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# EQUILATERAL TRIANGLE SKEW CONDITION FOR QUASICONFORMALITY

COLLEEN ACKERMANN, PETER HAÏSSINSKY, AND AIMO HINKKANEN

ABSTRACT. We characterize quasiconformal mappings in terms of the distortion of the vertices of equilateral triangles.

## 1. INTRODUCTION

Since quasiconformal mappings were first studied nearly a century ago, many diverse characterizations have been discovered. These have led to a wide variety of applications in many fields including Teichmüller theory, elliptic PDE's, hyperbolic geometry and complex dynamics. For an overview of these applications and the theory of quasiconformal mappings see [2], [4], and [6]. In this paper we will use the metric definition of quasiconformality to obtain a new formulation for the definition of planar quasiconformal mappings.

**Definition 1.** Let  $f : U \rightarrow V$  be a homeomorphism between planar domains. When  $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\} \subset U$ , define

$$M(z, r) = \sup\{|f(z) - f(w)| : |z - w| = r\}, \text{ and} \\ m(z, r) = \inf\{|f(z) - f(w)| : |z - w| = r\}.$$

Suppose that  $K \geq 1$ . The mapping  $f$  is said to be  $K$ -quasiconformal if

$$H(z) = \limsup_{r \rightarrow 0} \frac{M(z, r)}{m(z, r)} \leq K$$

for a.e.  $z \in U$ , and if  $H(z)$  is bounded in  $U$ .

In his book [6] John Hubbard obtained a new characterization of quasiconformal mappings. Let  $T$  be a closed topological triangle with specified vertices,  $L(T) = \max\{|a - b| : a, b \text{ are vertices of } T\}$  and  $l(T) = \min\{|a - b| : a, b \text{ are vertices of } T\}$ . We define

$$\text{skew}(T) = \frac{L(T)}{l(T)}.$$

Note that  $f(T)$  is also a topological triangle so the expression  $\text{skew}(f(T))$  is defined. Then  $f : U \rightarrow V$  is quasiconformal provided that there exists an increasing homeomorphism  $h : [0, \infty) \rightarrow [0, \infty)$  such that

$$\text{skew}(f(T)) \leq h(\text{skew}(T))$$

for all closed Euclidean triangles  $T \subset U$ . In fact, Hubbard showed that it is sufficient to only consider triangles with skew bounded above by  $\sqrt{7/3}$ . He then asked the question of whether it suffices to only consider equilateral triangles. Progress was made on this problem in a previous paper by Javier Aramayona and Peter Haïssinsky [3] in which they showed that there exists a constant  $\epsilon_0 > 0$  such that if  $\epsilon \in [0, \epsilon_0)$  and

$$\text{skew}(f(T)) \leq 1 + \epsilon$$

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for all equilateral triangles  $T \subset U$ , then  $f$  is quasiconformal.

Theorem 1 of this paper answers Hubbard's question in the affirmative.

**Theorem 1.** *Let  $U$  and  $V$  be two domains in the complex plane  $\mathbb{C}$ , and let  $f : U \rightarrow V$  be an orientation-preserving homeomorphism. For each  $\sigma \geq 1$  there exists  $H(\sigma) \geq 1$  with the following property. If  $\text{skew}(f(T)) \leq \sigma$  for all equilateral triangles  $T \subset U$ , then, for every  $z \in U$  and every  $r < \text{dist}(z, \mathbb{C} \setminus U)$ , the inequality  $M(z, r) \leq Hm(z, r)$  holds where  $H = H(\sigma)$ . In particular, the map  $f$  is quasiconformal.*

Our next result, Theorem 2, serves two purposes. On the one hand, the value  $H$  from Theorem 1 gives an upper bound for the dilatation of  $f$ . However, this estimate is far from optimal. We may use the fact that quasiconformal mappings are differentiable almost everywhere to obtain a sharp bound on the dilatation in terms of  $\sigma$ . On the other hand, quasiconformality is a local property and demanding, as in Theorem 1, a uniform bound on every equilateral triangle is unnecessarily strong.

To show that  $f$  is quasiconformal in  $U$ , it suffices to show that  $f$  satisfies two conditions. First we must show that each  $z \in U$  has a neighborhood  $W$  such that  $\text{skew}(f(T))$  is bounded for all equilateral triangles  $T \subset W$ . If the resulting upper bounds vary, this is not enough to get the result that  $f$  is quasiconformal in  $U$ . However, by formulating a new concept of skew that needs to be satisfied only almost everywhere and must be bounded uniformly in  $U$ , and combining it with the assumption that the skew as defined previously is bounded, not necessarily uniformly, in some neighborhood of each point of  $U$ , we obtain both quasiconformality and a better upper bound for the maximal dilatation of  $f$ . The details of how to define this new notion of skew are given below.

First note that quasiconformal maps are differentiable almost everywhere by Mori's theorem [9]. Let  $\text{Skew}(f)$  denote the supremum of  $\text{skew}(f(T))$  over all equilateral triangles  $T$  contained in  $U$ . Next, for  $z \in U$  and  $r > 0$ , let  $\text{skew}(f, z, r)$  denote the least upper bound of  $\text{skew}(f(T))$  over all equilateral triangles  $T \subset \{w \in U : |z - w| < r\}$ . Set  $\text{skew}(f, z) = \liminf_{r \rightarrow 0} \text{skew}(f, z, r)$  and  $\text{skew}(f) = \|\text{skew}(f, z)\|_\infty$ . Below, the assumption that  $\text{Skew}(f|W)$  is finite means that there is some upper bound for  $\text{skew}(f(T))$  over all equilateral triangles  $T$  contained in the appropriate domain  $W$ .

**Theorem 2.** *Let  $U$  be a domain in the complex plane  $\mathbb{C}$ , and let  $f : U \rightarrow f(U)$  be an orientation-preserving homeomorphism. Suppose that each  $z \in U$  has a neighborhood  $W$  such that  $\text{Skew}(f|W)$  is finite.*

*Suppose that  $\sigma \geq 1$ . If  $\text{skew}(f) \leq \sigma$  then  $f$  is  $K(\sigma)$ -quasiconformal where*

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

*In particular, if  $\text{skew}(f) = 1$  then  $f$  is a conformal mapping. The upper bound  $K(\sigma)$  for the maximal dilatation of  $f$  is the best possible and is attained at least for certain affine mappings.*

## 2. PROOF OF THE MAIN THEOREM

Throughout the rest of the paper we will use the following notation and conventions:

- (1) We define  $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\}$  and let  $C(z, r)$  be the boundary of  $D(z, r)$ .
- (2) By a curve we mean the image of a not necessarily one-to-one continuous function from a closed interval into  $\mathbb{C}$ .
- (3) All triangles (without the qualifier "topological") will be closed Euclidean triangles.
- (4) Let  $\mathcal{F}_\sigma$  denote the set of orientation-preserving homeomorphisms of any domain  $U \subset \mathbb{C}$  into any domain  $V \subset \mathbb{C}$  such that  $\text{skew}(f(T)) \leq \sigma$  for all closed equilateral triangles  $T \subset U$ .

Our strategy for the proof of Theorem 1 is to show first that the image of every equilateral triangle,  $T$ , contains a disk with radius proportional to  $L(f(T))$ . To do this, we naturally need the assumption of Theorem 1 for all equilateral triangles (at least in a suitable set containing this  $T$ ) and not only for this particular  $T$ . Once it is known that the images of equilateral triangles are “thick” in this sense, it is easier to obtain the quantitative estimates that are required to prove that  $f$  is quasiconformal, whether using the metric (Definition 1) or the analytic (Definition 2 in Section 6 below) definition of quasiconformality.

The most difficult part of the proof that the image  $f(T)$  is thick is to construct another equilateral triangle with suitable additional properties, to which the assumption of Theorem 1 can then be applied. This will be done in the proof of Proposition 2 below. That proof requires delicate geometric considerations. After these strategic comments, let us now move to the proof.

The proof of Theorem 1 relies on the following proposition.

**Proposition 1.** *Let  $U$  be a domain containing  $D(0, 1)$ , let  $f : U \rightarrow \mathbb{C}$  belong to  $\mathcal{F}_\sigma$ , and let  $T$  be the triangle with vertices  $0, 1$ , and  $\omega = 1/2 + (\sqrt{3}/2)i$ . Then there exists a disk  $D$  contained in  $f(T)$  such that*

- (1)  $D$  is centered at  $f(p)$  where  $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i \approx 0.5 + 0.29i$ , and
- (2) there exists a constant  $\alpha = \alpha(\sigma)$  such that the radius of  $D$  is at least  $\alpha L(f(T))$ .

We note that if  $f$  is to be quasiconformal, then, certainly, the image  $f(T)$  has to contain a disk of a definite size centered at the image of the centroid of the triangle, i.e., the point  $\xi = 1/2 + (\sqrt{3}/6)i$ . Unfortunately, its arithmetic properties make it difficult to relate this point  $\xi$  to the vertices of  $T$  using equilateral triangles. The point  $p$  was chosen, because it is both close to the centroid ( $|\xi - p| = \sqrt{3}/(2^9 \cdot 3)$ ), and it is a vertex of a tiling of the plane by equilateral triangles whose vertices include the vertices of  $T$ . Indeed, we have  $p = 1/2 - 85 \cdot 2^{-9} + 85 \cdot 2^{-8}\omega$ , cf. Lemma 1. We finally observe that  $p$  is closer to the horizontal side of  $T$  than  $\xi$ .

We first derive the proof of Theorem 1 from Proposition 1. We will then focus on the proof of the latter.

*Proof of Theorem 1.* Fix  $z \in U$  and  $r > 0$ . If  $D(z, r) \subset U$ , let  $M(z, r) = \max\{|f(z) - f(w)| : w \in C(z, r)\}$  and  $m(z, r) = \min\{|f(z) - f(w)| : w \in C(z, r)\}$ . Denote by  $z_M$  a point in  $C(z, r)$  such that  $|f(z_M) - f(z)| = M(z, r)$ .

Since  $\mathcal{F}_\sigma$  is invariant under pre- and post-composition by affine maps of the form  $z \mapsto az + b$  where  $a, b \in \mathbb{C}$  and  $a \neq 0$ , we assume that  $z = 0$ ,  $r = 1$ , and  $z_M = 1$ . Thus  $|f(0) - f(1)| = M(0, 1)$ . Let  $T_1$  be the equilateral triangle with vertices  $0, 1$ , and  $\omega$ . Since  $0$  and  $1$  are vertices of  $T_1$ , we have  $|f(0) - f(1)| \leq L(f(T_1))$ . Hence  $M(0, 1) \leq L(f(T_1))$ .

By Proposition 1, the image of  $T_1$  must contain a disk  $D$  centered at  $f(p)$  and of radius at least  $\alpha L(f(T_1))$ .

Let us consider the isometry  $A(z) = \overline{z - p}$ . Let  $T_2 = A(T_1)$  (see Figure 1). The triangle  $T_2$  is contained in the unit disk, and  $A$  maps  $p$  to  $0$  and  $1$  to  $p$ . Since the vertices of  $T_2$  other than  $p$  lie outside of  $T_1$ , their images lie outside of  $D$ , and so we have  $L(f(T_2)) \geq \alpha L(f(T_1))$ .

Moreover, another application of Proposition 1 implies that  $f(T_2)$  contains the disk  $D(f(0), \alpha L(f(T_2)))$ . Since all the points of  $f(C(0, 1))$  are outside the interior of  $f(T_2)$  and thus also outside the interior of  $D(f(0), \alpha L(f(T_2)))$ , it follows that  $m(0, 1) \geq \alpha L(f(T_2))$ .

Summing up these estimates, we obtain

$$m(0, 1) \geq \alpha L(f(T_2)) \geq \alpha^2 L(f(T_1)) \geq \alpha^2 M(0, 1).$$

□

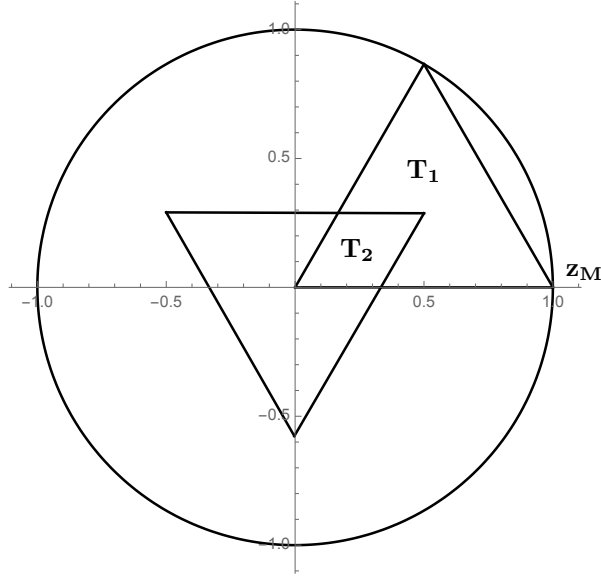


FIGURE 1. Configuration of  $C(0, 1)$ ,  $T_1$  and  $T_2$

### 3. CONSTRUCTION OF CERTAIN TRIANGLES

Proposition 1 is a consequence of the following proposition.

**Proposition 2.** *Let  $U$  be a neighborhood of  $D(0, 1)$ , and let  $f : U \rightarrow \mathbb{C}$  be a homeomorphism onto its image such that  $f \in \mathcal{F}_\sigma$ . Let  $T$  be the closed triangle with vertices  $0$ ,  $1$ , and  $\omega = 1/2 + (\sqrt{3}/2)i$ . Let  $q = p + 2^{-9}$ . Then there exist points  $t_1, t_2 \in T$  such that the points  $q, t_1, t_2$  form the vertices of an equilateral triangle and the inequalities  $|f(t_j) - f(p)| \leq C\mu$ , and  $|f(p) - f(q)| \geq cL(f(T))$  hold for some constants  $c = c(\sigma)$  and  $C = C(\sigma)$  where  $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$ . We permit the trivial triangle where we have  $t_1 = t_2 = q$ .*

*Proof of Proposition 1 assuming Proposition 2.* If  $t_1 = t_2 = q$ , then we have

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu.$$

Otherwise, by the triangle inequality we have

$$(1) \quad |f(p) - f(q)| \leq |f(t_1) - f(p)| + |f(t_1) - f(q)|.$$

Since  $f \in \mathcal{F}_\sigma$  and  $t_1, t_2, q$  are the vertices of an equilateral triangle, we obtain

$$(2) \quad |f(t_1) - f(q)| \leq \sigma |f(t_1) - f(t_2)|.$$

By Proposition 2, we have  $|f(t_1) - f(p)| \leq C\mu$  and  $|f(t_2) - f(p)| \leq C\mu$ , so that by this and the triangle inequality we get

$$|f(t_1) - f(t_2)| \leq |f(t_1) - f(p)| + |f(t_2) - f(p)| \leq 2C\mu.$$

By Proposition 2, we have  $|f(p) - f(q)| \geq cL(f(T))$ . Using this and appealing to  $|f(t_1) - f(p)| \leq C\mu$  again, we obtain from (1) and (2) that

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu + \sigma(2C\mu),$$

whence

$$\mu \geq \frac{c}{(2\sigma + 1)C} L(f(T)).$$

□

#### 4. PROOF OF PROPOSITION 2

Let the assumptions of Proposition 2 be satisfied. Recall that we write  $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$ . The idea of our proof of Proposition 2 is to define a curve  $\gamma'$  in  $T$  going through  $p$  such that

- (1) for all  $t \in \gamma'$  we have  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ ;
- (2) there are two points  $t_1, t_2 \in \gamma'$ , such that  $q, t_1, t_2$  form the vertices of an equilateral triangle.

The proof of Proposition 2 results from Lemma 1 and Lemma 3.

We first prove the following result. Recall that  $\omega = 1/2 + (\sqrt{3}/2)i$ .

**Lemma 1.** *Let  $T$  be the closed triangle with vertices  $0, 1$ , and  $\omega$ . Let  $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$  and  $q = p + 2^{-9}$ . Then  $|f(q) - f(p)| \geq cL(f(T))$  for some positive constant  $c = c(\sigma)$ .*

*Proof.* Let us first consider the tiling of the plane by equilateral triangles with vertices in  $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$ . Define a chain of triangles  $(T_j)_{0 \leq j \leq J}$  as a sequence of triangles with vertices in  $\Lambda$  such that  $T_j \cap T_{j+1}$  is an edge for all  $j$  with  $0 \leq j < J$ . Given two edges  $(v, w)$  and  $(v', w')$ , we may connect them by a chain of minimal length  $n \geq 0$ . A simple induction argument implies

$$(3) \quad |f(v) - f(w)| \leq \sigma^n |f(v') - f(w')|$$

if  $f \in \mathcal{F}_\sigma$  is defined in a neighborhood of the chain.

Let  $T$  be as defined in our hypotheses: it is tiled by  $N = 2^{18}$  triangles of  $2^{-9}\Lambda$ , and  $[p, q]$  is an edge of this tiling. Therefore, for every other edge  $[v, w]$ , it follows that

$$|f(v) - f(w)| \leq \sigma^N |f(p) - f(q)|.$$

But each side of  $T$  is the union of less than  $N$  edges of our tiling, therefore, the triangle inequality implies

$$L(f(T)) \leq N\sigma^N |f(p) - f(q)|.$$

□

We now prove a geometric lemma which will be used in the proof of Lemma 3.

**Lemma 2.** *Let  $|z| \leq 1/8$  and suppose that  $|\theta_\pm - (\pm\pi/3)| \leq 1/8$ . Then the angle  $\varphi$  between  $e^{i\theta_+} - z$  and  $e^{i\theta_-} - z$  which crosses the positive real axis belongs to  $(\pi/3, \pi)$ .*

*Proof.* We note that  $\cos \theta_\pm \geq 1/2 - 1/4 > 1/8 \geq |z|$  so that  $\varphi$  is less than  $\pi$ .

For the other inequality, we will estimate  $\tan |\arg(e^{i\theta_\pm} - z)|$  to obtain a lower bound of both angles with the horizontal line:

$$\begin{aligned} \tan |\arg(e^{i\theta_\pm} - z)| &\geq \frac{\sqrt{3}/2 - (|z| + 1/8)}{1/2 + (|z| + 1/8)} \geq \frac{\sqrt{3}/2 - 1/4}{1/2 + 1/4} \\ &\geq \frac{2\sqrt{3} - 1}{3} \geq \frac{2}{3} > \tan(\pi/6). \end{aligned}$$

Therefore  $\varphi$  is at least  $\pi/3$ .

□

Now we demonstrate how to find the curve  $\gamma'$  mentioned above.

**Lemma 3.** *Under the assumptions of Proposition 2, there exists a curve  $\gamma'$  in  $T$  going through  $p$  such that for all  $t \in \gamma'$  we have*

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$$

*and there are two points  $t_1, t_2 \in \gamma'$ , such that  $q, t_1, t_2$  form the vertices of an equilateral triangle. We permit the trivial triangle where we have  $t_1 = t_2 = q$ .*

*Proof.* We will do this in several steps. We first define a curve that will join two points of the boundary of a disk contained in  $T$  (Step 1). To make sure that we will be able to find two points that form an equilateral triangle with  $q$ , we will extend this curve so that it has end points in a slightly larger disk, and is only close to the boundary of the larger disk when it is also close to its end points (Step 2). Then we will use Lemma 2 to find our triangle (Step 3).

Since  $\sqrt{3} \geq 8/5$ , it follows that  $\text{dist}(p, \partial T) = 85\sqrt{3} \cdot 2^{-9} > 1/4 + 2^{-6}$ , so that  $D(p, 1/4 + 2^{-6})$  is contained in the interior of  $T$ .

Throughout the proof, for  $x \in \mathbb{C}$ ,  $R_x$  will denote the rotation centered at  $x$  by  $\pi/3$  radians, defined by  $R_x(z) = x + (z - x)\omega$  and  $\bar{R}_x$  the rotation centered at  $x$  by  $-\pi/3$  radians, defined by  $\bar{R}_x(z) = x + (z - x)\bar{\omega}$ . Recall that we set  $\omega = 1/2 + (\sqrt{3}/2)i$ .

**Step 1: There exists a curve  $\gamma_2$  that satisfies the following:**

- (1)  $\gamma_2 \subset D(p, 1/4)$ ,
- (2)  $\gamma_2$  has end points on  $C(p, 1/4)$  which are exactly  $2\pi/3$  radians apart, and
- (3) for all points  $t \in \gamma_2$  we have  $|f(t) - f(p)| \leq \sigma\mu$ .

Let  $p' \in \partial T$  be such that  $d(f(p), f(p')) = \mu$  and let  $\gamma = f^{-1}([f(p), f(p')])$ . Since  $D(p, 1/4)$  is contained in the interior of  $T$ , we may consider the component  $\gamma_1$  of  $\gamma \cap D(p, 1/4)$  that contains  $p$ , and we denote by  $w \in C(p, 1/4)$  the other end point of  $\gamma_1$ . We take  $w$  to be the first point of  $C(p, 1/4)$  encountered when moving along  $\gamma$  starting from  $p$ .

Now define

$$\gamma_2 = R_p(\gamma_1) \cup \bar{R}_p(\gamma_1).$$

Note that, for every  $s \in \gamma_1$ ,  $R_p(s)$  and  $\bar{R}_p(s)$  are two points in  $\gamma_2$  which make an angle of  $2\pi/3$  seen from  $p$ . Since  $f \in \mathcal{F}_\sigma$ , for all  $t \in \gamma_2$ , we have

$$|f(t) - f(p)| \leq \sigma|f(s) - f(p)| \leq \sigma\mu$$

where  $s \in \gamma_1$  denotes a point such that either  $t = R_p(s)$  or  $t = \bar{R}_p(s)$ .

**Step 2: Let  $a, b$  be the end points of  $\gamma_2$ . There exists a curve  $\gamma_3$  such that**

- (1)  $\gamma_3 \subset D(p, 1/4) \cup D(a, 2^{-6}) \cup D(b, 2^{-6})$ ;
- (2)  $\gamma_3$  has both end points on  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ ;
- (3) for all points  $t \in \gamma_3$ ,  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ .

Let  $D_a = D(a, 2^{-6})$  and  $D_b = D(b, 2^{-6})$ . Let  $\gamma_{2a}$  and  $\gamma_{2b}$  be the components of  $\gamma_2 \cap D_a$  and  $\gamma_2 \cap D_b$  that have end points at  $a$  and  $b$  respectively.

Clearly  $\gamma_{2a}$  also has an end point on the boundary of  $D_a$ . Let  $a'$  denote an end point of  $\gamma_{2a}$  on the boundary of  $D_a$ . Use the tangent line to  $D(p, 1/4)$  at  $a$  to divide  $D_a$  in half, and then divide each half into thirds. So we have divided  $D_a$  into closed sectors of  $\pi/3$  radians with three such sectors lying entirely outside of  $D(p, 1/4)$ . Let  $S_a$  denote the middle sector lying completely outside of  $D(p, 1/4)$ . Then there exists  $n \in \{2, 3\}$  such that when  $\gamma_{2a}$  is rotated  $n\pi/3$  radians in an appropriate direction about  $a$ , the image of  $a'$  under the rotation will lie in  $S_a$ . Let the image of  $\gamma_{2a}$  under this rotation be denoted by  $\gamma_{3a}$ .

Now we will bound the quantity  $|f(t) - f(p)|$  where  $t \in \gamma_{3a}$ . Fix  $t \in \gamma_{3a}$ . Let  $t_0$  be the point on  $\gamma_{2a}$  whose image under the rotation is  $t$ . Without loss of generality we will assume this rotation was clockwise. Let  $t_i$  denote the image of  $t_0$  under a clockwise rotation of  $i\pi/3$  radians where  $i = 1, \dots, n$  ( $t = t_n$ ). Then since  $a, t_{i-1}, t_i$  ( $1 \leq i \leq n$ ) form an equilateral triangle, we have

$$|f(t_i) - f(a)| \leq \sigma |f(t_{i-1}) - f(a)|.$$

Since  $a, t_0 \in \gamma_2$  we have

$$|f(a) - f(t_0)| \leq |f(a) - f(p)| + |f(p) - f(t_0)| \leq 2\sigma\mu$$

and

$$|f(a) - f(p)| \leq \sigma\mu.$$

Thus since  $n$  is at most 3 we may apply (3) with  $n \leq 3$ , which together with the triangle inequality yields

$$|f(t) - f(p)| \leq |f(a) - f(p)| + |f(a) - f(t)| \leq \sigma\mu + \sigma^n |f(a) - f(t_0)| \leq \sigma\mu(1 + 2\sigma^n) \leq \sigma\mu(1 + 2\sigma^3).$$

Furthermore  $\gamma_{3a}$  must intersect the circle  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ . This is because  $\gamma_{3a}$  has an end point in  $S_a$  and therefore the distance of the end point of  $\gamma_{3a}$  from  $D(p, 1/4)$  must be at least  $\cos(\pi/6) \cdot 2^{-6} = \sqrt{3} \cdot 2^{-7}$ . This is depicted in Figure 2.

We proceed similarly near  $b$  and define a curve  $\gamma_{3b}$  contained in  $D_b$  with end points at  $b$  and at some point on the intersection of the boundary of  $D_b$  and  $S_b$  (defined analogously to  $S_a$ ) such that for all  $t \in \gamma_{3b}$  we have  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ ; as above,  $\gamma_{3b}$  intersects the circle  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ .

Let  $\gamma_3$  be the connected component of  $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$  which includes points in both  $\gamma_{3a}$  and  $\gamma_{3b}$ . Then for all points  $t \in \gamma_3$ ,

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3).$$

The curve  $\gamma'$  in Lemma 3 can be chosen as  $\gamma' = \gamma_3$ .

**Step 3: Let  $q = p + 2^{-9}$ . There exist  $t_1, t_2 \in \gamma_3$  such that  $\{q, t_1, t_2\}$  form an equilateral triangle.**

Let  $D_q$  be the smallest disk centered at  $q$  which contains  $D(p, 1/4)$ . Then  $D_q \subset D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$  since  $|p - q| = 2^{-9} \leq \sqrt{3} \cdot 2^{-8}$ . Let  $\gamma_4$  be the connected component of  $\gamma_3 \cap D_q$  which has end points  $A \in D_a \cap D_q$  and  $B \in D_b \cap D_q$ .

Note that, if we write  $a = p + |a - p|e^{i\theta_a}$  and  $A = p + |A - p|e^{i\theta_A}$ , where  $|\theta_A - \theta_a|$  is chosen to be as small as possible modulo  $2\pi$ , then

$$|\theta_A - \theta_a| \leq 2|a - A|/(1/4) \leq 2 \cdot 2^{-6}/(1/4) \leq 1/8$$

and similarly for  $b$  and  $B$ . Note that  $|\arg(a - p)| + |\arg(b - p)| = 2\pi/3$  and  $|p - q|/(1/4) = 2^{-7} \leq 1/8$ . Therefore, by Lemma 2 applied in  $D(p, 1/4)$ , the angle between  $A - q$  and  $B - q$  lies in  $(\pi/3, \pi)$ . Hence,



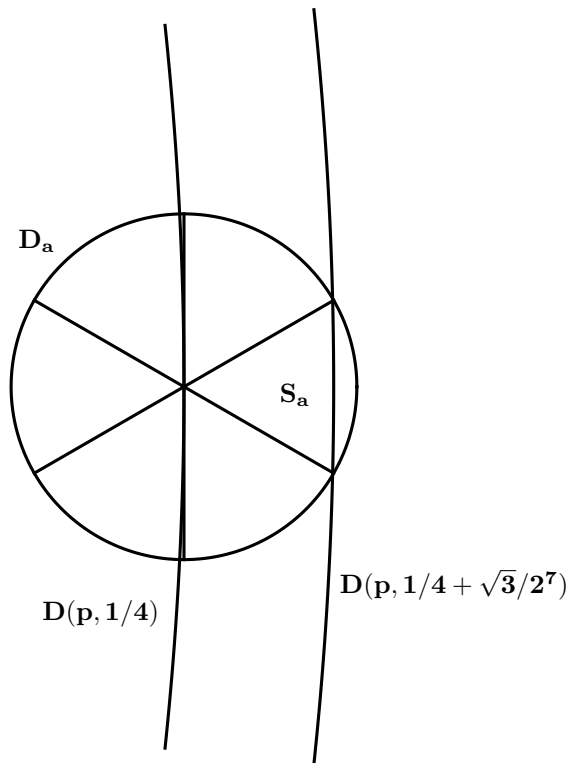


FIGURE 2

the images  $A_r$  and  $B_r$  of  $A$  and  $B$  respectively under  $\bar{R}_q$  will separate  $A$  and  $B$  on  $\partial D_q$ . Thus the image  $\bar{R}_q(\gamma_4)$  must intersect  $\gamma_4$  at a point  $q'$ , say. This gives us our desired equilateral triangle since  $q$ ,  $q'$ , and the pre-image  $q''$  of  $q'$  form an equilateral triangle.

This completes the proof of Lemma 3. □

To complete the proof of Proposition 2, we note that the points  $t_1$  and  $t_2$  required will be the points  $q'$  and  $q''$ . It remains to be shown that  $|f(q') - f(p)| \leq C\mu$  and  $|f(q'') - f(p)| \leq C\mu$  for some constant  $C$ . Since  $q'' \in \gamma_4 \subset \gamma_3$ , we have  $|f(q'') - f(p)| \leq \sigma(1 + 2\sigma^3)\mu$  by part (3) in Step 2. But also  $q' \in \gamma_4$ , so that for the same reason  $|f(q') - f(p)| \leq \sigma(1 + 2\sigma^3)\mu$ . This completes the proof of Proposition 2.

## 5. PROOF OF THEOREM 2

We prove Theorem 2 by approximating  $f$  by linear mappings at points where  $f$  is differentiable.

*Proof.* Let the assumptions of Theorem 2 be satisfied. Theorem 1 implies that  $f$  is locally quasiconformal and hence differentiable with  $f_z \neq 0$  at almost every point of  $U$ . Let  $z_0$  be a point of differentiability such that  $\text{skew}(f, z_0) \leq \sigma$ . We compute the maximum possible value for  $H(z_0)$ , which yields the same upper bound for the dilatation of  $f$  at  $z_0$ . Since  $H(z_0)$  is invariant under Möbius transformations we may compose with translations, a dilation and a rotation to assume that  $z_0 = f(z_0) = 0$ ,  $f_z(z_0) = 1$  and  $f_{\bar{z}}(z_0) = |f_{\bar{z}}(z_0)| < 1$ . Then  $f(z) = z + f_{\bar{z}}(z_0)\bar{z} + \epsilon(z)$  where  $\epsilon(z)/|z|$  tends to 0 as  $z$  tends to  $z_0$ , and thus  $\text{skew}(f, z_0) = \text{skew}(\tilde{f}, z_0)$  where  $\tilde{f}(z) = z + f_{\bar{z}}(z_0)\bar{z}$ . It now suffices to prove that for all affine mappings  $\tilde{f}$  that can arise in this way, under the assumptions of Theorem 2, the maximal dilatation of  $\tilde{f}$  is at most  $K(\sigma)$ . This then shows that also the maximal dilatation of our original mapping  $f$  is at most  $K(\sigma)$ , as required. Thus from now on we only consider  $\tilde{f}$ .

We will first compute the skew of  $\tilde{f}$ , and then express  $K(\tilde{f})$  in terms of  $\text{skew}(\tilde{f})$ .

Note that  $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(a+v) - \tilde{f}(b+v)|$ ,  $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(\bar{a}) - \tilde{f}(\bar{b})|$  and  $|\tilde{f}(a) - \tilde{f}(b)|/|\tilde{f}(a) - \tilde{f}(c)| = |\tilde{f}(ra) - \tilde{f}(rb)|/|\tilde{f}(ra) - \tilde{f}(rc)|$  for all  $a, b, c, v \in \mathbb{C}$  with  $a \neq c$  and all  $r > 0$ . This implies  $\text{skew}(\tilde{f}(T))$  where  $T$  is an equilateral triangle is invariant under translations, complex conjugation and dilations of  $T$ . Thus for all equilateral triangles  $T$ ,

$$\text{skew}(\tilde{f}(T)) \in \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\}.$$

Indeed, suppose  $T$  has vertices  $A, B$  and  $C$ , and  $\text{skew}(T) = \frac{|\tilde{f}(A) - \tilde{f}(B)|}{|\tilde{f}(A) - \tilde{f}(C)|}$ . First we translate  $A$  to the origin, and then we dilate  $T$  so its side lengths are equal to 1. If  $\overline{AB}$  is  $\pi/3$  radians clockwise from  $\overline{AC}$ , it is clear that our statement is true; otherwise we take the complex conjugate of  $T$  to change the orientation of  $T$  and then, since  $\text{skew}(\tilde{f}(T))$  is invariant under complex conjugation of  $T$ , our claim is true.

Hence we have

$$\text{skew}(\tilde{f}) = \max \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\} = \max \left\{ \frac{|\tilde{f}(z)|}{|\tilde{f}(ze^{i\pi/3})|} : |z| = 1 \right\}.$$

Let  $\mu = f_{\bar{z}}(z_0)$ . If  $\mu = 0$ , then  $\text{skew}(\tilde{f}) = 1$ , so we assume that  $0 < \mu < 1$ . Write  $\nu = \mu + \mu^{-1} > 2$  and  $\beta = e^{i\pi/6}$ . Let  $w \in \mathbb{C}$  with  $|w| = 1$ . We have

$$|\tilde{f}(w)|^2 = |w + \mu\bar{w}|^2 = (w + \mu\bar{w})(\bar{w} + \mu w) = 1 + \mu^2 + \mu(w^2 + \bar{w}^2) = \mu[\nu + (w^2 + \bar{w}^2)].$$

Now we are able to maximize  $|\tilde{f}(\beta w)|/|\tilde{f}(\bar{\beta} w)|$  with respect to  $w$ . Set  $z = w^2$  and  $\alpha = e^{i\pi/3}$ . Since we have assumed  $|w| = 1$ , we can instead maximize

$$\kappa = \left| \frac{\tilde{f}(\beta w)}{\tilde{f}(\bar{\beta} w)} \right|^2 = \frac{\nu + \alpha z + \bar{\alpha}\bar{z}}{\nu + \bar{\alpha}z + \alpha\bar{z}}.$$

We write  $z = e^{ix}$ ,  $x \in \mathbb{R}$ , so that  $z' \equiv dz/dx = iz$ ,  $\bar{z}' = -i\bar{z}$ . We may differentiate  $\kappa$  as a function of  $x$ . It follows that  $\kappa' = 0$  if, and only if,

$$(\alpha z - \bar{\alpha}\bar{z})(\nu + \bar{\alpha}z + \alpha\bar{z}) - (\nu + \alpha z + \bar{\alpha}\bar{z})(\bar{\alpha}z - \alpha\bar{z}) = 0.$$

Thus

$$\nu(\alpha z - \bar{\alpha}\bar{z} - \bar{\alpha}z + \alpha\bar{z}) = z^2 - \alpha^2 + \bar{\alpha}^2 - \bar{z}^2 - z^2 - \alpha^2 + \bar{\alpha}^2 + \bar{z}^2$$

which is equivalent to

$$\nu(z + \bar{z})(\alpha - \bar{\alpha}) = 2(\bar{\alpha} - \alpha)(\bar{\alpha} + \alpha).$$

Therefore

$$\cos x = -\frac{2}{\nu} \cos \frac{\pi}{3} = -\frac{1}{\nu}.$$

It follows that  $\sin^2 x = 1 - 1/\nu^2$  so  $\kappa' = 0$  for

$$z = \frac{1}{\nu} \left( -1 + i\varepsilon\sqrt{\nu^2 - 1} \right)$$

with  $\varepsilon \in \{\pm 1\}$ . For these values of  $z$ , one gets

$$\kappa = \frac{\nu + 2\operatorname{Re}(\alpha z)}{\nu + 2\operatorname{Re}(\bar{\alpha} z)} = \frac{\nu^2 - 1 - \varepsilon\sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 + \varepsilon\sqrt{3(\nu^2 - 1)}}$$

which is maximal for  $\varepsilon = -1$ . So we obtain

$$\operatorname{skew}(\tilde{f})^2 = \frac{\nu^2 - 1 + \sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 - \sqrt{3(\nu^2 - 1)}} = \frac{\sqrt{(\nu^2 - 1)/3} + 1}{\sqrt{(\nu^2 - 1)/3} - 1} > 1.$$

Note that  $\nu^2 - 1 > 3$  since  $\nu > 2$ . Let us write  $\tau = \operatorname{skew}(\tilde{f}) > 1$  so that

$$\sqrt{(\nu^2 - 1)/3} = \frac{\tau^2 + 1}{\tau^2 - 1}, \quad \nu^2 = 3 \left( \frac{\tau^2 + 1}{\tau^2 - 1} \right)^2 + 1 = \frac{4(\tau^4 + \tau^2 + 1)}{(\tau^2 - 1)^2}$$

and thus

$$\mu + \mu^{-1} = \nu = \frac{2\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1}.$$

Hence

$$\mu^2 - 2\mu \frac{\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1} + 1 = 0.$$

We compute the reduced discriminant

$$\Delta' = \frac{\tau^4 + \tau^2 + 1}{(\tau^2 - 1)^2} - 1 = \frac{3\tau^2}{(\tau^2 - 1)^2}$$

and we deduce from  $0 < \mu < 1$  that

$$\mu = \frac{\sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1}.$$

Thus

$$K(\tilde{f}) = \frac{1 + \mu}{1 - \mu} = \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1 - \sqrt{\tau^4 + \tau^2 + 1} + \sqrt{3}\tau} = \varphi(\tau),$$

say. Write  $t = \sqrt{\tau^4 + \tau^2 + 1}$  and note that  $t^2 = (\tau^2 - 1)^2 + 3\tau^2$  so that

$$\begin{aligned} \varphi(\tau) &= \frac{[(\tau^2 - 1 + t) - \sqrt{3}\tau][(\tau^2 - 1 + t) + \sqrt{3}\tau]}{[(\tau^2 - 1) + \sqrt{3}\tau]^2 - t^2} \\ &= \frac{[(\tau^2 - 1) + t]^2 - 3\tau^2}{2\sqrt{3}\tau(\tau^2 - 1)} = \frac{2[(\tau^2 - 1)^2 + t(\tau^2 - 1)]}{2\sqrt{3}\tau(\tau^2 - 1)} \\ &= \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1}}{\sqrt{3}\tau}. \end{aligned}$$

Differentiation shows that  $\varphi(\tau)$  is an increasing function of  $\tau$ , so that since  $\tau = \operatorname{skew}(\tilde{f}) = \operatorname{skew}(f, z_0) \leq \sigma$ , we have

$$K(\tilde{f}) = \varphi(\tau) \leq \varphi(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

Hence  $K(f) \leq \varphi(\sigma) = K(\sigma)$  as defined in Theorem 2.

If  $\mu \in (0, 1)$  is given and if  $f(z) = z + \mu\bar{z}$ , then we may take  $\tilde{f} = f$  and  $\sigma = \tau$  in the above argument, and we see that  $K(f) = K(\sigma)$ . Thus the upper bound in Theorem 2 is best possible.  $\square$

## 6. AN ALTERNATIVE PROOF OF THE QUASICONFORMALITY OF MAPPINGS SATISFYING THE HYPOTHESES OF THEOREM 1

From Proposition 1, there are several ways of establishing that a mapping  $f$  satisfying the hypotheses of Theorem 1 satisfies the analytic definition of quasiconformality which is equivalent to Definition 1. For  $|\xi| = 1$ , we denote by  $\partial_\xi f(z)$  the directional derivative  $\lim_{t \downarrow 0} (f(z + t\xi) - f(z))/(t\xi)$ .

**Definition 2.** We say that a homeomorphism  $f : U \rightarrow V$  is absolutely continuous on lines if for every rectangle  $R = \{(x, y) : a < x < b, c < y < d\}$  with  $\bar{R} \subset U$ ,  $f$  is absolutely continuous on a.e. interval  $I_x = \{(x, y) : c < y < d\}$  and a.e. interval  $I_y = \{(x, y) : a < x < b\}$ . An orientation-preserving homeomorphism  $f$  is called quasiconformal if  $f$  is absolutely continuous on lines and there exists  $K \geq 1$  such that

$$\max_{\xi} |\partial_\xi f(z)| \leq K \min_{\xi} |\partial_\xi f(z)| \text{ a.e.}$$

Proposition 1 tells us that the image of every equilateral triangle,  $T$ , contains a disk with radius proportional to  $L(f(T))$ . In [6, Section 4.5], Hubbard uses this to prove that the map belongs to the Sobolev space  $W_{loc}^{1,2}$  by an approximation argument. We propose another approach which shows directly that the map satisfies the ACL property.

*Proof.* First we show that  $f$  is absolutely continuous on lines. This part of the proof parallels Pfluger's proof that a mapping satisfying the geometric definition of quasiconformality is absolutely continuous on lines. His proof can be found in [10] and is reproduced in English in [8], p. 162. We fix a rectangle  $R = \{(x, y) : a < x < b, c < y < d\}$  and let  $I_y = \{(x, y) : a < x < b\}$  for  $y$  between  $c$  and  $d$ . Define  $A(y)$  to be the area in  $f(R)$  beneath the image of  $I_y$ . Since  $A$  is an increasing function of  $y$ , it is differentiable almost everywhere. We will show that  $f|_{I_y}$  is absolutely continuous for all  $y$  at which  $A$  is differentiable; a similar argument applies to vertical line segments. We select an arbitrary collection  $\{(z_k^*, z_k)\}_{k=1}^n$  of disjoint sub-intervals of  $I_y$  where  $z_k = (x_k, y)$  and  $z_k^* = (x_k^*, y)$ .

We consider the collection of rectangles  $\{R_k\}_{k=1}^n$  where each  $R_k$  has height  $\delta$  and has its bottom side contained in the  $k$ th sub-interval. More precisely, fix  $k$  and draw a rectangle of height  $\delta$  above  $(z_k^*, z_k)$ . Set  $N_k$  equal to the smallest integer greater than or equal to  $\frac{|x_k^* - x_k|}{\delta}$ . We draw  $N_k$  equilateral triangles in  $R_k$ . The triangles all have one side on the interval  $[z_k^*, z_k]$  and they overlap only at their vertices. The first  $N_k - 1$  triangles have width  $\delta$  and the last triangle has width  $x_k^* - x_k - \delta(N_k - 1)$ . Let  $\Delta_{k,i}$  denote the side length of the  $i$ th triangle for  $1 \leq i \leq N_k$  and set  $\Delta_{k,0} = 0$ . Set  $N = \sum_{k=1}^n N_k$ . Then  $N \leq \frac{\sum_{k=1}^n |x_k^* - x_k|}{\delta} + n$ .

Let  $\alpha$  be as in Proposition 1. By Proposition 1, the image of each of our triangles must contain a disk of radius comparable to the greatest distance between the images of the vertices of the triangle. Thus the total area of all the images of our rectangles of height  $\delta$  and width  $x_k^* - x_k$  is greater than or equal to

$$(4) \quad \sum_{k=1}^n \sum_{j=1}^{N_k} \pi \alpha^2 \left( \left| f\left(x_k + \sum_{i=1}^j \Delta_{k,i}, y\right) - f\left(x_k + \sum_{i=1}^{j-1} \Delta_{k,i-1}, y\right) \right| \right)^2.$$

Then by the Cauchy–Schwarz inequality, (4) is greater than or equal to

$$\begin{aligned}
& \frac{\pi\alpha^2}{N} \left( \sum_{k=1}^n \sum_{j=1}^{N_k} |f(x_k + \sum_{i=1}^j \Delta_{k,i}, y) - f(x_k + \sum_{i=1}^j \Delta_{k,i-1}, y)| \right)^2 \\
& \geq \frac{\pi\alpha^2}{N} \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \geq \frac{\pi\alpha^2}{\left( \frac{\sum_{k=1}^n |x_k^* - x_k|}{\delta} + n \right)} \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \\
& = \frac{\pi\alpha^2}{\left( \frac{\sum_{k=1}^n |x_k^* - x_k| + \delta n}{\delta} \right)} \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2.
\end{aligned}$$

Recall that  $A(y)$  is defined to be the area in  $f(R)$  beneath the image of the line segment  $I_y$ . Since the sum of the areas of the images of our rectangles of height  $\delta$  and width  $x_k^* - x_k$  is less than or equal to  $A(y + \delta) - A(y)$ , we have

$$\pi\alpha^2 \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq \left( \frac{A(y + \delta) - A(y)}{\delta} \right) \left( \sum_{k=1}^n |x_k^* - x_k| + \delta n \right).$$

Since we chose  $y$  at which  $A$  is differentiable, letting  $\delta \rightarrow 0$  gives

$$\pi\alpha^2 \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq A'(y) \left( \sum_{k=1}^n |x_k^* - x_k| \right).$$

Since  $A'(y)$  exists almost everywhere this gives absolute continuity on almost every horizontal line segment. The proof is analogous for vertical line segments.

To then conclude that  $f$  is quasiconformal we note that since  $f$  is open and absolutely continuous on lines,  $f$  is differentiable almost everywhere by a theorem of Gehring and Lehto [4]. Now the computations in the proof of Theorem 2 show that

$$\max_{\xi} |\partial_{\xi} f(z)| \leq K(\sigma) \min_{\xi} |\partial_{\xi} f(z)|$$

at points where  $f$  is differentiable and hence almost everywhere. According to the analytic definition of quasiconformality (Definition 2), the mapping  $f$  is quasiconformal.  $\square$

## 7. APPENDIX BY COLLEEN ACKERMANN: AN ANALOGUE OF THE MAIN THEOREM IN HILBERT SPACES OF DIMENSION AT LEAST THREE

In dimensions three and higher the proof of an analogue of Theorem 1 is surprisingly simpler than the proof of Theorem 1. Furthermore the proof itself gives an elegant bound on  $K(\sigma)$ .

**Theorem 3.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces with  $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \geq 3$  and let  $U \subset \mathcal{H}_1, V \subset \mathcal{H}_2$  be domains. Suppose that  $f : U \rightarrow V$  is a homeomorphism and that for all closed equilateral triangles  $T \subset U$ ,  $\text{skew}(f(T)) \leq \sigma$ . Then  $f$  is  $\sigma^3$ -quasiconformal when using the metric definition of quasiconformality.*

Note that Definition 1 can be used to define quasiconformal mappings also between Hilbert spaces, whether finite-dimensional or infinite-dimensional. The same discussion on the locality of quasiconformality made in the introduction applies here as well and enables us to weaken the hypothesis that  $\text{skew}(f(T)) \leq \sigma$  holds for all equilateral triangles.

*Proof.* Fix a point  $p \in U$ , a positive number  $r$  with  $r < \text{dist}(p, \partial U)$  and points  $a_1, a_2 \in \partial B(p, r)$ . We will prove  $|f(a_1) - f(p)| \leq \sigma^3 |f(p) - f(a_2)|$ . Let  $P \subset \mathcal{H}_1$  be the affine plane containing  $\{p, a_1, a_2\}$  and let us

fix a point  $q \in \mathcal{H}_1$  so that  $\vec{pq}$  is orthogonal to  $P$  and let us restrict ourselves to the three-dimensional space containing  $\{p, a_1, a_2, q\}$  that we may identify with  $\mathbb{R}^3$ . For the sake of convenience, we will consider spherical coordinates  $(r, \theta, \varphi)$  with origin  $p$  so that  $(r, \theta, \pi/2)$  reduces to polar coordinates in  $P$  and so that  $a_1$  and  $a_2$  have coordinates  $(r, \pm\theta/2, \pi/2)$  where  $\theta \in (0, \pi]$  denotes the non-oriented angle between  $\vec{pa}_1$  and  $\vec{pa}_2$ .

**Case 1:** If  $\theta \leq 2\pi/3$ , then we may find  $\varphi \in [0, \pi/2]$  such that  $2 \sin \varphi \cos(\theta/2) = 1$  and we set  $b = (r, 0, \varphi)$ . Both triangles  $T_j$  with vertices  $\{p, a_j, b\}$ ,  $j = 1, 2$ , are equilateral by construction and they share a common side with end points at  $b$  and  $p$ .

Thus

$$|f(p) - f(a_1)| \leq \sigma |f(p) - f(b)| \leq \sigma^2 |f(p) - f(a_2)|.$$

**Case 2:** If  $\theta > 2\pi/3$ , consider the equilateral triangle  $T_0$  with vertices  $p, a_1$  and  $b'$  where  $b'$  is the image of  $a_1$  under a rotation in  $P$  of angle  $\pi/3$  so that the smaller angle between  $b'$  and  $a_2$  is less than or equal to  $2\pi/3$ . Thus by Case 1

$$|f(p) - f(b')| \leq \sigma^2 |f(p) - f(a_2)|.$$

Then since the triangle  $T_0$  has sides with endpoints at  $p$  and  $a_1$ , and  $p$  and  $b'$  we have

$$|f(p) - f(a_1)| \leq \sigma |f(p) - f(b')| \leq \sigma^3 |f(p) - f(a_2)|.$$

□

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The key point of both of our proofs of Theorem 1 given here is contained in Proposition 1, where it is proved that the image of a triangle contains a disk of a definite size, exhibiting a certain length-area estimate. This approach goes back to Pfluger and was pushed forward by P. Koskela and S. Rogovin who proved that the ACL property of a homeomorphism  $f$  between open sets of  $\mathbb{R}^n$ ,  $n \geq 2$ , could be established from an  $L^1$ -control of

$$(5) \quad k_f = \liminf_{r \rightarrow 0} \left( \frac{\text{diam}(f(D(x, r)))^n}{|f(D(x, r))|} \right)^{1/(n-1)}$$

where  $|f(D(x, r))|$  denotes the Lebesgue measure of  $f(D(x, r))$ ; see [7] for details. The authors would like to thank Pekka Koskela for mentioning this similarity.

## REFERENCES

- [1] C. Ackermann, *Quasiconformal mappings on planar surfaces*, Ph.D. Thesis, 2016.
- [2] L. V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. MR0200442 (34 #336)
- [3] J. Aramayona and P. Haïssinsky, *A characterisation of plane quasiconformal maps using triangles*, Publ. Mat. **52** (2008), no. 2, 459–471. MR2436734 (2009j:30041)
- [4] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009. MR2472875 (2010j:30040)
- [5] F. W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I No. **272** (1959), 9. MR0124487 (23 #A1799)

- [6] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006. MR2245223 (2008k:30055)
- [7] P. Koskela and S. Rogovin, *Linear dilation and absolute continuity*, *Annales Academiae Scientiarum Fennicae Mathematica* **30** (2005), 385–392. MR2173371 (2006f:30017)
- [8] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Second, Springer-Verlag, New York-Heidelberg, 1973. Translated from the German by K. W. Lucas, *Die Grundlehren der mathematischen Wissenschaften, Band 126*. MR0344463 (49 #9202)
- [9] A. Mori, *On quasi-conformality and pseudo-analyticity*, *Trans. Amer. Math. Soc.* **84** (1957), 56–77. MR0083024
- [10] A. Pfluger, *Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen*, *Comment. Math. Helv.* **33** (1959), 23–33. MR0101307

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