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EQUILATERAL TRIANGLE SKEW CONDITION FOR QUASICONFORMALITY

COLLEEN ACKERMANN, PETER HAÏSSINSKY, AND AIMO HINKKANEN

ABSTRACT. We characterize quasiconformal mappings in terms of the distortion of the vertices of equilateral triangles.

1. INTRODUCTION

Since quasiconformal mappings were first studied nearly a century ago, many diverse characterizations have been discovered. These have led to a wide variety of applications in many fields including Teichmüller theory, elliptic PDE's, hyperbolic geometry and complex dynamics. For an overview of these applications and the theory of quasiconformal mappings see [2], [4], and [6]. In this paper we will use the metric definition of quasiconformality to obtain a new formulation for planar quasiconformal mappings.

Definition 1. Let $f : U \rightarrow V$ be a homeomorphism between planar domains. When $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\} \subset U$, define

$$M(z, r) = \sup\{|f(z) - f(w)| : |z - w| = r\}, \text{ and}$$
$$m(z, r) = \inf\{|f(z) - f(w)| : |z - w| = r\}.$$

The mapping f is said to be K -quasiconformal if

$$H(z) = \limsup_{r \rightarrow 0} \frac{M(z, r)}{m(z, r)} \leq K$$

for a.e. $z \in U$, and if $H(z)$ is bounded in U .

In his book [6] John Hubbard obtained a new characterization of quasiconformal mappings. Let T be a closed topological triangle with specified vertices, $L(T) = \max\{|a - b| : a, b \text{ are vertices of } T\}$ and $l(T) = \min\{|a - b| : a, b \text{ are vertices of } T\}$. We define

$$\text{skew}(T) = \frac{L(T)}{l(T)}.$$

Note that $f(T)$ is also a topological triangle so the expression $\text{skew}(f(T))$ is defined. Then $f : U \rightarrow V$ is quasiconformal provided that there exists an increasing homeomorphism $h : [0, \infty) \rightarrow [0, \infty)$ such that

$$\text{skew}(f(T)) \leq h(\text{skew}(T))$$

for all closed Euclidean triangles $T \subset U$. In fact, Hubbard showed that it is sufficient to only consider triangles with skew bounded above by $\sqrt{7/3}$. He then asks the question of whether it suffices to only consider equilateral triangles. Progress was made on this problem in a previous paper by Javier Aramayona and Peter Haïssinsky [3] in which they show there exists a constant $\epsilon_0 > 0$ such that if $\epsilon \in [0, \epsilon_0)$ and

$$\text{skew}(f(T)) \leq 1 + \epsilon$$

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for all equilateral triangles T , then f is quasiconformal.

Theorem 1 of this paper answers Hubbard's question in the affirmative.

Theorem 1. *Let U, V be two domains in the complex plane \mathbb{C} , and let $f : U \rightarrow V$ be an orientation-preserving homeomorphism. For each $\sigma \geq 1$ there exists $H(\sigma) \geq 1$ with the following property. If there exists σ such that $\text{skew}(f(T)) \leq \sigma$ for all equilateral triangles $T \subset U$, then, for any $z \in U$ and any $r < \text{dist}(z, \mathbb{C} \setminus U)$, the inequality $M(z, r) \leq Hm(z, r)$ holds where $H = H(\sigma)$. In particular, the map f is quasiconformal.*

Since quasiconformal maps are differentiable almost everywhere by Mori's theorem [9], we may improve the distortion bounds of quasiconformality. Let $\text{Skew}(f)$ denote the supremum of $\text{skew}(f(T))$ over all equilateral triangles contained in U ; for $z \in U$ and $r > 0$, let $\text{skew}(f, z, r)$ denote the least upper bound of $\text{skew}(f(T))$ over all $T \subset \{w \in U : |z - w| < r\}$. Set $\text{skew}(f, z) = \liminf_{r \rightarrow 0} \text{skew}(f, z, r)$ and $\text{skew}(f) = \|\text{skew}(f, z)\|_\infty$.

Corollary 1. *Let U be a domain in the complex plane \mathbb{C} , and let $f : U \rightarrow f(U)$ be an orientation-preserving homeomorphism with finite $\text{Skew}(f)$. If $\text{skew}(f) \leq \sigma$ then f is $K(\sigma)$ -quasiconformal where*

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

In particular, if $\text{skew}(f) = 1$ then f is a conformal mapping.

2. PROOF OF THE MAIN THEOREM

Throughout the rest of the paper we will use the following notation and conventions:

- (1) We define $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\}$ and let $C(z, r)$ to be the boundary of $D(z, r)$.
- (2) By a curve we mean the image of a not necessarily one-to-one continuous function from a closed interval into \mathbb{C} .
- (3) All triangles will be closed Euclidean triangles.
- (4) Let \mathcal{F}_σ denote the set of orientation-preserving homeomorphisms of any domain $U \subseteq \mathbb{C}$ into any domain $V \subseteq \mathbb{C}$ such that $\text{skew}(f(T)) \leq \sigma$ for all closed equilateral triangles $T \subset U$.

The proof of Theorem 1 relies on the following proposition.

Proposition 1. *Let U be a domain containing $D(0, 1)$, let $f : U \rightarrow \mathbb{C}$ belong to \mathcal{F}_σ , and let T be the triangle with vertices $0, 1$ and $\omega = 1/2 + (\sqrt{3}/2)i$. Then there exists a disk D contained in $f(T)$ such that*

- (1) D is centered at $f(p)$ where $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i \approx 0.5 + 0.29i$, and
- (2) there exists a constant $\alpha = \alpha(\sigma)$ such that the radius of D is at least $\alpha L(f(T))$.

We note that if f is to be quasiconformal, then, certainly, the image $f(T)$ has to contain a disk of definite size centered at the image of the centroid of the triangle, i.e., the point $\xi = 1/2 + (\sqrt{3}/6)i$. Unfortunately, its arithmetic properties make it difficult to relate this point to the vertices of T using equilateral triangles. The point p was chosen, because it is both close to the centroid ($|\xi - p| = \sqrt{3}/(2^9 \cdot 3)$), and it is a vertex of a tiling of the plane by equilateral triangles whose vertices include the vertices of T . Indeed, we have $p = 1/2 - 85 \cdot 2^{-9} + 85 \cdot 2^{-8}\omega$, cf. Lemma 1.

We first derive the proof of Theorem 1 from Proposition 1. We will then focus on the proof of the latter.

Proof of Theorem 1. Fix $z \in U$ and $r > 0$. If $D(z, r) \subset U$, let $M(z, r) = \max\{|f(z) - f(w)| : w \in C(z, r)\}$ and $m(z, r) = \min\{|f(z) - f(w)| : w \in C(z, r)\}$. Denote by z_M a point in $C(z, r)$ such that $|f(z_M) - f(z)| = M(z, r)$.

Since \mathcal{F}_σ is invariant under pre- and post-composition by affine maps, we may assume that $z = 0$, $r = 1$, and $z_M = 1$.

Let T_1 be the equilateral triangle with vertices $0, 1$ and ω . Then by Proposition 1, the image of T_1 must contain a disk centered at $f(p)$ and of radius at least $\alpha L(f(T_1))$.

Let us consider the isometry $A(z) = \bar{z} - p$. Let $T_2 = A(T_1)$. The triangle T_2 is contained in the unit disk, and A maps p to 0 and 1 to p . Since the other vertices of T_2 lie outside of T_1 , we have $L(f(T_2)) \geq \alpha L(f(T_1))$. Moreover, another application of Proposition 1 implies that $f(T_2)$ contains the disk $D(f(0), \alpha L(f(T_2)))$.

Summing up these estimates, we obtain

$$m(0, 1) \geq \alpha L(f(T_2)) \geq \alpha^2 L(f(T_1)) \geq \alpha^2 M(0, 1).$$

□

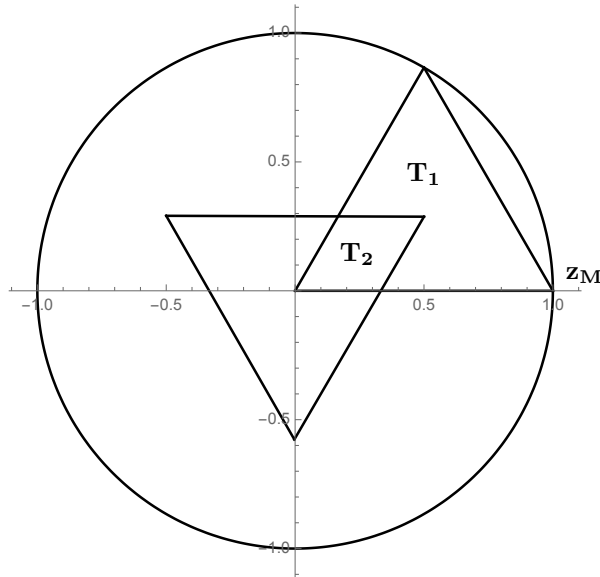


FIGURE 1. Configuration of $C(0, 1)$, T_1 and T_2

3. CONSTRUCTION OF CERTAIN TRIANGLES

Proposition 1 is a consequence of the following proposition.

Proposition 2. *Let U be a neighborhood of $D(0, 1)$, and let $f : U \rightarrow \mathbb{C}$ be a homeomorphism onto its image such that $f \in \mathcal{F}_\sigma$. Let T be the closed triangle with vertices $0, 1$ and $\omega = 1/2 + (\sqrt{3}/2)i$. Let $q = p + 2^{-9}$. Then there exist points $t_1, t_2 \in T$ such that the points q, t_1, t_2 form the vertices of an equilateral triangle and the inequalities $|f(t_j) - f(p)| \leq C\mu$, and $|f(p) - f(q)| \geq cL(f(T))$ hold for some constants $c = c(\sigma)$ and $C = C(\sigma)$ where $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$. We permit the trivial triangle where we have $t_1 = t_2 = q$.*

Proof of Proposition 1 assuming Proposition 2. If $t_1 = t_2 = q$, then we have

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu.$$

Otherwise, by the triangle inequality

$$|f(p) - f(q)| - |f(t_1) - f(p)| \leq |f(t_1) - f(q)| \leq \sigma |f(t_1) - f(t_2)| \leq \sigma(|f(t_1) - f(p)| + |f(p) - f(t_2)|)$$

so that by assumption,

$$cL(f(T)) - C\mu \leq 2\sigma C\mu \quad \text{whence} \quad \mu \geq \frac{c}{(2\sigma + 1)C} L(f(T)).$$

□

4. PROOF OF PROPOSITION 2

The idea of our proof of Proposition 2 is to define a curve γ' going through p such that

- (1) for all $t \in \gamma'$ we have $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$;
- (2) there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle.

The proof of Proposition 2 results from Lemma 1 and Lemma 3.

We first prove the following result.

Lemma 1. *Let T be the closed triangle with vertices $0, 1$ and ω . Let $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$ and $q = p + 2^{-9}$. Then $|f(q) - f(p)| \geq cL(f(T))$ for some positive constant $c = c(\sigma)$.*

Proof. Let us first consider the tiling of the plane by equilateral triangles with vertices in $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$. Define a chain of triangles $(T_j)_{0 \leq j \leq J}$ as a sequence of triangles with vertices in Λ such that $T_j \cap T_{j+1}$ is an edge for all j with $0 \leq j < J$. Given two edges (v, w) and (v', w') , we may connect them by a chain of minimal length $n \geq 0$. A simple induction argument implies

$$|f(v) - f(w)| \leq \sigma^n |f(v') - f(w')|$$

if $f \in \mathcal{F}_\sigma$ is defined in a neighborhood of the chain.

Let T be as defined in our hypotheses: it is tiled by $N = 2^{18}$ triangles of $2^{-9}\Lambda$, and $[p, q]$ is an edge of this tiling. Therefore, for any other edge $[v, w]$, it follows that

$$|f(v) - f(w)| \leq \sigma^N |f(p) - f(q)|.$$

But each side of T is the union of less than N edges of our tiling, therefore, the triangle inequality implies

$$L(f(T)) \leq N\sigma^N |f(p) - f(q)|.$$

□

We now prove a geometric lemma which will be used in the proof of Lemma 3.

Lemma 2. *Let $|z| \leq 1/8$ and suppose that $|\theta_\pm - (\pm\pi/3)| \leq 1/8$. Then the angle θ between $e^{i\theta_+} - z$ and $e^{i\theta_-} - z$ which crosses the positive real axis belongs to $(\pi/3, \pi)$.*

Proof. We note that $\cos\theta_\pm \geq 1/2 - 1/4 > 1/8 \geq |z|$ so that θ is less than π .

For the other inequality, we will estimate $\tan|\arg(e^{i\theta_\pm} - z)|$ to obtain a lower bound of both angles with the horizontal line:

$$\begin{aligned} \tan|\arg(e^{i\theta_\pm} - z)| &\geq \frac{\sqrt{3}/2 - (|z| + 1/8)}{1/2 + (|z| + 1/8)} \geq \frac{\sqrt{3}/2 - 1/4}{1/2 + 1/4} \\ &\geq \frac{2\sqrt{3} - 1}{3} \geq \frac{2}{3} > \tan(\pi/6). \end{aligned}$$

Therefore θ is at least $\pi/3$. □

Now we demonstrate how to find the curve mentioned above.

Lemma 3. *Under the assumptions of Proposition 2, there exists a curve γ' going through p such that for all $t \in \gamma'$ we have*

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$$

and there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle. We permit the trivial triangle where we have $t_1 = t_2 = q$.

Proof. We will do this in several steps. We first define a curve that will join two points of the boundary of a disk contained in T (Step 1). To make sure that we will be able to find two points that form an equilateral triangle with q , we will extend this curve so that it has end points in a slightly larger disk, and is only close to the boundary of the larger disk when it is also close to its end points (Step 2). Then we will use Lemma 2 to find our triangle (Step 3).

Since $\sqrt{3} \geq 8/5$, it follows that $\text{dist}(p, \partial T) = 85\sqrt{3} \cdot 2^{-9} > 1/4 + 2^{-6}$, so that $D(p, 1/4 + 2^{-6})$ is contained in the interior of T .

Throughout the proof, for $x \in \mathbb{C}$, R_x will denote the rotation centered at x by $\pi/3$ radians, defined by $R_x(z) = x + (z - x)\omega$ and \bar{R}_x the rotation centered at x by $-\pi/3$ radians, defined by $\bar{R}_x(z) = x + (z - x)\bar{\omega}$. Recall that we set $\omega = 1/2 + (\sqrt{3}/2)i$.

Step 1: There exists a curve γ_2 that satisfies the following:

- (1) $\gamma_2 \subset D(p, 1/4)$,
- (2) γ_2 has end points on $C(p, 1/4)$ which are exactly $2\pi/3$ radians apart, and
- (3) for all points $t \in \gamma_2$ we have $|f(t) - f(p)| \leq \sigma\mu$.

Let $p' \in \partial T$ be such that $d(f(p), f(p')) = \mu$ and let $\gamma = f^{-1}([f(p), f(p')])$. Since $D(p, 1/4)$ is contained in the interior of T , we may consider the component γ_1 of $\gamma \cap D(p, 1/4)$ that contains p , and we denote by $w \in C(p, 1/4)$ the other end point of γ_1 . We take w to be the first point of $C(p, 1/4)$ encountered when moving along γ starting from p .

Now define

$$\gamma_2 = R_p(\gamma_1) \cup \bar{R}_p(\gamma_1).$$

Note that, for any $s \in \gamma_1$, $R_p(s)$ and $\bar{R}_p(s)$ are two points in γ_2 which make an angle of $2\pi/3$ seen from p . Since $f \in \mathcal{F}_\sigma$, for all $t \in \gamma_2$, we have

$$|f(t) - f(p)| \leq \sigma|f(s) - f(p)| \leq \sigma\mu$$

where $s \in \gamma_1$ denotes a point such that either $t = R_p(s)$ or $t = \bar{R}_p(s)$.

Step 2: Let a, b be the end points of γ_2 . There exists a curve γ_3 such that

- (1) $\gamma_3 \subset D(p, 1/4) \cup D(a, 2^{-6}) \cup D(b, 2^{-6})$;
- (2) γ_3 has both end points on $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$;
- (3) for all points $t \in \gamma_3$, $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$.

Let $D_a = D(a, 2^{-6})$ and $D_b = D(b, 2^{-6})$. Let γ_{2a} and γ_{2b} be the components of $\gamma_2 \cap D_a$ and $\gamma_2 \cap D_b$ that have end points at a and b respectively.

Clearly γ_{2a} also has an end point on the boundary of D_a . Let a' denote an end point of γ_{2a} on the boundary of D_a . Use the tangent line to $D(p, 1/4)$ at a to divide D_a in half, and then divide each half into thirds. So we have divided D_a into closed sectors of $\pi/3$ radians with three such sectors lying entirely outside of $D(p, 1/4)$. Let S_a denote the middle sector lying completely outside of $D(p, 1/4)$. Then there exists $n \in \{2, 3\}$ such that when γ_{2a} is rotated $n\pi/3$ radians in an appropriate direction about a , the image of a' under the rotation will lie in S_a . Let the image of γ_{2a} under this rotation be denoted by γ_{3a} .

Now we will bound the quantity $|f(t) - f(p)|$ where $t \in \gamma_{3a}$. Fix $t \in \gamma_{3a}$. Let t_0 be the point on γ_{2a} whose image under the rotation is t . Without loss of generality we will assume this rotation was clockwise. Let t_i denote the image of t_0 under a clockwise rotation of $i\pi/3$ radians where $i = 1, \dots, n$ ($t = t_n$). Then since a, t_{i-1}, t_i ($1 \leq i \leq n$) form an equilateral triangle, we have

$$|f(t_i) - f(a)| \leq \sigma |f(t_{i-1}) - f(a)|.$$

Since $a, t_0 \in \gamma_2$ we have

$$|f(a) - f(t_0)| \leq |f(a) - f(p)| + |f(p) - f(t_0)| \leq 2\sigma\mu$$

and

$$|f(a) - f(p)| \leq \sigma\mu.$$

Thus since n is at most 3 we have

$$|f(t) - f(p)| \leq |f(a) - f(p)| + |f(a) - f(t)| \leq \sigma\mu + \sigma^n |f(a) - f(t_0)| \leq \sigma\mu(1 + 2\sigma^n) \leq \sigma\mu(1 + 2\sigma^3).$$

Furthermore γ_{3a} must intersect the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$. This is because γ_{3a} has an end point in S_a and therefore the distance of the end point of γ_{3a} from $D(p, 1/4)$ must be at least $\cos(\pi/6) \cdot 2^{-6} = \sqrt{3} \cdot 2^{-7}$. This is depicted in figure 2.

We proceed similarly near b and define a curve γ_{3b} contained in D_b with end points at b and at some point on the intersection of the boundary of D_b and S_b (defined analogously to S_a) such that for all $t \in \gamma_{3b}$ we have $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$; as above, γ_{3b} intersects the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$.

Let γ_3 be the connected component of $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ which includes points in both γ_{3a} and γ_{3b} . Then for all points $t \in \gamma_3$,

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3).$$

The curve γ' in Lemma 3 can be chosen as $\gamma' = \gamma_3$.

Step 3: Let $q = p + 2^{-9}$. There exist $t_1, t_2 \in \gamma_3$ such that $\{q, t_1, t_2\}$ form an equilateral triangle.

Let D_q be the smallest disk centered at q which contains $D(p, 1/4)$. Then $D_q \subset D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ since $|p - q| = 2^{-9} \leq \sqrt{3} \cdot 2^{-8}$. Let γ_4 be the connected component of $\gamma_3 \cap D_q$ which has end points $A \in D_a \cap D_q$ and $B \in D_b \cap D_q$.

Note that, if we write $a = p + |a - p|e^{i\theta_a}$ and $A = p + |A - p|e^{i\theta_A}$, where $|\theta_A - \theta_a|$ is chosen to be as small as possible modulo 2π , then

$$|\theta_A - \theta_a| \leq 2|a - A|/(1/4) \leq 2 \cdot 2^{-6}/(1/4) \leq 1/8$$

and similarly for b and B . Note that $|\arg(a - p)| + |\arg(b - p)| = 2\pi/3$ and $|p - q|/(1/4) = 2^{-7} \leq 1/8$. Therefore, by Lemma 2 applied in $D(p, 1/4)$, the angle between $A - q$ and $B - q$ lies in $(\pi/3, \pi)$. Hence,

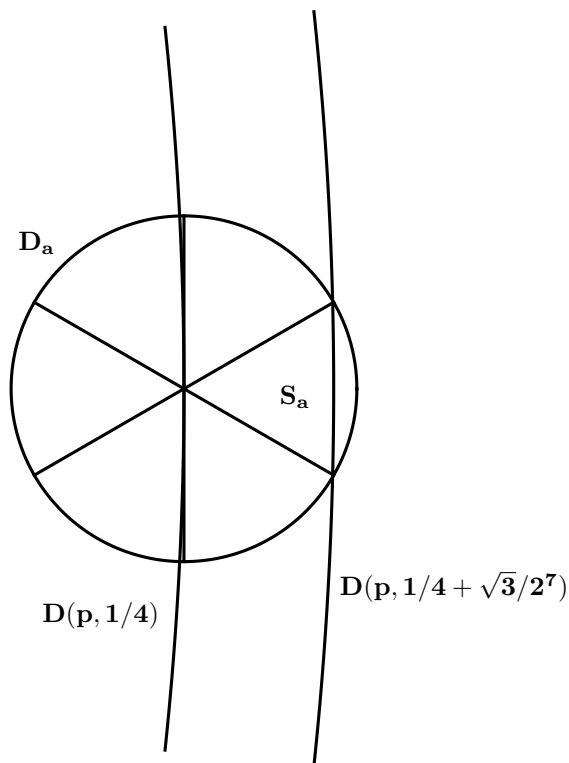


FIGURE 2

the images A_r and B_r of A and B respectively under \bar{R}_q will separate A and B on ∂D_q . Thus the image $\bar{R}_q(\gamma_4)$ must intersect γ_4 . This gives us our desired equilateral triangle since q , the intersection point, and the pre-image of the intersection point form an equilateral triangle.

This completes the proof of Lemma 3. □

5. PROOF OF COROLLARY 1

We prove Corollary 1 by approximating f by linear mappings at points where f is differentiable.

Proof. Theorem 1 implies that f is quasiconformal and hence differentiable at almost every point. Let z_0 be a point of differentiability such that $\text{skew}(f, z_0) \leq \sigma$. We will compute the maximum possible value for $H(z_0)$. Since $H(z_0)$ is invariant under Möbius transformations we may compose with translations, a dilation and a rotation to assume that $z_0 = f(z_0) = 0$, $f_z(z_0) = 1$ and $f_{\bar{z}}(z_0) = |f_{\bar{z}}(z_0)| < 1$. Then $f(z) = z + f_{\bar{z}}(z_0)\bar{z} + \epsilon(z)$ where $\epsilon(z)/|z|$ tends to 0 as z tends to z_0 , and thus $\text{skew}(f, z_0) = \text{skew}(\tilde{f}, z_0)$ where $\tilde{f}(z) = z + f_{\bar{z}}(z_0)\bar{z}$.

Note $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(a+v) - \tilde{f}(b+v)|$, $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(\bar{a}) - \tilde{f}(\bar{b})|$ and $|\tilde{f}(a) - \tilde{f}(b)|/|\tilde{f}(a) - \tilde{f}(c)| = |\tilde{f}(ra) - \tilde{f}(rb)|/|\tilde{f}(ra) - \tilde{f}(rc)|$ for all $a, b, c, v \in \mathbb{C}$ with $a \neq c$ and all $r > 0$. This implies $\text{skew}(\tilde{f}(T))$ where T is an equilateral triangle is invariant under translations, complex conjugation and dilations of T . Thus for all equilateral triangles T ,

$$\text{skew}(\tilde{f}(T)) \in \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\}.$$

Indeed, suppose T has vertices A, B and C , and $\text{skew}(T) = \frac{|\tilde{f}(A) - \tilde{f}(B)|}{|\tilde{f}(A) - \tilde{f}(C)|}$. First we translate A to the origin, and then we dilate T so its side lengths are equal to 1. If \overline{AB} is $\pi/3$ radians clockwise from \overline{AC} , it is clear that our statement is true, Otherwise we take the complex conjugate of T to change the orientation of T and then, since \tilde{f} is invariant under complex conjugation of T , our claim is true.

Hence we have

$$\text{skew}(\tilde{f}) = \max \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\} = \max \left\{ \frac{|\tilde{f}(z)|}{|\tilde{f}(ze^{i\pi/3})|} : |z| = 1 \right\}.$$

Let $\mu = f_{\bar{z}}$, $\nu = \mu + \mu^{-1}$ and $\beta = e^{i\pi/6}$. Let $w \in \mathbb{C}$ with $|w| = 1$. We have

$$|\tilde{f}(w)|^2 = |w + \mu\bar{w}|^2 = (w + \mu\bar{w})(\bar{w} + \mu w) = 1 + \mu^2 + \mu(w^2 + \bar{w}^2) = \mu[\nu + (w^2 + \bar{w}^2)].$$

Now we are able to maximize $|\tilde{f}(\beta w)|/|\tilde{f}(\bar{\beta} w)|$ with respect to w . Set $z = w^2$ and $\alpha = e^{i\pi/3}$. Since we have assumed $|w| = 1$, we can instead maximize

$$\kappa = \left| \frac{\tilde{f}(\beta w)}{\tilde{f}(\bar{\beta} w)} \right|^2 = \frac{\nu + \alpha z + \bar{\alpha}\bar{z}}{\nu + \bar{\alpha}z + \alpha\bar{z}}.$$

We write $z = e^{ix}$, $x \in \mathbb{R}$, so that $z' = iz$, $\bar{z}' = -i\bar{z}$. We may differentiate κ as a function of x . It follows that $\kappa' = 0$ if and only if

$$(\alpha z - \bar{\alpha}\bar{z})(\nu + \bar{\alpha}z + \alpha\bar{z}) - (\nu + \alpha z + \bar{\alpha}\bar{z})(\bar{\alpha}z - \alpha\bar{z}) = 0.$$

Thus

$$\nu(\alpha z - \bar{\alpha}\bar{z} - \bar{\alpha}z + \alpha\bar{z}) = z^2 - \alpha^2 + \bar{\alpha}^2 - \bar{z}^2 - z^2 - \alpha^2 + \bar{\alpha}^2 + \bar{z}^2$$

which is equivalent to

$$\nu(z + \bar{z})(\alpha - \bar{\alpha}) = 2(\bar{\alpha} - \alpha)(\bar{\alpha} + \alpha).$$

Therefore

$$\cos x = -\frac{2}{\nu} \cos \frac{\pi}{3} = -\frac{1}{\nu}.$$

It follows that $\sin^2 x = 1 - 1/\nu^2$ so $\kappa' = 0$ for

$$z = \frac{1}{\nu} \left(-1 + i\varepsilon\sqrt{\nu^2 - 1} \right)$$

with $\varepsilon \in \{\pm 1\}$. For these values of z , one gets

$$\kappa = \frac{\nu + 2\text{Re}(\alpha z)}{\nu + 2\text{Re}(\bar{\alpha}z)} = \frac{\nu^2 - 1 - \varepsilon\sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 + \varepsilon\sqrt{3(\nu^2 - 1)}}$$

which is maximal for $\varepsilon = -1$. So we obtain

$$\text{skew}(\tilde{f})^2 = \frac{\nu^2 - 1 + \sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 - \sqrt{3(\nu^2 - 1)}} = \frac{\sqrt{(\nu^2 - 1)/3} + 1}{\sqrt{(\nu^2 - 1)/3} - 1}.$$

Let us write $\tau = \text{skew}(\tilde{f})$ so that

$$\sqrt{(\nu^2 - 1)/3} = \frac{\tau^2 + 1}{\tau^2 - 1}, \quad \nu^2 = 3 \left(\frac{\tau^2 + 1}{\tau^2 - 1} \right)^2 + 1 = \frac{4(\tau^4 + \tau^2 + 1)}{(\tau^2 - 1)^2}$$

and

$$\nu = \frac{2\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1}.$$

Hence

$$\mu^2 - 2\mu \frac{\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1} + 1 = 0.$$

We compute the reduced discriminant

$$\Delta' = \frac{\tau^4 + \tau^2 + 1}{(\tau^2 - 1)^2} - 1 = \frac{3\tau^2}{(\tau^2 - 1)^2}$$

and we deduce from $0 < \mu < 1$ that

$$\mu = \frac{\sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1}.$$

Thus

$$K(\tilde{f}) = \frac{1 + \mu}{1 - \mu} = \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1 - \sqrt{\tau^4 + \tau^2 + 1} + \sqrt{3}\tau}.$$

By assumption, $\tau \leq \sigma$ so that

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1} - \sqrt{3}\sigma}{\sigma^2 - 1 - \sqrt{\sigma^4 + \sigma^2 + 1} + \sqrt{3}\sigma} = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

□

6. AN ALTERNATIVE PROOF OF THE QUASICONFORMALITY OF MAPPINGS SATISFYING THE HYPOTHESES OF THEOREM 1

From Proposition 1, there are several ways establishing that a mapping f satisfying the hypotheses of Theorem 1 satisfies the analytic definition of quasiconformality which is equivalent to Definition 1.

Definition 2. We say a homeomorphism $f : U \rightarrow V$ is absolutely continuous on lines if for every rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ with $\bar{R} \subset U$, f is absolutely continuous on a.e. interval $I_x = \{(x, y) : c < y < d\}$ and a.e. interval $I_y = \{(x, y) : a < x < b\}$. A mapping f is quasiconformal if it is absolutely continuous on lines and there exists $K \geq 1$ such that

$$\max_{\xi} |\partial_{\xi} f(z)| \leq K \min_{\xi} |\partial_{\xi} f(z)| \text{ a.e.}$$

Proposition 1 tells us that the image of every equilateral triangle, T , contains a disk with radius proportional to $L(f(T))$. In [6, Section 4.5], Hubbard uses this to prove that the map belongs to the Sobolev space $W_{loc}^{1,2}$ by an approximation argument. We propose another approach which shows directly that the map satisfies the ACL property. Below we will only give a brief sketch of the basic ideas of the proof. For full details please see Section 5.5 of [1].

Proof. First we show f is absolutely continuous on lines. This part of the proof parallels Pfluger's proof that a mapping satisfying the geometric definition of quasiconformality is absolutely continuous on lines. His proof can be found in [10] and is reproduced in English in [8], p. 162. We fix a rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ and let $I_y = \{(x, y) : a < x < b\}$ for y between c and d . Define $A(y)$ to be the area in $f(R)$ beneath the image of I_y . Since A is an increasing function of y , it is differentiable almost everywhere. We will

show $f|_{I_y}$ is absolutely continuous for all y at which A is differentiable; a similar argument applies to vertical line segments. We select an arbitrary collection $\{(z_k^*, z_k)\}_{k=1}^n$ of disjoint sub-intervals of I_y and consider the collection of rectangles $\{R_k\}_{k=1}^n$ where each R_k has height δ and bottom side on the k th sub-interval. Then we apply Proposition 1 to a set of non-intersecting equilateral triangles of height δ each contained in some R_k . We are able to conclude

$$\frac{\pi}{\alpha^2} \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq \left(\frac{A(y+\delta) - A(y)}{\delta} \right) \left(\sum_{k=1}^n |z_k^* - z_k| + \delta n \right).$$

Since we chose y where A is differentiable, letting δ go to 0 gives

$$\frac{\pi}{\alpha^2} \left(\sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq A'(y) \left(\sum_{k=1}^n |z_k^* - z_k| \right).$$

To then conclude that f is quasiconformal we once again use Proposition 1. Since f is open and absolutely continuous on lines, f is differentiable almost everywhere by Gehring and Lehto's theorem [4]. We look at a square S_δ with sides parallel to the coordinate axes, side-length 2δ and center at a point of differentiability. Let γ denote the pre-image under f of a curve of shortest length contained in $f(S_\delta)$ with end points on the images of the vertical sides of S_δ . We create a chain $\{T_i\}_{i=1}^4$ of non-intersecting triangles such that each T_i shares at least one vertex with T_{i+1} , T_1 has a vertex on one end point of γ , and T_4 has a vertex on the other end point of γ . By Proposition 1 we are able to conclude that

$$\frac{s_b(f(S_\delta))^2}{m(f(S_\delta))} \leq \frac{4}{(\pi\alpha)^2}$$

where $m(f(S_\delta))$ denotes the area of the image of S_δ and $s_b(f(S_\delta))$ denotes the length of the shortest curve between the images of the vertical sides of S_δ which is contained in $f(S_\delta)$. We then may use this inequality to replace the use of Rengel's inequality in Pfluger's proof and conclude

$$\max_{\xi} |\partial_{\xi} f(z)| \leq \frac{4}{(\pi\alpha)^2} \min_{\xi} |\partial_{\xi} f(z)|.$$

Note that once it is known that f is differentiable almost everywhere, the computations of Corollary 1 also show us that f is quasiconformal. \square

Remark 1. *The key point of both of our proofs given here is contained in Proposition 1, where it is proved that the image of a triangle contains a disk of a definite size, exhibiting a certain length-area estimate. This approach goes back to Pfluger and was pushed forward by Koskela and Rogovin who proved that the ACL property of a homeomorphism f between open sets of \mathbb{R}^n , $n \geq 2$, could be established from an L^1 -control of*

$$(1) \quad k_f = \liminf_{r \rightarrow 0} \left(\frac{\text{diam}(f(D(x, r)))^n}{|f(D(x, r))|} \right)^{1/(n-1)}$$

where $|f(D(x, r))|$ denotes the Lebesgue measure of $f(D(x, r))$; see [7] for details. The authors would like to thank Pekka Koskela for mentioning this similarity.

7. APPENDIX BY COLLEEN ACKERMANN: AN ANALOGUE OF THE MAIN THEOREM IN HILBERT SPACES OF DIMENSION AT LEAST THREE

In dimensions three and higher the proof of an analogue of Theorem 1 is surprisingly simpler than the proof of Theorem 1. Furthermore the proof itself gives an elegant bound on $K(\sigma)$.

Theorem 2. Let \mathcal{H}_1 and \mathcal{H}_2 be finite-dimensional Hilbert spaces with $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \geq 3$ and let $U \subset \mathcal{H}_1, V \subset \mathcal{H}_2$ be domains. Suppose $f : U \rightarrow V$ is a homeomorphism and that for all closed equilateral triangles $T \subset U$, $\text{skew}(f(T)) \leq \sigma$. Then f is σ^3 -quasiconformal when using the metric definition of quasiconformality.

Proof. It suffices to assume $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^n$ for some $n \geq 3$. We will show that f satisfies the metric definition of quasiconformality.

Fix a point $p \in U$, a positive number r with $r < \text{dist}(p, \partial U)$ and $a \in \partial B(p, r)$. Let $m \in \partial B(p, r)$ be such that

$$|f(p) - f(m)| = \min_{r=|z-p|} |f(z) - f(p)|.$$

We will prove $|f(a) - f(p)| \leq \sigma^3 |f(p) - f(m)|$. Let \mathbf{e}_i denote the unit vector in the i th direction. To simplify our calculations we will actually show $|f(a') - f(p')| \leq K^3 |f(p') - f(m')|$ where a', p' and m' are the images of the points a, p and m respectively under a sequence of conformal mappings, and where f is modified accordingly without changing notation. Namely, first apply a translation so that $p' = 0$, then a rotation so that $m' = r\mathbf{e}_1$ and $a' = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and finally a possible reflection so that $a_2 > 0$. From now on we will only work in the linear subspace spanned by the first three coordinates which we will identify with \mathbb{R}^3 . More precisely, we will identify \mathbf{e}_1 with the unit vector in the x direction, \mathbf{e}_2 with the unit vector in the y direction and another arbitrary coordinate with the z direction.

Case 1: If the smaller angle between a' and the positive x -axis is less than or equal to $2\pi/3$ then set

$$b = \left(r/2, r(1 - a_1)/(2a_2), r\sqrt{3/4 - [(1 - a_1)/(2a_2)]^2} \right).$$

Consider the triangles T_1 with vertices p', m' and b , and T_2 with vertices p', b and a' . T_1 and T_2 are equilateral triangles which share a common side with endpoints at b and p' .

Thus

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b)| \leq \sigma^2 |f(p') - f(m')|.$$

Case 2: If the smaller angle between a' and the x -axis is not less than or equal to $2\pi/3$, consider the equilateral triangle T_0 with vertices p', a' and b' where b' is the image of a' under a rotation of $\pi/3$ radians clockwise. The smaller angle between b' and the x -axis is less than or equal to $2\pi/3$. Thus by Case 1

$$|f(p') - f(b')| \leq \sigma^2 |f(p') - f(m')|.$$

Then since the triangle T_0 has sides with endpoints at p' and a' , and p' and b' we have

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b')| \leq \sigma^3 |f(p') - f(m')|.$$

□

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COLLEEN ACKERMANN, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, DEPARTMENT OF MATHEMATICS, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801, UNITED STATES

E-mail address: `cackerm@vt.edu`

PETER HAÏSSINSKY, UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, 118 ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX 9, FRANCE

Current address: Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 3453 Marseille, France

E-mail address: `phaissin@math.univ-toulouse.fr`

AIMO HINKKANEN, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, DEPARTMENT OF MATHEMATICS, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801, UNITED STATES

E-mail address: `aimo@math.uiuc.edu`