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EQUILATERAL TRIANGLE SKEW CONDITION FOR QUASICONFORMALITY

COLLEEN ACKERMANN, PETER HAÏSSINSKY, AND AIMO HINKKANEN

ABSTRACT. We characterize quasiconformal mappings in terms of the distortion of the vertices of equilateral triangles.

1. INTRODUCTION

Since quasiconformal mappings were first studied nearly a century ago, many diverse characterizations have been discovered. These have led to a wide variety of applications in many fields including Teichmüller theory, elliptic PDE's, hyperbolic geometry and complex dynamics. For an overview of these applications and the theory of quasiconformal mappings see [2], [4], and [6]. In this paper we will use the metric definition of quasiconformality to obtain a new formulation for the definition of planar quasiconformal mappings.

Definition 1. Let $f : U \rightarrow V$ be a homeomorphism between planar domains. When $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\} \subset U$, define

$$M(z, r) = \sup\{|f(z) - f(w)| : |z - w| = r\}, \text{ and} \\ m(z, r) = \inf\{|f(z) - f(w)| : |z - w| = r\}.$$

Suppose that $K \geq 1$. The mapping f is said to be K -quasiconformal if

$$H(z) = \limsup_{r \rightarrow 0} \frac{M(z, r)}{m(z, r)} \leq K$$

for a.e. $z \in U$, and if $H(z)$ is bounded in U .

In his book [6] John Hubbard obtained a new characterization of quasiconformal mappings. Let T be a closed topological triangle with specified vertices, $L(T) = \max\{|a - b| : a, b \text{ are vertices of } T\}$ and $l(T) = \min\{|a - b| : a, b \text{ are vertices of } T\}$. We define

$$\text{skew}(T) = \frac{L(T)}{l(T)}.$$

Note that $f(T)$ is also a topological triangle so the expression $\text{skew}(f(T))$ is defined. Then $f : U \rightarrow V$ is quasiconformal provided that there exists an increasing homeomorphism $h : [0, \infty) \rightarrow [0, \infty)$ such that

$$\text{skew}(f(T)) \leq h(\text{skew}(T))$$

for all closed Euclidean triangles $T \subset U$. In fact, Hubbard showed that it is sufficient to only consider triangles with skew bounded above by $\sqrt{7/3}$. He then asked the question of whether it suffices to only consider equilateral triangles. Progress was made on this problem in a previous paper by Javier Aramayona and Peter Haïssinsky [3] in which they showed that there exists a constant $\epsilon_0 > 0$ such that if $\epsilon \in [0, \epsilon_0)$ and

$$\text{skew}(f(T)) \leq 1 + \epsilon$$

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for all equilateral triangles $T \subset U$, then f is quasiconformal.

Theorem 1 of this paper answers Hubbard's question in the affirmative.

Theorem 1. *Let U and V be two domains in the complex plane \mathbb{C} , and let $f : U \rightarrow V$ be an orientation-preserving homeomorphism. For each $\sigma \geq 1$ there exists $H(\sigma) \geq 1$ with the following property. If $\text{skew}(f(T)) \leq \sigma$ for all equilateral triangles $T \subset U$, then, for every $z \in U$ and every $r < \text{dist}(z, \mathbb{C} \setminus U)$, the inequality $M(z, r) \leq Hm(z, r)$ holds where $H = H(\sigma)$. In particular, the map f is quasiconformal.*

Our next result, Theorem 2, serves two purposes. On the one hand, the value H from Theorem 1 gives an upper bound for the dilatation of f . However, this estimate is far from optimal. We may use the fact that quasiconformal mappings are differentiable almost everywhere to obtain a sharp bound on the dilatation in terms of σ . On the other hand, quasiconformality is a local property and demanding, as in Theorem 1, a uniform bound on every equilateral triangle is unnecessarily strong.

To show that f is quasiconformal in U , it suffices to show that f satisfies two conditions. First we must show that each $z \in U$ has a neighborhood W such that $\text{skew}(f(T))$ is bounded for all equilateral triangles $T \subset W$. If the resulting upper bounds vary, this is not enough to get the result that f is quasiconformal in U . However, by formulating a new concept of skew that needs to be satisfied only almost everywhere and must be bounded uniformly in U , and combining it with the assumption that the skew as defined previously is bounded, not necessarily uniformly, in some neighborhood of each point of U , we obtain both quasiconformality and a better upper bound for the maximal dilatation of f . The details of how to define this new notion of skew are given below.

First note that quasiconformal maps are differentiable almost everywhere by Mori's theorem [9]. Let $\text{Skew}(f)$ denote the supremum of $\text{skew}(f(T))$ over all equilateral triangles T contained in U . Next, for $z \in U$ and $r > 0$, let $\text{skew}(f, z, r)$ denote the least upper bound of $\text{skew}(f(T))$ over all equilateral triangles $T \subset \{w \in U : |z - w| < r\}$. Set $\text{skew}(f, z) = \liminf_{r \rightarrow 0} \text{skew}(f, z, r)$ and $\text{skew}(f) = \|\text{skew}(f, z)\|_\infty$. Below, the assumption that $\text{Skew}(f|W)$ is finite means that there is some upper bound for $\text{skew}(f(T))$ over all equilateral triangles T contained in the appropriate domain W .

Theorem 2. *Let U be a domain in the complex plane \mathbb{C} , and let $f : U \rightarrow f(U)$ be an orientation-preserving homeomorphism. Suppose that each $z \in U$ has a neighborhood W such that $\text{Skew}(f|W)$ is finite.*

Suppose that $\sigma \geq 1$. If $\text{skew}(f) \leq \sigma$ then f is $K(\sigma)$ -quasiconformal where

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

In particular, if $\text{skew}(f) = 1$ then f is a conformal mapping. The upper bound $K(\sigma)$ for the maximal dilatation of f is the best possible and is attained at least for certain affine mappings.

2. PROOF OF THE MAIN THEOREM

Throughout the rest of the paper we will use the following notation and conventions:

- (1) We define $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\}$ and let $C(z, r)$ be the boundary of $D(z, r)$.
- (2) By a curve we mean the image of a not necessarily one-to-one continuous function from a closed interval into \mathbb{C} .
- (3) All triangles (without the qualifier "topological") will be closed Euclidean triangles.
- (4) Let \mathcal{F}_σ denote the set of orientation-preserving homeomorphisms of any domain $U \subset \mathbb{C}$ into any domain $V \subset \mathbb{C}$ such that $\text{skew}(f(T)) \leq \sigma$ for all closed equilateral triangles $T \subset U$.

Our strategy for the proof of Theorem 1 is to show first that the image of every equilateral triangle, T , contains a disk with radius proportional to $L(f(T))$. To do this, we naturally need the assumption of Theorem 1 for all equilateral triangles (at least in a suitable set containing this T) and not only for this particular T . Once it is known that the images of equilateral triangles are “thick” in this sense, it is easier to obtain the quantitative estimates that are required to prove that f is quasiconformal, whether using the metric (Definition 1) or the analytic (Definition 2 in Section 6 below) definition of quasiconformality.

The most difficult part of the proof that the image $f(T)$ is thick is to construct another equilateral triangle with suitable additional properties, to which the assumption of Theorem 1 can then be applied. This will be done in the proof of Proposition 2 below. That proof requires delicate geometric considerations. After these strategic comments, let us now move to the proof.

The proof of Theorem 1 relies on the following proposition.

Proposition 1. *Let U be a domain containing $D(0, 1)$, let $f : U \rightarrow \mathbb{C}$ belong to \mathcal{F}_σ , and let T be the triangle with vertices $0, 1$, and $\omega = 1/2 + (\sqrt{3}/2)i$. Then there exists a disk D contained in $f(T)$ such that*

- (1) D is centered at $f(p)$ where $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i \approx 0.5 + 0.29i$, and
- (2) there exists a constant $\alpha = \alpha(\sigma)$ such that the radius of D is at least $\alpha L(f(T))$.

We note that if f is to be quasiconformal, then, certainly, the image $f(T)$ has to contain a disk of a definite size centered at the image of the centroid of the triangle, i.e., the point $\xi = 1/2 + (\sqrt{3}/6)i$. Unfortunately, its arithmetic properties make it difficult to relate this point ξ to the vertices of T using equilateral triangles. The point p was chosen, because it is both close to the centroid ($|\xi - p| = \sqrt{3}/(2^9 \cdot 3)$), and it is a vertex of a tiling of the plane by equilateral triangles whose vertices include the vertices of T . Indeed, we have $p = 1/2 - 85 \cdot 2^{-9} + 85 \cdot 2^{-8}\omega$, cf. Lemma 1. We finally observe that p is closer to the horizontal side of T than ξ .

We first derive the proof of Theorem 1 from Proposition 1. We will then focus on the proof of the latter.

Proof of Theorem 1. Fix $z \in U$ and $r > 0$. If $D(z, r) \subset U$, let $M(z, r) = \max\{|f(z) - f(w)| : w \in C(z, r)\}$ and $m(z, r) = \min\{|f(z) - f(w)| : w \in C(z, r)\}$. Denote by z_M a point in $C(z, r)$ such that $|f(z_M) - f(z)| = M(z, r)$.

Since \mathcal{F}_σ is invariant under pre- and post-composition by affine maps of the form $z \mapsto az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$, we assume that $z = 0$, $r = 1$, and $z_M = 1$. Thus $|f(0) - f(1)| = M(0, 1)$. Let T_1 be the equilateral triangle with vertices $0, 1$, and ω . Since 0 and 1 are vertices of T_1 , we have $|f(0) - f(1)| \leq L(f(T_1))$. Hence $M(0, 1) \leq L(f(T_1))$.

By Proposition 1, the image of T_1 must contain a disk D centered at $f(p)$ and of radius at least $\alpha L(f(T_1))$.

Let us consider the isometry $A(z) = \overline{z - p}$. Let $T_2 = A(T_1)$ (see Figure 1). The triangle T_2 is contained in the unit disk, and A maps p to 0 and 1 to p . Since the vertices of T_2 other than p lie outside of T_1 , their images lie outside of D , and so we have $L(f(T_2)) \geq \alpha L(f(T_1))$.

Moreover, another application of Proposition 1 implies that $f(T_2)$ contains the disk $D(f(0), \alpha L(f(T_2)))$. Since all the points of $f(C(0, 1))$ are outside the interior of $f(T_2)$ and thus also outside the interior of $D(f(0), \alpha L(f(T_2)))$, it follows that $m(0, 1) \geq \alpha L(f(T_2))$.

Summing up these estimates, we obtain

$$m(0, 1) \geq \alpha L(f(T_2)) \geq \alpha^2 L(f(T_1)) \geq \alpha^2 M(0, 1).$$

□

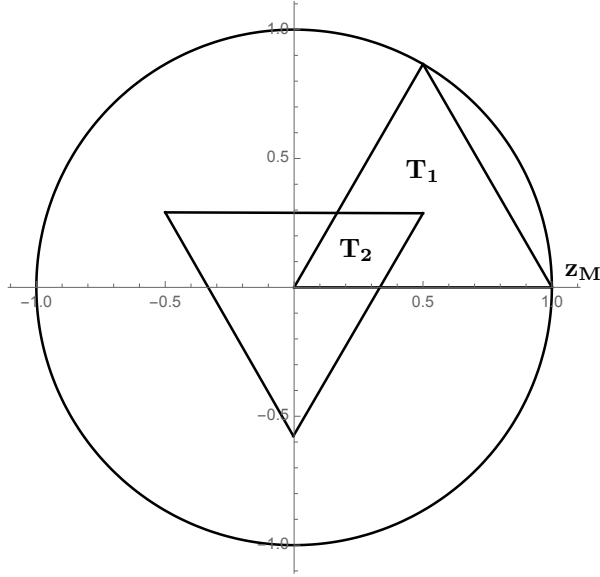


FIGURE 1. Configuration of $C(0, 1)$, T_1 and T_2

3. CONSTRUCTION OF CERTAIN TRIANGLES

Proposition 1 is a consequence of the following proposition.

Proposition 2. *Let U be a neighborhood of $D(0, 1)$, and let $f : U \rightarrow \mathbb{C}$ be a homeomorphism onto its image such that $f \in \mathcal{F}_\sigma$. Let T be the closed triangle with vertices 0 , 1 , and $\omega = 1/2 + (\sqrt{3}/2)i$. Let $q = p + 2^{-9}$. Then there exist points $t_1, t_2 \in T$ such that the points q, t_1, t_2 form the vertices of an equilateral triangle and the inequalities $|f(t_j) - f(p)| \leq C\mu$, and $|f(p) - f(q)| \geq cL(f(T))$ hold for some constants $c = c(\sigma)$ and $C = C(\sigma)$ where $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$. We permit the trivial triangle where we have $t_1 = t_2 = q$.*

Proof of Proposition 1 assuming Proposition 2. If $t_1 = t_2 = q$, then we have

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu.$$

Otherwise, by the triangle inequality we have

$$(1) \quad |f(p) - f(q)| \leq |f(t_1) - f(p)| + |f(t_1) - f(q)|.$$

Since $f \in \mathcal{F}_\sigma$ and t_1, t_2, q are the vertices of an equilateral triangle, we obtain

$$(2) \quad |f(t_1) - f(q)| \leq \sigma |f(t_1) - f(t_2)|.$$

By Proposition 2, we have $|f(t_1) - f(p)| \leq C\mu$ and $|f(t_2) - f(p)| \leq C\mu$, so that by this and the triangle inequality we get

$$|f(t_1) - f(t_2)| \leq |f(t_1) - f(p)| + |f(t_2) - f(p)| \leq 2C\mu.$$

By Proposition 2, we have $|f(p) - f(q)| \geq cL(f(T))$. Using this and appealing to $|f(t_1) - f(p)| \leq C\mu$ again, we obtain from (1) and (2) that

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu + \sigma(2C\mu),$$

whence

$$\mu \geq \frac{c}{(2\sigma + 1)C} L(f(T)).$$

□

4. PROOF OF PROPOSITION 2

Let the assumptions of Proposition 2 be satisfied. Recall that we write $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$. The idea of our proof of Proposition 2 is to define a curve γ' in T going through p such that

- (1) for all $t \in \gamma'$ we have $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$;
- (2) there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle.

The proof of Proposition 2 results from Lemma 1 and Lemma 3.

We first prove the following result. Recall that $\omega = 1/2 + (\sqrt{3}/2)i$.

Lemma 1. *Let T be the closed triangle with vertices $0, 1$, and ω . Let $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$ and $q = p + 2^{-9}$. Then $|f(q) - f(p)| \geq cL(f(T))$ for some positive constant $c = c(\sigma)$.*

Proof. Let us first consider the tiling of the plane by equilateral triangles with vertices in $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$. Define a chain of triangles $(T_j)_{0 \leq j \leq J}$ as a sequence of triangles with vertices in Λ such that $T_j \cap T_{j+1}$ is an edge for all j with $0 \leq j < J$. Given two edges (v, w) and (v', w') , we may connect them by a chain of minimal length $n \geq 0$. A simple induction argument implies

$$(3) \quad |f(v) - f(w)| \leq \sigma^n |f(v') - f(w')|$$

if $f \in \mathcal{F}_\sigma$ is defined in a neighborhood of the chain.

Let T be as defined in our hypotheses: it is tiled by $N = 2^{18}$ triangles of $2^{-9}\Lambda$, and $[p, q]$ is an edge of this tiling. Therefore, for every other edge $[v, w]$, it follows that

$$|f(v) - f(w)| \leq \sigma^N |f(p) - f(q)|.$$

But each side of T is the union of less than N edges of our tiling, therefore, the triangle inequality implies

$$L(f(T)) \leq N\sigma^N |f(p) - f(q)|.$$

□

We now prove a geometric lemma which will be used in the proof of Lemma 3.

Lemma 2. *Let $|z| \leq 1/8$ and suppose that $|\theta_\pm - (\pm\pi/3)| \leq 1/8$. Then the angle φ between $e^{i\theta_+} - z$ and $e^{i\theta_-} - z$ which crosses the positive real axis belongs to $(\pi/3, \pi)$.*

Proof. We note that $\cos \theta_\pm \geq 1/2 - 1/4 > 1/8 \geq |z|$ so that φ is less than π .

For the other inequality, we will estimate $\tan |\arg(e^{i\theta_\pm} - z)|$ to obtain a lower bound of both angles with the horizontal line:

$$\begin{aligned} \tan |\arg(e^{i\theta_\pm} - z)| &\geq \frac{\sqrt{3}/2 - (|z| + 1/8)}{1/2 + (|z| + 1/8)} \geq \frac{\sqrt{3}/2 - 1/4}{1/2 + 1/4} \\ &\geq \frac{2\sqrt{3} - 1}{3} \geq \frac{2}{3} > \tan(\pi/6). \end{aligned}$$

Therefore φ is at least $\pi/3$.

□

Now we demonstrate how to find the curve γ' mentioned above.

Lemma 3. *Under the assumptions of Proposition 2, there exists a curve γ' in T going through p such that for all $t \in \gamma'$ we have*

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$$

and there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle. We permit the trivial triangle where we have $t_1 = t_2 = q$.

Proof. We will do this in several steps. We first define a curve that will join two points of the boundary of a disk contained in T (Step 1). To make sure that we will be able to find two points that form an equilateral triangle with q , we will extend this curve so that it has end points in a slightly larger disk, and is only close to the boundary of the larger disk when it is also close to its end points (Step 2). Then we will use Lemma 2 to find our triangle (Step 3).

Since $\sqrt{3} \geq 8/5$, it follows that $\text{dist}(p, \partial T) = 85\sqrt{3} \cdot 2^{-9} > 1/4 + 2^{-6}$, so that $D(p, 1/4 + 2^{-6})$ is contained in the interior of T .

Throughout the proof, for $x \in \mathbb{C}$, R_x will denote the rotation centered at x by $\pi/3$ radians, defined by $R_x(z) = x + (z - x)\omega$ and \bar{R}_x the rotation centered at x by $-\pi/3$ radians, defined by $\bar{R}_x(z) = x + (z - x)\bar{\omega}$. Recall that we set $\omega = 1/2 + (\sqrt{3}/2)i$.

Step 1: There exists a curve γ_2 that satisfies the following:

- (1) $\gamma_2 \subset D(p, 1/4)$,
- (2) γ_2 has end points on $C(p, 1/4)$ which are exactly $2\pi/3$ radians apart, and
- (3) for all points $t \in \gamma_2$ we have $|f(t) - f(p)| \leq \sigma\mu$.

Let $p' \in \partial T$ be such that $d(f(p), f(p')) = \mu$ and let $\gamma = f^{-1}([f(p), f(p')])$. Since $D(p, 1/4)$ is contained in the interior of T , we may consider the component γ_1 of $\gamma \cap D(p, 1/4)$ that contains p , and we denote by $w \in C(p, 1/4)$ the other end point of γ_1 . We take w to be the first point of $C(p, 1/4)$ encountered when moving along γ starting from p .

Now define

$$\gamma_2 = R_p(\gamma_1) \cup \bar{R}_p(\gamma_1).$$

Note that, for every $s \in \gamma_1$, $R_p(s)$ and $\bar{R}_p(s)$ are two points in γ_2 which make an angle of $2\pi/3$ seen from p . Since $f \in \mathcal{F}_\sigma$, for all $t \in \gamma_2$, we have

$$|f(t) - f(p)| \leq \sigma|f(s) - f(p)| \leq \sigma\mu$$

where $s \in \gamma_1$ denotes a point such that either $t = R_p(s)$ or $t = \bar{R}_p(s)$.

Step 2: Let a, b be the end points of γ_2 . There exists a curve γ_3 such that

- (1) $\gamma_3 \subset D(p, 1/4) \cup D(a, 2^{-6}) \cup D(b, 2^{-6})$;
- (2) γ_3 has both end points on $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$;
- (3) for all points $t \in \gamma_3$, $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$.

Let $D_a = D(a, 2^{-6})$ and $D_b = D(b, 2^{-6})$. Let γ_{2a} and γ_{2b} be the components of $\gamma_2 \cap D_a$ and $\gamma_2 \cap D_b$ that have end points at a and b respectively.

Clearly γ_{2a} also has an end point on the boundary of D_a . Let a' denote an end point of γ_{2a} on the boundary of D_a . Use the tangent line to $D(p, 1/4)$ at a to divide D_a in half, and then divide each half into thirds. So we have divided D_a into closed sectors of $\pi/3$ radians with three such sectors lying entirely outside of $D(p, 1/4)$. Let S_a denote the middle sector lying completely outside of $D(p, 1/4)$. Then there exists $n \in \{2, 3\}$ such that when γ_{2a} is rotated $n\pi/3$ radians in an appropriate direction about a , the image of a' under the rotation will lie in S_a . Let the image of γ_{2a} under this rotation be denoted by γ_{3a} .

Now we will bound the quantity $|f(t) - f(p)|$ where $t \in \gamma_{3a}$. Fix $t \in \gamma_{3a}$. Let t_0 be the point on γ_{2a} whose image under the rotation is t . Without loss of generality we will assume this rotation was clockwise. Let t_i denote the image of t_0 under a clockwise rotation of $i\pi/3$ radians where $i = 1, \dots, n$ ($t = t_n$). Then since a, t_{i-1}, t_i ($1 \leq i \leq n$) form an equilateral triangle, we have

$$|f(t_i) - f(a)| \leq \sigma |f(t_{i-1}) - f(a)|.$$

Since $a, t_0 \in \gamma_2$ we have

$$|f(a) - f(t_0)| \leq |f(a) - f(p)| + |f(p) - f(t_0)| \leq 2\sigma\mu$$

and

$$|f(a) - f(p)| \leq \sigma\mu.$$

Thus since n is at most 3 we may apply (3) with $n \leq 3$, which together with the triangle inequality yields

$$|f(t) - f(p)| \leq |f(a) - f(p)| + |f(a) - f(t)| \leq \sigma\mu + \sigma^n |f(a) - f(t_0)| \leq \sigma\mu(1 + 2\sigma^n) \leq \sigma\mu(1 + 2\sigma^3).$$

Furthermore γ_{3a} must intersect the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$. This is because γ_{3a} has an end point in S_a and therefore the distance of the end point of γ_{3a} from $D(p, 1/4)$ must be at least $\cos(\pi/6) \cdot 2^{-6} = \sqrt{3} \cdot 2^{-7}$. This is depicted in Figure 2.

We proceed similarly near b and define a curve γ_{3b} contained in D_b with end points at b and at some point on the intersection of the boundary of D_b and S_b (defined analogously to S_a) such that for all $t \in \gamma_{3b}$ we have $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$; as above, γ_{3b} intersects the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$.

Let γ_3 be the connected component of $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ which includes points in both γ_{3a} and γ_{3b} . Then for all points $t \in \gamma_3$,

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3).$$

The curve γ' in Lemma 3 can be chosen as $\gamma' = \gamma_3$.

Step 3: Let $q = p + 2^{-9}$. There exist $t_1, t_2 \in \gamma_3$ such that $\{q, t_1, t_2\}$ form an equilateral triangle.

Let D_q be the smallest disk centered at q which contains $D(p, 1/4)$. Then $D_q \subset D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ since $|p - q| = 2^{-9} \leq \sqrt{3} \cdot 2^{-8}$. Let γ_4 be the connected component of $\gamma_3 \cap D_q$ which has end points $A \in D_a \cap D_q$ and $B \in D_b \cap D_q$.

Note that, if we write $a = p + |a - p|e^{i\theta_a}$ and $A = p + |A - p|e^{i\theta_A}$, where $|\theta_A - \theta_a|$ is chosen to be as small as possible modulo 2π , then

$$|\theta_A - \theta_a| \leq 2|a - A|/(1/4) \leq 2 \cdot 2^{-6}/(1/4) \leq 1/8$$

and similarly for b and B . Note that $|\arg(a - p)| + |\arg(b - p)| = 2\pi/3$ and $|p - q|/(1/4) = 2^{-7} \leq 1/8$. Therefore, by Lemma 2 applied in $D(p, 1/4)$, the angle between $A - q$ and $B - q$ lies in $(\pi/3, \pi)$. Hence,

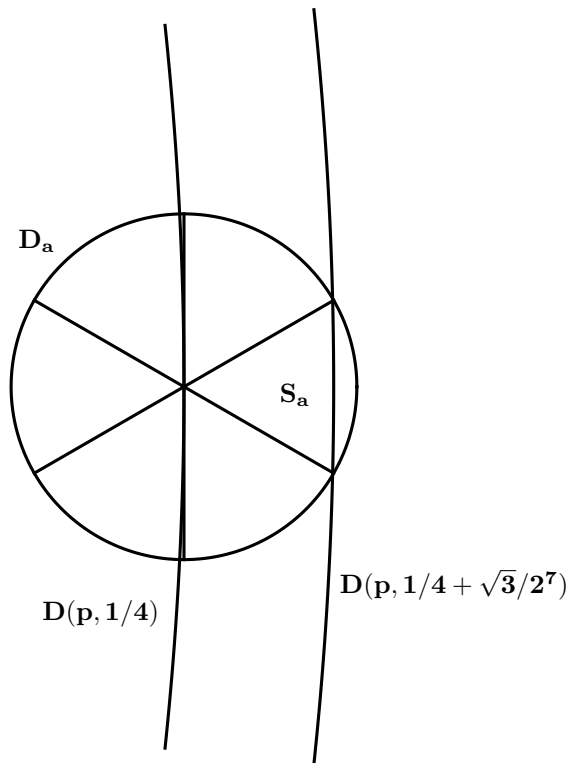


FIGURE 2

the images A_r and B_r of A and B respectively under \bar{R}_q will separate A and B on ∂D_q . Thus the image $\bar{R}_q(\gamma_4)$ must intersect γ_4 at a point q' , say. This gives us our desired equilateral triangle since q , q' , and the pre-image q'' of q' form an equilateral triangle.

This completes the proof of Lemma 3. □

To complete the proof of Proposition 2, we note that the points t_1 and t_2 required will be the points q' and q'' . It remains to be shown that $|f(q') - f(p)| \leq C\mu$ and $|f(q'') - f(p)| \leq C\mu$ for some constant C . Since $q'' \in \gamma_4 \subset \gamma_3$, we have $|f(q'') - f(p)| \leq \sigma(1 + 2\sigma^3)\mu$ by part (3) in Step 2. But also $q' \in \gamma_4$, so that for the same reason $|f(q') - f(p)| \leq \sigma(1 + 2\sigma^3)\mu$. This completes the proof of Proposition 2.

5. PROOF OF THEOREM 2

We prove Theorem 2 by approximating f by linear mappings at points where f is differentiable.

Proof. Let the assumptions of Theorem 2 be satisfied. Theorem 1 implies that f is locally quasiconformal and hence differentiable with $f_z \neq 0$ at almost every point of U . Let z_0 be a point of differentiability such that $\text{skew}(f, z_0) \leq \sigma$. We compute the maximum possible value for $H(z_0)$, which yields the same upper bound for the dilatation of f at z_0 . Since $H(z_0)$ is invariant under Möbius transformations we may compose with translations, a dilation and a rotation to assume that $z_0 = f(z_0) = 0$, $f_z(z_0) = 1$ and $f_{\bar{z}}(z_0) = |f_{\bar{z}}(z_0)| < 1$. Then $f(z) = z + f_{\bar{z}}(z_0)\bar{z} + \epsilon(z)$ where $\epsilon(z)/|z|$ tends to 0 as z tends to z_0 , and thus $\text{skew}(f, z_0) = \text{skew}(\tilde{f}, z_0)$ where $\tilde{f}(z) = z + f_{\bar{z}}(z_0)\bar{z}$. It now suffices to prove that for all affine mappings \tilde{f} that can arise in this way, under the assumptions of Theorem 2, the maximal dilatation of \tilde{f} is at most $K(\sigma)$. This then shows that also the maximal dilatation of our original mapping f is at most $K(\sigma)$, as required. Thus from now on we only consider \tilde{f} .

We will first compute the skew of \tilde{f} , and then express $K(\tilde{f})$ in terms of $\text{skew}(\tilde{f})$.

Note that $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(a+v) - \tilde{f}(b+v)|$, $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(\bar{a}) - \tilde{f}(\bar{b})|$ and $|\tilde{f}(a) - \tilde{f}(b)|/|\tilde{f}(a) - \tilde{f}(c)| = |\tilde{f}(ra) - \tilde{f}(rb)|/|\tilde{f}(ra) - \tilde{f}(rc)|$ for all $a, b, c, v \in \mathbb{C}$ with $a \neq c$ and all $r > 0$. This implies $\text{skew}(\tilde{f}(T))$ where T is an equilateral triangle is invariant under translations, complex conjugation and dilations of T . Thus for all equilateral triangles T ,

$$\text{skew}(\tilde{f}(T)) \in \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\}.$$

Indeed, suppose T has vertices A, B and C , and $\text{skew}(T) = \frac{|\tilde{f}(A) - \tilde{f}(B)|}{|\tilde{f}(A) - \tilde{f}(C)|}$. First we translate A to the origin, and then we dilate T so its side lengths are equal to 1. If \overline{AB} is $\pi/3$ radians clockwise from \overline{AC} , it is clear that our statement is true; otherwise we take the complex conjugate of T to change the orientation of T and then, since $\text{skew}(\tilde{f}(T))$ is invariant under complex conjugation of T , our claim is true.

Hence we have

$$\text{skew}(\tilde{f}) = \max \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\} = \max \left\{ \frac{|\tilde{f}(z)|}{|\tilde{f}(ze^{i\pi/3})|} : |z| = 1 \right\}.$$

Let $\mu = f_{\bar{z}}(z_0)$. If $\mu = 0$, then $\text{skew}(\tilde{f}) = 1$, so we assume that $0 < \mu < 1$. Write $\nu = \mu + \mu^{-1} > 2$ and $\beta = e^{i\pi/6}$. Let $w \in \mathbb{C}$ with $|w| = 1$. We have

$$|\tilde{f}(w)|^2 = |w + \mu\bar{w}|^2 = (w + \mu\bar{w})(\bar{w} + \mu w) = 1 + \mu^2 + \mu(w^2 + \bar{w}^2) = \mu[\nu + (w^2 + \bar{w}^2)].$$

Now we are able to maximize $|\tilde{f}(\beta w)|/|\tilde{f}(\bar{\beta} w)|$ with respect to w . Set $z = w^2$ and $\alpha = e^{i\pi/3}$. Since we have assumed $|w| = 1$, we can instead maximize

$$\kappa = \left| \frac{\tilde{f}(\beta w)}{\tilde{f}(\bar{\beta} w)} \right|^2 = \frac{\nu + \alpha z + \bar{\alpha}\bar{z}}{\nu + \bar{\alpha}z + \alpha\bar{z}}.$$

We write $z = e^{ix}$, $x \in \mathbb{R}$, so that $z' \equiv dz/dx = iz$, $\bar{z}' = -i\bar{z}$. We may differentiate κ as a function of x . It follows that $\kappa' = 0$ if, and only if,

$$(\alpha z - \bar{\alpha}\bar{z})(\nu + \bar{\alpha}z + \alpha\bar{z}) - (\nu + \alpha z + \bar{\alpha}\bar{z})(\bar{\alpha}z - \alpha\bar{z}) = 0.$$

Thus

$$\nu(\alpha z - \bar{\alpha}\bar{z} - \bar{\alpha}z + \alpha\bar{z}) = z^2 - \alpha^2 + \bar{\alpha}^2 - \bar{z}^2 - z^2 - \alpha^2 + \bar{\alpha}^2 + \bar{z}^2$$

which is equivalent to

$$\nu(z + \bar{z})(\alpha - \bar{\alpha}) = 2(\bar{\alpha} - \alpha)(\bar{\alpha} + \alpha).$$

Therefore

$$\cos x = -\frac{2}{\nu} \cos \frac{\pi}{3} = -\frac{1}{\nu}.$$

It follows that $\sin^2 x = 1 - 1/\nu^2$ so $\kappa' = 0$ for

$$z = \frac{1}{\nu} \left(-1 + i\varepsilon\sqrt{\nu^2 - 1} \right)$$

with $\varepsilon \in \{\pm 1\}$. For these values of z , one gets

$$\kappa = \frac{\nu + 2\operatorname{Re}(\alpha z)}{\nu + 2\operatorname{Re}(\bar{\alpha} z)} = \frac{\nu^2 - 1 - \varepsilon\sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 + \varepsilon\sqrt{3(\nu^2 - 1)}}$$

which is maximal for $\varepsilon = -1$. So we obtain

$$\operatorname{skew}(\tilde{f})^2 = \frac{\nu^2 - 1 + \sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 - \sqrt{3(\nu^2 - 1)}} = \frac{\sqrt{(\nu^2 - 1)/3} + 1}{\sqrt{(\nu^2 - 1)/3} - 1} > 1.$$

Note that $\nu^2 - 1 > 3$ since $\nu > 2$. Let us write $\tau = \operatorname{skew}(\tilde{f}) > 1$ so that

$$\sqrt{(\nu^2 - 1)/3} = \frac{\tau^2 + 1}{\tau^2 - 1}, \quad \nu^2 = 3 \left(\frac{\tau^2 + 1}{\tau^2 - 1} \right)^2 + 1 = \frac{4(\tau^4 + \tau^2 + 1)}{(\tau^2 - 1)^2}$$

and thus

$$\mu + \mu^{-1} = \nu = \frac{2\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1}.$$

Hence

$$\mu^2 - 2\mu \frac{\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1} + 1 = 0.$$

We compute the reduced discriminant

$$\Delta' = \frac{\tau^4 + \tau^2 + 1}{(\tau^2 - 1)^2} - 1 = \frac{3\tau^2}{(\tau^2 - 1)^2}$$

and we deduce from $0 < \mu < 1$ that

$$\mu = \frac{\sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1}.$$

Thus

$$K(\tilde{f}) = \frac{1 + \mu}{1 - \mu} = \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1 - \sqrt{\tau^4 + \tau^2 + 1} + \sqrt{3}\tau} = \varphi(\tau),$$

say. Write $t = \sqrt{\tau^4 + \tau^2 + 1}$ and note that $t^2 = (\tau^2 - 1)^2 + 3\tau^2$ so that

$$\begin{aligned} \varphi(\tau) &= \frac{[(\tau^2 - 1 + t) - \sqrt{3}\tau][(\tau^2 - 1 + t) + \sqrt{3}\tau]}{[(\tau^2 - 1) + \sqrt{3}\tau]^2 - t^2} \\ &= \frac{[(\tau^2 - 1) + t]^2 - 3\tau^2}{2\sqrt{3}\tau(\tau^2 - 1)} = \frac{2[(\tau^2 - 1)^2 + t(\tau^2 - 1)]}{2\sqrt{3}\tau(\tau^2 - 1)} \\ &= \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1}}{\sqrt{3}\tau}. \end{aligned}$$

Differentiation shows that $\varphi(\tau)$ is an increasing function of τ , so that since $\tau = \operatorname{skew}(\tilde{f}) = \operatorname{skew}(f, z_0) \leq \sigma$, we have

$$K(\tilde{f}) = \varphi(\tau) \leq \varphi(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

Hence $K(f) \leq \varphi(\sigma) = K(\sigma)$ as defined in Theorem 2.

If $\mu \in (0, 1)$ is given and if $f(z) = z + \mu\bar{z}$, then we may take $\tilde{f} = f$ and $\sigma = \tau$ in the above argument, and we see that $K(f) = K(\sigma)$. Thus the upper bound in Theorem 2 is best possible. \square

6. AN ALTERNATIVE PROOF OF THE QUASICONFORMALITY OF MAPPINGS SATISFYING THE HYPOTHESES OF THEOREM 1

From Proposition 1, there are several ways of establishing that a mapping f satisfying the hypotheses of Theorem 1 satisfies the analytic definition of quasiconformality which is equivalent to Definition 1. For $|\xi| = 1$, we denote by $\partial_\xi f(z)$ the directional derivative $\lim_{t \downarrow 0} (f(z + t\xi) - f(z))/(t\xi)$.

Definition 2. We say that a homeomorphism $f : U \rightarrow V$ is absolutely continuous on lines if for every rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ with $\bar{R} \subset U$, f is absolutely continuous on a.e. interval $I_x = \{(x, y) : c < y < d\}$ and a.e. interval $I_y = \{(x, y) : a < x < b\}$. An orientation-preserving homeomorphism f is called quasiconformal if f is absolutely continuous on lines and there exists $K \geq 1$ such that

$$\max_{\xi} |\partial_\xi f(z)| \leq K \min_{\xi} |\partial_\xi f(z)| \text{ a.e.}$$

Proposition 1 tells us that the image of every equilateral triangle, T , contains a disk with radius proportional to $L(f(T))$. In [6, Section 4.5], Hubbard uses this to prove that the map belongs to the Sobolev space $W_{loc}^{1,2}$ by an approximation argument. We propose another approach which shows directly that the map satisfies the ACL property.

Proof. First we show that f is absolutely continuous on lines. This part of the proof parallels Pfluger's proof that a mapping satisfying the geometric definition of quasiconformality is absolutely continuous on lines. His proof can be found in [10] and is reproduced in English in [8], p. 162. We fix a rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ and let $I_y = \{(x, y) : a < x < b\}$ for y between c and d . Define $A(y)$ to be the area in $f(R)$ beneath the image of I_y . Since A is an increasing function of y , it is differentiable almost everywhere. We will show that $f|_{I_y}$ is absolutely continuous for all y at which A is differentiable; a similar argument applies to vertical line segments. We select an arbitrary collection $\{(z_k^*, z_k)\}_{k=1}^n$ of disjoint sub-intervals of I_y where $z_k = (x_k, y)$ and $z_k^* = (x_k^*, y)$.

We consider the collection of rectangles $\{R_k\}_{k=1}^n$ where each R_k has height δ and has its bottom side contained in the k th sub-interval. More precisely, fix k and draw a rectangle of height δ above (z_k^*, z_k) . Set N_k equal to the smallest integer greater than or equal to $\frac{|x_k^* - x_k|}{\delta}$. We draw N_k equilateral triangles in R_k . The triangles all have one side on the interval $[z_k^*, z_k]$ and they overlap only at their vertices. The first $N_k - 1$ triangles have width δ and the last triangle has width $x_k^* - x_k - \delta(N_k - 1)$. Let $\Delta_{k,i}$ denote the side length of the i th triangle for $1 \leq i \leq N_k$ and set $\Delta_{k,0} = 0$. Set $N = \sum_{k=1}^n N_k$. Then $N \leq \frac{\sum_{k=1}^n |x_k^* - x_k|}{\delta} + n$.

Let α be as in Proposition 1. By Proposition 1, the image of each of our triangles must contain a disk of radius comparable to the greatest distance between the images of the vertices of the triangle. Thus the total area of all the images of our rectangles of height δ and width $x_k^* - x_k$ is greater than or equal to

$$(4) \quad \sum_{k=1}^n \sum_{j=1}^{N_k} \pi \alpha^2 \left(\left| f\left(x_k + \sum_{i=1}^j \Delta_{k,i}, y\right) - f\left(x_k + \sum_{i=1}^{j-1} \Delta_{k,i-1}, y\right) \right| \right)^2.$$

Then by the Cauchy–Schwarz inequality, (4) is greater than or equal to

$$\begin{aligned}
& \frac{\pi\alpha^2}{N} \left(\sum_{k=1}^n \sum_{j=1}^{N_k} |f(x_k + \sum_{i=1}^j \Delta_{k,i}, y) - f(x_k + \sum_{i=1}^j \Delta_{k,i-1}, y)| \right)^2 \\
& \geq \frac{\pi\alpha^2}{N} \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \geq \frac{\pi\alpha^2}{\left(\frac{\sum_{k=1}^n |x_k^* - x_k|}{\delta} + n \right)} \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \\
& = \frac{\pi\alpha^2}{\left(\frac{\sum_{k=1}^n |x_k^* - x_k| + \delta n}{\delta} \right)} \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2.
\end{aligned}$$

Recall that $A(y)$ is defined to be the area in $f(R)$ beneath the image of the line segment I_y . Since the sum of the areas of the images of our rectangles of height δ and width $x_k^* - x_k$ is less than or equal to $A(y + \delta) - A(y)$, we have

$$\pi\alpha^2 \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq \left(\frac{A(y + \delta) - A(y)}{\delta} \right) \left(\sum_{k=1}^n |x_k^* - x_k| + \delta n \right).$$

Since we chose y at which A is differentiable, letting $\delta \rightarrow 0$ gives

$$\pi\alpha^2 \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq A'(y) \left(\sum_{k=1}^n |x_k^* - x_k| \right).$$

Since $A'(y)$ exists almost everywhere this gives absolute continuity on almost every horizontal line segment. The proof is analogous for vertical line segments.

To then conclude that f is quasiconformal we note that since f is open and absolutely continuous on lines, f is differentiable almost everywhere by a theorem of Gehring and Lehto [4]. Now the computations in the proof of Theorem 2 show that

$$\max_{\xi} |\partial_{\xi} f(z)| \leq K(\sigma) \min_{\xi} |\partial_{\xi} f(z)|$$

at points where f is differentiable and hence almost everywhere. According to the analytic definition of quasiconformality (Definition 2), the mapping f is quasiconformal. \square

7. APPENDIX BY COLLEEN ACKERMANN: AN ANALOGUE OF THE MAIN THEOREM IN HILBERT SPACES OF DIMENSION AT LEAST THREE

In dimensions three and higher the proof of an analogue of Theorem 1 is surprisingly simpler than the proof of Theorem 1. Furthermore the proof itself gives an elegant bound on $K(\sigma)$.

Theorem 3. *Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces with $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \geq 3$ and let $U \subset \mathcal{H}_1, V \subset \mathcal{H}_2$ be domains. Suppose that $f : U \rightarrow V$ is a homeomorphism and that for all closed equilateral triangles $T \subset U$, $\text{skew}(f(T)) \leq \sigma$. Then f is σ^3 -quasiconformal when using the metric definition of quasiconformality.*

Note that Definition 1 can be used to define quasiconformal mappings also between Hilbert spaces, whether finite-dimensional or infinite-dimensional. The same discussion on the locality of quasiconformality made in the introduction applies here as well and enables us to weaken the hypothesis that $\text{skew}(f(T)) \leq \sigma$ holds for all equilateral triangles.

Proof. Fix a point $p \in U$, a positive number r with $r < \text{dist}(p, \partial U)$ and points $a_1, a_2 \in \partial B(p, r)$. We will prove $|f(a_1) - f(p)| \leq \sigma^3 |f(p) - f(a_2)|$. Let $P \subset \mathcal{H}_1$ be the affine plane containing $\{p, a_1, a_2\}$ and let us

fix a point $q \in \mathcal{H}_1$ so that \vec{pq} is orthogonal to P and let us restrict ourselves to the three-dimensional space containing $\{p, a_1, a_2, q\}$ that we may identify with \mathbb{R}^3 . For the sake of convenience, we will consider spherical coordinates (r, θ, φ) with origin p so that $(r, \theta, \pi/2)$ reduces to polar coordinates in P and so that a_1 and a_2 have coordinates $(r, \pm\theta/2, \pi/2)$ where $\theta \in (0, \pi]$ denotes the non-oriented angle between \vec{pa}_1 and \vec{pa}_2 .

Case 1: If $\theta \leq 2\pi/3$, then we may find $\varphi \in [0, \pi/2]$ such that $2 \sin \varphi \cos(\theta/2) = 1$ and we set $b = (r, 0, \varphi)$. Both triangles T_j with vertices $\{p, a_j, b\}$, $j = 1, 2$, are equilateral by construction and they share a common side with end points at b and p .

Thus

$$|f(p) - f(a_1)| \leq \sigma |f(p) - f(b)| \leq \sigma^2 |f(p) - f(a_2)|.$$

Case 2: If $\theta > 2\pi/3$, consider the equilateral triangle T_0 with vertices p, a_1 and b' where b' is the image of a_1 under a rotation in P of angle $\pi/3$ so that the smaller angle between b' and a_2 is less than or equal to $2\pi/3$. Thus by Case 1

$$|f(p) - f(b')| \leq \sigma^2 |f(p) - f(a_2)|.$$

Then since the triangle T_0 has sides with endpoints at p and a_1 , and p and b' we have

$$|f(p) - f(a_1)| \leq \sigma |f(p) - f(b')| \leq \sigma^3 |f(p) - f(a_2)|.$$

□

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The key point of both of our proofs of Theorem 1 given here is contained in Proposition 1, where it is proved that the image of a triangle contains a disk of a definite size, exhibiting a certain length-area estimate. This approach goes back to Pfluger and was pushed forward by P. Koskela and S. Rogovin who proved that the ACL property of a homeomorphism f between open sets of \mathbb{R}^n , $n \geq 2$, could be established from an L^1 -control of

$$(5) \quad k_f = \liminf_{r \rightarrow 0} \left(\frac{\text{diam}(f(D(x, r)))^n}{|f(D(x, r))|} \right)^{1/(n-1)}$$

where $|f(D(x, r))|$ denotes the Lebesgue measure of $f(D(x, r))$; see [7] for details. The authors would like to thank Pekka Koskela for mentioning this similarity.

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