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# Essential norms of Volterra and Cesàro operators on Müntz spaces

Ihab Al Alam,<sup>\*</sup> Georges Habib,<sup>†</sup> Pascal Lefèvre,<sup>‡</sup> Fares Maalouf<sup>§</sup>

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## Abstract

We study the properties of the Volterra and Cesàro operators viewed on the  $L^1$ -Müntz space  $M_\Lambda^1$  with range in the space of continuous functions. These operators are neither compact nor weakly compact. We estimate how far from being (weakly) compact they are by computing their (generalized) essential norm. It turns out that this latter does not depend on  $\Lambda$  and is equal to  $1/2$ .

**Key words:** Volterra operator, Cesàro operator, Müntz spaces, compact operator, essential norm.

**Mathematics Subject Classification:** 47B07, 47B38, 30H99

## 1 Introduction

Throughout this paper, we denote by  $C = C([0, 1])$  the space of continuous functions on  $[0, 1]$  equipped with the supremum norm, and  $L^p = L^p([0, 1])$  for  $p \geq 1$ , the usual spaces of integrable functions over  $[0, 1]$  with respect to the Lebesgue measure.

Let  $\Lambda$  be a strictly increasing sequence of positive numbers. We say that  $\Lambda$  satisfies the *Müntz condition* if and only if  $\sum_{\lambda \in \Lambda \setminus \{0\}} 1/\lambda < \infty$ . Let  $M_\Lambda$  be the linear space spanned by the monomials  $x^\lambda$  with  $\lambda \in \Lambda$ , and  $M_\Lambda^p$  the

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closure of  $M_\Lambda$  in  $L^p$  ( $1 \leq p < \infty$ ). The classical theorem of Müntz states that  $M_\Lambda^p$  is a proper subspace of  $L^p$  if and only if the Müntz condition holds (see [M], [BE]). This result remains true for the closure of  $M_\Lambda$  in  $C$ , denoted by  $M_\Lambda^\infty$ . The spaces  $M_\Lambda^p$  ( $1 \leq p \leq \infty$ ) are called *Müntz spaces* whenever the Müntz condition is fulfilled.

An interesting question in this framework is the study of operators acting on Müntz spaces. In this context, the composition and multiplication operators have been investigated in [A], [AL] and [N], where several results have been obtained for the compactness and the weak compactness. However, these operators do not map the Müntz spaces into themselves in general.

In this paper, we are interested in two classical operators of Hardy-Volterra type, namely the Volterra and Cesàro operators. Recall that the Volterra operator  $V$  is defined over the space  $L^1$  by  $V(f)(x) = \int_0^x f(t) dt$ , and is mapped to the space  $C$  by the usual Lebesgue theorem. The restriction  $V_\Lambda$  of  $V$  to the space  $M_\Lambda^1$  does not stabilize it, but almost, since  $V(M_\Lambda) = M_{1+\Lambda}$ . On the other hand, the Cesàro operator  $\Gamma$  is better-behaved relatively to this problem. Indeed, it is defined for  $x \in (0, 1]$  by  $\Gamma(f)(x) = \frac{1}{x}V(f)(x)$  for any  $f \in L^1$ , and therefore  $\Gamma(M_\Lambda) = M_\Lambda$ . Thanks to the Hardy inequality, it is well-known that the Cesàro operator maps the space  $L^p$  into  $L^p$  when  $p > 1$  but it does not map  $L^1$  into itself. By the Clarkson-Erdős theorem, any  $f \in M_\Lambda^1$  is equal to an analytic function  $\tilde{f}$  on  $[0, 1)$ , and the a priori ambiguous value  $f(0)$  will be defined by taking  $f(0) := \tilde{f}(0)$ . Therefore, the Cesàro operator on  $M_\Lambda^1$  is defined by

$$\Gamma_\Lambda(f)(x) = \begin{cases} f(0) & \text{if } x = 0 \\ \frac{1}{x} \int_0^x f(t) dt & \text{if } x \in ]0, 1]. \end{cases}$$

This can also be written as  $\Gamma_\Lambda(f)(x) = \int_0^1 f(xu) du$ . We easily see that  $\Gamma_\Lambda$  maps  $M_\Lambda^1$  to  $C$ .

This paper is divided into four parts. In section 2, we recall some preliminaries on the Müntz spaces. In section 3, we prove a general criterion for getting a lower bound for the essential norm of a bounded operator  $T$  (see Proposition 3.2). Recall that the essential norm of an operator  $T$  is the distance from  $T$  to the space of compact (or weakly compact) operators. Section 4 will be devoted to the study of some properties of the Volterra and Cesàro operators. We show that their essential norm is equal to  $\frac{1}{2}$ , and is thus independent from the sequence  $\Lambda$  (see Theorems 4.1 and 4.3). In the

last section, we study some further types of weak compactness, namely the strict singularity and the finite strict singularity. We show that  $V_\Lambda$  and  $\Gamma_\Lambda$  are finitely strictly singular, and that the growth of their Bernstein numbers is of order  $\frac{1}{n}$  (Theorem 5.3).

## 2 Preliminaries

In this section, we recall some basic ingredients of the geometry of Müntz spaces. These results can be found in detail in [BE] and [GL].

Given a strictly increasing sequence of non-negative real numbers  $\Lambda = (\lambda_k)_{k=0}^\infty$  such that  $\sum_{k=1}^\infty 1/\lambda_k < \infty$ , A. Clarkson and P. Erdős (see [GL, p.81] and [S1, S2]) proved that any element in the Müntz spaces can be represented as a series of power functions  $x^\lambda$  with  $\lambda \in \Lambda$ . The full statement of the result is the following:

**Proposition 2.1** *Let  $\Lambda = (\lambda_k)_{k=0}^\infty$  be a strictly increasing sequence of non-negative real numbers satisfying the Müntz condition. Assume that the gap condition  $\inf\{\lambda_{k+1} - \lambda_k : k \in \mathbb{N}\} > 0$  holds. For every function  $f \in L^p$ , we have  $f \in M_\Lambda^p$  if and only if  $f$  can be represented as*

$$f(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k}, \quad x \in [0, 1).$$

If the gap condition does not hold, any function  $f \in M_\Lambda^p$  can still be represented by an analytic function on  $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$  restricted to  $(0, 1)$ . Note that there is no information on the value of  $f$  at the point 1, but one can estimate the supremum norm of  $f$  over a compact set far from 1, by the  $L^1$ -norm of  $f$  over a compact set approaching 1 (see [BE, p.185]). Using this last result and the Arzela-Ascoli theorem, the following property of the Müntz spaces was established in [AL, Corollary 2.5] :

**Lemma 2.2** *Assume that  $(f_n)_n$  is a bounded sequence in  $M_\Lambda^1$ . There is a subsequence of  $(f_n)_n$  that converges uniformly on every compact subset of  $[0, 1)$ .*

We recall the definition of the essential norm of any bounded operator relatively to a closed subspace of operators.

**Definition 2.3** *Let  $X, Y$  be two Banach spaces, and  $\mathcal{I}$  a closed subspace of operators of the space  $B(X, Y)$  of bounded operators from  $X$  to  $Y$ . The essential norm of  $T \in B(X, Y)$ , relatively to  $\mathcal{I}$ , is the distance from  $T$  to  $\mathcal{I}$  defined by*

$$\|T\|_{e, \mathcal{I}} = \inf\{\|T - S\|; S \in \mathcal{I}\}.$$

*This is the canonical norm on the quotient space  $B(X, Y)/\mathcal{I}$ .*

With the same notations as in the above definition, if  $\mathcal{I}$  is the space  $\mathcal{K}(X, Y)$  of compact operators from  $X$  to  $Y$ , then  $\|T\|_{e, \mathcal{I}}$  will be denoted by  $\|T\|_e$ , and will be referred to as *the essential norm of  $T$* . If  $\mathcal{I}$  is the space  $\mathcal{W}(X, Y)$  of weakly compact operators from  $X$  to  $Y$ , then  $\|T\|_{e, \mathcal{I}}$  will be denoted by  $\|T\|_{e, w}$ . The following fact is a direct consequence of the definitions.

**Fact:** Let  $T : X \rightarrow Y$  be an operator, and let  $X_0$  and  $Y_1$  be subspaces of  $X$  and  $Y$  respectively, such that  $T(X) \subset Y_1$ . Let  $T_0$  be the operator obtained from  $T$  by restricting the domain to  $X_0$ , and  $T_1$  be the one obtained from  $T$  by restricting the codomain to  $Y_1$ . Then

$$\|T_0\|_e \leq \|T\|_e \leq \|T_1\|_e.$$

### 3 Lower bound of the essential norm

In this section, we will prove that the essential norm of any operator can be bounded from below by some term involving the “height of a discontinuity” of a function at some point. The idea is based on an argument used in [Z] for the classical essential norm, and in [L2] for the essential norm relatively to the weakly compact operators. For this, we say that a sequence  $(\tilde{x}_n)_{n \in \mathbb{N}}$  is a block-subsequence of  $(x_n)_{n \in \mathbb{N}}$  if there is a sequence of non empty finite subsets of integers  $(I_m)_{m \in \mathbb{N}}$  with  $\max I_m < \min I_{m+1}$  and  $c_i \in [0, 1]$  such that

$$\tilde{x}_m = \sum_{j \in I_m} c_j x_j \quad \text{and} \quad \sum_{j \in I_m} c_j = 1.$$

**Lemma 3.1** *Let  $T : X \rightarrow Y$  be a bounded operator from a Banach space  $X$  to another Banach space  $Y$ . Let  $(x_n)_{n \in \mathbb{N}}$  be a normalized sequence in  $X$  and  $\alpha > 0$ .*

1. *Assume that for any subsequence  $(x_{\varphi(n)})_{n \in \mathbb{N}}$  and any  $g \in Y$ , we have  $\overline{\lim} \|T(x_{\varphi(n)}) - g\| \geq \alpha$ . Then  $\|T\|_e \geq \alpha$ .*
2. *Assume that for any block-subsequence  $(\tilde{x}_n)_{n \in \mathbb{N}}$  and any  $g \in Y$ , we have  $\overline{\lim} \|T(\tilde{x}_n) - g\| \geq \alpha$ . Then  $\|T\|_{e, w} \geq \alpha$ .*

**Proof.** We prove only (2) since (1) is similar (and easier). Fix any weakly compact operator  $S : X \rightarrow Y$  and consider a normalized sequence  $(x_n)_{n \in \mathbb{N}}$ . From the weak compactness of  $S$ , we know that there exists a subsequence of  $(S(x_n))_{n \in \mathbb{N}}$  that converges weakly to some  $g \in Y$ . By the Banach-Mazur theorem, there exists a block-subsequence  $(\tilde{x}_n)_{n \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  with  $(S(\tilde{x}_n)) \rightarrow g$  in norm. Note that  $(\tilde{x}_n)_{n \in \mathbb{N}}$  lies in the unit ball of  $X$ . Hence,  $\|T - S\| \geq \overline{\lim} \|T(\tilde{x}_n) - g\| - \lim \|S(\tilde{x}_n) - g\| \geq \alpha$ . The lemma follows.  $\square$

Let  $K$  be a metric space. We will say that a function  $H : K \rightarrow \mathbb{C}$  has a discontinuity at the point  $t_0$  with height  $h > 0$  if  $\overline{\lim}_{\delta \rightarrow 0} \text{diam}(H(B(t_0, \delta))) \geq h$ , where  $B(t_0, \delta)$  denotes the open ball centered at  $t_0$  with radius  $\delta$ . Equivalently, one can find two sequences  $(s_n)_{n \in \mathbb{N}}$  and  $(s'_n)_{n \in \mathbb{N}}$  both converging to  $t_0$  and satisfying  $\overline{\lim} |H(s_n) - H(s'_n)| \geq h$ .

**Proposition 3.2** *Let  $T : X \rightarrow C(K)$  be a bounded operator from a Banach space  $X$  to the space  $C(K)$  of continuous functions over a compact  $K$ . We assume that there exists a normalized sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  such that  $T(x_n)$  converges pointwise to some function  $H$  with discontinuity of height  $h > 0$  at some point  $t_0 \in K$ . Then*

$$\|T\|_{e,w} \geq \frac{h}{2}.$$

**Proof.** Fix any  $g \in C(K)$  and any block-subsequence  $(\tilde{x}_n)_{n \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$ . With the previous notations, we write  $\tilde{x}_m = \sum_{j \in I_m} c_j x_j$ , with  $\sum_{j \in I_m} c_j = 1$ . For every  $t \in K$ , we have

$$T(\tilde{x}_m)(t) = \sum_{j \in I_m} c_j T(x_j)(t) \longrightarrow H(t)$$

since  $\min I_m \rightarrow +\infty$ . Thus, we obtain that

$$\overline{\lim} \|T(\tilde{x}_n) - g\| \geq \sup_{t \in K} \overline{\lim} |T(\tilde{x}_n)(t) - g(t)| = \sup_{t \in K} |H(t) - g(t)|.$$

Since the function  $H$  has a discontinuity at the point  $t_0$  with height  $h > 0$ , we can find two sequences  $s_n \rightarrow t_0$  and  $s'_n \rightarrow t_0$  such that

$$\overline{\lim} |H(s_n) - H(s'_n)| \geq h.$$

Therefore, we get that

$$\overline{\lim} \|T(\tilde{x}_n) - g\| \geq \max \{ \overline{\lim} |H(s_n) - g(t_0)|; \overline{\lim} |H(s'_n) - g(t_0)| \} \geq \frac{h}{2}$$

otherwise we would have

$$h \leq \overline{\lim} |H(s_n) - H(s'_n)| \leq \overline{\lim} |H(s_n) - g(t_0)| + \overline{\lim} |g(t_0) - H(s'_n)| < h$$

by continuity of  $g$  at point  $t_0$ . This finishes the proof of the proposition.  $\square$

**Application :** Let  $K$  be a (metric) connected compact set,  $\theta : K \rightarrow K$  be a non-constant continuous map, and  $T$  be the composition operator from  $C(K)$  into itself defined by  $f \in C(K) \mapsto f \circ \theta$ . If we consider the sequence

$$x_n(t) = \frac{nd(t, \alpha) - 1}{nd(t, \alpha) + 1}$$

where  $\alpha \in \text{Im}\theta$ , then

$$H(t) = \left( \lim_{n \rightarrow \infty} T x_n \right) (t) = \begin{cases} -1, & \text{if } t \in \theta^{-1}(\{\alpha\}); \\ 1, & \text{if not.} \end{cases}$$

The function  $H$  has a discontinuity at any point of the boundary of  $\theta^{-1}(\{\alpha\})$  (which is non empty by connectedness) with height 2. We get on one hand that  $\|T\|_e \geq \|T\|_{e,w} \geq 1$ . On the other hand, since  $\|T\|_e \leq \|T\| \leq 1$ , we deduce  $\|T\|_e = \|T\|_{e,w} = 1$ .

## 4 The Volterra and Cesàro operators

In this section, we will study some properties of the Volterra and Cesàro operators. These operators are neither compact nor weakly compact when restricted to the Müntz spaces. We will show that their (generalized) essential norm is equal to  $\frac{1}{2}$ , independently of the choice of the sequence  $\Lambda$ .

Recall first that the Volterra operator  $V$  is defined by

$$V(f) = \int_0^x f(t) dt$$

for any  $f \in L^1$ . The operator  $V$  is clearly well-defined from  $L^1$  to  $C$ , and is bounded with norm 1. It also acts as an isometry on the set of positive functions. If considered as an operator from  $L^p$  to  $C$  with  $p > 1$ , or from  $L^1$  to  $L^p$  with  $p$  finite, it is easy to see (via Ascoli's theorem) that  $V$  is compact. This implies that it is also compact, when viewed as a map from  $L^p$  to itself, for any  $p \geq 1$ . Considered as an operator from  $L^1$  to  $C$ , it is easy to see that  $V$  is not compact.

Actually the restriction of  $V$  to the Müntz space  $M_\Lambda^1$  is not weakly compact. Indeed, consider any increasing sequence  $(\lambda_n)$  in  $\Lambda$  and the associated normalized sequence  $x \mapsto (\lambda_n + 1) x^{\lambda_n}$  (for  $n \in \mathbb{N}$ ) in  $M_\Lambda^1 \subset L^1$ . The image of this sequence by  $V$  is the sequence  $x \mapsto x^{\lambda_n+1}$ , which admits no weakly convergent subsequence in  $C$ . We first compute the essential norm of the Volterra operator viewed as a map from  $L^1$  to  $C$ .

**Theorem 4.1** *Let  $V$  be the Volterra operator viewed as a map from  $L^1$  to  $C$ . Then*

$$\|V\|_e = \|V\|_{e,w} = \frac{1}{2}.$$

**Proof.** For the upper bound, we find a compact operator  $S : L^1 \rightarrow C$  such that  $\|V - S\| \leq \frac{1}{2}$ . For this, we set  $S(f) = \frac{1}{2} \int_0^1 f(t) dt$ , for  $f \in L^1$ . The

operator  $S$  is clearly compact since it has rank 1, and

$$\begin{aligned}
\|V - S\| &= \sup \{ \|V(f) - S(f)\|_\infty : \|f\|_1 = 1 \} \\
&= \sup \left\{ \left| \int_0^x f(t) dt - \frac{1}{2} \int_0^1 f(t) dt \right| : \|f\|_1 = 1; x \in [0, 1] \right\} \\
&= \sup \left\{ \left| \frac{1}{2} \int_0^x f(t) dt - \frac{1}{2} \int_x^1 f(t) dt \right| : \|f\|_1 = 1; x \in [0, 1] \right\} \\
&\leq \frac{1}{2} \sup \left\{ \int_0^1 |f(t)| dt : \|f\|_1 = 1 \right\} = \frac{1}{2}.
\end{aligned}$$

For the lower bound, consider any increasing sequence  $(\gamma_n)_n$  which tends to infinity and define a normalized sequence  $g_n(x) = (\gamma_n + 1)x^{\gamma_n}$  in  $L^1$ . The sequence  $V(g_n) = x^{\gamma_n+1}$  converges pointwise to the function  $H$  which is equal to 0 on  $[0, 1)$  and to 1 at the point 1. The function  $H$  has a jump of height 1 at 1, and the result follows by Proposition 3.2.  $\square$

In the sequel, we will study the properties of the Cesàro operator  $\Gamma$ . Recall that for any function  $f \in L^1$ ,  $\Gamma(f)$  is defined at any point  $x \in (0, 1]$  by

$$\Gamma(f)(x) = \frac{1}{x} V(f)(x).$$

As mentioned earlier, the Cesàro operator does not map  $L^1$  into itself. We will then study the restriction  $\Gamma_\Lambda$  of  $\Gamma$  to the Müntz spaces. Recall that for every  $f \in M_\Lambda^1$

$$\Gamma_\Lambda(f)(x) = \begin{cases} f(0) & \text{if } x = 0 \\ \frac{1}{x} \int_0^x f(t) dt & \text{if } x \in (0, 1] \end{cases}.$$

It is clear that  $\Gamma_\Lambda(f) \in M_\Lambda^\infty$ . Moreover, it has the Erdős decomposition  $\sum_{k=0}^\infty a_k \frac{x^{\lambda_k}}{\lambda_k + 1}$  when  $f$  has the decomposition  $\sum_{k=0}^\infty a_k x^{\lambda_k}$ .

**Proposition 4.2** *The operator  $\Gamma_\Lambda$  satisfies the following:*

1. *It is a bounded, one-to-one operator, and its image  $\text{Im } \Gamma_\Lambda$  is dense but not closed in  $M_\Lambda^\infty$ .*
2. *It is a Dunford-Pettis and non weakly compact operator.*

**Proof.** The operator  $\Gamma_\Lambda$  is the composition of the bounded operator  $V_\Lambda$ , and the division operator  $Q$  defined from  $M_{1+\Lambda}^\infty$  to  $M_\Lambda^\infty$  by

$$Q(f) = \frac{1}{x} f(x).$$



This latter operator is bounded by the closed graph theorem and Proposition 2.1. The operator  $\Gamma_\Lambda$  is clearly injective. The density of the image is given by the fact that  $M_\Lambda \subset \text{Im } \Gamma_\Lambda$ . Assume for a contradiction that  $\text{Im } \Gamma_\Lambda$  is closed. Then  $\Gamma_\Lambda : M_\Lambda^1 \mapsto M_\Lambda^\infty$  is onto, thus an isomorphism by the Banach theorem. Let  $S$  be a lacunary subsequence of  $\Lambda$ . The restriction of  $\Gamma_\Lambda$  to  $M_S^1$  is then an isomorphism onto  $M_S^\infty$ . On the other hand,  $M_S^1$  is isomorphic to the space  $\ell^1$ , and  $M_S^\infty$  is isomorphic to the space  $c$  of convergent sequences (see Theorem 9.2.2 of [GL]). So  $\ell^1$  is isomorphic to  $c$ , and this is a contradiction since, for instance, the Schur property holds in  $\ell^1$  and not in  $c$ .

Now we show that  $\Gamma_\Lambda$  is not weakly compact. Assume for a contradiction that it is not the case, and consider the normalized sequence  $(f_n)_n$  in  $M_\Lambda^1$  defined by  $f_n(x) = (\lambda_n + 1)x^{\lambda_n}$ . There exists a subsequence  $(f_{n_k})_k$  such that  $\Gamma_\Lambda(f_{n_k}) = x^{\lambda_{n_k}}$  converges weakly (*i.e.* pointwise) to  $f \in C$ . Hence,  $f = 0$  on  $[0, 1)$  and  $f(1) = 1$  which is a contradiction. Finally,  $\Gamma$  is a Dunford-Pettis operator, for  $M_\Lambda^1$  has the Schur property since it can be realized as an isomorphic copy of a subspace of  $\ell^1$ .  $\square$

**Remark 1** *We mention the following useful facts:*

1. For any  $p \geq 1$ , the Cesàro operator  $\Gamma_\Lambda$  viewed from  $M_\Lambda^1$  to  $M_\Lambda^p$  is a compact operator. Indeed, it is the composition of the identity map  $i_p$  from  $M_\Lambda^\infty$  to  $M_\Lambda^p$ , which is compact, with the operator  $\Gamma_\Lambda$ . The compactness of  $i_p$  is guaranteed by Lemma 2.2. Similarly, it follows from the Ascoli theorem that the Cesàro operator viewed from  $M_\Lambda^p$  to  $M_\Lambda^\infty$  is compact for  $p > 1$ .
2. The operator  $D : M_\Lambda^\infty \rightarrow M_\Lambda^1; f \mapsto (xf)'|_{[0,1]}$  is not a bounded operator, since this would give that  $\Gamma_\Lambda$  is an isomorphism and  $\Gamma_\Lambda^{-1} = D$ .

**Theorem 4.3** *Let  $\Lambda$  be an increasing sequence of positive numbers satisfying the Müntz condition. Let  $\Gamma_\Lambda$  and  $V_\Lambda$  be the Cesàro and Volterra operators, viewed as maps from  $M_\Lambda^1$  to  $M_\Lambda^\infty$  or  $C$ . Then*

1.  $\|\Gamma_\Lambda\|_e = \|\Gamma_\Lambda\|_{e,w} = \frac{1}{2}$ .
2.  $\|V_\Lambda\|_e = \|V_\Lambda\|_{e,w} = \frac{1}{2}$ .

To get the upper bound in the both preceding statements, we will use the following proposition, which gives a more general result for some weighted versions of the Volterra-Cesàro type operators. For this, fix a continuous function  $q$  on  $[0, 1]$ . We define an operator  $H_q$  on  $M_\Lambda^1$  as follows. For  $f \in M_\Lambda^1$ ,

$$H_q(f)(x) = \begin{cases} q(0)f(0) & \text{if } x = 0 \\ \frac{q(x)}{x} \int_0^x f(t) dt & \text{if } x \in (0, 1] \end{cases}.$$

If  $q(x) = x$ , we recover the Volterra operator  $V_\Lambda$ , and if  $q = \mathbb{1}$ , we recover the Cesàro operator  $\Gamma_\Lambda$ . We also point out that the range of  $H_q$  is included in the space  $Q_\Lambda = q.M_\Lambda^\infty$ .

**Proposition 4.4** *With the preceding notations, we have*

$$d(H_q, \mathcal{K}(M_\Lambda^1, Q_\Lambda)) \leq \frac{\|q\|_\infty}{2} \quad \text{and} \quad d(H_q, \mathcal{K}(M_\Lambda^1, C)) \leq \frac{|q(1)|}{2}.$$

**Proof.** For the first inequality, let  $\lambda \in \Lambda$  and  $\rho \in (0, 1)$ . Let  $R$  and  $T_\rho$  be the operators from  $M_\Lambda^1$  to  $Q_\Lambda$  defined for  $f \in M_\Lambda^1$  by

$$R(f)(x) = \frac{q(x)x^\lambda}{2} \int_0^1 f(t) dt$$

and

$$T_\rho(f)(x) = \begin{cases} \rho q(0)f(0) & \text{if } x = 0 \\ \frac{q(x)}{x} \int_0^{\rho x} f(t) dt & \text{if } x \in (0, 1] \end{cases}.$$

The operator  $R$  is compact since it has rank 1. The operator  $T_\rho$  is also compact since it is nuclear. Indeed, it can be written as the sum of an absolutely convergent series of rank one operators in the following way: consider the functionals  $e_n$  defined on  $M_\Lambda^1$  by

$$e_n : \sum_{n \geq 0} a_n x^{\lambda_n} \mapsto a_n.$$

By a result from [BE, p.178], the norm of the map  $e_n$  is bounded from above by  $C_\rho \rho^{-\lambda_n}$ , where  $C_\rho$  is a constant depending only on  $\rho$ . Hence for every polynomial  $f$  in  $M_\Lambda^1$ , we have

$$T_\rho(f)(x) = \sum_{n \geq 0} e_n(f) \frac{\rho^{\lambda_n+1}}{\lambda_n + 1} x^{\lambda_n} q(x).$$

The operator  $f \mapsto e_n(f) \frac{\rho^{\lambda_n+1}}{\lambda_n + 1} x^{\lambda_n} q(x)$  has rank one, and

$$\sum_{n \geq 0} \frac{\|e_n\| \rho^{\lambda_n+1}}{\lambda_n + 1} \leq \sum_{n \geq 0} \frac{C_\rho}{\lambda_n + 1} < \infty.$$

We show now that  $\lim_{\rho \rightarrow 1^-} \|H_q - (R + T_\rho)\| \leq \frac{\|q\|_\infty}{2}$ . For this, fix an arbitrary  $\varepsilon \in (0, 1)$ , and let  $c \in (0, 1)$  be such that  $2 - c^{\lambda+1} \leq (1 + \varepsilon)c$ . Let  $f$  be in the

unit ball of  $M_\Lambda^1$ , and  $x \in (0, 1)$ .

$$\begin{aligned} \left( H_q(f) - R(f) - T_\rho(f) \right)(x) &= \frac{q(x)}{x} \int_{\rho x}^x f(t) dt - \frac{x^\lambda q(x)}{2} \int_0^1 f(t) dt \\ &= \frac{q(x)}{2} \int_0^1 \varphi_x(t) f(t) dt \end{aligned} \tag{4.1}$$

with  $\varphi_x(t) = \frac{2}{x} - x^\lambda$  if  $t \in [\rho x, x]$  and  $\varphi_x(t) = -x^\lambda$  if  $t \notin [\rho x, x]$ . Now we give an upper bound to the value of  $|H_q(f) - R(f) - T_\rho(f)|(x)$  according to the position of  $x$ . If  $x \leq c$ , we apply the first equality in (4.1) and obtain the estimate

$$\begin{aligned} |H_q(f)(x) - R(f)(x) - T_\rho(f)(x)| &\leq (1 - \rho) \|q\|_\infty \sup_{t \in [0, c]} |f(t)| + \frac{\|q\|_\infty}{2} \|f\|_1 \\ &\leq \left( (1 - \rho) N_\varepsilon + \frac{1}{2} \right) \|q\|_\infty \|f\|_1 \\ &\leq \left( \varepsilon + \frac{1}{2} \right) \|q\|_\infty \end{aligned}$$

where we used [BE, p.185] to get  $N_\varepsilon \geq 1$  such that

$$\sup_{t \in [0, c]} |f(t)| \leq N_\varepsilon \|f\|_1$$

for any  $f \in M_\Lambda^1$ , and we chose  $\rho$  to be equal to  $1 - \frac{\varepsilon}{N_\varepsilon} \in (0, 1)$ . If  $x \geq c$ , we use the second equality in (4.1) which yields

$$|H_q(f)(x) - R(f)(x) - T_\rho(f)(x)| \leq \frac{1}{2} \|\varphi_x\|_\infty \|q\|_\infty \|f\|_1 \leq \frac{1 + \varepsilon}{2} \|q\|_\infty.$$

Therefore, since  $\varepsilon$  is arbitrary, the proof of the first inequality is finished.

The proof for the second inequality works as well replacing the operator  $R$  by

$$R_1(f)(x) = \frac{q(1)}{2} \int_0^1 f(t) dt.$$

Now the computation follows the lines of the first part but a new function  $\varphi_x$  appears which is given by  $\varphi_x(t) = \frac{2q(x)}{x} - q(1)$  if  $t \in [\rho x, x]$  and  $\varphi_x(t) = -q(1)$  if  $t \notin [\rho x, x]$ . A suitable choice of  $c$  relatively to the continuity of  $q$  at the point 1 gives  $\|\varphi_x\|_\infty \leq |q(1)| + \varepsilon$  for  $x \geq c$ . On the other hand, we manage the case  $x \leq c$  like before to get an upper bound  $\varepsilon + \frac{|q(1)|}{2}$ .  $\square$

We note that if  $0 \in \Lambda$ , the previous proposition is not needed for estimating the distance of the Volterra operator  $V_\Lambda$  to  $\mathcal{K}(M_\Lambda^1, M_\Lambda^\infty)$ . The short argument of Theorem 4.1 works for this case.

**Proof of Theorem 4.3.** The lower bound follows from Proposition 3.2, exactly as in Theorem 4.1. Just pick the exponents  $\gamma_n$  in  $\Lambda$ . The upper bound follows immediately from Proposition 4.4.  $\square$

## 5 Strict singularity

In this section, we will study other weak forms of compactness, namely the strict singularity and the finite strict singularity. We will show that  $V_\Lambda$  and  $\Gamma_\Lambda$  share these types of compactness. Moreover, we will estimate the  $n^{\text{th}}$  Bernstein numbers of these operators by showing that their growth are of order  $\frac{1}{n}$ . For this, let us recall the two definitions:

**Definition 5.1** *An operator  $T$  from a Banach space  $X$  to a Banach space  $Y$  is strictly singular if it never induces an isomorphism on an infinite dimensional (closed) subspace of  $X$ . That is for every  $\varepsilon > 0$  and every infinite dimensional subspace  $E$  of  $X$ , there exists  $v$  in the unit sphere of  $E$  such that  $\|T(v)\| \leq \varepsilon$ .*

This notion is now very classical and widely studied (see [LT, p.75] for instance).

**Definition 5.2** *An operator  $T$  from a Banach space  $X$  to a Banach space  $Y$  is finitely strictly singular if for every  $\varepsilon > 0$ , there exists  $N_\varepsilon \geq 1$  such that for every subspace  $E$  of  $X$  with dimension greater than  $N_\varepsilon$ , there exists  $v$  in the unit sphere of  $E$  such that  $\|T(v)\| \leq \varepsilon$ .*

This latter definition can be reformulated in terms of the so-called *Bernstein approximation numbers* (see [Pl] for instance). Recall that the  $n^{\text{th}}$  Bernstein number of an operator  $T$  is defined as

$$b_n(T) = \sup_{\substack{E \subset X \\ \dim(E)=n}} \inf_{\substack{v \in E \\ \|v\|=1}} \|T(v)\|.$$

Hence, with this terminology, the operator  $T$  is finitely strictly singular if and only if  $(b_n(T))_{n \geq 1}$  belongs to the space  $c_0$  of null sequences. This notion has appeared in the late sixties. For instance, in a paper of V. Milman [Mi], it is proved that the identity from  $\ell^p$  to  $\ell^q$  ( $p < q$ ) is finitely strictly singular (see [CFPTT], [Pl], [L], [LR] for recent results). It is also well-known that

compactness  $\implies$  finite strict singularity  $\implies$  strict singularity  
and that the reverse implications are not true. Moreover, complete continuity is not comparable to finite strict singularity in general. In [L], it is proved that the classical Volterra operator is finitely strictly singular (hence strictly singular) from  $L^1$  to  $C$  and moreover the Bernstein numbers satisfy

$$b_n(V) \approx \frac{1}{n}.$$

By restriction, the operator  $V_\Lambda$  is clearly finitely strictly singular as well. It is easy to see that the same property occurs for  $\Gamma_\Lambda$ . But a natural question then arises: how fast the Bernstein numbers vanish? Surprisingly, it turns out that it does not really depend on  $\Lambda$  and the answer is given in the following theorem

**Theorem 5.3** *The operators  $V_\Lambda$  and  $\Gamma_\Lambda$  are finitely strictly singular. Moreover the growth of their Bernstein numbers is of order  $\frac{1}{n}$ . For every  $n \geq 1$ , we have*

$$b_n(V_\Lambda) \geq \frac{1}{2n} \quad \text{and} \quad b_n(\Gamma_\Lambda) \geq \frac{1}{2n}.$$

**Proof.** Since  $V$  is finitely strictly singular ([L]), it is clear by restriction that  $V_\Lambda$  is also finitely strictly singular. Moreover by the fact that the map  $f \in M_{1+\Lambda}^\infty \mapsto \frac{1}{x}f \in M_\Lambda^\infty$  is bounded, the operator  $\Gamma_\Lambda$  is again finitely strictly singular. Moreover, we have that  $b_n(V_\Lambda) \leq b_n(V) \leq \frac{C}{2n-1}$  where  $C$  is equal to 1 or  $\sqrt{2}$  according to the fact that we consider real or complex valued functions ([L]). Now, let us estimate the lower bound. We mainly follow the ideas of Newman in [Ne, Lemma 2]. Fix  $n \geq 1$  and  $\varepsilon > 0$  and let  $(\lambda'_n)_{n \in \mathbb{N}}$  be a subsequence of  $\Lambda$  of positive numbers going very fast to the infinity such that it satisfies the following condition

$$\prod \left[ 1 - \frac{2\lambda_j'^2(1 + \ln(\lambda_{j+1}'))}{\lambda_{j+1}'} \right] \geq 1 - \varepsilon.$$

It is straightforward from [Ne, Lemma 2] to get for every  $a_1, \dots, a_n \in \mathbb{C}$  that

$$\left\| \sum_{k=1}^n a_k x^{\lambda'_k} \right\|_\infty \geq (1 - \varepsilon) \max_{1 \leq m \leq n} \left\| \sum_{k=1}^m a_k x^{\lambda'_k} \right\|_\infty \geq (1 - \varepsilon) \max_{1 \leq m \leq n} \left| \sum_{k=1}^m a_k \right|.$$

Next, we consider the space  $E$  spanned by  $x^{\lambda'_1}, \dots, x^{\lambda'_n}$ . Hence, we may write any  $f \in E$  as  $\sum_{k=1}^n a_k (\lambda'_k + 1) x^{\lambda'_k}$  and we have  $\Gamma_\Lambda(f) = \sum_{k=1}^n a_k x^{\lambda'_k}$ . Therefore, by denoting  $a = (a_k)_{k=1, \dots, n}$ , we find

$$\|\Gamma_\Lambda(f)\|_\infty \geq (1 - \varepsilon) \max_{1 \leq m \leq n} \left| \sum_{k=1}^m a_k \right| \geq \frac{(1 - \varepsilon)}{2} \max_{1 \leq k \leq n} |a_k| \geq \frac{(1 - \varepsilon)}{2n} \|a\|_{\ell^1}.$$

Finally, the inequality  $\|a\|_{\ell^1} \geq \|f\|_1$  finishes the proof of the theorem. The same argument works as well for the operator  $V_\Lambda$ .  $\square$

**Remark 2** *We point out that the lower bound for the Bernstein numbers of the Volterra operator  $V_\Lambda$  turns out to be the same one than the lower bound for  $V$  (see [L]). Hence the speed of the Bernstein numbers is independent of  $\Lambda$  (up to uniform constants).*

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