant

Proof. Let g^t be a one parametric group of diffeomorphism of A corresponding to the flow $v(t)$. Then from Euler-Helmholtz equation follows, see [AK],

$$\omega(x, 0) = \omega(g^t, t).$$

Thus

$$\frac{\partial \nabla \omega(g^t(x), t)}{\partial t}(0) = J_v^T \nabla \omega(x, t), \quad (23)$$

where J_v is the Jacobian matrix of the vector field $v(x, 0)$. Notice that

$$J_{\tilde{v}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},$$

where $a > 1$.

Let u, \tilde{u} be the stream functions of $v(\cdot, 0)$ and \tilde{v} . Denote $u' = u - \tilde{u}$, $\omega' = \omega(\cdot, 0) - \tilde{\omega}$. Then u' satisfies the equation,

$$\Delta u' = \omega'.$$

On the boundary circles of A u' equal to constants bounded by δ . From the standard estimates for the solutions of the Poisson equation it follows, that $|u'_{12}| < K\delta$ or

$$|v_1^1| < K\delta,$$

where K is a positive constant. For u'_θ we have the equation

$$\Delta u'_\theta = \omega'_\theta \text{ in } A,$$

$$u'_\theta = 0 \text{ on } \partial A.$$

Then $(u'_\theta)_{|C^1(A)} \leq K(\omega'_\theta)_{|C(A)}$. Since $|\omega'_\theta| < \delta\omega^+ + \omega^-$ we get

$$|\nabla(v^2 - \tilde{v}^2)| < \delta\omega^+ + \omega^-.$$

Thus for sufficiently small $\delta > 0$ we have

$$|v_1^2| > 1, |v_2^2| < K\delta\omega^+ + \omega^-, |v_2^1| < K\delta\omega^+ + \omega^-, |v_1^1| < K\delta, \quad (24)$$

Let $x_0, x_1 \in A$ and

$$\frac{\partial \omega(x_0, 0)/\partial \theta}{|\nabla \omega(x_0, 0)|} = \omega^+, \quad \frac{\partial \omega(x_1, 0)/\partial \theta}{|\nabla \omega(x_1, 0)|} = \omega^-.$$

Assume first that $\omega^+ \geq -\omega^-$. Then from (24) we get

$$\omega_1(x_0, 0) > c, \quad |\omega_2(x_0, 0)| < C\omega^+. \quad (25)$$

Denote

$$(a, b) = \frac{\partial \nabla \omega(g^t(x), t)}{\partial t}(0).$$

Then from (23), (24), (25) we get $a > c - C\delta\omega^+$, $|b| < C\delta\omega^+$, where C is a positive constant. Since

$$\frac{\omega_2(x_0, 0)}{\omega_1(x_0, 0)} = \omega^+$$

we get that $\partial\omega^+(0)/\partial t > c_0\omega^+$.

If we assume now that $\omega^+ \leq -\omega^-$. Then after the similar computations at the point x_1 we get that $\partial\omega^-(0)/\partial t > -c_0\omega^-$. The lemma is proved.

Proof of Theorem 1.5. Assume by contradiction that for all $t > 0$ $\|v(0, t) - \tilde{v}(x)\|_{C^{2,\alpha}(A)} < \delta$.

Assume that $\omega^+(t_0) = -\omega^-(t_0)$, $t_0 \in \mathbb{R}$. Then by Lemma 4.1 $\partial\omega^+(t_0)/\partial t > 0$, $\partial\omega^-(t_0)/\partial t > 0$ and hence for all $t > t_0$ $\omega^+(t) > -\omega^-(t)$. Hence for $t > t_0$ we have $\partial\omega^+/\partial t > c_0\omega^+$. Then there is $T > t_0$ such that for $t > T$ $\|v(0, t) - \tilde{v}(x)\|_{C^{2,\alpha}(A)} > \delta$.

Assume now that for all $t > 0$, $\omega^+(t) < -\omega^-(t)$. Then we have $\partial\omega^-/\partial t < -c_0\omega^-$ and therefore $\omega^+(t), \omega^-(t)$ tend to 0 as $t \rightarrow \infty$. Hence $\omega(\cdot, t) \rightarrow \bar{\omega}$, where $\bar{\omega}$ depends only on r . Let \bar{v} be the velocity corresponding to the vorticity $\bar{\omega}$. Then \bar{v} is a steady flow corresponding to the vorticity $\bar{\omega}$. Since by our assumption $\|\bar{v} - \tilde{v}(x)\|_{C^{2,\alpha}(A)} < \delta$ the flow \bar{v} is Arnold stable and hence by the theorem of Arnold, [A], the trajectory $v(\cdot, t)$ can not tend to \bar{v} . Thus we got a contradiction. Theorem 1.5 is proved.

5 Variational solutions of Euler equation

In this section we study Arnold stable steady state flows and prove Theorem 1.4 and 1.6.

Theorem 5.1. *Let $v \in C^2(\Omega)$ be an Arnold stable solution of (2), (3). Then ω is of a constant sign. In any compact subdomain of Ω the critical set of v is in a union of a finite collection of C^2 curves. If Ω is a convex domain then the critical set of v is a single point.*

Proof. Let u be a stream function of v , $u = 0$ on $\partial\Omega$. Denote by Σ the set of critical points of v , and by Σ_0 the interior of Σ .

On the set $\Omega \setminus \Sigma$ v is a solution of equation (20), where

$$c(x) = \nabla \Delta u / \nabla u,$$

$c \in L_\infty$, $c > 0$. Since $v \in C^2$ one can define equation (20) on $\Sigma \setminus \Sigma_0$ by continuity with $c \in L_\infty$.

If Σ_0 is nonempty then, since Σ_0 is an open set and $v = 0$ on Σ_0 we have by unique continuation theorem, see [H1], $v \equiv 0$. Hence Σ_0 is empty. Therefore,

v satisfies equation (20) in the whole domain Ω . The critical set of v is the intersection of the nodal lines of v^1 and v^2 . Since solutions of (20) have isolated second order zeros, see [HO], the second part of the theorem follows.

From the boundary condition (2) it follows that $\omega = m = \text{const}$ on $\partial\Omega$. We show that $m \neq 0$. Assume by contradiction that $m = 0$. Denote by N the nodal set of the function u . We show first that there is a nodal domain $G \subset \Omega \setminus N$ which has limit points on $\partial\Omega$. If there is no such domain it implies that any curve in Ω with the end point on $\partial\Omega$ has infinitely many intersection with N . Hence u has 3-d order zero on $\partial\Omega$. Thus v has the second order zero on $\partial\Omega$ and by the uniqueness of the Cauchy problem for equation (20), [H1], it follows that $v \equiv 0$.

Thus there exists a nodal domain of u G which has limit points on $\partial\Omega$. Thus the vorticity ω vanishes on the exterior component of the boundary of G . Assume without loss that $u > 0$ in G . From the structure of the critical points of v in Ω easily follows the existence of a rectifiable curve $\gamma : [0, 1] \rightarrow G$ such that $(\dot{\gamma}, \nabla u) \geq 0$, $\gamma(0) \in \partial\Omega$ and $\gamma(1)$ is a local supremum of u . From the inequality (5) follows that $\omega(\gamma(1)) > 0$. Since $\gamma(1)$ is a local supremum of u the last inequality contradicts the maximum principle.

Thus we prove that $m \neq 0$. We assume without loss that $m < 0$. We prove that $\omega < 0$ in Ω . Assume by contradiction that there is a nodal domain D of ω where $\omega(x) > 0$. Assume that G is non empty domain. Let at a point $z \in \bar{D}$ u attains its supremum over \bar{D} . From inequality (5) it follows, that $\omega(z) > 0$. Thus $z \in D$. By the maximum principle point z can not be a point of a local supremum of u . Thus it follows that D is an empty set, and hence the first part of the theorem is proved.

Assume now that Ω is a convex domain. We prove that then v has a single critical point. We will use some arguments suggested in [CC]. Let $e \in \mathbb{R}^2$, $|e| = 1$. Denote by e_1, e_2 two points on $\partial\Omega$ where e is tangent to the boundary. Consider the derivative of stream function u_e . Then u_e is a solution of equation (20). Denote by γ_e the nodal line of u_e . Since $c \geq 0$ the maximum principle holds for the solutions of (20). Therefore γ_e can not enclosed any subdomain in Ω . Then it follows that γ_e is a simple arc with the end points e_1, e_2 . Let $z_1, z_2 \in \Omega$ be two different critical points of v . Then $z_1, z_2 \in \gamma_e$ for all $e \in S^1$. Notice that γ_e continuously depends on $e \in S^1$. Thus order of the points e_1, z_1, z_2, e_2 along the arc γ_e is independent on e . On the other hand when e is changing to $-e$ the curve γ_e changes its orientation. That leads to a contradiction with the existence of the second critical point of v . The theorem is proved.

From Theorem 5.1 follows Theorem 1.5.

Lemma 5.2. *Let v be a smooth steady flow. Assume that in a neighborhood of 0,*

$$\omega = c + l_1 l_2 l_3 + o(|x|^3), \quad (26)$$

where l_i are three linear functions, each two are linear independent. Then $c = 0$.

Proof. Let u be a stream function of v . Vorticity ω is a constant on connected components of the level sets of u . By our assumption that is impossible in a neighborhood of 0 if $Du(0) \neq 0$ or $D^2u(0) \neq 0$. Thus $D^2u(0) = 0$ and

hence $c = 0$. The lemma is proved.

Proof of Theorem 1.6. By theorem of Burton, [B2], there exists a steady state solution v with $\omega \in \bar{S}(\omega)$ with the stream function u such that $\Delta u = f(u)$, with $f \in L_\infty$ being a monotonically non-decreasing function. Thus $u \in C^{1,a}$, $a > 0$. By Theorem 2.1 and Theorem 1.4 the level lines of u are smooth curves. Let m_1, m_2 be functions of distribution of ω and u ,

$$m_1(t) = |\omega^{-1}(0, t)|, \quad m_2(t) = |u^{-1}(0, t)|.$$

Set $z = m_1^{-1}$, $y = m_2^{-1}$. Functions z, y are defined on $(0, m)$, where $m = |\Omega|$. Function z is smooth on $(0, 1)$ by the assumptions, $y \in C^{1,a}$ since $\partial u / \partial \nu$ does not vanish on the level curves of u . Since of functional equation $z(t) = f(y(t))$ we get by Lemma 3.1 $f \in C^{1,a}$. Hence from equation (6) we get $u \in C^{3,a}$ in $\Omega \setminus s$, where s is a point of supremum of u . Iterating the argument we get the smoothness of u in $\Omega \setminus s$. The theorem is proved.

Remark. Let h be a smooth positive function in Ω , $h = 0$ on $\partial\Omega$ and h has a singularity of type (26) at $0 \in \Omega$. Assume that variational problem (A) has a smooth extremal $\omega = h(g)$, where $g \in S \text{Diff}(\Omega)$. Then ω has a singularity of the type (26) at $g(0)$. Hence, by Lemma 5.2 $\omega(g(0)) = 0$. That contradicts the assumption on the positivity of h and therefore the variational problem (A) has no smooth extremals.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Let u_1, u_2 solutions of the Dirichlet problem*

$$\begin{cases} \Delta u_i = f_i(u_i), & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \quad (27)$$

where f_i are negative increasing C^1 -functions. Then $f_1(u_1) \notin \bar{S}(f_2(u_2))$.

Proof. Assume by contradiction that

$$f_1(u_1) \in \bar{S}(f_2(u_2))$$

$S^*(f_2(u_2))$ be a weak closure of $\bar{S}(f_2(u_2))$. Then $S^*(f_2(u_2))$ is a convex set and hence if u^* be a global minimizer of the variational problem (K) on $S^*(f_2(u_2))$ then u^* also a global minimizer of (K) on $S^*(f_2(u_2))$, [B1]. We may assume without loss that

$$u_2 = u^*. \quad (28)$$

Set $w = u_2 - u_1$. Compute variation of the kinetic energy E along w at u_1 ,

$$\delta E|_w(u_1) = \int_{\Omega} \nabla u_1 \nabla w dx = - \int_{\Omega} w f_1(u_1) dx.$$

Since $f_1 < 0$, $f_1' > 0$ and by (27), (28) we have

$$\int_{\Omega} u_1 f_1(u_1) dx \geq \int_{\Omega} u_2 f_1(u_2) dx \geq \int_{\Omega} u_2 f_1(u_1) dx$$

Hence

$$\delta E|_w(u_1) \geq 0.$$

By assumption (28)

$$\delta E|_{-w}(u_2) = 0.$$

Since the energy E is a convex function on the line connecting points u_1 and u_2 we get a contradiction. The proposition is proved.

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