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To cite this version:

Robert Coquereaux, Jean-Bernard Zuber. Conjugation Properties of Tensor Product and Fusion Coefficients. Letters in Mathematical Physics, 2017, 107 (2), pp.291-299. 10.1007/s11005-016-0901-3 . hal-01335622

HAL Id: hal-01335622

https://hal.science/hal-01335622v1

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arXiv:1606.06442v1 [math-ph] 21 Jun 2016

Conjugation Properties of Tensor Product and Fusion Coefficients

Robert Coquereaux

Aix Marseille Univ, Univ Toulon, CNRS, CPT, Marseille, France

Jean-Bernard Zuber

Laboratoire de Physique Théorique et des Hautes Énergies, CNRS UMR 7589,
Université Pierre et Marie Curie, Sorbonne Universités, 4 Place Jussieu, 75252 Paris Cedex 05

June 22, 2016

Abstract

We review some recent results on properties of tensor product and fusion coefficients under complex conjugation of one of the factors. Some of these results have been proven, some others are conjectures awaiting a proof, one of them involving hitherto unnoticed observations on ordinary representation theory of finite simple groups of Lie type.

Keywords: Lie groups; Lie algebras; fusion categories; conformal field theories; quantum symmetries; Drinfeld doubles

Mathematics Subject Classification (2010):81R50; 81T40; 20C99; 18D10

1 Notations and Results

1.1 Notations

In the following λ, μ, etc label either finite dimensional irreps of a simple Lie algebra $\mathfrak{g}$ or of the corresponding simply connected compact Lie group G; or (for the sake of comparison) irreps of a finite group Γ; or integrable irreps of an affine algebra $\widehat{\mathfrak{g}}_k$ at a finite integral level k.
Following a standard abuse of notations, for the Lie groups and algebras, λ denotes both the highest weight of the representation and the representation itself.

$N_{\lambda\mu}^{\nu}$ denotes respectively the coefficients of decomposition of the tensor product $\lambda \otimes \mu$ into inequivalent irreps ν (Littlewood–Richardson coefficients) of G or Γ; or the coefficients of decomposition of the fusion product denoted $\lambda \star \mu$ into irreps ν of $\widehat{\mathfrak{g}}_k$.

It is often convenient to regard this set of coefficients as elements of matrices, thus

$$N_{\lambda\mu}^{\nu} = (N_\lambda)_\mu^\nu \,. \tag{1}$$

These coefficients satisfy the sum rule

$$\dim_\lambda \dim_\mu = \sum_\nu N_{\lambda\mu}^{\nu} \dim_\nu \tag{2}$$

where $\dim_\alpha$ denotes the dimension, resp. the *quantum dimension*, of the irrep α of G, $\mathfrak{g}$ or Γ, resp. of $\widehat{\mathfrak{g}}$. When λ refers to a representation of complex type, we denote by $\bar{\lambda}$ the (equivalence class of its) complex conjugate. Recall that among the simple Lie algebras, only those of type A_r, any r; D_r, r odd; and E_6 admit complex representations.

For a given pair (λ, μ), consider the moments of the N's

$$m_r := \sum_\nu (N_{\lambda\mu}^{\nu})^r \qquad\qquad r \in \mathbb{N} \,.$$

In particular, m_0 counts the number of distinct (i.e., non equivalent) ν's appearing in the decomposition of $\lambda \otimes \mu$, resp. $\lambda \star \mu$.

For non real λ and μ, we want to compare m_r and $\overline{m}_r := \sum_\nu (N_{\lambda\bar{\mu}}^{\nu})^r$.

Call $\mathfrak{P}$ the property that the *multisets* $\{N_{\lambda\mu}^{\nu}\}$ and $\{N_{\lambda\bar{\mu}}^{\nu'}\}$ are identical. Since for given λ and μ these multisets are finite, there is an equivalence

$$m_r = \overline{m}_r \quad \forall r \in \mathbb{N} \qquad \Leftrightarrow \qquad \mathfrak{P} \,. \tag{3}$$

1.2 A list of results and open questions

We start with a fairly obvious statement

Proposition 1 [3] *For any Lie group, any finite group or any affine Lie algebra,* $m_2 = \overline{m}_2$, i.e., $\sum_\nu (N_{\lambda\mu}^\nu)^2 = \sum_{\nu'} (N_{\lambda\bar{\mu}}^{\nu'})^2$,
see below the (easy) proof in sect. 1.3. Much more surprising is the following

Proposition 2 [1] *For any simple Lie algebra, or any affine simple Lie algebra,* $m_1 = \overline{m}_1$, i.e., $\sum_\nu N_{\lambda\mu}^\nu = \sum_{\nu'} N_{\lambda\bar{\mu}}^{\nu'}$. *This is not generally true for finite groups* Γ. *This is not generally true for quantum doubles of finite groups either* [2].

Problem 1 *For a given finite group* Γ, *find a criterion on* (Γ, λ, μ) *for Prop. 2 to hold.*

Proposition 3 [3] *For the Lie group SU(3),* $m_r = \overline{m}_r$ *for all* r, i.e., *we have property* $\mathfrak{P}$. *Moreover we know a (non-canonical and non-unique) piece-wise linear bijection* $(\nu, \alpha) \leftrightarrow (\nu', \alpha')$, *where* α *is a multiplicity index running over* $N_{\lambda\mu}^\nu$ *values. This property* $\mathfrak{P}$ *is not true in general for higher rank SU(N) nor for other Lie groups .*

Proposition 4 [3] *For the affine algebra* $\widehat{su}(3)$ *at finite level* k, $m_r = \overline{m}_r$ *for all* r, i.e., *we have property* $\mathfrak{P}$. *This is not true in general for higher rank* $\widehat{su}(N)$ *or other affine algebras.*

This is, however, satisfied by low-level representations.
Problem 2 *For each* $\widehat{\mathfrak{g}}_k$, *find a criterion on* (λ, μ, k) *for Prop. 4 to hold.*

Also missing in $\widehat{su}(3)$ is a general mapping $\nu \leftrightarrow \nu'$ compatible with the level. Although we found one in a few particular cases, a general expression is still missing.
Problem 3 *For each level in* $\widehat{su(3)}_k$, *find a piece-wise linear bijection* $\nu \leftrightarrow \nu'$.

A weaker property than property $\mathfrak{P}$, which follows from it, is that $m_0 = \overline{m}_0$.

Proposition 5 [3] *For the affine algebra* $\widehat{su}(3)$ *at finite level* k, $m_0 = \overline{m}_0$. *This seems to be also true for* $\widehat{su}(4)$, *but this is not true in general for higher rank* $\widehat{su}(N)$, $N \geq 5$, *or other affine algebras.*

This is, however, satisfied by low-level representations.
Problem 4 *For each* $\widehat{\mathfrak{g}}_k$, *find a criterion on* (λ, μ, k) *for Prop. 5 to hold.*

These results on the equality of various m_k and $\bar{m}_k$ are summarized in Table 1.

	$SU(3)$ or $su(3)$ or $\widehat{su}(3)$	$SU(4)$ or $su(4)$ or $\widehat{su}(4)$	G or $\mathfrak{g}$ other simple Lie group or Lie algebra	$\widehat{\mathfrak{g}}$ other affine Lie algebra	Γ finite simple group of Lie type	Γ other finite group
$m_2 = \bar{m}_2$	✓	✓	✓	✓	✓	✓
$m_1 = \bar{m}_1$	✓	✓	✓	✓	✓?	X
$m_0 = \bar{m}_0$	✓	✓?	X	X	X	X
$m_r = \overline{m}_r \ \forall r \Leftrightarrow \mathfrak{P}$	✓	X	X	X	X	X

Table 1: ✓ means that the property is true and proven ; X that it is not true in general and there are counter-examples ; ✓? that the property has been checked in many cases (see text) but that a general proof is still missing.

1.3 Comments, remarks, examples and counter-examples

- The equality $m_2 = \bar{m}_2$ is the easiest to interpret and to prove. More explicitly it asserts that

$$\sum_{\nu} (N_{\lambda\mu}^{\nu})^2 = \sum_{\nu'} (N_{\lambda\bar{\mu}}^{\nu'})^2 \,. \tag{4}$$

Proof. The number of invariants $N_{\lambda\mu\bar{\lambda}\bar{\mu}}^{0}$ in $\lambda \otimes \mu \otimes \bar{\lambda} \otimes \bar{\mu}$ may be written as

$$\begin{aligned} N_{\lambda\mu\bar{\lambda}\bar{\mu}}^{0} &\overset{\text{(i)}}{=} \sum_{\nu,\nu'} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\nu'} N_{\nu\nu'}^{0} \overset{\text{(ii)}}{=} \sum_{\nu,\nu'} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\nu'} \delta_{\nu'\bar{\nu}} \overset{\text{(iii)}}{=} \sum_{\nu} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\bar{\nu}} = \sum_{\nu} (N_{\lambda\mu}^{\nu})^2 \\ &\overset{\text{(iv)}}{=} N_{\lambda\bar{\mu}\bar{\lambda}\mu}^{0} = \sum_{\nu} N_{\lambda\bar{\mu}}^{\nu} N_{\bar{\lambda}\mu}^{\bar{\nu}} = \sum_{\nu} (N_{\lambda\bar{\mu}}^{\nu})^2 \end{aligned} \tag{5}$$

where we have made use of (i) associativity of the tensor or fusion product, (ii) $N_{\nu\nu'}^{0} = \delta_{\nu'\bar{\nu}}$, (iii) invariance under conjugation $N_{\bar{\lambda}\bar{\mu}}^{\bar{\nu}} = N_{\lambda\mu}^{\nu}$, and (iv) commutativity $N_{\lambda\mu\bar{\lambda}\bar{\mu}}^{0} = N_{\lambda\bar{\mu}\bar{\lambda}\mu}^{0}$.

Graphically, this may be represented as in Fig. 1. In physical terms, and in the context of particle physics, it expresses the fact that the numbers of independent amplitudes in the "s channel" $\lambda \otimes \mu \to \lambda \otimes \mu$ and in the "crossed u channel" $\lambda \otimes \bar{\mu} \to \lambda \otimes \bar{\mu}$ are the same.

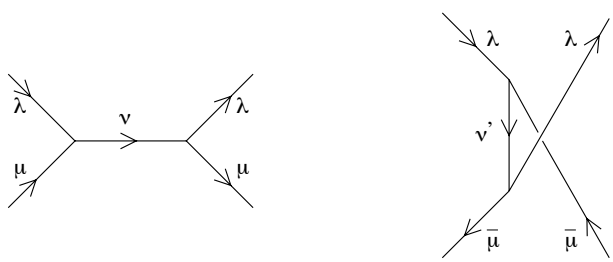

Figure 1: Graphical representation of $m_2 = \bar{m}_2$. Each $\lambda\mu\nu$ vertex carries the multiplicity $N_{\lambda\mu}^{\nu}$, and likewise for $\lambda\bar{\mu}\nu'$ on the right. Sums over ν, resp. ν' are equal.

- In contrast, the equality $m_1 = \bar{m}_1$ is neither natural nor general. While it is valid for all simple Lie algebras, either finite dimensional or affine, (see the discussion and elements of proofs in the next section), it is known not to be true for general finite groups. Counter-examples are provided by some finite subgroups of SU(3), see below in sect. 1.5, and also [1], and the detailed discussion in [2].

- Even more elusive and exceptional is the equality $m_0 = \bar{m}_0$, which happens to be true in SU(3) or for the affine algebra $\widehat{su}(3)$, as a particular case of the more general property $\mathfrak{P}$ that they satisfy. Curiously we have found evidence (but no proof yet) that it also holds true for SU(4) and $\widehat{su}(4)$ (this was tested in $\widehat{su}(4)_k$ up to level $k = 15$), but it fails in general for higher rank SU(N) or $\widehat{su}(N)$.

- Finally the equality $m_r = \bar{m}_r$ for all r, or equivalently property $\mathfrak{P}$, is satisfied in SU(3) [3] and in $\widehat{su}(3)$ at all levels [4].

Example 1. In SU(3), for the ten-dimensional representations,

$$\begin{aligned} (2,1) \otimes (2,1) &= 1(4,2) + 1(5,0) + 1(2,3) + 2(3,1) + 1(0,4) + 2(1,2) + 1(2,0) + 1(0,1) \\ (2,1) \otimes (1,2) &= 1(3,3) + 1(4,1) + 1(1,4) + 2(2,2) + 1(3,0) + 1(0,3) + 2(1,1) + 1(0,0) \end{aligned} \tag{6}$$

on which we do observe all the above properties: $m_2 = \overline{m}_2 = 14$, $m_1 = \overline{m}_1 = 10$, $m_0 = \overline{m}_0 = 8$ and the multisets of multiplicities are both $\{1,1,1,1,1,1,2,2\}$, or in short, $\{1^6 2^2\}$ (where we note the number n of occurrences of multiplicity m by m^n).

Example 2. In SU(4), with $\lambda = \mu = (1,2,2)$, we find for the multiplicities $N^\nu_{\lambda\mu}$ the multiset $\{1^{17}2^8 3^9 4^8 5^1 6^3 7^3\}$ while for those for $\lambda \otimes \bar{\mu}$ it is $\{1^{16}2^{12}3^6 4^3 5^8 6^3 8^1\}$, whence $m_2 = \overline{m}_2 = 538$, $m_1 = \overline{m}_1 = 136$, $m_0 = \overline{m}_0 = 49$ but the multisets are clearly different.

Example 3. In SU(5), for $\lambda = (1,1,1,0)$, $\mu = (1,1,0,1)$, we find that the list of $N^\nu_{\lambda\mu}$ reads $\{1^{12}2^6 3^3 4^3\}$ while that of $N^{\nu'}_{\lambda\bar{\mu}}$ reads $\{1^{15}2^3 3^4 4^3\}$. We check that $m_2 = \overline{m}_2 = 111$ and $m_1 = \overline{m}_1 = 45$ but $m_0 = 24 \neq \overline{m}_0 = 25$.

Example 4. In SO(10) (Lie algebra D_5), with $\lambda = \mu = (1,1,0,1,0)$, the two multisets are respectively $\{1^{17}2^{10}3^3 4^8 5^6 6^3 7^2 8^2 12^1\}$ and $\{1^{15}2^{11}3^8 4^7 5^2 6^3 7^1 8^2 9^2 10^1\}$ from which we check that $m_2 = \overline{m}_2 = 840$, $m_1 = \overline{m}_1 = 168$, $m_0 = \overline{m}_0 = 52$ while the two multisets are manifestly different.

Example 5. In SO(10) (Lie algebra D_5), with $\lambda = \mu = (1,1,1,1,0)$, we find $m_1 = \overline{m}_1 = 4456$ et $m_2 = \overline{m}_2 = 184216$ but $m_0 = 240$ and $\overline{m}_0 = 243$, hence a counter-example to the property of Prop. 5.

Example 6. In E_6, likewise, we may find pairs of λ, μ which violate Prop. 4 and 5. Take $\lambda = \mu = (1,1,0,0,0;1)$; $\bar{\mu} = (0,0,0,1,1;1)$[1]; we find $m_1 = \overline{m}_1 = 947$, $m_2 = \overline{m}_2 = 14163$ but $m_0 = 119$, $\overline{m}_0 = 123$. (Incidentally, the reducible representation encoded by $\lambda \otimes \mu$ in that case has dimension $63631071504 = 252252^2$.)

1.4 Related properties of the modular S-matrix

In the case of an affine algebra $\widehat{\mathfrak{g}}_k$, it is well known that the fusion coefficients are given by Verlinde formula [5]

$$N^\nu_{\lambda\mu} = \sum_\kappa \frac{S_{\lambda\kappa} S_{\mu\kappa} S^*_{\nu\kappa}}{S_{0\kappa}} \,. \tag{7}$$

Proposition 6 [1] *For the affine algebra $\widehat{\mathfrak{g}}_k$ at finite level k, $\Sigma(\kappa) := \sum_\nu S_{\kappa\nu}$ vanishes if the irrep κ is either of complex or of quaternionic type.*

For κ complex, $\kappa \neq \bar{\kappa}$, this implies immediately Proposition 2, since, using the fact that $S_{\mu\bar{\kappa}} = S_{\bar{\mu}\kappa} = S^*_{\mu\kappa}$,

$$\sum_\nu N^\nu_{\lambda\mu} = \sum_\kappa \frac{S_{\lambda\kappa} S_{\mu\kappa} \sum_\nu S^*_{\nu\kappa}}{S_{0\kappa}} = \sum_{\nu,\kappa=\bar{\kappa}} \frac{S_{\lambda\kappa} S_{\mu\kappa} S^*_{\nu\kappa}}{S_{0\kappa}} = \sum_{\nu,\kappa=\bar{\kappa}} \frac{S_{\lambda\kappa} S_{\bar{\mu}\kappa} S^*_{\nu\kappa}}{S_{0\kappa}} = \sum_\nu N^\nu_{\lambda\bar{\mu}} \,. \tag{8}$$

But conversely, as shown in [1] by a fairly simple argument, Prop. 2 implies that $\Sigma(\kappa) = 0$ if $\kappa \neq \bar{\kappa}$.

[1] We use the common convention that the component of the vertex located on the short branch of the Dynkin diagram is written at the end

The fact that the sum $\Sigma(\kappa)$ also vanishes for κ, a representation of quaternionic type, though of no direct relevance for the present discussion, is also a curious observation and was proved in [1] as the result of a case by case analysis.

1.5 The case of finite groups

Finite groups (admitting complex representations) do not generally satisfy Prop. 2. Consider for example the finite subgroup $\Gamma = \Sigma(3 \times 360)$ of SU(3) [6, 7]. A simple way to show that the equality of m_1 and $\overline{m}_1$ is not satisfied is to draw the oriented graph whose vertices are the irreps of Γ and whose adjacency matrix is the matrix $N^{\nu}_{f\mu}$, where f denotes one of the 3-dimensional irreducible representations, see Fig. 2. In Fig. 2, pairs of complex conjugate representations are images under a reflection through the horizontal axis. The sum $\sum_\nu N^{\nu}_{f\mu}$ counts the number of oriented edges exiting vertex μ. It is clear that the sums relative to $\mu =$ the outmost upper and lower vertices are different.
Ultimately, we found among subgroups of SU(3) the following counter-examples [1] to Prop. 2: $\Sigma(3 \times 72)$, $\Sigma(3 \times 360)$, and the subgroups of the type $F_{3m} = \mathbb{Z}_m \rtimes \mathbb{Z}_3$, where m should be a prime of the type $6p + 1$.

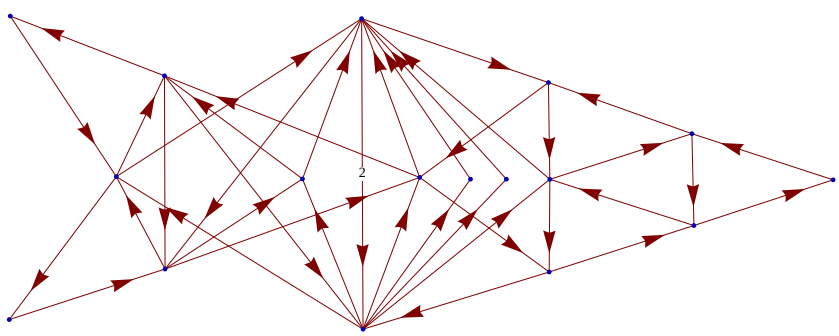

Figure 2: Tensor product graph for the subgroup $\Sigma(3 \times 360)$ of SU(3). Note: the middle vertical edge carries a multiplicity 2.

Discussion. Could the validity of Prop. 2 be related to the *modularity* of the tensor (or fusion) category, which holds true for Lie groups and affine algebras, but not generally for finite groups? In [2] we explored that possibility by constructing the Drinfeld doubles of subgroups of SU(2) and SU(3). While tensor product in Drinfeld doubles is known to be modular, we found again many counter-examples to Prop. 2, in particular for the double of the same group $\Sigma(3 \times 360)$. We conclude that the property encapsulated in Prop. 2 is not a modular property but rather seems to be a Lie theory property. See below in sect. 2 a remark on the role of the Weyl group in the proof.

The validity of Prop. 2 is not directly related, either, to the simplicity of the group considered; indeed, the Mathieu groups M_{11}, M_{12}, M_{21} M_{22}, M_{23}, M_{24} are simple finite groups, but Prop. 2 is only valid for M_{12} and M_{21} (the latter, although simple, does not appear in the list of sporadic simple groups because it is isomorphic with $PSL(3, 4)$).

We also considered those Chevalley groups that admit complex representations – otherwise Prop. 2 would be trivially verified. For small values of $n \geq 1$ and q (a power of a prime) we looked at examples from the families $A_n(q) = SL(n+1,q)$, $B_n(q) = O(2n+1,q)$, $C_n(q) = Sp(2n,q)$, $D_n(q) = \Omega^+(2n,q)$, $G_2(q)$, and also from the families called $2A_n(q) = SU(n+1,q)$, $2B_n(q) = Sz(q)$ (Suzuki), $2D_n(q) = \Omega^-(2n,q)$, $2G_2(q)$ (Ree), $3D_4(q)$, with the notations used in MAGMA [8]. We could not explicitly study examples from the families $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, or $2F_4(q)$ (Ree), $2E_6(q)$, because of the size of their character table. The largest simple group of Lie type that we considered (and obeying Prop. 2) was $G_2(5)$, with 5859000000 elements, 44 conjugacy classes (or irreps), and only four complex irreps. All together we tested about 70 Chevalley groups, 33 of them being simple, and 37 had complex irreps, so that testing the sum rule (Prop. 2) for them was meaningful. Among those 37 groups with complex irreps, 21 were simple and the sum rule was obeyed by all of them; among the $37 - 21 = 16$ non-simple groups with complex irreps, we found 4 cases for which the sum rule fails. In all cases where this sum rule failed for a non-simple Chevalley group, it turned out to hold for the corresponding projective group (a simple quotient of the latter): for instance the rule fails for the non-simple group $A_2(7) = SL(3,7)$ but it holds for the simple group $PSL(3,7)$ (and also holds for the non-isomorphic simple group $2A_2(7) = SU(3,7) = PSU(3,7)$). Although the obtained results may not be statistically significant they seem to indicate that Prop. 2 is valid for simple groups of Lie type. We did not try to prove this property but if it happens to be true, one may expect, for finite groups of Lie type, that the Weyl group could play a role in the proof, like in the case of simple Lie groups (see below).

2 A sketch of proofs

The proof of Prop. 1 has been given above. We shall content ourselves with sketches of proofs for the other propositions.

The proof of Prop. 2 may be split in two steps.
Lemma 1. *Prop. 2 holds for $\lambda = \omega_p$, a fundamental weight.*
Lemma 2. *Prop. 2 holds for any product of the fundamental representations.*

Proof. The first lemma was established for the A_r, $D_{r\,\mathrm{odd}}$, E_6 simple Lie algebras (the others do not have complex irreps) making use of the Racah-Speiser formula, or of its affine extension. The latter expresses the tensor coefficient $N_{\lambda\mu}^{\nu}$ as a weighted sum over suitable elements of the (classical or affine) Weyl group, see [1] for details. Restricting λ to be a fundamental weight makes the discussion amenable to a fairly simple analysis of a finite number of cases.

The second lemma follows simply from the associativity and commutativity of the tensor or fusion product, using the outcome of Lemma 1:

$$\sum_{\nu} \left(N_{\omega_p}\right)_{\mu}^{\ \nu} = \sum_{\nu} \left(N_{\bar{\omega}_p}\right)_{\mu}^{\ \nu} \ .$$

This, together with the commutativity of the N matrices, entails that for any monomial $N_{\omega_{j_1}} \cdots N_{\omega_{j_q}}$

$$\begin{aligned}
\sum_{\nu} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_q}} \right)_{\mu}^{\nu} &= \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu'} \sum_{\nu} \left(N_{\omega_{j_q}} \right)_{\nu'}^{\nu} \\
&\overset{\text{(Lemma 1)}}{=} \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu'} \sum_{\nu} \left(N_{\bar{\omega}_{j_q}} \right)_{\nu'}^{\nu} \\
&= \sum_{\nu} \left(N_{\bar{\omega}_{j_q}} N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}} \right)_{\mu}^{\nu} = \cdots \\
&= \sum_{\nu} \left(N_{\bar{\omega}_{j_1}} \cdots N_{\bar{\omega}_{j_q}} \right)_{\mu}^{\nu} \qquad (9)
\end{aligned}$$

This completes the proof of the two lemmas. As any N_λ is a polynomial in the commuting N_{ω_p}, $p = 1, \cdots, r$, $N_\lambda = P_\lambda(N_{\omega_1}, \cdots, N_{\omega_r})$ and $N_{\bar{\lambda}} = P_\lambda(N_{\bar{\omega}_1}, \cdots, N_{\bar{\omega}_r})$, this also establishes Prop. 2. □

The salient feature of this approach is the crucial role played by the (classical or affine) Weyl group.

Prop. 3 and 4, which deal with the explicit case of the classical or affine $su(3)$ algebra, have been established through a detailed and laborious analysis which will not be repeated here. We only mention that a variety of graphical representations of the determination of the $N_{\lambda\mu}^{\nu}$ coefficients has been used. We refer the reader to [3] and [4] for details.

3 Conclusion

In this letter, we have reviewed some recent results on conjugation properties of tensor product (or fusion) multiplicities. Although quite simple to state, it appears that these results were not previously known, and that some are fairly difficult to prove. In particular we feel that our proofs of Propositions 2, 3 and 4 lack elegance and may miss some essential concept. Hopefully some inspired reader will come with new insight in these matters.

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