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STOCHASTIC APPROXIMATION OF QUASI-STATIONARY DISTRIBUTIONS ON COMPACT SPACES AND APPLICATIONS

MICHEL BENAÏM, BERTRAND CLOEZ, FABIEN PANLOUP

ABSTRACT. In the continuity of a recent paper ([6]), dealing with finite Markov chains, this paper proposes and analyzes a recursive algorithm for the approximation of the quasi-stationary distribution of a general Markov chain living on a compact metric space killed in finite time. The idea is to run the process until extinction and then to bring it back to life at a position randomly chosen according to the (possibly weighted) empirical occupation measure of its past positions. General conditions are given ensuring the convergence of this measure to the quasi-stationary distribution of the chain. We then apply this method to the numerical approximation of the quasi-stationary distribution of a diffusion process killed on the boundary of a compact set and to the estimation of the spectral gap of irreducible Markov processes. Finally, the sharpness of the assumptions is illustrated through the study of the algorithm in a non-irreducible setting.

Keywords. Quasi-stationary distributions ; stochastic approximation ; reinforced random walks ; random perturbations of dynamical systems; extinction rate; spectral gap; Euler scheme.

AMS-MSC. 65C20; 60B12; 60J10, Secondary 34F05; 60J20; 60J60.

1. INTRODUCTION

Numerous models, in ecology and elsewhere, describe the temporal evolution of a system by a Markov process which eventually gets killed in finite time. In population dynamics, for instance, extinction in finite time is a typical effect of finite population sizes. However, when populations are large, extinction usually occurs over very large time scales and the relevant phenomena are given by the behavior of the process conditionally to its non-extinction.

More formally, let $(\xi_t)_{t \geq 0}$ be a Markov process with values in $\mathcal{E} \cup \{\partial\}$ where $\partial \notin \mathcal{E}$ denotes an absorbing point (typically, the extinction set or the complementary of a domain). Under appropriate assumptions, there exists a distribution ν on \mathcal{E} (possibly depending on the initial distribution of ξ) such that

$$\lim_{t \rightarrow \infty} \mathbb{P}(\xi_t \in \cdot | \xi_t \neq \partial) = \nu(\cdot). \quad (1)$$

Such a distribution well describes the behavior of the process before extinction, and is necessary (see e.g [34]) a *quasi-stationary distribution* (QSD) in the sense that

$$\mathbb{P}_\nu(\xi_t \in \cdot | \xi_t \neq \partial) = \nu(\cdot).$$

We refer the reader to the survey paper [34] or the book [21] for general background and a comprehensive introduction to the subject.

The simulation and numerical approximation of quasi-stationary distributions have received a lot of attention in the recent years and led to the development and analysis of a class of particle systems algorithms known in the literature as *Fleming-Viot algorithms* (see [14, 20, 22, 41]). The principle of these algorithms is to run a large number of particles independently until one

is killed and then to replace the killed particle by an offspring whose location is randomly (and uniformly) chosen among the locations of the other (alive) particles. In the limit of an infinite number of particles, the (spatial) empirical occupation measure of the particles approaches the law of the process conditioned to never be absorbed. Combined with (1), this gives a method for estimating the QSD of the process.

Recently, Benaïm and Cloez [6] (see also [1, 12]) analyzed, in the setting of finite state Markov chains, an alternative approach in which the **spatial occupation measure** of a set of particles is replaced by the **temporal occupation measure** of a single particle. Each time the particle is killed it is risen at a location randomly chosen according to its temporal occupation measure. The details of the construction are recalled in Section 2.

The objective of this paper is twofold: on the one hand, we aim at extending the results of [6] to the setting of Markov chains with values in a general space, being killed when leaving a compact domain. On the other hand, we also explore various applications: we propose and investigate a numerical procedure, based on an Euler discretization, for approximating QSD of diffusions and we also focus on the estimation of the extinction rate that we apply as a way to get numerical bounds for the spectral gap of Markov processes.

Outline. The paper is organized as follows: In Section 2 we detail the general framework, the hypotheses and state our main results. In Section 3, we first discuss our assumptions in the simple case of finite Markov chains and then focus on the series of applications introduced previously: numerical approximation of QSDs for diffusions (including theoretical results and numerical tests), estimation of (bounds for) general spectral gap (we recover in particular the curvature of the sphere). The sequel of the paper (Sections 4, 5, 6, and 7) is mainly devoted to the proofs and the details about their sequencing will be given at the end of Section 3. We end the paper by some potential extensions of this work to some more general settings such as non-compact domains or continuous-time reinforced strategies.

2. SETTING AND MAIN RESULTS

2.1. Notation and Setting. Let \mathcal{E} be a compact metric space¹ equipped with its Borel σ -field $\mathcal{B}(\mathcal{E})$. Throughout, we let $\mathcal{B}(\mathcal{E}, \mathbb{R})$ denote the set of real valued bounded measurable functions on \mathcal{E} and $\mathcal{C}(\mathcal{E}, \mathbb{R}) \subset \mathcal{B}(\mathcal{E}, \mathbb{R})$ the subset of continuous functions. For all $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$ we let $\|f\|_\infty = \sup_{x \in \mathcal{E}} |f(x)|$ and we let $\mathbf{1}$ denote the constant map $x \mapsto 1$. We let $\mathcal{P}(\mathcal{E})$ denote the space of (Borel) probabilities over \mathcal{E} equipped with the topology of weak* convergence. For all $\mu \in \mathcal{P}(\mathcal{E})$ and $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$, or f nonnegative measurable, we write $\mu(f)$ (or μf) for $\int_{\mathcal{E}} f d\mu$. Recall that $\mu_n \rightarrow \mu$ in $\mathcal{P}(\mathcal{E})$ provided $\mu_n(f) \rightarrow \mu(f)$ for all $f \in \mathcal{C}(\mathcal{E}, \mathbb{R})$, and that (by compactness of \mathcal{E} and Prohorov Theorem), $\mathcal{P}(\mathcal{E})$ is a compact metric space (see e.g. [23, Chapter 11]).

A *sub-Markovian kernel* on \mathcal{E} is a map $Q : \mathcal{E} \times \mathcal{B}(\mathcal{E}) \mapsto [0, 1]$ such that for all $x \in \mathcal{E}$, $A \mapsto Q(x, A)$ is a nonzero measure (i.e $Q(x, \mathcal{E}) > 0$) and for all $A \in \mathcal{B}(\mathcal{E})$, $x \mapsto Q(x, A)$ is measurable. If furthermore $Q(x, \mathcal{E}) = 1$ for all $x \in M$, then Q is called a *Markov (or Markovian) kernel*.

¹For comments about a possible extension to the non-compact case, see Section 8

Let Q be a sub-Markovian (respectively Markovian) kernel. For every $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$ and $\mu \in \mathcal{P}(\mathcal{E})$, we let Qf and μQ respectively denote the map and measure defined by

$$Qf(x) = \int_{\mathcal{E}} f(y)Q(x, dy) \quad \text{and} \quad \mu Q(\cdot) = \int_{\mathcal{E}} \mu(dx)Q(x, \cdot).$$

If $Qf \in \mathcal{C}(\mathcal{E}, \mathbb{R})$ whenever $f \in \mathcal{C}(\mathcal{E}, \mathbb{R})$, then Q is said to be *Feller*. For all $n \in \mathbb{N}$, we let Q^n denote the sub-Markovian (respectively Markovian) kernel recursively defined by

$$Q^{n+1}(x, \cdot) = \int_{\mathcal{E}} Q(y, \cdot)Q^n(x, dy) \quad \text{and} \quad Q^0(x, \cdot) = \delta_x.$$

A probability $\mu \in \mathcal{P}(\mathcal{E})$ is called a *quasi-stationary distribution* (QSD) for Q if μ and μQ are proportional or equivalently if,

$$\mu = \frac{\mu Q}{\mu Q \mathbf{1}}.$$

The number

$$\Theta(\mu) := \mu Q \mathbf{1} \tag{2}$$

is called the *extinction rate* of μ .

Note that when Q is Markovian, a quasi stationary distribution is *stationary* (or *invariant*) in the sense that $\mu = \mu Q$. In this case $\Theta(\mu) = 1$, otherwise $\Theta(\mu) < 1$.

From now on and throughout the remainder of the paper we assume given a Feller sub-Markovian kernel K on \mathcal{E} .

Let $\partial \notin \mathcal{E}$ be a *cemetery point*. Associated to K is the Markov kernel \bar{K} on $\mathcal{E} \cup \{\partial\}$ defined, for all $x \in \mathcal{E}$, $A \in \mathcal{B}(\mathcal{E})$, by

$$\begin{cases} \bar{K}(x, A) = K(x, A), \\ \bar{K}(x, \{\partial\}) = 1 - K(x, \mathcal{E}), \quad \text{and} \\ \bar{K}(\partial, \{\partial\}) = 1. \end{cases}$$

The kernel \bar{K} can be understood as the transition kernel of a Markov chain $(Y_n)_{n \geq 0}$ on $\mathcal{E} \cup \{\partial\}$ whose transitions in \mathcal{E} are given by K and which is "killed" when it leaves \mathcal{E} .

Let $\delta : \mathcal{E} \mapsto [0, 1]$ be the function defined by

$$\delta = \mathbf{1} - K \mathbf{1}.$$

That is, for every $x \in \mathcal{E}$,

$$\delta(x) = \bar{K}(x, \{\partial\}) = 1 - K(x, \mathcal{E}). \tag{3}$$

For a given $\mu \in \mathcal{P}(\mathcal{E})$, we let K_μ denote the Markov kernel on \mathcal{E} defined by

$$K_\mu(x, A) = K(x, A) + \delta(x)\mu(A)$$

for all $x \in \mathcal{E}$ and $A \in \mathcal{B}(\mathcal{E})$. Equivalently, for every $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$,

$$K_\mu f(x) = Kf(x) + \delta(x)\mu(f).$$

The chain induced by K_μ behaves like (Y_n) until it is killed and then is redistributed in \mathcal{E} according to μ . Note that K_μ inherits the Feller continuity from K . For the sequel, an important feature of K_μ is that μ is a QSD for K if and only if it is invariant for K_μ (see Lemma 4.2 for details).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$ (i.e an increasing family of σ -fields). We now consider an \mathcal{E} -valued random process $(X_n)_{n \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to $\{\mathcal{F}_n\}_{n \geq 0}$ such that :

$$X_0 = x \in \mathcal{E},$$

and for every $n \geq 0$,

$$\mathbb{P}(X_{n+1} \in dy | \mathcal{F}_n) = K_{\mu_n}(X_n, dy), \quad (4)$$

where

$$\mu_n = \frac{\sum_{k=0}^n \eta_k \delta_{X_k}}{\sum_{k=0}^n \eta_k}$$

is a *weighted occupation measure*. Here $(\eta_n)_{n \geq 0}$ is a sequence of positive numbers satisfying certain conditions that will be specified below (see Hypothesis 2.1).

With the definition of K_μ , this means that whenever the original process $(Y_n)_{n \geq 0}$ is killed, it is redistributed in \mathcal{E} according to its weighted empirical occupation measure μ_n . Note that such a process is a type of *reinforced random walk* (see e.g [37]). It is reminiscent of *interacting particle systems* algorithms used for the simulation of QSDs such as the so-called *Fleming-Viot algorithm* (see [14, 20, 41] and [6, section 3]). However, while these latter algorithms involve a large number of particles whose individual dynamics depend on the *spatial occupation measure* of the particles, here there is a single particle whose dynamics depends on its *temporal occupation measure*. From a simulation point of view, this is of potential interest, suggesting less calculus (but more memory) and leading to a recursive method which avoids (at least in name) the trade-off between the number of particles and the time horizon induced by Fleming-Viot algorithm.

Set, for $n \geq 0$,

$$\gamma_n = \frac{\eta_n}{\sum_{k=0}^n \eta_k}.$$

The occupation measure can then be computed recursively as follows:

$$\mu_{n+1} = (1 - \gamma_{n+1})\mu_n + \gamma_{n+1}\delta_{X_{n+1}}. \quad (5)$$

Under appropriate irreducibility assumptions (see Hypothesis 2.2 below), K_μ admits a unique invariant probability Π_μ . Owing to the above characterization of QSDs as fixed points of $\mu \mapsto \Pi_\mu$, we choose to rewrite the evolution of (μ_n) as:

$$\mu_{n+1} = \mu_n + \gamma_{n+1}(-\mu_n + \Pi_{\mu_n}) + \gamma_{n+1}\varepsilon_n \quad (6)$$

where $\varepsilon_n = \delta_{X_{n+1}} - \Pi_{\mu_n}$. The process (μ_n) is therefore a *stochastic approximation algorithm* associated to the ordinary differential equation (ODE) (for which rigorous sense will be given in Section 5):

$$\dot{\mu} = -\mu + \Pi_\mu. \quad (7)$$

The almost sure convergence of (μ_n) towards μ^* (the QSD of K) will then be achieved by proving that :

- (i) The asymptotic dynamics of $(\mu_n)_{n \geq 0}$ matches with that of solutions of the above ODE: more precisely, $(\mu_n)_{n \geq 0}$ is (at a different scale) an *asymptotic pseudo-trajectory* of the ODE (in the sense of Benaïm and Hirsch [7], see [5] for background).
- (ii) The set $\text{Fix}(\Pi) = \{\mu \in \mathcal{P}(\mathcal{E}), \mu = \Pi_\mu\}$ reduces to μ^* and is a global attractor of the ODE.

This strategy was applied in [6] when \mathcal{E} is a finite set. However, the proofs in [6], strongly rely on finite dimensional arguments that cannot be applied in this more general setting and the new proofs will require a careful study of the kernels family $(K_\mu)_\mu$.

2.2. Main results. We first summarize the standing assumptions under which our main results will be proved. We begin by the assumptions on $(\gamma_n)_{n \geq 1}$.

Hypothesis 2.1 (Standing assumption on (γ_n)). *The sequence $(\gamma_n)_{n \geq 0}$ appearing in equation (5) is a non-increasing sequence such that*

$$\sum_{n \geq 0} \gamma_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \gamma_n \ln(n) = 0. \quad (8)$$

The typical sequence is given by $\gamma_n = \frac{1}{n+1}$, which corresponds to $\eta_n = 1$ for all $n \geq 1$.

Now, let us focus on the assumptions on the sub-Markovian kernel K . We say that a non-empty set $A \in \mathcal{B}(\mathcal{E}) \cup \{\partial\}$ is *accessible* if for all $x \in \mathcal{E}$

$$\sum_{n \geq 1} \bar{K}^n(x, A) > 0.$$

It is called a *weak² small set* if $A \subset \mathcal{E}$ and there exists a probability measure Ψ on \mathcal{E} and $\epsilon > 0$ such that for all $x \in A$

$$\sum_{n \geq 1} K^n(x, dy) \geq \epsilon \Psi(dy). \quad (9)$$

Hypothesis 2.2 (Standing assumptions on K).

- **(H₁)** K is Feller.
- **(H₂)** The cemetery point $\{\partial\}$ is accessible.

Assumptions **H₁** and **H₂** imply the existence of a quasi-stationary distribution but are not sufficient to ensure its uniqueness (see the example developed in Subsection 3.1). For this, we require the supplementary assumptions below

Hypothesis 2.3 (Additional assumptions on K).

- **(H₃)** There exists an open accessible weak small set U .
- **(H₄)** There exists a non increasing convex function $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying

$$\int_0^\infty C(s) ds = \infty \quad (10)$$

such that

$$\frac{\Psi(K^n \mathbf{1})}{\sup_{x \in \mathcal{E}} K^n \mathbf{1}(x)} \geq C(n)$$

where Ψ is like in equation (9).

This latter hypothesis is motivated by the recent work of Champagnat and Villemonais [16] in which it is proved that under mildly stronger versions of **H₃** (namely, $K^l(x, \cdot) \geq \epsilon \Psi$ for some l independent of x) and **H₄** (namely $C(t) \geq c > 0$) the conditioned law

$$\left(\mathbb{P}_x(Y_n \in \cdot | Y_n \in \mathcal{E}) = \frac{K^n(x, \cdot)}{K^n \mathbf{1}(x)} \right)$$

²this is a mildly weaker definition than the usual definition of small or petite sets (see e.g [24, 35])

converges, as $n \rightarrow \infty$, exponentially fast to a (unique) QSD. Roughly speaking, this assumption stipulates that the speed at which the process dies can be uniformly controlled. Here it will be shown that under \mathbf{H}_3 and \mathbf{H}_4 , the semiflow induced by (7) is globally asymptotically stable (i.e. $\text{Fix}(\Pi)$ has cardinal one and is a global attractor). In particular, there exists a unique QSD.

Remark 2.4 (Sufficient condition). A simple condition ensuring Hypothesis 2.3 is that, for some $l \geq 1$, constants $c_1, c_2 > 0$ and some $\Psi \in \mathcal{P}(\mathcal{E})$

$$c_1 \Psi(dy) \leq K^\ell(x, dy) \leq c_2 \Psi(dy). \quad (11)$$

Indeed, under (11), for $n \geq \ell$,

$$c_2 \Psi(K^{n-\ell} \mathbf{1}) \geq K^n \mathbf{1} \geq c_1 \Psi(K^{n-\ell} \mathbf{1})$$

while for $n \leq \ell$, $\mathbf{1} \geq K^n \mathbf{1} \geq K^\ell \mathbf{1} \geq c_1$. Hence $C(t) = \min\left(\frac{c_1}{c_2}, c_1\right) > 0$. Note that (11), which is usual in the literature (see *e.g.* [9, Theorem 3.2]), is satisfied if K^ℓ admits a continuous and positive density with respect to a positive reference measure.

Finally, note that in Hypotheses 2.2 and 2.3 there is no aperiodicity assumption on K .

We are now able to state our main general result about the convergence of the empirical measure $(\mu_n)_{n \geq 0}$ towards the QSD.

Theorem 2.5 (Convergence of the algorithm). *Assume Hypotheses 2.1, 2.2 and 2.3. Then, K has a unique QSD μ^* and the sequence $(\mu_n)_{n \geq 0}$ converges a.s. in $\mathcal{P}(\mathcal{E})$ towards μ^* .*

In fact, the previous setting also leads to the convergence in distribution of the reinforced random walk:

Theorem 2.6 (Convergence in distribution of $(X_n)_{n \geq 0}$). *Suppose that the assumptions of Theorem 2.5 hold. Then, for any starting distribution α , $(X_n)_{n \geq 0}$ converges in distribution to μ^* .*

The two above results thus show that the algorithm both produces a way to approximate μ^* and also to sample a random variable with this distribution. The convergence in law of $(X_n)_{n \geq 0}$ may appear surprising due to the lack of aperiodicity assumption for $(Y_n)_{n \geq 0}$. To overcome this problem, we prove in fact that $(X_n)_{n \geq 0}$ gets asymptotically this property.

The extinction rate $\Theta(\mu^*)$, defined in (2), can be estimated through the same procedure. For this, we need to keep track of the times at which a "resurrection" occurs. We then construct (X_n) as follows. Let $((U_n, X_n))$ be a process adapted to $\{\mathcal{F}_n\}$, with $U_n \in \mathcal{E} \cup \{\partial\}$, $X_n \in \mathcal{E}$, satisfying $X_0 = U_0 = x$,

$$\mathbb{P}(U_{n+1} \in dy | \mathcal{F}_n) = \bar{K}(X_n, dy)$$

and

$$X_{n+1} = V_{n+1} \mathbf{1}_{\{U_{n+1} = \partial\}} + U_{n+1} \mathbf{1}_{\{U_{n+1} \in \mathcal{E}\}},$$

where (V_n) is a sequence of independent variables such that $V_{n+1} \sim \mu_n$, conditionally on $\sigma(\mathcal{F}_n, U_{n+1})$. Clearly, (X_n) satisfies (4) and the times at which $U_n = \partial$ are the "resurrection" times (at which X_n is redistributed).

Theorem 2.7 (Extinction rate estimation). *Suppose that the assumptions of Theorem 2.5 are satisfied. Then,*

$$\bar{\theta}_n := \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{U_k = \partial\}} \xrightarrow{n \rightarrow +\infty} 1 - \Theta(\mu^*).$$

Proof. Since $\mathbb{P}(U_{n+1} = \partial | \mathcal{F}_n) = \delta(X_n)$, we can decompose $\bar{\theta}_n$ as

$$\bar{\theta}_n = \frac{M_n}{n} + \mu_n(\delta)$$

where (M_n) is the martingale defined by $M_n = \sum_{k=1}^n (1_{\{U_k = \partial\}} - \delta(X_{k-1}))$. Since the increments of (M_n) are uniformly bounded and $\langle M \rangle_n \leq Cn$, it follows from the strong law of large numbers for martingales, that $\frac{M_n}{n} \rightarrow 0$ a.s. as $n \rightarrow +\infty$. On the other hand, $\mu_n(\delta) \xrightarrow{n \rightarrow +\infty} \mu^*(\delta) = 1 - \Theta(\mu^*)$ a.s. This ends the proof. \square

3. EXAMPLES AND APPLICATIONS

3.1. Finite Markov Chains. We consider here the simple situation where \mathcal{E} is a finite set in order to discuss our main assumptions.

We use the notation

$$K(x, y) = K(x, \{y\}), \quad \forall x, y \in \mathcal{E}.$$

We say that x leads to y , written $x \hookrightarrow y$, if $\sum_{n \geq 0} K^n(x, y) > 0$. If $A, B \subset \mathcal{E}$ we write $A \hookrightarrow B$ whenever there exist $x \in A$ and $y \in B$ such that $x \hookrightarrow y$.

Kernel K is called *indecomposable* if there exists $x_0 \in \mathcal{E}$ such that $x \hookrightarrow x_0$ for all $x \in \mathcal{E}$, and *irreducible* if $x \hookrightarrow y$ for all $x, y \in \mathcal{E}$.

Note that Hypothesis \mathbf{H}_1 is automatically satisfied (endow \mathcal{E} with the discrete topology) and that \mathbf{H}_3 is equivalent to indecomposability (choose $U = \{x_0\}$ and $\Psi = \delta_{x_0}$). From now on, we investigate separately the irreducible and non-irreducible cases.

Irreducible setting. When K is irreducible Hypothesis \mathbf{H}_4 holds with $C(n) = c > 0$. This follows from the following lemma.

Lemma 3.1. *There exists $c > 0$ such that*

$$K^n \mathbf{1}(x) = K^n(x, \mathcal{E}) \geq c K^n(y, \mathcal{E}) = c K^n \mathbf{1}(y)$$

for all $n \in \mathbb{N}$ and $x, y \in \mathcal{E}$ such that $x \hookrightarrow y$.

Proof. Let $c_1 = \min\{K(x, y) : K(x, y) > 0\}$. From the relation

$$K^n(x, \mathcal{E}) \geq K^{n+1}(x, \mathcal{E}) = \sum_y K(x, y) K^n(y, \mathcal{E})$$

it comes that

$$K^n(x, \mathcal{E}) \geq c_1 K^n(y, \mathcal{E})$$

whenever $K(x, y) > 0$. This proves the result with $c = c_1^{|\mathcal{E}|}$. \square

As a consequence, except for the rate of convergence, we retrieve of [6, Theorem 1.2] (see also [1, 12] for the convergence result in the case $\gamma_n = \frac{1}{n+1}$).

Theorem 3.2. *Suppose K is irreducible and $K(x_0, \mathcal{E}) < 1$ for some $x_0 \in \mathcal{E}$. Then K has a unique QSD μ^* and under Hypothesis 2.1, (μ_n) converges almost surely to μ^* .*

Bottleneck effect and condition \mathbf{H}_4 . Here we discuss an example demonstrating the necessity of condition \mathbf{H}_4 for non irreducible chains. Note that this example can also be understood as a benchmark of more general processes admitting several QSDs such as general indecomposable Markov chains.

Suppose $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ where \mathcal{E}_1 and \mathcal{E}_2 are nonempty disjoint sets such that

- (1) $\forall x, y \in \mathcal{E}_i \ x \leftrightarrow y$;
- (2) $\mathcal{E}_1 \leftrightarrow \mathcal{E}_2$;
- (3) $\mathcal{E}_2 \nleftrightarrow \mathcal{E}_1$;
- (4) $\mathcal{E}_2 \leftrightarrow \partial$, (that is $\exists x \in \mathcal{E}_2 \ K(x, \mathcal{E}) < 1$) and $\mathcal{E}_1 \nleftrightarrow \partial$.

Let K_i be the kernel K restricted to \mathcal{E}_i . That is

$$K_i = (K(x, y))_{x, y \in \mathcal{E}_i}.$$

Let μ_i^* be the (unique) QSD of K_i and Θ_i the associated extinction rate. Note that, by irreducibility of K_i , and Perron Frobenius Theorem, Θ_i is nothing but the spectral radius of K_i .

We see μ_i^* as an element of $\mathcal{P}(\mathcal{E})$ by identifying $\mathcal{P}(\mathcal{E}_i)$ with the set of $\mu \in \mathcal{P}(\mathcal{E})$ supported by \mathcal{E}_i .

As previously noticed, \mathbf{H}_1 , \mathbf{H}_2 and \mathbf{H}_3 are always true. However, assumption \mathbf{H}_4 might fail to hold. More precisely, we have the following result.

Proposition 3.3 (Sharpness of \mathbf{H}_4). *Condition \mathbf{H}_4 holds if and only if $\Theta_1 \leq \Theta_2$. In this case the unique QSD of K is μ_2^* and, under hypothesis 2.1 $\mu_n \rightarrow \mu_2^*$.*

Proof. Fix $x_0 \in \mathcal{E}_2$ and let $\Psi = \delta_{x_0}$, so that \mathbf{H}_1 , \mathbf{H}_2 and \mathbf{H}_3 hold. By Lemma 3.1 there exists $c > 0$ such that $\Psi(K^n \mathbf{1}) = K^n \mathbf{1}(x_0) \geq c K^n \mathbf{1}(x)$ for all $x \in \mathcal{E}_2$ and $n \geq 0$. Thus \mathbf{H}_4 is equivalent to

$$\forall x \in \mathcal{E}_1, \quad K^n \mathbf{1}(x_0) \geq C(n) K^n \mathbf{1}(x) \quad (12)$$

with C satisfying (10). Let $\tau_1 = \min\{n \geq 0 : Y_n \notin \mathcal{E}_1\}$ and $\tau_2 = \min\{n \geq 0 : Y_n = \partial\}$. By Lemma 3.1 applied to each of the kernel K_i , and from the relation $\Theta_i^n = \mu_i K_i^n \mathbf{1}_{\mathcal{E}_i}$, we get that for all $x \in \mathcal{E}_i$

$$\frac{1}{c} \Theta_i^n \geq \mathbb{P}_x(\tau_i > n) \geq c \Theta_i^n \quad (13)$$

for some $c > 0$. Thus for all $x \in \mathcal{E}_1$,

$$\begin{aligned} \mathbb{P}_x(\tau_2 > n) &= \mathbb{E}_x [\mathbb{P}(\tau_2 > n | \mathcal{F}_{\tau_1})] = \mathbb{E}_x [\mathbb{P}_{Y_{\tau_1}}(\tau_2 > n - \tau_1)] \\ &\leq \frac{1}{c} \mathbb{E}_x [\Theta_2^{n-\tau_1} \mathbf{1}_{\tau_1 \leq n} + \mathbf{1}_{\tau_1 > n}] \end{aligned}$$

by (13). Thus,

$$\begin{aligned} \mathbb{P}_x(\tau_2 > n) &\leq \frac{1}{c} \Theta_2^n \sum_{k=1}^n \Theta_2^{-k} (\mathbb{P}_x(\tau_1 > k-1) - \mathbb{P}_x(\tau_1 > k)) + \frac{1}{c} \mathbb{P}_x(\tau_1 > n) \\ &= \frac{1}{c} \Theta_2^n (\Theta_2^{-1} + \sum_{k=1}^{n-1} (\Theta_2^{-k-1} - \Theta_2^{-k}) \mathbb{P}_x(\tau_1 > k)) \\ &= \frac{1}{c} \Theta_2^{n-1} (1 + (1 - \Theta_2) \sum_{k=1}^{n-1} \Theta_2^{-k} \mathbb{P}_x(\tau_1 > k)) \end{aligned}$$

Then, by (13) again, we get

$$\mathbb{P}_x(\tau_2 > n) \leq O(\Theta_2^{n-1}) = O(\mathbb{P}_{x_0}(\tau_2 > n))$$

when $\Theta_1 < \Theta_2$ and

$$\mathbb{P}_x(\tau_2 > n) \leq O(\Theta_2^{n-1}(1+n)) = O(\mathbb{P}_{x_0}(\tau_2 > n)(1+n))$$

when $\Theta_2 = \Theta_1$. This proves that (12) holds with $C(t) = C$ when $\Theta_1 < \Theta_2$ and $C(t) = C/(1+t)$ when $\Theta_1 = \Theta_2$. If now, $\Theta_1 > \Theta_2$, it follows from Theorem 3.4 below that another QSD $\mu^* \neq \mu_2^*$ exists, so that \mathbf{H}_4 fails. \square

For $\Theta_1 \leq \Theta_2$, μ_2^* is a global attractor of the dynamics induced by (7), but when Θ_1 exceeds Θ_2 a *transcritical bifurcation* occurs: μ_2^* becomes a saddle point whose stable manifold is $\mathcal{P}(\mathcal{E}_2)$, while there is another linearly stable point μ^* whose basin of attraction is $\mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2)$.

This behavior will be shown in section 7 and combined with standard technics from stochastic approximation, it will be used to prove the following result.

Theorem 3.4 (Behavior of the algorithm without Assumption \mathbf{H}_4). *Suppose $\Theta_1 > \Theta_2$. Then there is another QSD μ^* having full support (i.e $\mu^*(x) > 0$ for all $x \in \mathcal{E}$). Under Hypothesis 2.1,*

- (i) $(\mu_n)_{n \geq 0}$ converges, almost surely to $\mu_\infty \in \{\mu_2^*, \mu^*\}$.
- (ii) If $X_0 \in \mathcal{E}_2$, $X_n \in \mathcal{E}_2$ for all n and $\mu_\infty = \mu_2^*$ with probability one.
- (iii) If $X_0 \in \mathcal{E}_1$, the event $\{\mu_\infty = \mu^*\}$ has positive probability.
- (iv) If $\sum_n \prod_{i=1}^n (1 - \gamma_i) < +\infty$, the event $\{\exists N \in \mathbb{N} : X_n \in \mathcal{E}_2 \text{ for all } n \geq N\}$ has positive probability, and on this event $\mu_\infty = \mu_2^*$.

Example 3.5 (Two points space). The previous results are in particular adapted to the case where $\mathcal{E}_i = \{i\}$, $i = 1, 2$ and

$$K = \begin{pmatrix} a & 1-a \\ 0 & b \end{pmatrix}$$

with $a, b \in (0, 1)$. Write $\mu \in \mathcal{P}(\mathcal{E})$ as $\mu = (x, 1-x)$, $0 \leq x \leq 1$. Then

$$K_\mu = \begin{pmatrix} a & 1-a \\ (1-b)x & b + (1-b)(1-x) \end{pmatrix}$$

and the ODE (7) writes

$$\dot{x} = -x + \frac{(1-b)x}{(1-a) + (1-b)x}. \quad (14)$$

In this case, one can check that $\Theta_1 = a$ and $\Theta_2 = b$, $\mu_2^* = \delta_2$ and when $a > b$, $\mu^* = \frac{a-b}{1-b}\delta_1 + \frac{1-a}{1-b}\delta_2$. In Figure 1, for a fixed value of b , we draw the phase portrait of the ODE (14) in terms of a and especially the bifurcation which appears when $a > b$.

Remark 3.6 (Open problem). Suppose $\gamma_n = \frac{A}{n}$. Although μ_2^* is a saddle point when $\Theta_1 > \Theta_2$, Theorem 3.4 shows that the event $\mu_n \rightarrow \mu_2^*$ has positive probability when $A > 1$. A challenging question would be to prove (or disprove) that this event has zero probability when $A \leq 1$. This is reminiscent of the situation thoroughly analyzed for two-armed bandit problems in [29, 30].

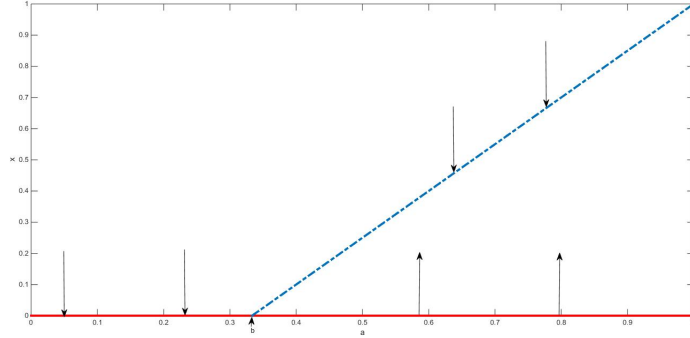


FIGURE 1. Transcritical bifurcation associated to Equation (14); $b = 1/3$, Continuous line: $a \mapsto \mu_2^*(1)$, dotted line: $a \mapsto \mu^*(1)$.

Remark 3.7 (Conditioned dynamics). Note that by mimicking the proof of Lemma 7.2 later, one is also able to compute the limit of the conditioned dynamics:

$$\lim_{n \rightarrow \infty} \mathbb{P}_y(Y_n \in \cdot | Y_n \in \mathcal{E}) = \lim_{n \rightarrow \infty} \frac{K^n(y, \cdot)}{K^n \mathbf{1}(y)} = \nu,$$

where $\nu = \mu_2^*$ if $\Theta_1 \leq \Theta_2$ or $y \in \mathcal{E}_2$ and $\nu = \mu^*$ when $\Theta_1 > \Theta_2$ and $y \in \mathcal{E}_1$. Furthermore, at least for Example 3.5 before, it is worth noting that the convergence is not exponential when $\Theta_1 = \Theta_2$.

Remark 3.8 (Fleming-Viot algorithm). Theorem 3.4 shows that, with positive probability, our algorithm asymptotically matches with the behavior of the dynamics conditioned to the non-absorption. Surprisingly, this is not the case for the discrete-time (or continuous-time) Fleming-Viot particle system (see [6, Section 3], for the definition) which always converges to μ_2^* . Actually, let us recall that this algorithm has two parameters: the (current) time $t \geq 0$ and the number of particles $N \geq 1$. When the number of particles goes to infinity, it is known that the empirical measure (induced by the particles at a fixed time) converges to the laws conditioned to not be extinct; see for instance [20, 41] in the continuous-time setting. However, if we keep constant the number of particles and let first the time t tend to infinity then, one obtains the convergence to μ_2^* in place of μ^* . This comes from the fact that, the state where all the particles are in \mathcal{E}_2 is absorbing and attainable. In this case, the commutation of the limits established in [6, Section 3] fails. Finally, note that the study of the rate of convergence of Fleming-Viot processes in a two-points space is investigated in [19].

3.2. Approximation of QSD of diffusions. A potential application of this work is to generate a way to simulate QSD of continuous-time Markov dynamics. To this end, the natural idea is to apply the procedure to a discretized version (Euler scheme in the sequel) of the process. Here, we focus on the case of non-degenerate diffusions $(\xi_t)_{t \geq 0}$ in \mathbb{R}^d killed when leaving a bounded connected open set D . More precisely, let $(\xi_t)_{t \geq 0}$ be the unique solution to the d -dimensional SDE

$$d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dW_t, \quad \xi_0 \in D,$$

where b and $\sigma : \mathbb{R}^d \mapsto \mathbb{M}_{d,d}$ are defined on \mathbb{R}^d with values in \mathbb{R}^d and $\mathbb{M}_{d,d}$. One assumes below that the diffusion is uniformly elliptic and that b and σ belong to $\mathcal{C}^2(\mathbb{R}^d)$ (see Remark 3.10).

For a given step $h > 0$, we denote by $(\xi_t^h)_{t \geq 0}$, the stepwise constant Euler scheme defined by $\xi_0 = y \in D$,

$$\forall n \in \mathbb{N}, \quad \xi_{(n+1)h}^h = \xi_{nh}^h + hb(\xi_{nh}^h) + \sigma(\xi_{nh}^h)(W_{(n+1)h} - W_{nh}),$$

and for all $t \in [nh, (n+1)h)$, $\xi_t^h = \xi_{nh}^h$. Under the ellipticity assumptions on the diffusion, the Markov chain $(Y_n := \xi_{nh}^h)_n$ satisfies the assumptions of Theorem 2.5 (with $\mathcal{E} = \bar{D}$) (in particular (11)) and thus admits a unique QSD that we denote by μ_h^* .

This QSD can be approximated through the procedure defined above and the natural question is: does $(\mu_h^*)_h$ converge to μ^* when $h \rightarrow 0$, where μ^* denotes the unique QSD of $(\xi_t)_{t \geq 0}$ killed when leaving D ? A positive answer is given below.

Theorem 3.9 (Euler scheme approximation). *Assume that $(\xi_t)_{t \geq 0}$ is a uniformly elliptic diffusion and that D is a bounded domain (i.e. connected open set) with \mathcal{C}^3 -boundary. Then, $((\xi_t)_{t \geq 0}, \partial D)$ admits a unique QSD and $(\mu_h^*)_{h > 0}$ converges weakly to μ^* when $h \rightarrow 0$.*

Remark 3.10 (Smoothness assumptions). *The uniqueness of μ^* is given by Theorem 5.5 of Chapter 3 in [38]. Also note that in the proof of the above theorem, one makes use of some results of [27] about the discretization of killed diffusions. The \mathcal{C}^2 -assumption on b and σ is adapted to the setting of these papers but could be probably relaxed in our context.*

We propose to illustrate the previous results by some simulations. We consider an Ornstein-Uhlenbeck process

$$d\xi_t = -\xi_t dt + dB_t, \quad t \geq 0$$

killed outside an interval $[a, b]$ and thus compute the sequence $(\mu_n^h)_{n \geq 1}$ with step h .

We will assume that $a = 0$ and $b = 3$. In Figure 2, we represent on the left the approximated density of μ_n^h (obtained by a convolution with a Gaussian kernel) for a fixed value of h and different values of n . Then, on the right, n is fixed ($n = 10^7$) and h decreases to 0. Unfortunately, even though the convergence in n seems to be fast, the convergence of μ_n^h towards μ^* is very slow: the discretization of the problem underestimates the probability to be killed between two discretization times. The slow convergence means in fact that this probability decreases slowly to 0 with h .

However, it is now well-known that, under some conditions on the domain and/or on the dimension, it is possible to compute a sharp estimation of this probability. More precisely, let $(\tilde{\xi}_t^h)_t$ denote the refined continuous-time Euler scheme $\tilde{\xi}_{nh} = \xi_{nh}$ and for all $t \in [nh, (n+1)h)$,

$$\tilde{\xi}_t = \tilde{\xi}_{nh} + (t - nh)b(\tilde{\xi}_{nh}) + \sigma(\tilde{\xi}_{nh})(W_t - W_{nh}).$$

It can be shown that

$$\mathcal{L} \left((\tilde{\xi}_t)_{t \in [nh, (n+1)h]} \mid \tilde{\xi}_{nh} = x, \tilde{\xi}_{(n+1)h} = y \right) = \mathcal{L} \left(x + \frac{t - nh}{h}(y - x) + \sigma(\tilde{\xi}_{nh})\mathbf{B}_t^h \right)$$

where for a given $T > 0$, \mathbf{B}^T denotes the Brownian Bridge on the interval $[0, T]$ defined by: $\mathbf{B}_t^T = W_t - \frac{t}{T}W_T$. In dimension 1, the law of the infimum and the supremum of the Brownian

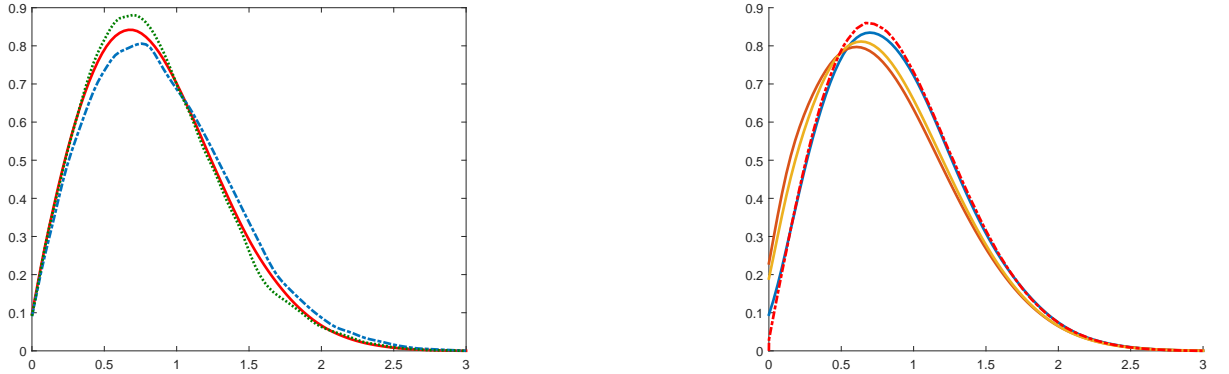


FIGURE 2. Left: Approximated density of (μ_n^h) with $h = 0.01$ and $n = 5 \cdot 10^4, 10^5, 10^6$ (green, blue, red) Right: Comparison of μ_h^* for $h = 0.05, 0.01, 0.001$, (red, orange, blue) with μ^* (red, dotted-line)

Bridge can be computed (see [27] for details and a discussion about higher dimension). One has for every $z \geq \max(x, y)$,

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left(x + (y - x) \frac{t}{T} + \lambda \mathbf{B}^T \right) \leq z \right) = 1 - \exp \left(-\frac{2}{T\lambda^2} (z - x)(z - y) \right).$$

Thus, this means that at each step n , if $\xi_{(n+1)h} \in D$, one can compute, with the help of the above properties, a Bernoulli random variable V with parameter

$$p = \mathbb{P}(\exists t \in (nh, (n+1)h), \xi_t^h \in D^c | \xi_{nh} = x, \xi_{(n+1)h} = y) \quad (\text{If } V = 1, \text{ the particle is killed}).$$

This refined algorithm has been tested numerically and illustrated in Figure 3.

Remark 3.11. *Here, the effect of the Brownian Bridge method is only considered from a numerical viewpoint. The theoretical consequences on the rate of convergence are outside of the scope of this paper. Also remark that in order to get only one asymptotic for the algorithm, it would be natural to replace the constant step h by a decreasing sequence as in [31, 36]. Once again, such a theoretical extension is left to a future work.*

3.3. Spectral gap estimate. Recall that a QSD μ is an eigenvector of the transition kernel and that the extinction rate $\Theta(\mu)$ corresponds to the associated eigenvalue. Using the discretization argument introduced in the last subsection, Theorem 2.7 thus gives a way to approximate eigenelements of elliptic operators with Dirichlet-type boundary condition. In this section, we show that it can be also used to estimate the first non trivial eigenvalue of non-absorbed processes.

Namely, we focus on a potential application of our results to the estimation of a bound of the spectral gap of geometrically ergodic Markov chains. Once again, it could also be extended to the spectral gap of continuous-time Markov processes such as diffusion processes through

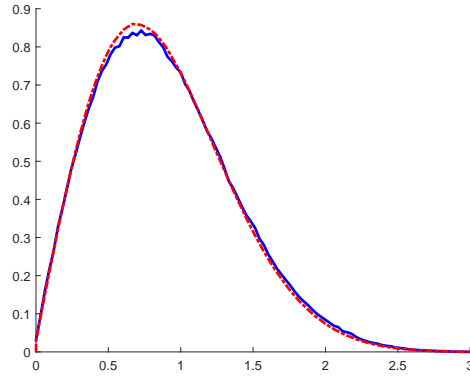


FIGURE 3. Approximation of μ_h^* with the Brownian Bridge method for $h = 0.1$ (blue) compared with the reference density (red, dotted-line)

the discretization introduced previously. Let us consider a geometrically ergodic Markov chain $(Z_n)_{n \geq 0}$, namely, a process satisfying

$$\forall n \geq 0, \|\mathcal{L}(Z_n) - \pi\|_{\text{TV}} \leq C\rho^n, \quad (15)$$

for some $C > 0$ and $\rho \in (0, 1)$. A way to bound its rate of convergence is to build a coalescent coupling $(Z_n^1, Z_n^2)_{n \geq 0}$, whose marginals have different starting distributions, (one of them being the invariant measure of $(Z_n)_{n \geq 0}$) but evolve under the same dynamics as $(Z_n)_{n \geq 0}$ (see for instance [17, 33] and [18, Chapter 5]). We set

$$T_\Delta = \inf\{n \geq 0 \mid Z_n^1 = Z_n^2\} = \inf\{n \geq 0 \mid (Z_n^1, Z_n^2) \in \Delta\},$$

where $\Delta := \{(x, x), x \in E\}$ with E denoting the state space of $(Z_n)_{n \geq 0}$. It is classical background that

$$\|\mathcal{L}(Z_n) - \pi\|_{\text{TV}} \leq \mathbb{P}(T_\Delta > n).$$

But, by construction, T_Δ is the first time when (Z_n^1, Z_n^2) is absorbed by the diagonal Δ . Thus, if the coupling is Markovian, *i.e.* if (Z_n^1, Z_n^2) is a Markov chain, the rate of convergence corresponds to the extinction rate and our method could thus be considered as a way to bound this rate.

Furthermore, let us point out that in many situations, it is possible to exhibit explicit Markovian couplings which are optimal (see for instance [17] or [18, Chapter 9]). Optimal here means that $\mathbb{P}(T_\Delta > n)$ decays exponentially fast, at rate ρ^* , where ρ^* is defined by

$$\ln(\rho^*) = \limsup_{n \rightarrow \infty} \frac{\ln(\|\mathcal{L}(Z_n) - \pi\|_{\text{TV}})}{n}.$$

For a large class of diffusions processes, such an optimal Markovian construction can be achieved through the help of the *mirror coupling* (see *e.g.* [17] or [25]). Using the Euler discretization introduced before, this can thus provide some ways to obtain some numerical bounds for the

spectral gap of diffusions which could in turn be used to bound the rate of convergence to equilibrium (in Wasserstein distance) of the diffusion. Due to the links between spectral gap and geometry, it can also yield some bounds on the Ricci curvature of smooth Riemannian manifolds or discrete spaces (see *e.g.* [28, 42]).

We propose below an illustration with the Brownian motion on the sphere with radius R in \mathbb{R}^d with $d = 3$. In this case, the spectral gap λ_1 is known and is equal to $\lambda_1 = \frac{1}{2} \frac{d-1}{R^2}$ (The Brownian motion being up to a factor 2, the Laplace-Beltrami operator, this result can be deduced from *e.g.* [2, p.214]). In this illustration, we simulate simply (an approximation of) the Brownian motion on the sphere by the following algorithm

$$Z_{n+1}^1 = R \frac{Z_n^1 + \sqrt{h}U_{n+1}}{\|Z_n^1 + \sqrt{h}U_{n+1}\|} \quad (16)$$

where $(U_n)_{n \geq 1}$ is a sequence of *i.i.d.* random variables with distribution $\mathcal{N}(0, I_3)$. To compute the mirror coupling, it is then enough, given two initial conditions z_1 and z_2 on the sphere (which are generated after each offspring) to compute the evolution of (Z_n^2) by a reflection of the one of (Z_n^1) with respect to the hyperplan $\mathcal{H} = \{z \in \mathbb{R}^3, z \cdot (z_1 - z_2) = 0\}$. In Figure 4, we show that we are able to obtain an approximation of the spectral gap with our algorithm. We compute the algorithm for several values of R and draw the path of the approximation of λ_1 after $N = 10^5$ iterations with $h = 0.01$ (so that the “absolute” time is equal to $T = 1000$).

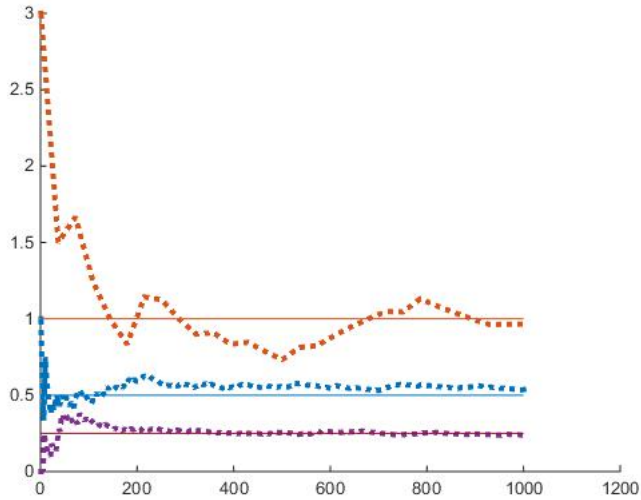


FIGURE 4. Approximation of $\lambda_1 = \frac{1}{R^2}$ induced by the mirror coupling for the Brownian motion on the sphere for $R = 1$, $R = \sqrt{2}$ and $R = 2$, $N = 10^5$, $h = 0.01$

Remark 3.12 (Spherical Brownian motion). *Through the Stroock representation, the Brownian motion on the sphere \mathbb{S}_{d-1} can be achieved as the unique solution of the SDE*

$$dX_t^i = \sum_{j=1}^d (\delta_{i,j} - X_t^i X_t^j) dW_t^j - \frac{d-1}{2} X_t^i dt, \quad i = 1, 2, 3.$$

The convergence of the discretization (16) when $h \rightarrow 0$ towards the solution of the associated martingale problem can be obtained by simple computations.

Remark 3.13 (Compactness and the diagonal set). *It is worth noting that even though the state space E of $(Z_n)_{n \geq 0}$ is assumed to be compact, this is usually not the case for $\mathcal{E} = E^2 \setminus \Delta$. This implies that, except in the discrete case, the theoretical convergence of the procedure induced by the Markov chain (Z_n^1, Z_n^2) absorbed in Δ , can not be directly deduced from our main theorems. To overcome this problem, one way is to study the stability of the extinction rate to an approximation of the absorption space. More precisely, set $\Delta_m = E^2 \setminus \{\|x - y\| < \frac{1}{m}\}$ and denote by ρ_m the associated extinction rate. In the diffusion setting, [40, Proposition 3.3] shows that under reasonable assumptions, $\rho_m \rightarrow \rho$ as m goes to infinity (where ρ denotes the extinction rate related to Δ). Then, let us denote by $\bar{\rho}_{h,m,n}$ the estimator related to Theorem 2.7 where h is the Euler scheme step, m the approximation of the diagonal and n the number of iterations of our algorithm. Under the assumptions of [40, Proposition 3.3] and of Theorem 2.7, it could be shown that*

$$\lim_{m \rightarrow \infty} \lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \bar{\rho}_{h,m,n} = \rho.$$

Outline of the proofs. In Section 4, we begin by some preliminaries: the starting point is to show that the QSD is a fixed point for the application $\mu \mapsto \Pi_\mu$ (on $\mathcal{P}(\mathcal{E})$) where Π_μ denotes the invariant distribution of K_μ (see Lemma 4.2 below). Then, in order to give a rigorous sense to the ODE (7), we prove that this application is Lipschitz continuous for the total variation norm (Proposition 4.4) by taking advantage of the exponential ergodicity of the kernel transition K_μ and the control of the exit time τ (see Lemma 4.1 and Lemma 4.3). In Section 5, we define the solution of the ODE and prove its global asymptotic stability. In Section 6, we then show that (a scaled version of) $(\mu_n)_{n \geq 0}$ is an asymptotic pseudo-trajectory for the ODE. The proofs of Theorems 2.5 and 2.6 are finally achieved at the beginning of Section 7. In this section, we also prove the main results of Section 3: Theorems 3.4 and 3.9. We end the paper by some possible extensions of our present work.

4. PRELIMINARIES

We begin the proof by a series of preliminary lemmas. The first one provides uniform estimates on the extinction time

$$\tau = \min\{n \geq 0 : Y_n = \partial\} \tag{17}$$

Lemma 4.1 (Expectation of the extinction time). *Assume \mathbf{H}_1 and \mathbf{H}_2 . Then*

- (i) *There exist $N \in \mathbb{N}$ and $\delta_0 > 0$ such that for all $x \in \mathcal{E}$, $\bar{K}^N(x, \{\partial\}) \geq \delta_0$.*
- (ii)

$$\sup_{x \in \mathcal{E}} \mathbb{E}_x[\tau] < +\infty.$$

Proof. (i) By \mathbf{H}_1 the map $x \mapsto \bar{K}^N(x, \partial) = 1 - K^N \mathbf{1}(x)$ is continuous on \mathcal{E} . It then suffices to show that there exists $N \in \mathbb{N}$ such that $\bar{K}^N(x, \partial) > 0$ for all $x \in \mathcal{E}$. Suppose to the contrary that for all $N \in \mathbb{N}$ there exists $x_N \in \mathcal{E}$ such that $\bar{K}^N(x_N, \partial) = 0$. Hence $\bar{K}^k(x_N, \partial) = 0$ for all $k \leq N$. By compactness of \mathcal{E} , we can always assume (by replacing (x_N) by a subsequence) that $x_N \rightarrow x^* \in \mathcal{E}$. Thus $\bar{K}^k(x_N, \partial) \xrightarrow{N \rightarrow \infty} \bar{K}^k(x^*, \partial) = 0$ for all $k \in \mathbb{N}$. A contradiction with assumption \mathbf{H}_2 .

(ii) Let N and δ_0 be like in (i). By the Markov property, for all $k \in \mathbb{N}^*$

$$\mathbb{P}_x(\tau > kN) = \mathbb{E}_x \left[\mathbb{P}_{Y_{(k-1)N}}(\tau > N) \mathbf{1}_{\tau > (k-1)N} \right] \leq (1 - \delta_0) \mathbb{P}_x(\tau > (k-1)N)$$

Thus, for all $k \in \mathbb{N}$

$$\mathbb{P}_x(\tau > kN) \leq (1 - \delta_0)^k$$

and, consequently,

$$\frac{1}{N} \mathbb{E}_x[\tau] \leq \frac{1}{\delta_0} + 1.$$

□

Lemma 4.2 (Invariant distributions and QSD). *Assume \mathbf{H}_1 . Then,*

- (i) For every $\mu \in \mathcal{P}(\mathcal{E})$, K_μ is a Feller kernel and admits at least one invariant probability.
- (ii) A probability μ^* is a QSD for K if and only if it is an invariant probability of K_{μ^*} .
- (iii) Assume that for every μ , K_μ has a unique invariant probability Π_μ . Then $\mu \mapsto \Pi_\mu$ is continuous in $\mathcal{P}(\mathcal{E})$ (i.e for the topology of weak convergence) and then there exists $\mu^* \in \mathcal{P}(\mathcal{E})$ such that $\mu^* = \Pi_{\mu^*}$ or, equivalently, a QSD μ^* for K .

Proof. (i) The Feller property is obvious under \mathbf{H}_1 and it is well known that a Feller Markov chain on a compact space has an invariant probability (since any weak limit of the sequence $(\frac{1}{n} \sum_{k=1}^n \nu K_\mu^k)_{n \geq 0}$ is an invariant probability).

(ii) Since $\delta = \mathbf{1} - K\mathbf{1}$, for every $A \in \mathcal{B}(\mathcal{E})$, we have

$$\mu^*(A) = (\mu^* K_{\mu^*})(A) \Leftrightarrow \mu^*(A) = \frac{(\mu^* K)(A)}{(\mu^* K)\mathbf{1}}.$$

But, by definition μ^* is a QSD if and only if the right-hand side is satisfied for every $A \in \mathcal{B}(\mathcal{E})$.

(iii) Let $(\mu_n)_{n \geq 0}$ be a probability sequence converging to some μ in $\mathcal{P}(\mathcal{E})$. Replacing $(\mu_n)_{n \geq 0}$ by a subsequence, we can always assume, by compactness of $\mathcal{P}(\mathcal{E})$, that $(\Pi_{\mu_n})_{n \geq 0}$ converges to some ν . For every $n \geq 0$ and $f \in \mathcal{C}(\mathcal{E}, \mathbb{R})$, we have

$$\Pi_{\mu_n}(f) = \Pi_{\mu_n}(K_{\mu_n} f) = \Pi_{\mu_n}(K f) + \Pi_{\mu_n}(\delta) \mu_n(f).$$

By \mathbf{H}_1 , the maps $K f$ and δ are continuous so that by letting $n \rightarrow \infty$, one obtains

$$\nu(f) = \nu(K f) + \nu(\delta) \mu(f),$$

namely ν is an invariant for K_μ . By uniqueness $\nu = \Pi_\mu$. This proves the continuity of the map $\mu \mapsto \Pi_\mu$. Now, since $\mathcal{P}(\mathcal{E})$ is a convex compact subset of a locally convex topological space (the space of signed measures equipped with the weak* topology) every continuous mapping from $\mathcal{P}(\mathcal{E})$ into itself has a fixed point by Leray-Schauder-Tychonoff fixed point theorem.

□

For all $\mu \in \mathcal{P}(M)$ and $t \geq 0$ we let P_t^μ denote the Markov kernel on \mathcal{E} defined by

$$P_t^\mu(x, \cdot) := e^{-t} \sum_n \frac{t^n}{n!} K_\mu^n(x, \cdot). \quad (18)$$

It is classical (and easy to verify) that

- (a) $(P_t^\mu)_{t \geq 0}$ is a semigroup (i.e. $P_{t+s}^\mu f = P_t^\mu P_s^\mu f$ for all $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$);
- (b) Every invariant probability for K_μ is invariant for P_t^μ ;
- (c) P_t^μ is Feller whenever K_μ is (in particular under \mathbf{H}_1).

If $(X_n^\mu)_{n \geq 0}$ is a Markov chain with transition K_μ , $(P_t^\mu)_{t \geq 0}$ denotes the semi-group of $(X_{N_t}^\mu)_{t \geq 0}$ where $(N_t)_{t \geq 0}$ is an independent Poisson process with intensity 1.

For any finite signed measure ν on M recall that the *total variation norm* of ν is defined as

$$\begin{aligned} \|\nu\|_{\text{TV}} &= \sup\{|\nu f| : f \in \mathcal{B}(\mathcal{E}, \mathbb{R}), \|f\|_\infty \leq 1\} \\ &= \nu^+(\mathcal{E}) + \nu^-(\mathcal{E}) \end{aligned} \quad (19)$$

where $\nu = \nu^+ - \nu^-$ is the Hahn Jordan decomposition of ν . Let us recall that if P is a Markov kernel on M and $\alpha, \beta \in \mathcal{P}(\mathcal{E})$, then

$$\|\alpha P - \beta P\|_{\text{TV}} \leq \|\alpha - \beta\|_{\text{TV}} \quad (20)$$

since $\|Pf\|_\infty \leq \|f\|_\infty$.

Lemma 4.3 (Uniform exponential ergodicity). *Assume \mathbf{H}_1 and \mathbf{H}_2 . Then there exists $0 < \varepsilon < 1$ such that for all $\alpha, \beta, \mu \in \mathcal{P}(\mathcal{E})$ and $t \geq 0$*

$$\|\alpha P_t^\mu - \beta P_t^\mu\|_{\text{TV}} \leq (1 - \varepsilon)^{\lfloor t \rfloor} \|\alpha - \beta\|_{\text{TV}}.$$

In particular, if Π_μ denotes an invariant probability for K_μ ,

$$\|\alpha P_t^\mu - \Pi_\mu\|_{\text{TV}} \leq (1 - \varepsilon)^{\lfloor t \rfloor} \|\alpha - \Pi_\mu\|_{\text{TV}}.$$

As a consequence, K_μ has a unique invariant probability.

Proof. (i). Set $P_\mu = P_1^\mu$. Let $\delta_0 > 0$ and $N \in \mathbb{N}$ be given by Lemma 4.1 (i). It easily seen by induction that for all $k \geq 1$ and $f : \mathcal{E} \mapsto [0, \infty[$ measurable,

$$K_\mu^k f \geq \mu(f) K^{k-1} \delta.$$

Thus,

$$\begin{aligned} P_\mu f &\geq \frac{1}{e} \mu(f) \sum_{k=1}^N \frac{1}{k!} K^{k-1} \delta \geq \frac{1}{eN!} \mu(f) \sum_{k=1}^N K^{k-1} \delta \\ &= \frac{1}{eN!} \mu(f) (\mathbf{1} - K^N \mathbf{1}) \geq \varepsilon \mu(f) \end{aligned} \quad (21)$$

where

$$\varepsilon = \frac{1}{eN!} \delta_0.$$

Let R_μ be the kernel on \mathcal{E} defined by

$$\forall x \in \mathcal{E}, \quad P_\mu(x, \cdot) = \varepsilon \mu(\cdot) + (1 - \varepsilon) R_\mu(x, \cdot). \quad (22)$$

Inequality (21) makes R_μ a Markov kernel. Thus for all $\alpha, \beta \in \mathcal{P}(\mathcal{E})$

$$\|\alpha P_\mu - \beta P_\mu\|_{\text{TV}} = (1 - \varepsilon) \|\alpha R_\mu - \beta R_\mu\|_{\text{TV}} \leq (1 - \varepsilon) \|\alpha - \beta\|_{\text{TV}},$$

(where the last inequality follows from (20)) and, by induction,

$$\|\alpha P_\mu^n - \beta P_\mu^n\|_{\text{TV}} \leq (1 - \varepsilon)^n \|\alpha - \beta\|_{\text{TV}}.$$

Now, for all $t \geq 0$ write $t = n + r$ with $n \in \mathbb{N}$ and $0 \leq r < 1$. Then,

$$\begin{aligned} \|\alpha P_t^\mu - \beta P_t^\mu\|_{\text{TV}} &= \|\alpha P_r^\mu P_\mu^n - \beta P_r^\mu P_\mu^n\|_{\text{TV}} \\ &\leq (1 - \varepsilon)^n \|\alpha P_r^\mu - \beta P_r^\mu\|_{\text{TV}} \leq (1 - \varepsilon)^n \|\alpha - \beta\|_{\text{TV}}. \end{aligned}$$

As mentioned before, if Π_μ is an invariant probability for K_μ , Π_μ is also an invariant probability for $(P_t^\mu)_{t \geq 0}$. The second inequality is thus obtained by setting $\beta = \Pi_\mu$ and uniqueness of the invariant probability is a consequence of the convergence of $(\alpha P_t^\mu)_{t \geq 0}$ towards Π_μ . \square

4.1. Explicit form for Π_μ . Let us denote by \mathcal{A} the transition kernel on \mathcal{E} defined by

$$\mathcal{A}(x, \cdot) = \sum_{n \geq 0} K^n(x, \cdot)$$

and set

$$\|\mathcal{A}\|_\infty = \sup\{\|\mathcal{A}f\|_\infty : f \in \mathcal{B}(\mathcal{E}, \mathbb{R}), \|f\|_\infty \leq 1\}.$$

Remark that

$$\|\mathcal{A}\|_\infty = \sup_{x \in \mathcal{E}} \mathcal{A}(x, \mathcal{E}) \in [0, \infty].$$

Proposition 4.4. *Assume \mathbf{H}_1 and \mathbf{H}_2 . Then:*

(i) For all $x \in \mathcal{E}$,

$$1 \leq \mathcal{A}(x, \mathcal{E}) = \mathcal{A}\mathbf{1}(x) = \mathbb{E}_x[\tau] \leq \|\mathcal{A}\|_\infty < \infty.$$

(ii) For all $\mu \in \mathcal{P}(M)$,

$$\Pi_\mu = \frac{\mu \mathcal{A}}{(\mu \mathcal{A})(\mathbf{1})}. \quad (23)$$

(iii) The map $\mu \mapsto \Pi_\mu$ is Lipschitz continuous for the total variation distance.

Proof. (i) For all $x \in \mathcal{E}$

$$\mathcal{A}(x, \mathcal{E}) = \sum_{n \geq 0} K^n(x, \mathcal{E}) = \sum_{n \geq 0} \mathbb{P}_x(\tau > n) = \mathbb{E}_x[\tau] \leq \sup_x \mathbb{E}_x(\tau) < \infty$$

where the last inequality follows from Lemma 4.1.

(ii) For any $f \in \mathcal{B}(\mathcal{E}, \mathbb{R})$,

$$\mu \mathcal{A} K_\mu(f) = \mu \left(\sum_{n \geq 0} (K^{n+1} f + K^n \delta \mu(f)) \right) = \sum_{n \geq 0} \mu K^{n+1} f + \mu(f) \mu \left(\sum_{n \geq 0} K^n(\delta) \right).$$

Since $\sum_{n \geq 0} K^n \delta(x) = \sum_{n \geq 0} K^n(x, \mathcal{E}) - K^{n+1}(x, \mathcal{E}) = \mathcal{A}(x, \mathcal{E}) - (\mathcal{A}(x, \mathcal{E}) - 1) = 1$, it follows that

$$(\mu \mathcal{A}) K_\mu(f) = \mu(f) + \sum_{n \geq 1} \mu K^n f = (\mu \mathcal{A})(f).$$

As a consequence, $\mu \mathcal{A}$ is an invariant measure and it remains to divide by its mass to obtain an invariant probability.

(iii) It follows from (i) that $\|\mu \mathcal{A}\|_{\text{TV}} \leq \|\mu\|_{\text{TV}} \|\mathcal{A}\|_\infty$ and $\mu \mathcal{A} \mathbf{1} \geq 1$. Thus, reducing the fraction, it easily follows from (ii) that $\|\Pi_\mu - \Pi_\nu\|_{\text{TV}} \leq 2 \|\mathcal{A}\|_\infty \|\mu - \nu\|_{\text{TV}}$. \square

5. THE LIMITING ODE

As mentioned before, the idea of the proof of Theorem 2.5 is to show that, the long time behavior of $(\mu_n)_{n \geq 0}$ can be precisely related to the long term behavior of a deterministic dynamical system $\mathcal{P}(\mathcal{E})$ induced by the "ODE"

$$“\dot{\mu} = -\mu + \Pi_{\mu}.” \quad (24)$$

The purpose of this section is to define rigorously this dynamical system and to investigate some of its asymptotic properties.

Throughout the section, hypotheses \mathbf{H}_1 and \mathbf{H}_2 are implicitly assumed. Recall that $\mathcal{P}(\mathcal{E})$ is a compact metric space equipped with a distance metrizing the weak* convergence.

A *semi-flow* on $\mathcal{P}(\mathcal{E})$ is a continuous map

$$\begin{aligned} \Phi : \mathbb{R}^+ \times \mathcal{P}(\mathcal{E}) &\rightarrow \mathcal{P}(\mathcal{E}), \\ (t, \mu) &\mapsto \Phi_t(\mu) \end{aligned}$$

such that

$$\Phi_0(\mu) = \mu \text{ and } \Phi_{t+s}(\mu) = \Phi_t \circ \Phi_s(\mu).$$

We call such a semi-flow *injective* if each of the maps Φ_t is injective.

A *weak solution* to (24) with initial condition $\mu \in \mathcal{P}(\mathcal{E})$, is a continuous map $\xi : \mathbb{R}^+ \mapsto \mathcal{P}(\mathcal{E})$ such that

$$\xi(t)f = \mu f + \int_0^t (-\xi(s)f + \Pi_{\xi(s)}f) ds$$

for all $f \in C(\mathcal{E})$ and $t \geq 0$.

We shall now show that there exists an injective semi-flow Φ on $\mathcal{P}(\mathcal{E})$ such that the trajectory $t \mapsto \Phi_t(\mu)$ is the unique weak solution to (24) with initial condition μ .

Let $\mathcal{M}_s(\mathcal{E})$ be the space of finite signed measures on \mathcal{E} equipped with the total variation norm $\|\cdot\|_{\text{TV}}$ (defined by equation (19)). By a Riesz type theorem, $\mathcal{M}_s(\mathcal{E})$ is a Banach space which can be identified with the dual space of $C(\mathcal{E}, \mathbb{R})$ equipped with the uniform norm (see e.g [23, chapter 7]). In particular, the supremum in the definition of $\|\cdot\|_{\text{TV}}$ can be taken over continuous functions.

Proposition 23 (i) shows that the operator $f \rightarrow \mathcal{A}f$ is a bounded operator on $C(\mathcal{E}, \mathbb{R})$ with bounded adjoint $\mu \rightarrow \mu\mathcal{A}$ on $\mathcal{M}_s(\mathcal{E})$. Thus, by standard results on linear differential equations in Banach spaces, $e^{t\mathcal{A}}$ is a well defined bounded operator and the mappings $(t, f) \rightarrow e^{t\mathcal{A}}f$ and $(t, \mu) \rightarrow \mu e^{t\mathcal{A}}$ are C^∞ mappings satisfying the differential equations

$$\frac{d}{dt}(e^{t\mathcal{A}}f) = (e^{t\mathcal{A}}\mathcal{A}f) = \mathcal{A}e^{t\mathcal{A}}f$$

and

$$\frac{d}{dt}(\mu e^{t\mathcal{A}}) = \mu(e^{t\mathcal{A}}\mathcal{A}) = \mu\mathcal{A}e^{t\mathcal{A}}.$$

For $\mu \in \mathcal{P}(\mathcal{E})$ and $t \geq 0$ set

$$\begin{aligned} g_t &= e^{t\mathcal{A}}\mathbf{1} \in C(\mathcal{E}), \\ \tilde{\Phi}_t(\mu) &:= \frac{\mu e^{t\mathcal{A}}}{\mu g_t} \in \mathcal{P}(\mathcal{E}), \end{aligned} \quad (25)$$

and

$$s_\mu(t) = \int_0^t \tilde{\Phi}_s(\mu)\mathcal{A}\mathbf{1} ds.$$

Note that, by Proposition 4.4 (i), $\dot{s}_\mu(t) = \tilde{\Phi}_t(\mu)\mathcal{A}\mathbf{1} = \frac{\mu e^{t\mathcal{A}}\mathcal{A}\mathbf{1}}{\mu e^{t\mathcal{A}}\mathbf{1}} \geq 1$, so that s_μ maps diffeomorphically \mathbb{R}^+ onto itself. We let τ_μ denote its inverse and

$$\Phi_t(\mu) = \tilde{\Phi}_{\tau_\mu(t)}(\mu) \quad (26)$$

Proposition 5.1. *The map Φ defined by (26) is an injective semi-flow on $\mathcal{P}(\mathcal{E})$ and for all $\mu \in \mathcal{P}(\mathcal{E})$, $t \mapsto \Phi_t(\mu)$ is the unique weak solution to (24) with initial condition μ .*

Proof. Step 1 (Continuity of Φ): Let $\mu_n \rightarrow \mu$ in $\mathcal{P}(\mathcal{E})$ and $t_n \rightarrow t$. Then for all $f \in C(\mathcal{E})$

$$\begin{aligned} |\mu_n e^{t_n \mathcal{A}} f - \mu e^{t \mathcal{A}} f| &\leq |\mu_n e^{t_n \mathcal{A}} f - \mu_n e^{t \mathcal{A}} f| + |\mu_n e^{t \mathcal{A}} f - \mu e^{t \mathcal{A}} f| \\ &\leq \|e^{t_n \mathcal{A}} - e^{t \mathcal{A}}\|_\infty \|f\|_\infty + |\mu_n e^{t \mathcal{A}} f - \mu e^{t \mathcal{A}} f|. \end{aligned}$$

The second term goes to zero because $\mu_n \rightarrow \mu$ and the first one by strong continuity of $t \mapsto e^{t\mathcal{A}}$. This easily implies that the maps $(t, \mu) \rightarrow \tilde{\Phi}_t(\mu)$ and $(t, \mu) \rightarrow s_\mu(t)$ are continuous. This latter continuity combined with the relation $s_{\mu_n} \circ \tau_{\mu_n}(t_n) = t_n$ implies that every limit point of $\{\tau_{\mu_n}(t_n)\}$ equals $\tau_\mu(t)$; but since $\tau_\mu(t) \leq t$ (because $s_\mu(t) \geq t$) the sequence $\{\tau_{\mu_n}(t_n)\}$ is bounded and this proves the continuity of $(t, \mu) \rightarrow \tau_\mu(t)$. Continuity of Φ follows.

Step 2 (Injectivity of Φ): Suppose $\Phi_t(\mu) = \Phi_t(\nu)$ for some $t \geq 0$, $\mu, \nu \in \mathcal{P}(\mathcal{E})$. Set $\tau = \tau_\mu(t)$ and $\sigma = \tau_\nu(t)$. Assume $\sigma \geq \tau$. Multiplying the equality $\tilde{\Phi}_\tau(\mu) = \tilde{\Phi}_\sigma(\nu)$ by $e^{-\tau\mathcal{A}}$ shows that $\mu = \tilde{\Phi}_{\sigma-\tau}(\nu)$. Thus

$$\begin{aligned} t = s_\mu(\tau) &= \int_0^\tau \tilde{\Phi}_{s+\sigma-\tau}(\nu)\mathcal{A}\mathbf{1} ds = \int_0^\sigma \tilde{\Phi}_s(\nu)\mathcal{A}\mathbf{1} ds - \int_0^{\sigma-\tau} \tilde{\Phi}_s(\nu)\mathcal{A}\mathbf{1} ds \\ &= t - s_\nu(\sigma - \tau) \end{aligned}$$

This implies that $\tau = \sigma$, hence $\mu = \nu$.

Step 3 ($t \rightarrow \Phi_t(\mu)$ is a weak solution): The mappings $t \rightarrow \tilde{\mu}_t := \tilde{\Phi}_t(\mu)$ and $t \rightarrow \mu_t := \Phi_t(\mu)$ are C^∞ from \mathbb{R}^+ into $\mathcal{M}_s(M)$. Furthermore,

$$\dot{\tilde{\mu}}_t = \tilde{\mu}_t \mathcal{A} - (\tilde{\mu}_t \mathcal{A} \mathbf{1}) \tilde{\mu}_t = \dot{s}_\mu(t) (-\tilde{\mu}_t + \Pi_{\tilde{\mu}_t}).$$

so that

$$\dot{\mu}_t = -\mu_t + \Pi_{\mu_t}$$

and, in particular,

$$\mu_t f - \mu_0 f = \int_0^t (-\mu_s f + \Pi_{\mu_s} f) ds$$

for all $f \in C(\mathcal{E})$.

Step 4 (Uniqueness and flow property): Let $\{\mu_t\}$ and $\{\nu_t\}$ be two weak solutions of (24). By separability of $C(\mathcal{E})$, $\|\mu_t - \nu_t\|_{\text{TV}} = \sup_{f \in \mathcal{H}} |\mu_t f - \nu_t f|$ for some countable set $\mathcal{H} \subset C(\mathcal{E})$. This shows that $t \rightarrow \|\mu_t - \nu_t\|_{\text{TV}}$ is measurable, as a countable supremum of continuous functions. Thus, by Lipschitz continuity of $\mu \mapsto \Pi_\mu$ with respect to the total variation distance (see Lemma 4.4) we get that

$$\|\mu_t - \nu_t\|_{\text{TV}} \leq \|\mu_0 - \nu_0\|_{\text{TV}} + L \int_0^t \|\mu_s - \nu_s\|_{\text{TV}} ds$$

for some $L > 0$. Hence, by the measurable version of Gronwall's inequality ([26, Theorem 5.1 of the Appendix])

$$\|\mu_t - \nu_t\|_{\text{TV}} \leq e^{Lt} \|\mu_0 - \nu_0\|_{\text{TV}},$$

so that there is at most one weak solution with initial condition μ_0 . This, combined with (ii) above shows that $t \rightarrow \Phi_t(\mu)$ is the unique weak solution to (24). The semi-flow property $\Phi_{t+s} = \Phi_t \circ \Phi_s$ follows directly from this uniqueness. \square

5.1. Attractors and attractor free sets. A set $\mathcal{K} \subset \mathcal{P}(\mathcal{E})$ is called *invariant* under Φ (respectively *positively invariant*) if $\Phi_t(\mathcal{K}) = \mathcal{K}$ (respectively $\Phi_t(\mathcal{K}) \subset \mathcal{K}$), for all $t \geq 0$.

If \mathcal{K} is compact and invariant, then by injectivity of Φ and compactness, each map Φ_t maps homeomorphically \mathcal{K} onto itself. In this case we set

$$\Phi_t^{\mathcal{K}} = \Phi_t|_{\mathcal{K}}$$

for $t \geq 0$ and

$$\Phi_t^{\mathcal{K}} = (\Phi_{-t}|_{\mathcal{K}})^{-1}$$

for all $t \leq 0$. It is not hard to check that $\Phi^{\mathcal{K}} : \mathbb{R} \times \mathcal{K} \mapsto \mathcal{K}$ is a flow. That is a continuous map such that $\Phi_t^{\mathcal{K}} \circ \Phi_s^{\mathcal{K}} = \Phi_{t+s}^{\mathcal{K}}$ for all $t, s \in \mathbb{R}$.

An *attractor* for Φ is a non empty compact invariant set A having a neighborhood U_A (called a fundamental neighborhood) such that for every neighborhood V of A there exists $t \geq 0$ such that

$$s \geq t \Rightarrow \Phi_s(U_A) \subset V.$$

Equivalently, if d is a distance metrizing $\mathcal{P}(\mathcal{E})$

$$\lim_{t \rightarrow \infty} d(\Phi_t(\mu), A) = 0,$$

uniformly in $\mu \in U_A$.

The *basin of attraction* of A is the set $\text{Bas}(A)$ consisting of points $\mu \in \mathcal{P}(\mathcal{E})$ such that $\lim_{t \rightarrow \infty} d(\Phi_t(\mu), A) = 0$.

Attractor A is called *global* if its basin is the full space $\mathcal{P}(\mathcal{E})$. It is not hard to verify that there is always a (unique) *global attractor* for Φ given as

$$A = \bigcap_{t \geq 0} \Phi_t(\mathcal{P}(\mathcal{E})).$$

If \mathcal{K} denotes a compact invariant set, an *attractor* for $\Phi^{\mathcal{K}}$ is a non empty compact invariant set $A \subset \mathcal{K}$ having a neighborhood U_A such that for every neighborhood V of A there exists $t \geq 0$ such that

$$s \geq t \Rightarrow \Phi_s(U_A \cap \mathcal{K}) \subset V.$$

If furthermore $A \neq \mathcal{K}$, A is called a *proper attractor*.

\mathcal{K} is called *attractor free* provided \mathcal{K} is compact invariant and $\Phi^{\mathcal{K}}$ has no proper attractors. Attractor free sets coincide with *internally chain transitive sets* and characterize the limit sets of *asymptotic pseudo trajectories* (see [7, 5]). Recall that the limit set of (μ_n) is defined by

$$L = \bigcap_{n \geq 0} \overline{\{\mu_k \mid k \geq n\}}.$$

In the present context, by Theorem 6.4 of Section 6, this implies that

Theorem 5.2 (Characterisation of L). *Under Hypotheses 2.2 and 2.1, the limit set of $\{\mu_n\}$ is almost surely attractor free for Φ .*

This theorem, combined with elementary properties of attractor free sets, gives the following (more tractable) result.

Corollary 5.3 (Limit set and attractors). *Assume Hypotheses 2.2 and 2.1. Let L be the limit set of $\{\mu_n\}$. With probability one,*

- (i) L is a compact connected invariant set.
- (ii) If A is an attractor and $L \cap \text{Bas}(A) \neq \emptyset$, then $L \subset A$. In particular, L is contained in the global attractor of Φ .

Note that in the two previous theorems, we do not assume Hypothesis 2.3. In particular, the previous result may be true in some settings with several QSDs. This flexibility is, for instance, used in the proof of Theorem 3.4.

5.2. Global Asymptotic Stability. The flow Φ is called *globally asymptotically stable* if its global attractor reduces to a singleton $\{\mu^*\}$. Observe that, in such a case, μ^* is necessary the unique equilibrium of Φ , hence the unique QSD of K .

We shall give here sufficient conditions ensuring global asymptotic stability. The main idea is to relate the (nonlinear) dynamics of Φ to the (linear) Fokker-Planck equation of a nonhomogeneous Markov process on \mathcal{E} . This idea is due to Champagnat and Villemonais in [16] where it was successfully used to prove the exponential convergence of the conditioned laws and the exponential ergodicity of the Q -process for a general almost surely absorbed Markov process.

For all $t \geq 0$ and $s \in \mathbb{R}$ let $R_{t,s}$ be the bounded operator defined on $C(\mathcal{E})$ by

$$R_{t,s}f = \frac{e^{(t-s)\mathcal{A}}(fg_s)}{g_t} = \frac{e^{(t-s)\mathcal{A}}(fe^{s\mathcal{A}}\mathbf{1})}{e^{t\mathcal{A}}\mathbf{1}}$$

where g is defined by (25). It is easily checked³ that $R_{t,t} = Id$ and $R_{t,s} \circ R_{s,u} = R_{t,u}$ for all $t, s \geq 0$ and $u \in \mathbb{R}$. Furthermore, for all $t \geq s \geq 0$ $R_{t,s}$ is a Markov operator. That is $R_{t,s}\mathbf{1} = \mathbf{1}$ and $R_{t,s}f \geq 0$ whenever $f \geq 0$.

To shorten notation we set

$$R_t = R_{t,0}.$$

The flow $\tilde{\Phi}$ and the family $\{R_t\}_{t \geq 0}$ are linked by the relation

$$\tilde{\Phi}_t(\delta_x) = \delta_x R_t$$

for all $t \geq 0$ and $x \in \mathcal{E}$. However, note that for an arbitrary $\mu \in \mathcal{P}(\mathcal{E})$ $\tilde{\Phi}_t(\mu)$ and μR_t are not equal. Indeed, recall that $\mu R_t f = \int_{\mathcal{E}} R_t f(x) \mu(dx)$.

Lemma 5.4. *Let d_ω be any distance on $\mathcal{P}(\mathcal{E})$ metrizing the weak* convergence. Assume that*

$$\Delta_t := \sup_{x,y \in \mathcal{E}} d_\omega(\delta_x R_t, \delta_y R_t) \rightarrow 0$$

as $t \rightarrow \infty$. Then Φ is globally asymptotically stable.

Proof. By compactness of \mathcal{E} the condition $\Delta_t \rightarrow 0$ is independent of the choice of d_ω . We can then assume that d_{FM} is the *Fortet-Mourier distance* (see e.g [23, 39]) given as

$$d_\omega(\mu, \nu) = \sup\{|\mu f - \nu f| : \|f\|_\infty + \text{Lip}(f) \leq 1\}. \quad (27)$$

where $\text{Lip}(f)$ stands for $\sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}$.

³One can also note that $R_{t,s}$ is the resolvent of the linear differential equation on $C(\mathcal{E})$ $\dot{u} = \frac{1}{g_t}(\mathcal{A}(ug_t) - (\mathcal{A}u)g_t)$. This explains the unusual order for the indices of R (w.r.t. the standard notations of in-homogeneous Markov processes).

Since

$$|\mu R_t f - \nu R_t f| = \left| \int (R_t(x) - R_t f(y)) d\mu(x) d\nu(y) \right| \leq \sup_{x,y \in \mathcal{E}} |R_t f(x) - R_t f(y)|,$$

it follows from (27) that

$$\Delta_t = \sup_{\mu, \nu \in \mathcal{P}(M)} d_\omega(\mu R_t, \nu R_t). \quad (28)$$

Fix $\nu \in \mathcal{P}(\mathcal{E})$. Then

$$\sup_{s \geq 0} d_\omega(\nu R_{t+s}, \nu R_t) = \sup_{s \geq 0} d_\omega((\nu R_{t+s,t}) R_t, \nu R_t) \leq \Delta_t.$$

This shows that $\{\nu R_t\}_{t \geq 0}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{E})$. Then $\nu R_t \rightarrow \mu^*$ for some μ^* and for all $\mu \in \mathcal{P}(\mathcal{E})$

$$d_\omega(\mu R_t, \mu^*) \leq \Delta_t.$$

Now, for all $f \in C(\mathcal{E})$

$$|\tilde{\Phi}_t(\mu) f - \mu^* f| = \left| \frac{\mu((R_t f - \mu^* f) g_t)}{\mu g_t} \right| \leq \|R_t f - \mu^* f\|_\infty = \sup_x |\delta_x R_t f - \mu^* f|.$$

Therefore $d_\omega(\tilde{\Phi}_t(\mu), \mu^*) \leq \Delta_t$ and

$$d_\omega(\Phi_t(\mu), \mu^*) \leq \Delta_{\tau_\mu(t)} \leq \sup \left\{ \Delta_s : s \geq \frac{t}{\|\mathcal{A}\|_\infty} \right\}$$

where the last inequality follows from the fact that $\dot{s}_\mu(t) \leq \|\mathcal{A}\|_\infty$. This proves that $\{\mu^*\}$ is a global attractor for Φ . \square

Recall that $g_t(x) = e^{t\mathcal{A}} \mathbf{1}$ (see equation (25)).

Lemma 5.5. *Assume $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$. Assume furthermore that*

$$\sum_n \frac{\Psi(g_n)}{\|g_n\|_\infty} = \infty$$

where Ψ is the probability measure given by (9). Then Φ is globally asymptotically stable.

Proof. We first assume that $U = \mathcal{E}$ in condition \mathbf{H}_3 . That is $\mathcal{A}(x, dy) \geq \epsilon \Psi(dy)$ for all $x \in \mathcal{E}$. Then, for all $f \geq 0$ and $n \in \mathbb{N}$

$$R_{n+1,n} f = \frac{e^{\mathcal{A}}(f g_n)}{e^{\mathcal{A}} g_n} \geq \frac{\mathcal{A}(f g_n)}{e^{\|\mathcal{A}\|_\infty} \|g_n\|_\infty} \geq \epsilon \frac{\Psi(f g_n)}{e^{\|\mathcal{A}\|_\infty} \|g_n\|_\infty}.$$

Let $\Psi_n \in \mathcal{P}(\mathcal{E})$ be defined as $\Psi_n(f) = \frac{\Psi(f g_n)}{\Psi(g_n)}$. We get

$$R_{n+1,n}(x, \cdot) \geq \epsilon_n \Psi_n(\cdot)$$

with $\epsilon_n = \epsilon e^{-\|\mathcal{A}\|_\infty} \frac{\Psi(g_n)}{\|g_n\|_\infty}$. Thus, reasoning exactly like in the proof of Lemma 4.3, for all $\mu, \nu \in \mathcal{P}(\mathcal{E})$

$$\| \mu R_{n+1,n} - \nu R_{n+1,n} \|_{\text{TV}} \leq (1 - \epsilon_n) \| \mu - \nu \|_{\text{TV}}$$

and, consequently,

$$\| \delta_x R_{n+1} - \delta_y R_{n+1} \|_{\text{TV}} \leq 2 \prod_{k=0}^n (1 - \epsilon_k).$$

The condition $\sum_n \epsilon_n = \infty$ then implies that $\|\delta_x R_{n+1} - \delta_y R_{n+1}\|_{\text{TV}} \rightarrow 0$ uniformly in x, y as $t \rightarrow \infty$. In particular, the assumption, hence the conclusion, of Lemma 5.4 is satisfied.

To conclude the proof it remains to show that there is no loss of generality in assuming that $U = \mathcal{E}$ in \mathbf{H}_3 . By Feller continuity, and Portmanteau's theorem, for all $n \in \mathbb{N}$ and $\delta > 0$ the set

$$U(n, \delta) = \{x \in \mathcal{E} : K^n(x, U) > \delta\}$$

is open. Thus by \mathbf{H}_3 and compactness of \mathcal{E} , there exist $\delta > 0$ and $n_1, \dots, n_k \in \mathbb{N}$ such that

$$\mathcal{E} = \bigcup_{i=1}^k U(n_i, \delta).$$

Let now $x \in \mathcal{E}$. Then $x \in U(n_i, \delta)$ for some i and

$$\mathcal{A}(x, dy) \geq \sum_{n \geq 0} K^{n_i+n}(x, dy) = \int_U K^{n_i}(x, dz) \mathcal{A}(z, dy) \geq \epsilon \delta \Psi(dy).$$

□

The next proposition shows that under $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and \mathbf{H}_4 , the assumptions of the preceding lemma are satisfied.

Proposition 5.6 (Convergence of Φ). *Assume $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and \mathbf{H}_4 . Then the assumptions of Lemma 5.5 are satisfied. In particular, Φ is globally asymptotically stable.*

Proof. By Lemma 4.1 (i) there exists $N \in \mathbb{N}^*$ and $\Theta < 1$ such that

$$K^N(x, \mathcal{E}) \leq \Theta$$

for all $x \in \mathcal{E}$. Let $(Z_n)_{n \geq 1}$ be a sequence of i.i.d random variable on \mathbb{N} having a geometric distribution,

$$\mathbb{P}(Z_n = k) = \Theta^k(1 - \Theta), \quad k \geq 0.$$

Let (U_n) be a sequence of i.i.d random variables on $\{0, \dots, N-1\}$ having a uniform distribution,

$$\mathbb{P}(U_n = k) = \frac{1}{N}, \quad k = 0, \dots, N-1,$$

and let $(N_t)_{t \geq 0}$ be a standard Poisson process with parameter 1. We assume that $(Z_n)_{n \geq 1}, (U_n)_{n \geq 1}, (N_t)_{t \geq 0}$ are mutually independent.

By independence we get that

$$\begin{aligned} \mathbb{E} \left[\frac{K^{\sum_{i=1}^{N_t} (NZ_i + U_i)}}{\Theta^{\sum_{i=1}^{N_t} Z_i}} \right] &= \sum_{n \geq 0} \frac{t^n}{n!} e^{-t} \mathbb{E} \left[\frac{K^{NZ_1 + U_1}}{\Theta^{Z_1}} \right]^n \\ &= \sum_{n \geq 0} \frac{t^n}{n!} e^{-t} \left(\frac{(1-\Theta)}{N} \sum_{k \geq 0} \sum_{r=0, \dots, N-1} K^{Nk+r} \right)^n = e^{-t} e^{t \frac{(1-\Theta)}{N} \mathcal{A}}. \end{aligned}$$

To shorten notation, set $s = t \frac{(1-\Theta)}{N}$. Then, for all $x \in \mathcal{E}$

$$\Psi(g_s) = \Psi(e^{s\mathcal{A}} \mathbf{1}) = e^t \mathbb{E} \left[\frac{\Psi(K^{\sum_{i=1}^{N_t} (NZ_i + U_i)} \mathbf{1})}{\Theta^{\sum_{i=1}^{N_t} Z_i}} \right]$$

$$\geq e^t \mathbb{E} \left[C \left(\sum_{i=1}^{N_t} (NZ_i + U_i) \right) \frac{K^{\sum_{i=1}^{N_t} (NZ_i + U_i)} \mathbf{1}(x)}{\Theta^{\sum_{i=1}^{N_t} Z_i}} \right)$$

where the last inequality comes from hypothesis \mathbf{H}_4 .

For all $n \in \mathbb{N}$, $\mathbf{k} = (k_i) \in \mathbb{N}^{\mathbb{N}^*}$ and $\mathbf{r} = (r_i) \in \{0, \dots, N-1\}^{\mathbb{N}^*}$ set

$$F(n, \mathbf{k}, \mathbf{r}) = \frac{K^{\sum_{i=1}^n (Nk_i + r_i)} \mathbf{1}(x)}{\Theta^{\sum_{i=1}^n k_i}}$$

and

$$G(n, \mathbf{k}, \mathbf{r}) = C \left(\sum_{i=1}^n (Nk_i + r_i) \right)$$

so that, the preceding inequality rewrites,

$$\Psi(g_s) \geq e^t \mathbb{E} [G(N_t, Z, U) F(N_t, Z, U)].$$

Write $(n, \mathbf{k}, \mathbf{r}) \leq (n', \mathbf{k}', \mathbf{r}')$ when $n \leq n'$, $k_i \leq k'_i$ and $r_i \leq r'_i$. The relations $\frac{K^N(x, \mathcal{E})}{\Theta} \leq 1$ and $K(x, \mathcal{E}) \leq 1$ on one hand, and the monotonicity of C on the other hand, imply that

$$F(n, \mathbf{k}, \mathbf{r}) \leq F(n', \mathbf{k}', \mathbf{r}')$$

and

$$G(n, \mathbf{k}, \mathbf{r}) \leq G(n', \mathbf{k}', \mathbf{r}')$$

whenever $(n, \mathbf{k}, \mathbf{r}) \leq (n', \mathbf{k}', \mathbf{r}')$. Then, by tensorisation of the classical FKG inequality, and Jensen inequality, we get that

$$\begin{aligned} \Psi(g_s) &\geq e^t \mathbb{E} [G(N_t, Z, U)] \mathbb{E} [F(N_t, Z, U)] \\ &\geq e^t C \left(t \frac{N}{1-\Theta} + \frac{N-1}{2} \right) \mathbb{E} [F(N_t, Z, U)] = C \left(t \frac{N}{1-\Theta} + \frac{N-1}{2} \right) g_s(x). \end{aligned}$$

That is

$$\Psi(g_s) \geq C \left(\frac{N^2}{(1-\Theta)^2} s + \frac{N-1}{2} \right) g_s(x),$$

so that the assumptions of Lemma 5.5 are fulfilled. \square

6. ASYMPTOTIC PSEUDO-TRAJECTORY

Our aim is now to prove that $(\mu_n)_{n \geq 0}$, correctly normalized, is an asymptotic pseudo-trajectory of the flow Φ defined by (26).

6.1. Background. To prove that our procedure has asymptotically the dynamics of an ODE, we first need to embed it in a continuous-time process at the good scale. Let us add some notations to explain this point. For $n \geq 0$ and $t \geq 0$, set $\tau_n = \sum_{k=1}^n \gamma_k$ and $m(t) = \sup\{k \geq 0, t \geq \tau_k\}$. Let $(\hat{\mu}_t)_{t \geq 0}$, $(\bar{\mu}_t)_{t \geq 0}$, $(\bar{\epsilon}_t)_{t \geq 0}$, $(\bar{\gamma}_t)_{t \geq 0}$ defined for all $n \geq 0$ and $s \in [0, \gamma_{n+1})$ by

$$\hat{\mu}_{\tau_n+s} = \left(1 - \frac{s}{\gamma_{n+1}} \right) \mu_n + \frac{s}{\gamma_{n+1}} \mu_{n+1}, \quad \bar{\mu}_{\tau_n+s} = \mu_n,$$

$\bar{\epsilon}_{\tau_n+s} = \epsilon_n$ and $\bar{\gamma}(\tau_n + s) = \gamma_n$. With these notations, Equation (6) can be written as follows:

$$\hat{\mu}_t = \mu_0 + \int_0^t h(\bar{\mu}_s) ds + \int_0^t \bar{\epsilon}_s ds.$$

The aim of this section is now to show that $\hat{\mu}$ is a pseudo-trajectory of Φ defined in (26). Let d_ω be a metric on \mathcal{P} whose the topology corresponds to the convergence in law (as for instance the Fortet-Mourier distance defined in (27)). A continuous map $\zeta : \mathbb{R}_+ \rightarrow \mathcal{P}$ is called an asymptotic pseudo-trajectory for Φ if

$$\forall T > 0, \quad \lim_{t \rightarrow \infty} \left(\sup_{0 \leq s \leq T} d_\omega(\zeta(t+s), \Phi(s, \zeta(t))) \right) = 0.$$

Note that this definition makes an explicit reference to d_ω but is in fact purely topological (see [5, Theorem 3.2]). In our setting, the asymptotic pseudo-trajectory property can be obtained by the following characterization:

Theorem 6.1 (Asymptotic pseudo-trajectories). *The following assertions are equivalent.*

- (1) *The function $\hat{\mu}$ is (almost surely) an asymptotic pseudo-trajectory for Φ .*
- (2) *For all continuous and bounded f and $T > 0$,*

$$\lim_{t \rightarrow \infty} \sup_{0 \leq s \leq T} \left| \int_t^{t+s} \bar{\epsilon}_u f du \right| = 0 \quad a.s. \quad (29)$$

Proof. This is a consequence of [8, Proposition 3.5]. □

The previous theorem is one of the main differences with the previous article [6]. Indeed, in finite state space, the topology of the total variation distance is not stronger than the weak topology.

As in [6] and older works on reinforced random walks (see references therein), we now need some properties of solutions of Poisson equations to prove that (29) holds. However, in contrast with the finite-space setting of [6], the associated bounds are intricate.

6.2. Poisson Equation related to K_μ . For a fixed μ and a given function $f : \mathcal{E} \rightarrow \mathbb{R}$, let us consider the Poisson equation

$$f - \Pi_\mu f = (I - K_\mu)(Q_\mu f). \quad (30)$$

The existence of a solution $Q_\mu f$ to this equation and the smoothness of $\mu \mapsto Q_\mu f$ play an important role for the study of our algorithm. We have the following result:

Lemma 6.2 (Poisson equation). *Assume Hypothesis 2.2. For any $\mu \in \mathcal{P}(\mathcal{E})$ and any measurable function $f : \mathcal{E} \rightarrow \mathbb{R}$, the Poisson equation (30) admits a solution denoted by $Q_\mu f$ and defined by*

$$Q_\mu f(x) = \sum_{n \geq 0} (P_\mu^n f(x) - \Pi_\mu(f)), \quad (31)$$

where $P_\mu = P_1^\mu$; see Equation (18). Furthermore,

- (i) *for every μ , $\|Q_\mu f\|_\infty \leq C \|f\|_\infty$.*
- (ii) *for every $\mu, \alpha \in \mathcal{P}(\mathcal{E})$, $|\alpha Q_\mu f| \leq C \|f\|_\infty \|\alpha - \Pi_\mu\|_{TV}$.*
- (iii) *for every $\mu \in \mathcal{P}(\mathcal{E})$, f , $\|Q_\mu f - Q_\nu f\|_\infty \leq C_2 \|f\|_\infty \|\mu - \nu\|_{TV}$*
- (iv) *for every $\mu, \alpha \in \mathcal{P}(\mathcal{E})$, $\|\alpha Q_\mu - \alpha Q_\nu\|_{TV} \leq C_2 \|\mu - \nu\|_{TV}$*

Note that our work is closely related to [8] which also investigates the pseudo-trajectory property of a measure-valued sequence. Nevertheless, the scheme of proof for the smoothness of the Poisson solutions really differs. Indeed, in contrast with [8, Lemma 5.1], which is proved using classical functional results (such as the Bakry-Emery criterion), the above lemma (especially (iii) and (iv)) is obtained using a refinement of the ergodicity result provided by Lemma 4.3.

Proof. First, by Lemma 4.3, the sum in (31) is well defined. Then, the fact that it is a solution to the Poisson equation follows directly from its construction. Statements (i) and (ii) are also straightforward consequences of Lemma 4.3. Thus, in the sequel of the proof, we only focus on the "Lipschitz" properties (iii) and (iv).

Without loss of generality, we assume in the sequel that $\|f\|_\infty \leq 1$. By (22), for every $n \geq 0$

$$(\alpha - \beta)P_\mu^n = (1 - \varepsilon)^n(\alpha - \beta)R_\mu^n \quad (32)$$

where, with the notations of Lemma 4.3, R_μ is given by

$$R_\mu = \frac{1}{(1 - \varepsilon)} \left(e^{-1} \sum_{j \geq 0} \frac{1}{j!} K_\mu^j - \varepsilon \mu \right) \quad (33)$$

The Kernels K_μ^j are Lipschitz continuous with respect to the total variation norm, uniformly in $\alpha \in \mathcal{P}(\mathcal{E})$, as it can be checked in Lemma 6.3 below. Set

$$\Xi_n(\mu, \nu) = \sup_{\alpha \in \mathcal{P}(\mathcal{E})} \|\alpha R_\mu^n - \alpha R_\nu^n\|_{\text{TV}}.$$

From (33) and (35), we have

$$\Xi_1(\mu, \nu) \leq \frac{e}{(1 - \varepsilon)} \|\mu - \nu\|_{\text{TV}}.$$

Now,

$$\begin{aligned} \Xi_{n+1}(\mu, \nu) &\leq \sup_{\alpha \in \mathcal{P}(\mathcal{E})} \|(\alpha R_\mu^n)R_\mu - (\alpha R_\mu^n)R_\nu\|_{\text{TV}} + \sup_{\alpha \in \mathcal{P}(\mathcal{E})} \|(\alpha R_\mu^n)R_\nu - (\alpha R_\nu^n)R_\nu\|_{\text{TV}} \\ &\leq \Xi_1(\mu, \nu) + \Xi_n(\mu, \nu), \end{aligned}$$

where for the second term, we used that for some laws α and β and for a kernel transition P , $\|\alpha P - \beta P\|_{\text{TV}} \leq \|\alpha - \beta\|_{\text{TV}}$. By an induction, it follows that

$$\Xi_n(\mu, \nu) \leq \frac{ne}{(1 - \varepsilon)} \|\mu - \nu\|_{\text{TV}}.$$

As a consequence, there exists a constant C such that

$$\|R_\mu^n f - R_\nu^n f\|_\infty \leq Cn \|\mu - \nu\|_{\text{TV}}.$$

and for every $\alpha \in \mathcal{P}(\mathcal{E})$,

$$|\alpha(R_\mu^n)^n f - \alpha(R_\nu^n)^n f| \leq Cn \|\mu - \nu\|_{\text{TV}}. \quad (34)$$

Let us now prove that $\mu \rightarrow Q_\mu f(x)$ is Lipschitz continuous. From the definition of Q_μ and from (32), we have

$$\begin{aligned} Q_\mu f(x) - Q_\nu f(x) &= \sum_{n=0}^{+\infty} ((P_\mu^n f(x) - \Pi_\mu P_\mu^n(f)) - (P_\nu^n f(x) - \Pi_\nu P_\nu^n(f))) \\ &= \sum_{n=0}^{+\infty} (1 - \varepsilon)^n ((\delta_x - \Pi_\mu)R_\mu^n - (\delta_x - \Pi_\nu)R_\nu^n f). \end{aligned}$$

Now, for every $n \geq 0$,

$$\begin{aligned} & \left| (\delta_x - \Pi_\mu) R_\mu^n f - (\delta_x - \Pi_\nu) R_\nu^n f \right| \\ & \leq |\delta_x R_\mu^n f - \delta_x R_\nu^n f| + |\Pi_\mu R_\mu^n f - \Pi_\mu R_\nu^n f| + |\Pi_\mu R_\nu^n f - \Pi_\nu R_\nu^n f|. \end{aligned}$$

The two first terms can be controlled by (34), whereas the second and third ones can be managed by (35). The last term can be bounded using Proposition 4.4 (ii) and the fact that $\sup_{\|g\|_\infty \leq 1} \|R_\mu^n g\|_\infty \leq 1$.

One deduces that, for some constants $C_1, C_2 > 0$,

$$|Q_\mu f(x) - Q_\nu f(x)| \leq \sum_{n=0}^{+\infty} (1 - \varepsilon)^n [2Cn\|\mu - \nu\|_{\text{TV}} + C_1\|\mu - \nu\|_{\text{TV}}] \leq C_2\|\mu - \nu\|_{\text{TV}}.$$

Since the Lipschitz constant C_2 does not depend on x , the statements (iii) and (iv) easily follow. \square

Lemma 6.3 (Lipschitz property of $\mu \rightarrow K_\mu^j$). *For every $\mu, \nu \in \mathcal{P}(\mathcal{E})$ and $j \in \mathbb{N}$, we have*

$$\sup_{\alpha \in \mathcal{P}(\mathcal{E})} \|\alpha K_\mu^j - \alpha K_\nu^j\|_{\text{TV}} \leq 2^j \|\mu - \nu\|_{\text{TV}}, \quad (35)$$

and for every bounded function f then

$$\sup_{x \in \mathcal{E}} \|K_\mu^j(f) - K_\nu^j(f)\|_\infty \leq 2^j \|f\|_\infty \|\mu - \nu\|_{\text{TV}}.$$

Proof. By the definition of the total variation, the second part follows from the first one. We thus only focus on the first statement. For every $j \in \mathbb{N}$, one sets

$$\kappa_j(\mu, \nu) = \sup_{\alpha \in \mathcal{P}} \|\alpha K_\mu^j - \alpha K_\nu^j\|_{\text{TV}}.$$

We have $\kappa_0(\mu, \nu) = 0$ and since $K_\mu(\cdot) = K(\cdot) + \delta(\cdot)\mu$ and $\alpha(\delta) \leq 1$,

$$\kappa_1(\mu, \nu) = \sup_{\alpha \in \mathcal{P}} \|\alpha K_\mu - \alpha K_\nu\|_{\text{TV}} = \|\alpha(\delta)(\mu - \nu)\|_{\text{TV}} \leq \|\mu - \nu\|_{\text{TV}}.$$

Furthermore, for every $j \geq 0$,

$$\begin{aligned} \|\alpha(K_\mu)^{j+1} - \alpha(K_\nu)^{j+1}\|_{\text{TV}} &= \|\alpha(K + \delta\mu)(K_\mu)^j - \alpha(K + \delta\nu)(K_\nu)^j\|_{\text{TV}} \\ &= \|\alpha K(K_\mu^{(j)} - K_\nu^{(j)}) + \alpha(\delta)\mu(K_\mu)^j - \alpha(\delta)\nu(K_\nu)^j\|_{\text{TV}} \\ &\leq \|\alpha K K_\mu^{(j)} - \alpha K K_\nu^{(j)}\|_{\text{TV}} + \|\alpha(\delta)\mu(K_\mu)^j - \alpha(\delta)\nu(K_\nu)^j\|_{\text{TV}} \\ &\quad + \|\alpha(\delta)\nu(K_\mu)^j - \alpha(\delta)\nu(K_\nu)^j\|_{\text{TV}} \\ &\leq \kappa_j(\mu, \nu) + \|\mu - \nu\|_{\text{TV}} + \kappa_j(\mu, \nu) = 2\kappa_j(\mu, \nu) + \|\mu - \nu\|_{\text{TV}}. \end{aligned}$$

Note that for the last inequality, we again used that $\alpha(\delta) \leq 1$ and that for every probabilities α, β and every kernel transition P , $\|\alpha P - \beta P\|_{\text{TV}} \leq \|\alpha - \beta\|_{\text{TV}}$. An induction of the previous inequality then leads to:

$$\forall j \in \mathbb{N}, \quad \kappa_j(\mu, \nu) \leq 2^j \|\mu - \nu\|_{\text{TV}}$$

This yields (35). \square

6.3. Asymptotic Pseudo-trajectories.

Theorem 6.4. *Under Hypotheses 2.1 and 2.2 $(\hat{\mu}_t)_{t \geq 0}$ is an asymptotic pseudo-trajectory of Φ as defined by (26).*

Remark 6.5. Limit of precompact asymptotic pseudo trajectories being internally chain transitive (see [7], [5]), this theorem implies Theorem 5.2, hence Corollary 5.3.

Proof. By Theorem 6.1, it is enough to show that for any bounded continuous function f , for any $T > 0$,

$$\limsup_{t \rightarrow +\infty} \sup_{s \in [0, T]} \left| \int_t^{t+s} \bar{\varepsilon}_s(f) ds \right| = 0$$

which in turn is equivalent to, for every $m \geq 0$

$$\limsup_{n \rightarrow +\infty} \max_{j \leq m} \left| \sum_{k=n}^{n+j} \varepsilon_k(f) \right| = 0.$$

Here,

$$\varepsilon_n(f) = f(X_{n+1}) - \Pi_{\mu_n}(f) = Q_{\mu_n}f(X_{n+1}) - K_{\mu_n}Q_{\mu_n}f(X_{n+1}).$$

We decompose this term as follows:

$$\varepsilon_n(f) = \gamma_{n+1}\Delta M_{n+1}(f) + \Delta R_{n+1}(f) + \gamma_{n+1}\Delta D_{n+1}(f) \quad (36)$$

with

$$\begin{aligned} \Delta M_{n+1}(f) &= Q_{\mu_n}f(X_{n+1}) - K_{\mu_n}Q_{\mu_n}f(X_n) \\ \Delta R_{n+1}(f) &= (\gamma_{n+1} - \gamma_n)K_{\mu_n}Q_{\mu_n}f(X_n) + (\gamma_n K_{\mu_n}Q_{\mu_n}f(X_n) - \gamma_{n+1}K_{\mu_{n+1}}Q_{\mu_{n+1}}f(X_{n+1})) \\ \Delta D_{n+1}(f) &= (K_{\mu_{n+1}}Q_{\mu_{n+1}}f(X_{n+1}) - K_{\mu_n}Q_{\mu_n}f(X_{n+1})). \end{aligned}$$

First, let us focus on ΔR_{n+1} . Using for the first part that $(\gamma_n)_{n \geq 0}$ is decreasing and that $(x, \mu) \mapsto K_{\mu}Q_{\mu}f(x)$ is (uniformly) bounded (Lemma 6.2 (i)), and a telescoping argument for the second part yields for any positive integer m :

$$\left| \sum_{k=n}^m \Delta R_k(f) \right| \leq C\gamma_n.$$

Second (ΔM_k) is a sequence of (\mathcal{F}_n) -martingale increments. From Lemma 6.2, $\Delta M_n(f)$ is bounded (and thus subgaussian). As a consequence, using that $\lim_{n \rightarrow +\infty} \gamma_n \log(n) = 0$, one can adapt the arguments of [5, Proposition 4.4] (based on exponential martingales) to obtain that

$$\limsup_{n \rightarrow +\infty} \max_{j \leq m} \left| \sum_{k=n}^{n+j} \gamma_k \Delta M_k(f) \right| = 0.$$

Finally, for the last term, one uses that $\mu \mapsto K_\mu$ and $\mu \mapsto Q_\mu$ are Lipschitz continuous. More precisely, using Lemma 6.2 (i), (iii) and Lemma 6.3, we see that there exists $C > 0$ such that

$$\begin{aligned} & |K_{\mu_{n+1}}Q_{\mu_{n+1}}f(X_{n+1}) - K_{\mu_n}Q_{\mu_n}f(X_{n+1})| \\ \leq & |K_{\mu_{n+1}}Q_{\mu_{n+1}}f(X_{n+1}) - K_{\mu_{n+1}}Q_{\mu_n}f(X_{n+1})| \\ & + |K_{\mu_{n+1}}Q_{\mu_n}f(X_{n+1}) - K_{\mu_n}Q_{\mu_n}f(X_{n+1})| \\ \leq & \|Q_{\mu_{n+1}}f - Q_{\mu_n}f\|_\infty \\ & + \|(K_{\mu_{n+1}} - K_{\mu_n})(Q_{\mu_n}f)\|_\infty \\ \leq & C\|f\|_\infty\|\mu_{n+1} - \mu_n\|_{\text{TV}} \\ \leq & C\|f\|_\infty\gamma_{n+1}, \end{aligned}$$

where for the last line, we simply used (5). This ends the proof. \square

7. PROOF OF THE MAIN RESULTS

7.1. Proof of Theorem 2.5. By Proposition 5.6, $\{\mu^*\}$ is a global attractor for Φ . The result then follows from Corollary 5.3.

7.2. Proof of Theorem 2.6. Let us assume for the moment that there exist $C > 0$ and $\rho \in (0, 1)$ such that for any any starting distribution α ,

$$\forall n \geq 0, \|\alpha K_{\mu^*}^n - \mu^*\|_{\text{TV}} \leq C\rho^n. \quad (37)$$

With this assumption, μ^* is a global attractor for the discrete time dynamical system on \mathcal{P} induced by the map $\mu \mapsto \mu K_{\mu^*}$. Let ν_n be the law of X_n , for $n \geq 0$; namely $\nu_n(A) = \mathbb{P}(X_n \in A)$, for every Borel set A . To prove that $\nu_n \rightarrow \mu^*$, it is then enough to prove that the sequence $(\nu_n)_{n \geq 0}$ is an asymptotic pseudo-trajectory of this dynamics; namely that $d_\omega(\nu_n K_{\mu^*}, \nu_{n+1}) \rightarrow 0$. Indeed, the limit set of a bounded asymptotic pseudo-trajectory is contained in every global attractor (see e.g [5, Theorem 6.9] or [5, Theorem 6.10].) So, let us firstly show that for every continuous and bounded function f ,

$$\lim_{n \rightarrow \infty} (\nu_{n+1}(f) - \nu_n K_{\mu^*} f) = 0. \quad (38)$$

By definition of the algorithm, for every $n \geq 0$,

$$\mathbb{E}[f(X_{n+1}) | \mathcal{F}_n] = K_{\mu_n} f(X_n) = K f(X_n) + \mu_n(f)\delta(X_n)$$

Taking the expectation, we find

$$\nu_{n+1}(f) = \nu_n K(f) + \mathbb{E}[\mu_n(f)\delta(X_n)] = \nu_n K_{\mu^*}(f) + \mathbb{E}[(\mu_n(f) - \mu^*)(f))\delta(X_n)].$$

But by Theorem 2.5 and dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \mathbb{E}[(\mu_n(f) - \mu^*)(f))\delta(X_n)] = 0,$$

so that (38) holds.

We are now free to choose any metric on \mathcal{P} embedded with the weak topology. Let $(f_k)_{k \geq 0}$ be a sequence of C^∞ functions dense in the space of continuous and bounded (by 1) functions (with respect to the uniform convergence). Consider the distance d_ω defined by

$$d_\omega(\mu, \nu) = \sum_{k \geq 0} \frac{1}{2^k} |\mu(f_k) - \nu(f_k)|.$$

It is well known that d_ω is a metric on \mathcal{P} which induces the convergence in law. From (38) and dominated convergence Theorem, we have that $d_\omega(\nu_n K_{\mu^*}, \nu_{n+1}) \rightarrow 0$.

It remains to prove inequality (37). The proof is similar to Lemma 4.3. Indeed, by Lemma 4.1, there exists $N \geq 0$ and δ_0 such that for all $x \in \mathcal{E}$ such that $\bar{K}^N(x, \partial) \geq \delta_0$. Using that $\mu^* K_{\mu^*} = \theta_* \mu^*$, we have

$$K_{\mu^*}^N = K^N + \bar{K}^N(\cdot, \partial) \mu^*$$

so that the following lower-bound holds: $\inf_{x \in \mathcal{E}} K_{\mu^*}^N(x, \cdot) \geq \delta_0 \mu^*(\cdot)$. It then implies bound (37) with the same argument as that of Lemma 4.3.

Remark 7.1 (Periodicity). *Note that the previous argument shows in particular that the uniform ergodicity of K_{μ^*} is preserved in a non-aperiodic setting. This is the reason why the convergence in distribution of $(X_n)_{n \geq 1}$ also holds in this case.*

7.3. Proof of Theorem 3.4. The proof relies on the following lemma.

Lemma 7.2. *Suppose $\Theta_1 > \Theta_2$. Then*

- (i) $\mathcal{P}(\mathcal{E}_2)$ is positively invariant under Φ and $\Phi|_{\mathcal{P}(\mathcal{E}_2)}$ is globally asymptotically stable with attractor $\{\mu_2^*\}$.
- (ii) There exists another equilibrium for Φ (i.e another QSD for K) μ^* having full support (i.e $\mu^*(x) > 0$ for all $x \in \mathcal{E}$). Furthermore $\{\mu^*\}$ is an attractor whose basin of attraction is $\mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2)$.

Proof. (i) It easily follows from the assumption $\mathcal{E}_2 \leftrightarrow \mathcal{E}_1$, and from the definitions of K_μ and Π_μ that $\mathcal{P}(\mathcal{E}_2)$ is positively invariant under Φ . By irreducibility of K_2 , Lemma 3.1 and Proposition 5.6, μ_2^* is then a global attractor for $\Phi|_{\mathcal{P}(\mathcal{E}_2)}$.

(ii) Let d_i be the cardinal of \mathcal{E}_i and $d = d_1 + d_2$. Identifying $\mathcal{B}(\mathcal{E}_i, \mathbb{R})$ (respectively $\mathcal{B}(\mathcal{E}, \mathbb{R})$) with column vectors of \mathbb{R}^{d_i} (respectively \mathbb{R}^d) and $\mathcal{M}(\mathcal{E}_i)$ (respectively $\mathcal{M}(\mathcal{E})$) with row vectors of \mathbb{R}^{d_i} (respectively \mathbb{R}^d), K can be written as a $d \times d$ block triangular matrix

$$K = \begin{pmatrix} K_1 & K_{12} \\ 0 & K_2 \end{pmatrix},$$

where for each $i = 1, 2$, K_i is a $d_i \times d_i$ irreducible matrix.

Let $E_{\Theta_1}^l$ and $E_{\Theta_1}^r$ be the left and right eigenspaces associated to Θ_1 . That is

$$E_{\Theta_1}^l = \{\mu \in \mathcal{M}(\mathcal{E}) : \mu K = \Theta_1 \mu\}$$

and

$$E_{\Theta_1}^r = \{f \in \mathcal{B}(\mathcal{E}, \mathbb{R}) : K f = \Theta_1 f\}.$$

We claim that

$$E_{\Theta_1}^l = \mathbb{R} \mu^* \tag{39}$$

for some $\mu^* \in \mathcal{P}(\mathcal{E})$ having full support (i.e $\mu(x) > 0$ for all x); and

$$E_{\Theta_1}^r = \mathbb{R} f^* \tag{40}$$

for some $f^* \in \mathcal{B}(\mathcal{E}, \mathbb{R}^+)$ satisfying

$$f^*(x) > 0 \Leftrightarrow x \in \mathcal{E}_1,$$

and $\mu^*(f^*) = 1$.

Actually, by irreducibility of K_1 and Perron Frobenius Theorem (for irreducible matrices), Θ_1 is a simple eigenvalue of K_1 and there exists $g \in \mathcal{B}(\mathcal{E}_1, \mathbb{R}) := \mathbb{R}^{d_1}$ with positive entries

such that $K_1 g = \Theta_1 g$. Θ_1 being strictly larger than the spectral radius Θ_2 of K_2 , Θ_1 is not an eigenvalue of K_2 . Thus, it is also simple for K and (40) holds with f^* defined by $f^*(x) = g(x)$ for $x \in \mathcal{E}_1$ and $f^*(x) = 0$ for $x \in \mathcal{E}_2$.

Again by Perron Frobenius theorem (but this time for non irreducible matrices) there exists $\mu^* \in \mathcal{P}(\mathcal{E}) \cap E_{\Theta_1}^l$, so that, by simplicity of Θ_1 , (39) holds. It remains to check that μ^* has full support. First, observe that μ^* cannot be supported by \mathcal{E}_2 for otherwise μ^* would be a left eigenvector of K_2 and Θ_1 an eigenvalue of K_2 . Thus there exists $x \in \mathcal{E}_1$ such that $\mu^*(x) > 0$, but since $x \hookrightarrow y$, then for all $y \in \mathcal{E}$, we have $\mu^*(y) > 0$.

Replacing f^* by $\frac{f^*}{\mu^*(f^*)}$ we can always assume that

$$\mu^*(f^*) = 1.$$

This ends the proof of the claim.

Let $(f^*)^\perp = \{\nu \in \mathcal{M}(\mathcal{E}) : \nu(f^*) = 0\}$. It follows from what precedes that the splitting

$$\mathcal{M}(\mathcal{E}) = \mathbb{R}\mu^* \oplus (f^*)^\perp$$

is invariant by the map $\nu \mapsto \nu K$, hence also by $\nu \mapsto \nu \mathcal{A}$, and $\nu \mapsto \nu e^{t\mathcal{A}}$. Let \mathcal{A}^\perp denote the operator on $(f^*)^\perp$ defined by

$$\nu \mathcal{A}^\perp = \nu \mathcal{A} - \frac{1}{1 - \Theta_1} \nu.$$

For all $\mu \in \mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2)$, $\mu(f^*) \neq 0$ and μ decomposes as $\mu = \mu(f^*)\mu^* + \tilde{\mu}$ with $\tilde{\mu} = \mu - \mu(f^*)\mu^* \in (f^*)^\perp$. Therefore

$$\mu e^{t\mathcal{A}} = \mu(f^*) e^{\frac{t}{1-\Theta_1}} \mu^* + \tilde{\mu} e^{t\mathcal{A}} = \mu(f^*) e^{\frac{t}{1-\Theta_1}} \left(\mu^* + \frac{\tilde{\mu}}{\mu(f^*)} e^{t\mathcal{A}^\perp} \right)$$

and

$$\tilde{\Phi}_t(\mu) := \frac{\mu e^{t\mathcal{A}}}{\mu e^{t\mathcal{A}} \mathbf{1}} = \frac{\mu^* + \frac{\tilde{\mu}}{\mu(f^*)} e^{t\mathcal{A}^\perp}}{1 + \frac{\tilde{\mu}}{\mu(f^*)} e^{t\mathcal{A}^\perp} \mathbf{1}}.$$

Now, remark that any eigenvalue Λ of \mathcal{A}^\perp writes

$$\Lambda = \frac{1}{1 - \lambda} - \frac{1}{1 - \Theta_1}$$

where $\lambda = a + ib$ is an eigenvalue of K distinct from Θ_1 . In particular, $a < \Theta_1 < 1$. Then,

$$\operatorname{Re}(\Lambda) = \frac{1 - a}{(1 - a)^2 + b^2} - \frac{1}{1 - \Theta_1} \leq \frac{1}{1 - a} - \frac{1}{1 - \Theta_1} < 0.$$

The fact that all eigenvalues of \mathcal{A}^\perp have negative real part implies that $\|e^{t\mathcal{A}^\perp}\| \rightarrow 0$ as $t \rightarrow \infty$. This proves that $\lim_{t \rightarrow \infty} \tilde{\Phi}_t(\mu) = \mu^*$ and that for every compact set $\mathcal{K} \subset \mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2)$ the convergence is uniform in $\mu \in \mathcal{K}$ (because $\mu \mapsto \mu(f^*)$ is bounded below from zero on such a compact). This shows that μ^* is an attractor for $\tilde{\Phi}$ whose basin is $\mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2)$. Proceeding like in the end of the proof of Lemma 5.4 we conclude that the same is true for Φ . \square

We now pass to the proof of Theorem 3.4.

(i) Let L be the limit set of $\{\mu_n\}$. If $L \subset \mathcal{P}(\mathcal{E}_2)$, then $L = \{\mu_2^*\}$ because L is compact invariant and, by Lemma 7.2 (i), $\{\mu_2^*\}$ is the only compact invariant subset of $\mathcal{P}(\mathcal{E}_2)$. If $L \cap \mathcal{P}(\mathcal{E}) \setminus \mathcal{P}(\mathcal{E}_2) \neq \emptyset$, then by Lemma 7.2 (ii) and Corollary 5.3, $L = \{\mu^*\}$.

(ii) If $X_0 \in \mathcal{E}_2$, then $\mu_0 = \delta_{X_0} \in \mathcal{P}(\mathcal{E}_2)$ and, by the definition of (X_n) (see equation (4)), (X_n) lives in \mathcal{E}_2 . This implies that $\mu_n \rightarrow \mu_2^*$ by assertion (i).

(iii) If $X_0 \in \mathcal{E}_1$, the point μ^* is straightforwardly attainable (in the sense of [5, Definition 7.1]) and then, by [5, Theorem 7.3], we have

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \mu_n = \mu^*\right) > 0.$$

(iv) Let us now prove the last point of Theorem 3.4 using an *ad hoc* argument under the additional assumption:

$$\sum_{n \geq 0} \prod_{i=0}^n (1 - \gamma_i) < +\infty. \quad (41)$$

If $X_0 \in \mathcal{E}_2$ there is nothing to prove. We then suppose that $X_0 = x \in \mathcal{E}_1$. Clearly there exists $n_0 \geq 1$ such that $X_{n_0} \in \mathcal{E}_2$ with positive probability. Using the estimate (13), the definition of (X_n) and the recursive formula (5) we get that for all $n > n_0$:

$$\mathbb{P}(X_{n+1} \in \mathcal{E}_1 | \mathcal{F}_n) \leq (1 - c\Theta_2)\mu_n(\mathcal{E}_1)$$

almost surely on the event $\{X_n \in \mathcal{E}_2\}$ and

$$\mu_n(\mathcal{E}_1) = \mu_{n_0}(\mathcal{E}_1) \prod_{i=n_0+1}^n (1 - \gamma_i) \leq \prod_{i=n_0+1}^n (1 - \gamma_i)$$

almost surely on the event

$$\{X_{n_0}, X_{n_0+1}, \dots, X_n \in \mathcal{E}_2\}.$$

Therefore

$$\begin{aligned} \mathbb{P}(X_{n_0}, X_{n_0+1}, \dots, X_{n+1} \in \mathcal{E}_2) &= \mathbb{E}\left(\mathbb{P}(X_{n+1} \in \mathcal{E}_2 | \mathcal{F}_n) \mathbf{1}_{\{X_{n_0}, X_{n_0+1}, \dots, X_n \in \mathcal{E}_2\}}\right) \\ &\geq \left(1 - (1 - c\Theta_2) \prod_{i=n_0+1}^n (1 - \gamma_i)\right) \mathbb{P}(X_{n_0}, X_{n_0+1}, \dots, X_n \in \mathcal{E}_2); \end{aligned}$$

and, consequently,

$$\mathbb{P}(\forall n \geq n_0 X_n \in \mathcal{E}_2) \geq \prod_{n \geq n_0+1} \left(1 - (1 - c\Theta_2) \prod_{i=n_0+1}^n (1 - \gamma_i)\right) \mathbb{P}(X_{n_0} \in \mathcal{E}_2).$$

The right hand side of the previous bound is finally positive if and only if (41) holds.

7.4. Proof of Theorem 3.9. We begin by recalling a classical lemma about the L^2 -control of the distance between the Euler scheme and the diffusion (see *e.g.* [13]).

Lemma 7.3. *Assume that b and σ are Lipschitz continuous functions. Then, for every positive T , there exists a constant $C(T)$ such that for every starting point x of \mathbb{R}^d ,*

$$\mathbb{E}_x \left[\sup_{t \in [0, T]} |\xi_t^h - \xi_t|^2 \right] \leq C(T)(1 + |x|^2)h.$$

We continue with some uniform controls of the exit time of D . For a given set A and a path $w : \mathbb{R}_+ \mapsto \mathbb{R}^d$, we denote by $\tau_A(w)$ the exit time of A defined by:

$$\tau_A(w) = \inf\{t > 0, w(t) \in A^c\}.$$

Lemma 7.4. *Assume that b and σ are Lipschitz continuous functions and that $\sigma\sigma^* \geq \rho_0 I_d$. Then,*

(i) *Let $\delta > 0$ and set $D_\delta = \{x \in \mathbb{R}^d, d(x, D) \leq \delta\}$. For each $t_0 > 0$, we have*

$$\sup_{x \in D} \mathbb{P}_x(\tau_{D_\delta}(\xi) > t_0) < 1.$$

(ii) *There exist some compact sets \mathcal{K} and $\tilde{\mathcal{K}}$ of D such that $\mathcal{K} \subset \tilde{\mathcal{K}}$ and such that there exist some positive t_0, t_1 , such that for all $h > 0$,*

$$\forall x \in D, \mathbb{P}_x(\xi_{t_0} \in \mathcal{K}, \tau_D(\xi^h) > t_0) > 0$$

and

$$\inf_{x \in \mathcal{K}} \mathbb{P}_x(\xi_{t_1} \in \mathcal{K} \text{ and } \forall t \in [0, t_1], \xi_t \in \tilde{\mathcal{K}}) > 0.$$

Proof. (i) Using that for every $t_0 > 0$, $\sup_{s \in [0, t_0]} |\xi_s^x - \xi_s^{x_0}| \rightarrow 0$ in probability when $x \rightarrow x_0$, one deduces from the dominated convergence theorem that $x \mapsto \mathbb{P}_x(\tau_{D_\delta}(\xi) > t_0)$ is continuous on \bar{D} . As a consequence, it is enough to show that for every $x \in D$, $\mathbb{P}_x(\tau_{D_\delta}(\xi) > t_0) < 1$. This last point is a consequence of the ellipticity condition.

(ii) Let us begin by the first statement. Let $t_0 > 0$. Since the Euler scheme is stepwise constant, it is enough to show that $\mathbb{P}_x(\xi_{[t_0/h]h}^h \in \mathcal{K}, \xi_{\ell h}^h \in D, \ell \in \{0, \dots, [t_0/h]\}) > 0$. This follows easily from the fact that, under the ellipticity condition, the transition kernel of the discrete Euler scheme is a (uniformly) non-degenerated Gaussian with bounded bias (on compact sets).

For the second statement, let $x_0 \in D$, $\mathcal{K} = \bar{B}(x_0, r)$ and $\tilde{\mathcal{K}} = \bar{B}(x_0, 2r)$ where $r = \frac{1}{4}d(x_0, \partial D)$. For every $x \in \mathcal{K}$, let $\psi^{x, x_0} : [0, 1] \rightarrow \mathbb{R}$ denote the function defined by $\psi^{x, x_0}(t) = tx_0 + (1-t)x$, $t \in [0, 1]$. Let $t_1 > 0$. Since ψ^{x, x_0} is \mathcal{C}^1 and that $\sup_{x \in \mathcal{K}, t \in [0, 1]} |\partial_t \psi^{x, x_0}| < +\infty$, it is classical background (see e.g. [4, Theorem 8.5]) that for all $\varepsilon > 0$, there exists a positive c_ε such that

$$\forall x \in \mathcal{K}, \mathbb{P}_x(\sup_{t \in [0, t_1]} |\xi_t - \psi^{x, x_0}(t)| \leq \varepsilon) \geq c_\varepsilon.$$

Taking $\varepsilon = \frac{r}{2}$, the result follows. □

Proof of Theorem 3.9. First, let us remark that by the ellipticity condition, we have for every starting point x of D , $\tau_D(\xi^h) = \tau_{\bar{D}}(\xi^h)$ a.s. Actually, as mentioned before, $\mathcal{L}(Y_{k+1}^h | Y_k^h = x)$ is a non-degenerate Gaussian random variable, which implies that $\mathbb{P}(Y_{k+1}^h \in \partial D | Y_k^h \in D) = 0$. Furthermore, if $x \in \partial D$, $\tau_{\bar{D}}(\xi^h) = 0$. Then, if one denotes by μ_h^* the unique QSD of Y^h killed when leaving \bar{D} , it follows that $\mu_h^*(\partial D) = 0$. Without loss of generality, one can thus work with D instead of \bar{D} in the sequel.

Let K^h denote the submarkovian kernel related to the discrete-time Euler scheme killed when it leaves D : for every bounded borelian function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$K^h f(x) = \mathbb{E}[f(x + hb(x) + \sqrt{h}\sigma(x)Z)1_{\{x + hb(x) + \sqrt{h}\sigma(x)Z \in D\}}]$$

where $Z \sim \mathcal{N}(0, I_d)$. Let ρ_h denote the extinction rate related to μ_h^* : we have

$$\int K^h f(x) \mu_h^*(dx) = \rho_h \mu_h^*(f). \quad (42)$$

Setting $\lambda_h = \log(\rho_h)/h$, it easily follows from an induction that for every positive t , for every bounded measurable function $f : D \rightarrow \mathbb{R}$,

$$\mathbb{E}_{\mu_h^*} [f(\xi_t^h) 1_{\tau_D(\xi^h) > t}] = C_h(t) \exp(-\lambda_h t) \mu_h^*(f) \quad (43)$$

where $C_h(t) = \exp(\frac{t}{h} - \lfloor \frac{t}{h} \rfloor)$.

The aim is now to first show that $(\mu_h^*)_h$ is tight on the open set D and then, to prove that every weak limit μ (for the weak topology induced by the usual topology on D) is a QSD. The convergence will follow from the uniqueness of μ^* given in Theorem 5.5 of [38, Chapter 3]. This task is divided in three steps:

Step 1 (Bounds for λ_h): We prove that there exist some positive λ_{\min} , λ_{\max} and h_0 such that for any $h \in (0, h_0)$, λ_h defined in (43) satisfies $\lambda_{\min} \leq \lambda_h \leq \lambda_{\max}$. Let us begin by the lower-bound. For every $x \in D$ and $\delta > 0$,

$$\begin{aligned} \mathbb{P}_x(\tau_D(\xi^h) > t_0) &\leq \mathbb{P}_x(\{\tau_{D_\delta}(\xi) > t_0\} \cup \{\sup_{t \in [0, t_0]} |\xi^h - \xi| \geq \delta\}) \\ &\leq \mathbb{P}_x(\tau_{D_\delta}(\xi) > t_0) + \mathbb{P}_x\left(\sup_{t \in [0, t_0]} |\xi^h - \xi| \geq \delta\right). \end{aligned}$$

By Lemma 7.4(i) and Lemma 7.3, one easily deduces that for h small enough,

$$a := \sup_{x \in D} \mathbb{P}_x(\tau_D(\xi^h) > t_0) < 1.$$

Reminding that ξ^h is stepwise constant, note that t_0 can be replaced by $t_0^h = \lfloor t_0/h \rfloor h$ in the previous inequality. Then,

$$\begin{aligned} \mathbb{P}_x(\tau_D(\xi^h) > kt_0^h) &= \mathbb{P}_x(\tau_D(\xi^h) > kt_0^h | \tau_D(\xi^h) > (k-1)t_0^h) \mathbb{P}_x(\tau_D(\xi^h) > (k-1)t_0^h) \\ &\leq a \mathbb{P}_x(\tau_D(\xi^h) > (k-1)t_0^h). \end{aligned}$$

By an induction, it follows that there exists $h_0 \in (0, t_0)$ such that for every $t > 0$ and $h \in (0, h_0)$,

$$\mathbb{P}_x(\tau_D(\xi^h) > t) \leq a^{\lfloor t/t_0^h \rfloor} \leq C \exp(-\lambda_{\min} t)$$

with $\lambda_{\min} = -\log(a)/(t_0 - h_0)$. Since the right-hand side does not depend on x , one deduces that $\mathbb{P}_{\mu_h}(\tau_D(\xi^h) > t) \leq C \exp(-\lambda_{\min} t)$. Since this inequality holds for every t (with C not depending on t), it follows from (43) that $\lambda_{\min} < \lambda_h$ (using that $C(t) \leq e$). This yields the lower-bound. As concerns the upper-bound, one first deduces from Lemmas 7.3 and 7.4(ii) that there exists a compact set \mathcal{K} of D such that there exist some positive t_0, t_1, h_0 and ε , such that for every $x \in D$, and $h \in (0, h_0)$, $\mathbb{P}_x(\xi_{t_0}^h \in \mathcal{K}, \tau_D(\xi^h) > t_0) > 0$ and

$$\inf_{y \in \mathcal{K}} \mathbb{P}_y(\xi_{t_1}^h \in \mathcal{K} \text{ and } \forall t \in [0, t_1], \xi_t^h \in D) \geq \varepsilon. \quad (44)$$

Once again, in order to manage the fact that ξ^h is stepwise constant, one sets $t_0^h = \lfloor t_0/h \rfloor h$ and $t_1^h = \lfloor t_1/h \rfloor h$. We have

$$\mathbb{P}_x(\tau_D(\xi^h) > t) \geq \mathbb{P}_x(\xi_{t_0^h + \ell t_1^h}^h \in \mathcal{K}, \ell \in \{0, \dots, N_h(t)\}, \tau_D(\xi^h) > t)$$

with $N_h(t) = \inf\{\ell, t_0^h + \ell t_1^h > t\}$. Using the Markov property and an induction, it follows from (44) that, for h small enough, for every $t > t_0^h$, for every $x \in D$,

$$\begin{aligned} \mathbb{P}_x(\tau_D(\xi^h) > t) &\geq \mathbb{P}_x(\xi_{t_0}^h \in \mathcal{K}, \tau_D(\xi^h) > t_0) \varepsilon^{N_h(t)} \\ &\geq C \mathbb{P}_x(\xi_{t_0}^h \in \mathcal{K}, \tau_D(\xi^h) > t_0) \exp\left(\frac{\log(\varepsilon)}{2t_1} t\right), \end{aligned}$$

where in the last inequality, we used that for h small enough

$$N_h(t) = \left\lfloor \frac{t - t_0^h}{t_1^h} \right\rfloor + 1 \geq \frac{t}{t_1^h} - \frac{t_0^h}{t_1^h} \geq \frac{t}{2t_1} - \delta, \quad \delta > 0.$$

Set $\lambda_{\max} = -\frac{\log(\varepsilon)}{2t_1}$. Since $\mathbb{P}_x(\xi_{t_0}^h \in \mathcal{K}, \tau_D(\xi^h) > t_0) > 0$ for every $x \in D$, it follows from what precedes that for every $t > t_0$,

$$\mathbb{P}_{\mu_h}(\tau_D(\xi^h) > t) \geq c \exp(-\lambda_{\max} t),$$

where c is a positive constant (which does not depend on t). By (43), one can conclude that $\lambda_h < \lambda_{\max}$ (using that $C(t) \geq 1$).

Step 2 (Tightness of (μ_h^*)): We show that $(\mu_h^*)_{h \in (0, h_0]}$ is tight on D . For $\delta > 0$, set $B_\delta := \{x \in D, d(x, \partial D) \leq \delta\}$. We need to prove that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for every $h \in (0, h_0)$, $\mu_h^*(B_{\delta_\varepsilon}) \leq \varepsilon$. First, by (43) (applied with $t = 1$) and Step 1,

$$\mu_h^*(B_\delta) \leq \mathbb{E}_{\mu_h^*} \left[\mathbf{1}_{\{\xi_1^h \in B_\delta\} \cap \{\tau_D(\xi^h) > 1\}} \right] \leq \mathbb{P}_{\mu_h^*}(\xi_1^h \in B_\delta).$$

But, under the ellipticity condition, $\mathcal{L}(\xi_1^h | \xi_0^h = x)$ admits a density $p_1^h(x, \cdot)$ w.r.t. the Lebesgue measure λ_d and by [32, Theorem 2.1] (for instance),

$$\sup_{x, x'} p_1^h(x, x') \leq C$$

where C does not depend on h . As a consequence, for every $x \in D$,

$$\mathbb{P}_{\mu_h^*}(\xi_1^h \in B_\delta) \leq \int p_1^h(x, x') \lambda(dx') \leq C \lambda_d(B_\delta).$$

The tightness follows.

Step 3 (Identification of the limit): Let $(\mu_{h_n}^*)_n$ denote a convergent subsequence to μ . One wants to show that $\mu = \mu^*$ (where μ^* stands for the unique QSD of the diffusion killed when leaving D). To this end, it remains to show that there exists $\lambda \in \mathbb{R}_+^*$ such that for any positive t and any bounded continuous function $f : D \rightarrow \mathbb{R}$,

$$\mathbb{E}_\mu[f(X_t) \mathbf{1}_{\tau_D(X) > t}] = \exp(-\lambda t) \mu(f). \quad (45)$$

With standard arguments, one can check that this is enough to prove this statement when f is \mathcal{C}^2 with compact support in D .

Let us consider Equation (43). First, up to a potential extraction, one can deduce from Step 1 that $\lambda_{h_n} \rightarrow \lambda \in \mathbb{R}$. By the weak convergence of $(\mu_{h_n}^*)_n$, it follows that the right-hand side of (43) satisfies:

$$C_h(t) \exp(-\lambda_h t) \mu_{h_n}^*(f) = \exp(-\lambda t) \mu(f). \quad (46)$$

Second, by [27] (Theorem 2.4 and remarks of Section 6 therein about the hypothesis of this

theorem), there exists a constant $C_1(t)$ such that for all $x \in D$,

$$|\mathbb{E}_x[f(\xi_t^h)1_{\tau_D(\xi^h)>t}] - \mathbb{E}_x[f(\xi_t)1_{\tau_D(\xi)>t}]| \leq C_f C_1(t)\sqrt{h}.$$

It follows that

$$\sup_{x \in D} |\mathbb{E}_x[f(\xi_t^h)1_{\tau_D(\xi^h)>t}] - \mathbb{E}_x[f(\xi_t)1_{\tau_D(\xi)>t}]| \xrightarrow{h \rightarrow 0} 0.$$

As a consequence,

$$\mathbb{E}_{\mu_h^*}[f(\xi_t^h)1_{\tau_D(\xi^h)>t}] - \mathbb{E}_{\mu_h^*}[f(\xi_t)1_{\tau_D(\xi)>t}] \xrightarrow{h \rightarrow 0} 0.$$

Now, by a dominated convergence argument (using that $\sup_{s \in [0, t]} |\xi_s^x - \xi_s^{x_0}| \rightarrow 0$ in probability when $x \rightarrow x_0$), one remarks that $x \mapsto \mathbb{E}_x[f(\xi_t)1_{\tau_D(\xi)>t}]$ is (bounded) continuous on D . As a consequence, $\mathbb{E}_{\mu_{h_n}^*}[f(\xi_t)1_{\tau_D(\xi)>t}] \rightarrow \mathbb{E}_\mu[f(\xi_t)1_{\tau_D(\xi)>t}]$ so that

$$\mathbb{E}_{\mu_{h_n}^*}[f(\xi_t^{h_n})1_{\tau_D(\xi^{h_n})>t}] \xrightarrow{h \rightarrow 0} \mathbb{E}_\mu[f(\xi_t)1_{\tau_D(\xi)>t}].$$

Equality (45) follows by plugging the above convergence and (46) into (43).

8. EXTENSIONS

8.1. Non-compact case: Processes coming down infinity. In the main results, we chose to restrain our purpose to compact spaces. When \mathcal{E} is only locally compact, the results of this paper could be extended to the class of processes which *come down from infinity* (CDFI), *i.e.* which have the ability to come back to a compact set in a bounded time with a uniformly lower-bounded probability (for more details, see *e.g.* [3, 15, 21]). First, note that (CDFI)-condition is a usual and sharp assumption which ensures uniqueness of the QSD in the locally compact setting. Second, the (CDFI)-condition is in particular ensured if \mathcal{E} is locally compact and if there exists a map V , such that $\{V \leq C\}$ is compact for every $C > 0$ and $M := \sup_{x \in \mathcal{E}} KV(x)$ is finite. Also, let us remark that $\bar{\mathcal{P}} := \{\mu \in \mathcal{P}(\mathcal{E}), \mu(V) \leq M\}$ is compact for the weak convergence topology (owing to the coercivity condition on V) and is invariant under the action of the kernel K . Then, on this subspace $\bar{\mathcal{P}}$, the main arguments of the proof of the main results could be adapted to obtain the convergence of the algorithm.

8.2. Non-compact space: the minimal QSD. In Theorem 3.4, we have seen that when a process admits several QSDs, our algorithm may select all of its QSDs with positive probabilities. When (CDFI)-condition fails in the non-compact setting (think for instance to the real Ornstein-Uhlenbeck process killed when leaving \mathbb{R}_+), uniqueness generally fails and one can not expect the algorithm to select only one QSD. However, if the aim is to approximate the so-called minimal QSD, namely the one associated to the minimal eigenvalue and appearing in the *Yaglom limit*, then, one can use a compact approximation method in the spirit of [40]. More precisely, consider for instance a diffusion process $(\xi_t)_{t \geq 0}$ on \mathbb{R}^d killed when leaving an unbounded domain D and denote by μ^* the related minimal QSD (when exists). Let also $(K_n)_{n \geq 1}$ be an increasing sequence of compact spaces such that $\bigcup_{n \geq 1} K_n = D$. Then, under some non-degeneracy assumptions (see *e.g.* Theorem 3.9), the QSD μ_n^* related to K_n is unique for every n and by [40, Theorem 3.1], $\lim_{n \rightarrow +\infty} \mu_n^* = \mu^*$. Then, using our algorithm for an approximation of μ_n^* would lead to an approximation of μ^* for n large enough.

8.3. Continuous-time algorithm. In view of the approximation of the QSD of a diffusion process $(\xi_t)_{t \geq 0}$ on a bounded domain D satisfying the assumptions of Theorem 3.9, it may be of interest to study the convergence of a continuous-time equivalent of our algorithm (instead of considering an Euler scheme with constant step). Of course, without discretization, such a problem is mainly theoretical but it is worth noting that the difficulties mentioned below should be very similar if one investigated an algorithm with decreasing step (on this topic, see also Remark 3.11). The continuous-time algorithm is defined as follows:

- let $x \in D$ and $(X_t^1)_{t \geq 0}$ be as $(\xi_t)_{t \geq 0}$ with initial condition $\xi_0 = x$;
- let $\tau^1 = \inf\{t \geq 0 \mid X_t^1 \notin D\}$, for all $t < \tau^1$ we set $X_t = X_t^1$;
- Let $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$ be the occupation measure of X ;
- Let U be random variable distributed as μ_{τ^1} (conditionally on \mathcal{F}_{τ^1} , defined classically);
- we set $X_{\tau^1} = U$;
- the process evolves then as before starting from U .

We denote by $(\tau^k)_{k \geq 1}$ the sequence of jumping times. At the n^{th} time, it jumps uniformly over all the past and not only on $[\tau^n, \tau^{n+1}]$. This sequence of stopping times is increasing and almost surely converges to some $\tau^\infty \in \mathbb{R}_+^* \cup \{+\infty\}$. The process $(X_t)_{t \geq 0}$ is well defined until the time τ^∞ . It is not trivial that $\tau^\infty = \infty$ because the process $(X_t)_{t \geq 0}$ can be closer and closer to the boundary and the inter-jumping times then becoming shorter and shorter. Nonetheless, we have

Lemma 8.1 (Non-explosion of the continuous-time algorithm). *Under the assumptions of Theorem 3.9, we have $\tau^\infty = +\infty$ a.s.*

The proof is given below. This type of problem is reminiscent of the Fleming-Viot particle system [10, 11, 41]. However, the comparison stops here because as our procedure is not "Markovian", their proofs can not be adapted.

Proof of Lemma 8.1. Let $x \in D$ be the starting point of $(X_t)_{t \geq 0}$. Fix $\varepsilon > \delta > 0$ such that $d(x, \partial D) < \varepsilon$. Let $(B_t)_{t \geq 0}$ a Brownian motion, z in the ball $B(x, \delta)$ of center x and radius δ . We let $(\xi_t^z)_{t \geq 0}$ be the solution of

$$d\xi_t^z = b(\xi_t^z)dt + \sigma(\xi_t^z)dB_t, \quad \xi_0 = z,$$

and

$$\tau_{\varepsilon, \delta, x} = \inf_{z \in B(x, \delta)} \inf\{t \geq 0 \mid \xi_t^z \notin B(x, \varepsilon)\}.$$

The variable $\tau_{\varepsilon, \delta, x}$ is almost-surely positive. On $\{\tau^\infty < +\infty\}$, we have, for every $t \in [\tau^1, \tau^\infty)$,

$$\mu_t(B(x, \delta)) \geq \frac{\tau^1}{\tau^\infty}.$$

As a consequence on $\{\tau^\infty < +\infty\}$, the process $(X_t)_{t \geq 0}$ jumps infinitely often in $B(x, \delta)$. But if it starts from a point $z \in B(x, \delta)$, its absorption time can be bounded from below by a random variable σ (independent from the past) such that σ has the same law as $\tau_{\varepsilon, \delta, x}$. In application, we have

$$\tau^\infty \geq \sum_{n \geq 1} \sigma_n,$$

on $\{\tau^\infty < +\infty\}$, where $(\sigma_n)_{n \geq 1}$ is a sequence of i.i.d. random variable distributed as $\tau_{\varepsilon, \delta, x}$. As they are positive the strong law of large numbers ensures that $\sum_{n \geq 0} \sigma_n = +\infty$ almost surely and then $\mathbb{P}(\tau^\infty < +\infty) = 0$. \square

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