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JORDAN-PRODUCT COMMUTING NONLINEAR MAPS WITH λ -ALUTHGE TRANSFORM

F. CHABBABI AND M. MBEKHTA

ABSTRACT. Let H and K be two complex Hilbert spaces and $\mathcal{B}(H)$ be the algebra of bounded linear operators from H into itself.

The main purpose in this paper is to obtain a characterization of bijective maps $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ satisfying the following condition

$$\Delta_\lambda(\Phi(A) \circ \Phi(B)) = \Phi(\Delta_\lambda(A \circ B)) \text{ for all } A, B \in \mathcal{B}(H),$$

where $\Delta_\lambda(T)$ stands the λ -Aluthge transform of the operator $T \in \mathcal{B}(H)$ and $A \circ B = \frac{1}{2}(AB + BA)$ is the Jordan product of A and B . We prove that a bijective map Φ satisfies the above condition, if and only if there exists a unitary operator $U : H \rightarrow K$, such that Φ has the form $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$.

1. INTRODUCTION

Let H and K be two complex Hilbert spaces and $\mathcal{B}(H, K)$ be the Banach space of all bounded linear operators from H into K . In the case $K = H$, $\mathcal{B}(H, H)$ is simply denoted by $\mathcal{B}(H)$ and is a Banach algebra.

For an arbitrary operator $T \in \mathcal{B}(H, K)$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the null subspace and the operator adjoint of T respectively. For $T \in \mathcal{B}(H)$, the spectrum of T is denoted by $\sigma(T)$.

An operator $T \in \mathcal{B}(H, K)$ is a *partial isometry* when T^*T is an orthogonal projection (or, equivalently $TT^*T = T$). In particular T is an *isometry* if $T^*T = I_H$, and *unitary* if T is a surjective isometry.

As usually, for $T \in \mathcal{B}(H)$ we denote the module of T by $|T| = (T^*T)^{1/2}$ and we shall always write, without further mention, $T = V|T|$ to be the unique polar decomposition of T , where V is the appropriate partial isometry satisfying $\mathcal{N}(V) = \mathcal{N}(T)$. The Aluthge transform was introduced in [1] as

$$\Delta(T) = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(H),$$

to extend some properties of hyponormal operators. Later, in [13], Okubo introduced a more general notion called λ -Aluthge transform which has also been studied in detail.

For $\lambda \in [0, 1]$, the λ -Aluthge transform is defined by,

$$\Delta_\lambda(T) = |T|^\lambda V |T|^{1-\lambda}, \quad T \in \mathcal{B}(H).$$

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Notice that $\Delta_0(T) = V|T| = T$, and $\Delta_1(T) = |T|V$ which is known as Duggal's transform. It has since been studied in many different contexts and considered by a number of authors (see for instance, [1, 2, 3, 4, 11, 10, 9] and some of the references there). One of the interests of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example,

$$(1.1) \quad \sigma_*(\Delta_\lambda(T)) = \sigma_*(T), \text{ for every } T \in \mathcal{B}(H),$$

where σ_* runs over a large family of spectra. See [9, Theorems 1.3, 1.5]. Another important property is that $\text{Lat}(T)$, the lattice of T -invariant subspaces of H , is nontrivial if and only if $\text{Lat}(\Delta(T))$ is nontrivial (see [9, Theorem 1.15]).

In [5], the authors described the linear bijective mappings on von Neumann algebras which commute with the λ -Aluthge transform

$$\Delta_\lambda(\Phi(T)) = \Phi(\Delta_\lambda(T)) \text{ for every } T \in \mathcal{B}(H).$$

In [6] the first author gives a complete description of the bijective maps $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ which satisfies following condition,

$$(1.2) \quad \Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \text{ for every } A, B \in \mathcal{B}(H),$$

for some $\lambda \in]0, 1[$.

In this paper we will be concerned with the Jordan product commuting maps with the λ -Aluthge transform in the following sense,

$$(1.3) \quad \Delta_\lambda(\Phi(A) \circ \Phi(B)) = \Phi(\Delta_\lambda(A \circ B)) \text{ for all } A, B \in \mathcal{B}(H),$$

where $A \circ B = \frac{1}{2}(AB + BA)$ is the Jordan product of A and B .

Our main result gives a complete description of the bijective maps $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ which satisfies Condition (1.3), the precise theorem being as follows

Theorem 1.1. Let H and K be two complex Hilbert space, such that H is of dimension greater than 2. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map. Then Φ satisfies (1.3) for some $\lambda \in]0, 1[$ if and only if there exists a unitary operator $U : H \rightarrow K$ such that

$$\Phi(A) = UAU^* \quad \text{for every } A \in \mathcal{B}(H).$$

The paper is organized as follows. In the second section, we establish some useful results on the Aluthge transform. These results are needed for proving our main theorem in the section 3.

2. SOME PROPERTIES OF THE ALUTHGE TRANSFORM

In this section we establish some results, properties of the Aluthge transform. These results are necessary for the proof of the main theorem.

We first recall some basic notions that are used in the sequel. An operator $T \in \mathcal{B}(H)$ is normal if $T^*T = TT^*$, and is quasi-normal, if it commutes with T^*T (i.e. $TT^*T = T^*T^2$), or equivalently $|T|$ and V commutes ($T = V|T|$ a polar decomposition of T). In finite dimensional spaces every quasi-normal operator is normal. It is easy to see that

if T is quasi-normal, then T^2 is also quasi-normal, but the converse is false as shown by nonzero nilpotent operators .

Also, it is well known that the quasi-normal operators are exactly the fixed points of Δ_λ (see [9, Proposition 1.10]). That is,

$$(2.1) \quad T \text{ quasi-normal} \iff \Delta_\lambda(T) = T.$$

For nonzero $x, y \in H$, we denote by $x \otimes y$ the rank one operator defined by

$$(x \otimes y)u = \langle u, y \rangle x \text{ for } u \in H.$$

It is easy to show that every rank one operator has the previous form and that $x \otimes y$ is an orthogonal projection, if and only if $x = y$ and $\|x\| = 1$.

We start with the following proposition. It is found in [6]. To facilitate the reading we include the proof.

Proposition 2.1. ([6]) Let $x, y \in H$ be nonzero vectors. We have

$$\Delta_\lambda(x \otimes y) = \frac{\langle x, y \rangle}{\|y\|^2} (y \otimes y) \text{ for every } \lambda \in]0, 1[.$$

Proof. Denote $T = x \otimes y$, then

$$T^*T = |T|^2 = \|x\|^2 (y \otimes y) = \left(\frac{\|x\|}{\|y\|} (y \otimes y) \right)^2 \text{ and } |T| = \frac{\|x\|}{\|y\|} (y \otimes y).$$

It follows that

$$|T|^2 = \|x\| \|y\| |T| \text{ and } |T|^\gamma = (\|x\| \|y\|)^{\gamma-1} |T| \text{ for every } \gamma > 0.$$

Now, let $T = V|T|$ be the polar decomposition of T . We have

$$\begin{aligned} \Delta_\lambda(x \otimes y) &= \Delta_\lambda(T) = |T|^\lambda V |T|^{1-\lambda} \\ &= (\|x\| \|y\|)^{\lambda-1} (\|x\| \|y\|)^{-\lambda} |T| V |T| \\ &= \frac{1}{\|x\| \|y\|} |T| T = \frac{1}{\|y\|^2} (y \otimes y) \circ (x \otimes y) \\ &= \frac{\langle x, y \rangle}{\|y\|^2} (y \otimes y). \end{aligned}$$

Thus , $\Delta_\lambda(x \otimes y) = \frac{\langle x, y \rangle}{\|y\|^2} (y \otimes y)$ as desired □

Proposition 2.2. Let $A \in \mathcal{B}(H)$ and $P = x \otimes x$ be a rank one projection on H . Then

$$\Delta_\lambda(A \circ P) = P \text{ if and only if } PA = P.$$

Proof. The "if" part follows directly from the previous proposition. We show the "only if" part. Suppose that $\Delta_\lambda(A \circ P) = P$. First, it is easy to see the following

$$PA = P \iff A^*x = x.$$

So, to complete the proof of the proposition, it suffices to show that $A^*x = x$.

Put $T = A \circ P$. Then $T = \frac{1}{2}(Ax \otimes x + x \otimes A^*x)$, and thus

$$(2.2) \quad T^2 = \frac{1}{4}(\langle Ax, x \rangle Ax \otimes x + Ax \otimes A^*x + \langle A^2x, x \rangle x \otimes x + \langle Ax, x \rangle x \otimes A^*x).$$

In the other hand, we have

$$\sigma(T) = \sigma(\Delta_\lambda(T)) = \sigma(P) = \{0, 1\} \quad \text{and thus } \sigma(T^2) = \{0, 1\}.$$

Since the rank of T is at most 2, $\text{tr}(T^2) = \text{tr}(T) = 1$. We also have $\text{tr}(T) = \langle Ax, x \rangle$ and $\text{tr}(T^2) = \frac{1}{2}(\langle A^2x, x \rangle + \langle Ax, x \rangle)$. Therefore

$$\langle Ax, x \rangle = \langle A^2x, x \rangle = 1.$$

It follows that $Tx = \frac{1}{2}(Ax + x)$ and $T^*x = \frac{1}{2}(A^*x + x)$. Hence

$$\langle Tx, x \rangle = 1 \quad \text{and} \quad \langle T^*x, x \rangle = 1.$$

From (2.2) we get

$$(2.3) \quad T^2 = \frac{1}{4}(Ax + x) \otimes (A^*x + x) = Tx \otimes T^*x.$$

Let $T = U|T|$ be the polar decomposition of T . It holds

$$\begin{aligned} T^2 &= U|T|U|T| = U|T|^{1-\lambda}(|T|^\lambda U|T|^{1-\lambda})|T|^\lambda \\ &= U|T|^{1-\lambda}\Delta_\lambda(T)|T|^\lambda = U|T|^{1-\lambda}(x \otimes x)|T|^\lambda \\ &= U|T|^{1-\lambda}x \otimes |T|^\lambda x. \end{aligned}$$

From (2.3) we get

$$(2.4) \quad Tx \otimes T^*x = U|T|^{1-\lambda}x \otimes |T|^\lambda x.$$

Since $\langle T^*x, x \rangle = 1$, we get that

$$Tx = \langle x, |T|^\lambda x \rangle U|T|^{1-\lambda}x.$$

Thus

$$|T|^\lambda |T|x = |T|^\lambda U^*Tx = \langle x, |T|^\lambda x \rangle U^*U|T|x = \langle x, |T|^\lambda x \rangle |T|x.$$

By function calculus, we obtain

$$|T||T|x = (\langle x, |T|^\lambda x \rangle)^{1/\lambda} |T|x.$$

It follows that

$$\langle |T|^2x, x \rangle = |||T|x|||^2 = (\langle x, |T|^\lambda x \rangle)^{1/\lambda} \langle |T|x, x \rangle.$$

Now, by Holder-McCarthy inequality (see [7] page 123), we deduce

$$|||T|x|||^2 = (\langle x, |T|^\lambda x \rangle)^{1/\lambda} \langle |T|x, x \rangle \leq (\langle |T|x, x \rangle)^2 \leq |||T|x|||^2.$$

Hence $|||T|x||| = \langle |T|x, x \rangle$. Therefore there exists $\alpha \in \mathbb{C}$ such that $|T|x = \alpha x$.

The last equality and Equation (2.4) imply

$$Tx \otimes T^*x = \alpha Ux \otimes x.$$

Thus, $T^*x = \beta x$ for some $\beta \in \mathbb{C}$. Since, $\langle T^*x, x \rangle = 1$, it holds $\beta = 1$. On the other hand $T^*x = \frac{1}{2}(A^*x + x)$. Then $\frac{1}{2}(A^*x + x) = x$. Finally we conclude that $A^*x = x$ and hence $PA = P$. The proof is completed. \square

The following lemma gives a property of rank one projections

Proposition 2.3. Let $A \in \mathcal{B}(H)$ and $P = x \otimes x$ be a rank one projection on H . Then

$$\Delta_\lambda(A \circ P) = A \text{ if and only if } A = \alpha P \text{ for some } \alpha \in \mathbb{C}.$$

Proof. The "if" part is obvious. We show the "only if" part. Put $T = A \circ P$. Then

$$T = \frac{1}{2}(Ax \otimes x + x \otimes A^*x) \text{ and } T^* = \frac{1}{2}(x \otimes Ax + A^*x \otimes x).$$

By assumption, $A = \Delta_\lambda(T)$. Since $\mathcal{R}(\Delta_\lambda(T)), \mathcal{R}(\Delta_\lambda(T)^*) \subseteq \mathcal{R}(T^*)$, we have

$$\mathcal{R}(A), \mathcal{R}(A^*) \subseteq \mathcal{R}(T^*) \subseteq H_0 := \text{span}\{x, A^*x\}.$$

Note that H_0 is an invariant subspace of A and A^* .

Claim: there exists $\delta \in \mathbb{C}$ such that $A^*x = \delta x$. In this case, A and A^* are rank one operators. Moreover, their ranges $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are generated by x . It is easy to deduce from this, that $A = \bar{\delta}P$.

We prove the claim by contradiction. Assume on contrary that A^*x and x are linearly independent. Let $\{x, e\}$ be an orthonormal basis of H_0 . We can choose $e \in H_0$ such that $\langle e, Ax \rangle \geq 0$. In this basis A has the following form

$$(2.5) \quad A = x \otimes A^*x + e \otimes A^*e.$$

Let us consider the following cases :

Case 1 : $A^*e = 0$. In this case, we show A^*x and x are linearly dependent, which is a contradiction.

From (2.5), we have $A = x \otimes A^*x$. Hence

$$T = P \circ A = 1/2(x \otimes A^*x + \langle Ax, x \rangle x \otimes x) = 1/2x \otimes (A^*x + \langle A^*x, x \rangle x).$$

Since $\mathcal{R}(A) \subseteq \mathcal{R}(T^*)$, then x and $A^*x + \langle A^*x, x \rangle x$ are linearly dependent it follows that x and A^*x are also linearly dependent, which is a contradiction.

Case 2 : $A^*e \neq 0$. In this case, we show that $A = 2T$ which is a contradiction with the fact that $\Delta_\lambda(T) = \Delta_\lambda(A/2) = A$, and

$$\|A\| = \|\Delta_\lambda(A/2)\| \leq \frac{\|A\|}{2}.$$

Thus $A = 0$, which is not possible.

Note that since $A = \Delta_\lambda(T)$ it holds

$$\|A\| = \|\Delta_\lambda(T)\| \leq \|T\| = \left\| \frac{1}{2}(Ax \otimes x + x \otimes A^*x) \right\| \leq \frac{1}{2}(\|Ax\| + \|A^*x\|) \leq \|A\|.$$

Hence

$$(2.6) \quad \|A\| = \|Ax\| = \|A^*x\|.$$

Now, by (2.5), we have

$$A = 2T \Leftrightarrow e \otimes A^*e = Ax \otimes x \Leftrightarrow \begin{cases} Ax = ae; \\ A^*e = \bar{a}x, \end{cases} \text{ for some } a \in \mathbb{C}.$$

So, we just need to prove the equivalence. First, we show that

$$A^*e = \|A^*e\|x.$$

Indeed by assumption, we get

$$\text{tr}(A) = \text{tr}(\Delta_\lambda(T)) = \text{tr}(T) = \langle Ax, x \rangle.$$

By (2.5), we get

$$\text{tr}(A) = \langle Ax, x \rangle + \langle Ae, e \rangle.$$

Thus $\langle Ae, e \rangle = 0$. Since $A^*e \in H_0$ and $\langle e, Ax \rangle \geq 0$, it follows that

$$(2.7) \quad A^*e = \langle A^*e, x \rangle x = \|A^*e\|x.$$

Second, we show

$$Ax = \|A^*e\|e.$$

Indeed From (2.5), we get

$$AA^* = x \otimes AA^*x + e \otimes AA^*e.$$

Thus

$$AA^*x = \|A^*x\|^2x + \langle x, AA^*e \rangle e,$$

moreover, by (2.6) it follows that

$$\|AA^*x\|^2 = \|A^*x\|^4 + |\langle x, AA^*e \rangle|^2 \leq \|AA^*\|^2 = \|A\|^4 = \|A^*x\|^4.$$

Therefore

$$\langle x, AA^*e \rangle = 0.$$

This, together with the fact $A^*e = \|A^*e\|x$, show that

$$AA^*x = \|A\|^2x \quad \text{and} \quad AA^*e = \|A^*e\|^2e = \|A^*e\|Ax.$$

It follows that

$$(2.8) \quad Ax = \|A^*e\|e.$$

This completes the proof. \square

Remark 2.1. If $T = V|T| \in \mathcal{B}(H)$ is the polar decomposition of T , then V is unitary if and only if T and T^* are one-to-one.

Lemma 2.1. Let $T \in \mathcal{B}(H)$ and $\lambda \in]0, 1[$. Suppose that T and T^* are one-to-one. Then,

$$\Delta_\lambda(T^2) = T \implies T^2 = T^*.$$

Proof. Let us consider $T^2 = U|T^2|$ the polar decomposition of T^2 . Since T and T^* are injective, then T^2 and $(T^2)^*$ are also injective. Thus U is unitary operator. We have

$$\begin{aligned}\Delta_\lambda(T^2) = |T^2|^\lambda U|T^2|^{1-\lambda} = T &\implies |T^2|^\lambda U|T^2|^{1-\lambda}|T^2|^\lambda = T|T^2|^\lambda \\ &\implies |T^2|^\lambda U|T^2| = T|T^2|^\lambda \\ &\implies |T^2|^\lambda T^2 = T|T^2|^\lambda.\end{aligned}$$

On the other hand, $T\Delta_\lambda(T^2) = T^2$, hence

$$T|T^2|^\lambda U|T^2|^{1-\lambda} = T^2 = U|T^2| = U|T^2|^\lambda |T^2|^{1-\lambda}$$

Since T^2 is injective, we get

$$T|T^2|^\lambda U = U|T^2|^\lambda.$$

Thus,

$$|T^2|^\lambda T^2 U = U|T^2|^\lambda \quad \text{and} \quad |T^2|^\lambda T^2 = U|T^2|^\lambda U^* \geq 0.$$

It follows that

$$\begin{aligned}|T^2|^\lambda T^2 &= T^{*2}|T^2|^\lambda = |T^2|U^*|T^2|^\lambda \\ &= |T^2|^\lambda |T^2|^{1-\lambda} U^* |T^2|^\lambda \\ &= |T^2|^\lambda \Delta_\lambda(T^2)^* \\ &= |T^2|^\lambda T^*.\end{aligned}$$

Finally we obtain $T^2 = T^*$, as claimed. \square

Lemma 2.2. Let $S \in \mathcal{B}(H)$ and $\lambda \in]0, 1[$. Suppose that S and S^* are one-to-one. Then,

$$\Delta_\lambda(S) = S^* \implies S = S^*.$$

Proof. Let us consider $S = U|S|$ the polar decomposition of S . Since S and S^* are injective, U is unitary operator. We have,

$$\Delta_\lambda(S)\Delta_\lambda(S)^* = S^*S = |S|^2 = |S|^\lambda |S|^{1-\lambda} |S|^{1-\lambda} |S|^\lambda.$$

By a simple calculation, we deduce

$$U|S|^{2(1-\lambda)} = |S|^{2(1-\lambda)}U$$

By the continuous functional calculus, we obtain S quasi-normal.

Consequently, $S = \Delta_\lambda(S) = S^*$, as desired. \square

Remark 2.2. It is well known that T is quasi-normal does not imply that his adjoint T^* is always quasi-normal.

For example : take the right Shift operator S on a separable Hilbert space H , with orthonormal bas is $(e_n)_n$. We have

$$S^*S = I \quad \text{and} \quad SS^* = I - P_1$$

where P_1 is the projection onto the span of the first vector e_1 . Hence

$$S^*SS^* = S^* \quad \text{and} \quad S(S^*)^2 = S^* - P_1S^* \neq S^*.$$

This shows that S^* is not quasi-normal.

Now, we replace the condition S and S^* are injective in preceding lemma by S is quasi-normal. We have the following result

Lemma 2.3. Let $S \in \mathcal{B}(H)$ and $\lambda \in]0, 1[$. Suppose that S is quasi-normal. Then

$$\Delta_\lambda(S^*) = S \implies S = S^*.$$

Proof. Let us consider the polar decomposition of S and S^* , $S = U|S|$ and $S^* = U^*|S^*|$ respectively. It is well known that $|S^*|^p = U|S|^p U^*$ for all $p > 0$.

Now, since S is quasi-normal, $U|S|^p = |S|^p U$ and $U^*|S|^p = |S|^p U^*$ for every $p > 0$. Hence, we have the following equalities

$$\begin{aligned} S = \Delta_\lambda(S^*) &= |S^*|^\lambda U^* |S^*|^{1-\lambda} \\ &= U|S|^\lambda U^* U|S|^{1-\lambda} U^* \\ &= U(U^*)^2 |S|. \end{aligned}$$

It follows that

$$(2.9) \quad U(U^*)^2 |S| = U|S|.$$

After multiplying this equality by U^* , we get

$$(U^*)^2 |S| = |S|.$$

Hence

$$(U^*)^2 |S| = |S| U^2 = U|S|U = |S|.$$

Thus

$$S = U|S| = |S|U = U^* U|S|U = U^* |S| = |S|U^* = S^*.$$

Thus $S = S^*$ and the lemma is proved. \square

Combining Lemmas 2.3 and 2.1, we obtain the following corollary which plays an important role in the proof of $\Phi(I) = I$.

Corollary 2.1. Let $T \in \mathcal{B}(H)$ and $\lambda \in]0, 1[$. Suppose that T and T^* are one-to-one. Then,

$$\Delta_\lambda(T^2) = T \text{ if and only if } T = I.$$

The next lemma, was established by S. Garcia in the case $\lambda = \frac{1}{2}$, see [8]. It identifies the kernel of the λ -Aluthge transform as the set of all operators which are nilpotent of order two. We need this lemma later.

Lemma 2.4. Let $\lambda \in]0, 1]$ and $T \in \mathcal{B}(H)$. Then

$$\Delta_\lambda(T) = 0 \text{ if and only if } T^2 = 0.$$

Proof. Let $T = V|T|$ be the polar decomposition of T .

If $\Delta_\lambda(T) = 0$, then we have

$$T^2 = V|T|V|T| = V|T|^{1-\lambda}\Delta_\lambda(T)|T|^\lambda = 0, \text{ and thus } T^2 = 0.$$

Conversely, suppose that $T^2 = 0$. Then $V|T|V|T| = T^2 = 0$. Hence $V^*V|T|V|T| = 0$, which implies $|T|V|T| = 0$, since V^*V is the projection onto $\overline{\mathcal{R}(|T|)}$. Whence $|T|V$ vanishes on $\overline{\mathcal{R}(|T|)}$. Furthermore $\mathcal{N}(V) = \mathcal{N}(|T|)$, from which one concludes $|T|V$ vanishes on $\mathcal{N}(|T|)$. Consequently, $|T|V = 0$. Thus $\Delta_\lambda(T) = |T|^\lambda V|T|^{1-\lambda} = 0$. \square

3. PROOF OF THE MAIN THEOREM

An idempotent self adjoint operator $P \in \mathcal{B}(H)$ is said to be an orthogonal projection. Clearly quasi-normal idempotents are orthogonal projections. Two projections $P, Q \in \mathcal{B}(H)$ are said to be orthogonal if

$$PQ = QP = 0,$$

in this case, we denote $P \perp Q$. A partial ordering between orthogonal projections is defined as follows,

$$Q \leq P \text{ if } PQ = QP = Q.$$

Remark 3.1. We have

$$P \perp Q \iff P + Q \text{ orthogonal projection,}$$

and

$$Q \leq P \iff PQ + QP = 2Q \iff P \circ Q = Q.$$

Proposition 3.1. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map satisfying (1.3). Then

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(I) = I.$$

Proof. Since Φ is onto, let $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. By (1.3) we have

$$\Phi(0) = \Delta_\lambda(\Phi(0) \circ \Phi(A)) = \Delta_\lambda(0) = 0.$$

This proves $\Phi(0) = 0$.

Now, we prove that $\Phi(I) = I$. For the sake of simplicity, write $T = \Phi(I)$. First, we show that T and T^* are injective. Indeed, let $y \in K$ such that $Ty = 0$. Since Φ is onto, there exists $B \in \mathcal{B}(H)$ such that $\Phi(B) = y \otimes y$. By (1.3) we get

$$\Delta_\lambda((y \otimes y) \circ T) = \Delta_\lambda(\Phi(B) \circ T) = \Delta_\lambda(\Phi(B) \circ \Phi(I)) = \Phi(\Delta_\lambda(B)).$$

This can be rewritten in the other terms,

$$\frac{1}{2}\Delta_\lambda(Ty \otimes y + y \otimes T^*y) = \frac{1}{2}\Delta_\lambda(y \otimes T^*y) = \Phi(\Delta_\lambda(B)).$$

In the other hand $(y \otimes T^*y)^2 = \langle Ty, y \rangle y \otimes T^*y = 0$, since $Ty = 0$. By Lemma 2.4, $\Delta_\lambda(y \otimes T^*y) = 0$ and thus $\Phi(\Delta_\lambda(B)) = 0$. Therefore $\Delta_\lambda(B) = 0$, because Φ is bijective and $\Phi(0) = 0$. Again, by Lemma 2.4, $B^2 = 0$.

Now, using Condition (1.3), we get

$$\|y\|^2 y \otimes y = \Phi(B)^2 = \Delta_\lambda(\Phi(B) \circ \Phi(B)) = \Phi(\Delta_\lambda(B^2)) = 0,$$

which implies that $y = 0$ and whence T is injective. With a similar argument we prove that T^* is injective.

Again from (1.3), we get

$$\Delta_\lambda(T^2) = \Delta_\lambda(\Phi(I) \circ \Phi(I)) = \Phi(\Delta_\lambda(I)) = \Phi(I) = T.$$

From Corollary 2.1, we deduce that $T = \Phi(I) = I$, and completes the proof. \square

Theorem 3.1. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map satisfying (1.3). Then

- (i) $\Delta_\lambda(\Phi(A)) = \Phi(\Delta_\lambda(A))$, for all $A \in \mathcal{B}(H)$. In particular Φ preserves the set of quasi-normal operators in both directions.
- (ii) $\Phi(A^2) = (\Phi(A))^2$ for all A quasi-normal.
- (iii) Φ preserves the set of orthogonal projections.
- (iv) Φ preserves the orthogonality between the projections ;

$$P \perp Q \Leftrightarrow \Phi(P) \perp \Phi(Q).$$

- (v) Φ preserves the order relation on the set of orthogonal projections in both directions ;

$$Q \leq P \Leftrightarrow \Phi(Q) \leq \Phi(P).$$

- (vi) $\Phi(P + Q) = \Phi(P) + \Phi(Q)$ for all orthogonal projections P, Q such that $P \perp Q$.
- (vii) Φ preserves the set of rank one orthogonal projections in both directions.

Proof. (i) Since, from Proposition 3.1, $\Phi(I) = I$, for $B = I$ in (1.3) we obtain $\Delta_\lambda(\Phi(A)) = \Phi(\Delta_\lambda(A))$, for all $A \in \mathcal{B}(H)$. Now, by (2.1) we get that Φ preserves the set of quasi-normal operators in both directions.

(ii) Let $A \in \mathcal{B}(H)$ be a quasi-normal operator. Since Φ preserves the set of quasi-normal operators, $\Phi(A)$, $\Phi(A^2)$, $(\Phi(A))^2$ are quasi-normal. By (1.3), we get $\Delta_\lambda((\Phi(A))^2) = \Phi(\Delta_\lambda(A^2))$. Hence $(\Phi(A))^2 = \Phi(A^2)$ follows from (2.1).

(iii) It is an immediate consequence of (i) and (ii).

In the rest of the proof P, Q are two orthogonal projections.

(iv) Assume that P, Q are orthogonal ($P \perp Q$). Since Φ preserves the set of orthogonal projections, $\Phi(P) \circ \Phi(Q)$ is self-adjoint operator. Hence by (1.3) and (2.1), we get

$$\Phi(P) \circ \Phi(Q) = \Delta_\lambda(\Phi(P) \circ \Phi(Q)) = \Phi(\Delta_\lambda(P \circ Q)) = 0.$$

Thus

$$(3.1) \quad \Phi(P)\Phi(Q) + \Phi(Q)\Phi(P) = 0$$

Multiplying (3.1) by $\Phi(P)$ on both sides, we get

$$(3.2) \quad \Phi(P)\Phi(Q) + \Phi(P)\Phi(Q)\Phi(P) = 0 \quad \text{and} \quad \Phi(P)\Phi(Q)\Phi(P) + \Phi(Q)\Phi(P) = 0.$$

It follows that $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$.

(v) Suppose that $Q \leq P$. Using Remark 3.1 and (1.3), we get

$$\Phi(P) \circ \Phi(Q) = \Delta_\lambda(\Phi(P) \circ \Phi(Q)) = \Phi(\Delta_\lambda(P \circ Q)) = \Phi(Q).$$

Hence,

$$\Phi(Q) \leq \Phi(P).$$

(vi) Since $P \perp Q$, $P + Q$ is an orthogonal projection and $P, Q \leq P + Q$. By (v), $\Phi(P), \Phi(Q) \leq \Phi(P + Q)$. Thus $\Phi(P) + \Phi(Q) \leq \Phi(P + Q)$. Since Φ^{-1} satisfies the same assumptions as Φ , it follows that $\Phi(P) + \Phi(Q) = \Phi(P + Q)$.

(vii) Let $P = x \otimes x$ be a rank one projection. Then $\Phi(P)$ is a non zero projection. Let $y \in K$ be an unit vector such that $y \otimes y \leq \Phi(P)$. Thus $\Phi^{-1}(y \otimes y) \leq P$. Since P is a minimal projection and $\Phi^{-1}(y \otimes y)$ is a non zero projection, $\Phi^{-1}(y \otimes y) = P$. Therefore $\Phi(P) = y \otimes y$ is a rank one projection. \square

In the following proposition we introduce a function $h : \mathbb{C} \rightarrow \mathbb{C}$, which will be used later to prove the linearity of Φ .

Proposition 3.2. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map satisfying (1.3). Then there exists a bijective function $h : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$h(0) = 0 \quad \text{and} \quad h(1) = 1,$$

and it satisfies the following properties :

- (i) $\Phi(\alpha I) = h(\alpha)I$ for all $\alpha \in \mathbb{C}$.
- (ii) $h(\alpha\beta) = h(\alpha)h(\beta)$ for all $\alpha, \beta \in \mathbb{C}$.
- (iii) $h(-\alpha) = -h(\alpha)$ for all $\alpha \in \mathbb{C}$.
- (iv) $\Phi(\alpha A) = h(\alpha)\Phi(A)$ for all A quasi-normal and $\alpha \in \mathbb{C}$.

Proof. First, we show that there exists a function

$$h : \mathbb{C} \rightarrow \mathbb{C} \text{ such that } \Phi(\alpha I) = h(\alpha)I \text{ for all } \alpha \in \mathbb{C}.$$

When $\alpha \in \{0, 1\}$ the function h is defined by $h(\alpha) = \alpha$ since $\Phi(\alpha I) = \alpha I$.

Now, we suppose that $\alpha \neq 0$. Let $y \in K$ an arbitrary unit vector. By Theorem 3.1 (iii), there exists $P = x \otimes x \in \mathcal{B}(H)$ with $\|x\| = 1$, such that $\Phi(P) = y \otimes y$. By (1.3) with $A = P$ and $B = \alpha P$ we get

$$\Delta_\lambda(\Phi(\alpha P) \circ \Phi(P)) = \Phi(\Delta_\lambda(\alpha P)) = \Phi(\alpha P).$$

From Proposition 2.3, there exists $h_P(\alpha) \in \mathbb{C}$ such that

$$\Phi(\alpha P) = h_P(\alpha)P.$$

Since Φ is bijective and $\alpha \neq 0$, $h_P(\alpha) \neq 0$.

Again using (1.3) with $A = \alpha I$ and $B = P$, we get

$$\Delta_\lambda(\Phi(\alpha I) \circ \Phi(P)) = \Phi(\alpha P) = h_P(\alpha)\Phi(P).$$

From Proposition 2.2, we get that

$$\Phi(P)\Phi(\alpha I) = h_P(\alpha)\Phi(P) = y \otimes \Phi(\alpha I)^*y = h_P(\alpha)y \otimes y$$

Then $\Phi(\alpha I)^*y = \overline{h_P(\alpha)}y$ for every $y \in K$. It follows that $\Phi(\alpha I) = h_P(\alpha)I$. Thus $h := h_P$ does not depend on P .

On the other hand, since Φ is bijective and Φ^{-1} satisfies the same properties as Φ , we conclude that the function $h : \mathbb{C} \rightarrow \mathbb{C}$ is well defined, bijective and $\Phi(\alpha I) = h(\alpha)I$.

(ii) Let $\alpha, \beta \in \mathbb{C}$, taking $A = \alpha I$ and $B = \beta I$ in (1.3), we get

$$h(\alpha)h(\beta)I = \Delta_\lambda(\Phi(\alpha I) \circ \Phi(\beta I)) = \Phi(\Delta_\lambda(\alpha I \circ \beta I)) = h(\alpha\beta)I.$$

Therefore h satisfies the point (ii).

From (ii), we have $(h(-1))^2 = 1$. Hence $h(-1) = -1$, since $h(1) = 1$ and h is bijective. Hence

$$h(-\alpha) = h(-1)h(\alpha) = -h(\alpha) \text{ for all } \alpha \in \mathbb{C}.$$

Thus (iii) is satisfied.

To prove (iv), let us consider A quasi-normal and $\alpha \in \mathbb{C}$. By (1.3), we have

$$h(\alpha)\Phi(A) = \Delta_\lambda(h(\alpha)\Phi(A)) = \Delta_\lambda(\Phi(A) \circ \Phi(\alpha I)) = \Phi(\Delta_\lambda(\alpha A)) = \Phi(\alpha A).$$

Thus (iv) holds. □

Lemma 3.1. Let $P = x \otimes x$, $P' = x' \otimes x'$ be two rank one orthogonal projections such that $P \perp P'$ (that is $\langle x, x' \rangle = 0$). Then

$$\Phi(\alpha P + \beta P') = h(\alpha)\Phi(P) + h(\beta)\Phi(P'), \text{ for every } \alpha, \beta \in \mathbb{C}$$

Proof. If $\alpha = 0$ or $\beta = 0$ the result follows from preceding Proposition.

Suppose that $\alpha \neq 0$ and $\beta \neq 0$. Note that $(\alpha P + \beta P') \circ (P + P') = \alpha P + \beta P'$. By (1.3), applied to $B = \alpha P + \beta P'$ and $A = P + P'$, we obtain

$$\begin{aligned} \Phi(\alpha P + \beta P') &= \Phi(\Delta_\lambda(\alpha P + \beta P')) \\ &= \Phi(\Delta_\lambda((\alpha P + \beta P') \circ (P + P'))) \\ &= \Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(P + P')) \\ &= \Delta_\lambda(\Phi(\alpha P + \beta P') \circ (\Phi(P) + \Phi(P'))). \end{aligned}$$

It follows that

$$(3.3) \quad \Phi(\alpha P + \beta P') = \Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(P) + \Phi(\alpha P + \beta P') \circ \Phi(P'))$$

Again by (1.3) with $B = \alpha P + \beta P'$ and $A = P$, we get that

$$\begin{aligned} \Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(P)) &= \Delta_\lambda(\Phi((\alpha P + \beta P') \circ P)) \\ &= \Phi(\Delta_\lambda(\alpha P)) \quad (\text{since } (\alpha P + \beta P') \circ P = \alpha P) \\ &= \Phi(\alpha P) = h(\alpha)\Phi(P). \end{aligned}$$

By Proposition 2.2 and $h(\alpha) \neq 0$, it follows that

$$(3.4) \quad \Phi(P)\Phi(\alpha P + \beta P') = h(\alpha)\Phi(P).$$

In the same manner, we also have

$$(3.5) \quad \Phi(P')\Phi(\alpha P + \beta P') = h(\beta)\Phi(P').$$

Now, we denote by $T = \Phi(\alpha P + \beta P')$, $\Phi(P) = y \otimes y$ and $\Phi(P') = y' \otimes y'$. Since P and P' are orthogonal, then y and y' are orthogonal too. By (3.4) and (3.5), we get that

$$T^*y = \overline{h(\alpha)}y \quad \text{and} \quad T^*y' = \overline{h(\beta)}y',$$

and also

$$(3.6) \quad \Phi(\alpha P + \beta P') \circ \Phi(P) = \frac{1}{2}(h(\alpha)\Phi(P) + \Phi(\alpha P + \beta P')\Phi(P)) = \frac{1}{2}((h(\alpha)y + Ty) \otimes y),$$

and

$$(3.7) \quad \Phi(\alpha P + \beta P') \circ \Phi(P') = \frac{1}{2}(h(\beta)\Phi(P') + \Phi(\alpha P + \beta P')\Phi(P')) = \frac{1}{2}((h(\alpha)y' + Ty') \otimes y').$$

By (3.7), it follows that

$$(3.8) \quad T = \frac{1}{2}\Delta_\lambda((h(\alpha)y + Ty) \otimes y + (h(\alpha)y' + Ty') \otimes y').$$

This implies that the rank of T and T^* are less than two. Moreover $\mathcal{R}(T), \mathcal{R}(T^*) \subseteq \text{span}\{y, y'\}$. Since $T^*y = \overline{h(\alpha)}y$ and $T^*y' = \overline{h(\beta)}y'$, then $T = h(\alpha)y \otimes y + h(\beta)y' \otimes y'$. Therefore

$$T = \Phi(\alpha P + \beta P') = h(\alpha)\Phi(P) + h(\beta)\Phi(P'),$$

which is the desired equality. \square

If Φ satisfies (1.3), then Φ preserves the set of rank one projection and preserves also the orthogonality between projections (see Theorem 3.1 (iii) and (iv)). Then for two vectors $x, x' \in H$ such that $\|x\| = \|x'\| = 1$ and $\langle x, x' \rangle = 0$, there exist two vectors $y, y' \in K$ such that

$$\|y\| = \|y'\| = 1, \quad \langle y, y' \rangle = 0, \quad \text{and} \quad \Phi(x \otimes x) = y \otimes y, \quad \Phi(x' \otimes x') = y' \otimes y'.$$

With the preceding notations, we have the following lemma

Lemma 3.2. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map satisfying (1.3). Then there exists $\mu \in \mathbb{C}$ such that $|\mu| = 1$ and

$$\Phi(x \otimes x' + x' \otimes x) = \mu y \otimes y' + \bar{\mu} y' \otimes y.$$

In addition, we obtain

$$h(2) = 2 \quad \text{and} \quad h\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Proof. Let us denote by $A = x \otimes x' + x' \otimes x$. First, note that A is self adjoint operator of rank two, and we have

$$A^2 = x \otimes x + x' \otimes x',$$

which is an non-trivial orthogonal projection. Therefore, the spectrum of A is $\sigma(A) = \{-1, 0, 1\}$. Hence we can find f_1, f_2 two unit and orthogonal vectors from H ($\|f_1\| = \|f_2\| = 1$ and $\langle f_1, f_2 \rangle = 0$) such that

$$(3.9) \quad A = f_1 \otimes f_1 - f_2 \otimes f_2.$$

Let us denote $T = \Phi(A)$. By Lemma 3.1, we have

$$T = \Phi(f_1 \otimes f_1 - f_2 \otimes f_2) = \Phi(f_1 \otimes f_1) + h(-1)\Phi(f_2 \otimes f_2) = \Phi(f_1 \otimes f_1) - \Phi(f_2 \otimes f_2),$$

Now, since $\Phi(f_1 \otimes f_1)$ and $\Phi(f_2 \otimes f_2)$ are two orthogonal projections, then we deduce that T is a self adjoint operator of rank two, and

$$\text{tr}(T) = \text{tr}(\Phi(f_1 \otimes f_1)) - \text{tr}(\Phi(f_2 \otimes f_2)) = 0.$$

In the other hand, since A is self adjoint, by Theorem 3.1 (ii) and (vi) , we get

$$T^2 = \Phi(A)^2 = \Phi(A^2) = \Phi(x \otimes x + x' \otimes x') = y \otimes y + y' \otimes y'.$$

Moreover,

$$(3.10) \quad T^2 = y \otimes y + y' \otimes y'.$$

It follows that $y, y' \in \mathcal{R}(T)$, and thus

$$\mathcal{R}(T) = \text{span}\{y, y'\}.$$

Clearly, $A = 2A \circ (x \otimes x)$. By (1.3), we get

$$\begin{aligned} T = \Delta_\lambda(T) &= \Delta_\lambda(\Phi(2A \circ (x \otimes x))) \\ &= h(2)\Phi(\Delta_\lambda(A \circ (x \otimes x))) \\ &= h(2)\Delta_\lambda(\Phi(A) \circ \Phi(x \otimes x)) \\ &= h(2)\Delta_\lambda(T \circ (y \otimes y)). \end{aligned}$$

Therefore

$$(3.11) \quad T = \frac{h(2)}{2}\Delta_\lambda(Ty \otimes y + y \otimes T^*y) = \frac{h(2)}{2}(Ty \otimes y + y \otimes Ty).$$

It follows that

$$\text{tr}(T) = h(2) \langle Ty, y \rangle = 0.$$

Hence $\langle Ty, y \rangle = 0$. Since $\{y, y'\}$ is an orthonormal basis of $\mathcal{R}(T)$ then there exists $\mu \in \mathbb{C}$ such that $Ty = \mu y'$. From (3.11) we get

$$(3.12) \quad T = \frac{h(2)}{2}(\mu y' \otimes y + \bar{\mu} y \otimes y').$$

In particular, we also have $Ty = \frac{h(2)}{2}\mu y' = \mu y'$, it follows that

$$h(2) = 2 \quad \text{and thus} \quad h\left(\frac{1}{2}\right) = \frac{1}{2} \quad (\text{since } h \text{ multiplicative and } h(1) = 1).$$

Hence

$$T = \mu y' \otimes y + \bar{\mu} y \otimes y',$$

and we have

$$T^2 = |\mu|^2(y \otimes y + y' \otimes y') = y \otimes y + y' \otimes y'.$$

Hence $|\mu| = 1$. This completes the proof. \square

As a direct consequence from the preceding lemma, we have the following corollary.

Corollary 3.1. The function $h : \mathbb{C} \rightarrow \mathbb{C}$ defined in Proposition 3.2 is additive.

Proof. Let $x, x' \in H$ be two unit and orthogonal vectors for H and $\alpha, \beta \in \mathbb{C}$. Denote

$$P = x \otimes x, P' = x' \otimes x', Q = \Phi(P) = y \otimes y, Q' = \Phi(P') = y' \otimes y'.$$

Put $A = x \otimes x' + x' \otimes x$, and $B = \alpha P + \beta P'$. Observe that

$$A \circ P = \frac{1}{2}A \quad \text{and} \quad A \circ P' = \frac{1}{2}A,$$

thus

$$A \circ B = \frac{\alpha + \beta}{2}A.$$

By Lemma 3.2, there exists $\mu \in \mathbb{C}$ such that $|\mu| = 1$ and

$$\Phi(A) = \mu y \otimes y' + \bar{\mu} y' \otimes y.$$

Then, by a simple calculation, we get

$$\Phi(P) \circ \Phi(A) = Q \circ \Phi(A) = \frac{1}{2}\Phi(A).$$

Hence, using (1.3), we immediately obtain the following

$$\begin{aligned} h\left(\frac{\alpha + \beta}{2}\right)\Phi(A) &= \Phi\left(\frac{\alpha + \beta}{2}A\right) = \Phi(\Delta_\lambda\left(\frac{\alpha + \beta}{2}(A)\right)) \\ &= \Phi(\Delta_\lambda(B \circ A)) = \Delta_\lambda(\Phi(B) \circ \Phi(A)) \\ &= \Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(A)) \\ &= \Delta_\lambda(\Phi(\alpha P) \circ \Phi(A) + \Phi(\beta P') \circ \Phi(A)) \\ &= \Delta_\lambda(h(\alpha)\Phi(P) \circ \Phi(A) + h(\beta)\Phi(P') \circ \Phi(A)) \\ &= \frac{1}{2}(h(\alpha) + h(\beta))\Phi(A). \end{aligned}$$

Since $h(1/2) = 1/2$, we have

$$\frac{1}{2}h(\alpha + \beta) = h\left(\frac{\alpha + \beta}{2}\right) = \frac{1}{2}(h(\alpha) + h(\beta)).$$

Thus $h(\alpha + \beta) = h(\alpha) + h(\beta)$. It follows that h is additive. \square

Now, we are in position to prove our main Theorem.

Proof of Theorem 1.1.

Proof. The "if" part is immediate.

We show the "only if" part. Assume that $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is bijective and satisfies (1.3). The proof of theorem is organized in several steps.

Step. 1 For every $A \in \mathcal{B}(H)$, for all x, y such that $\|x\| = \|y\| = 1$ and $\Phi(x \otimes x) = y \otimes y$, we have

$$(3.13) \quad \langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle),$$

Put $T = 2(A \circ (x \otimes x)) = Ax \otimes x + x \otimes A^*x$. Let α_1, α_2 be the non zero eigenvalues of T . By the Schur decomposition of T , there exist two unit and orthogonal vectors e_1, e_2 such that

$$T = \alpha_1 e_1 \otimes e_1 + \alpha_2 e_2 \otimes e_2 + \beta e_1 \otimes e_2.$$

First, we show that

$$(3.14) \quad \text{tr}(\Delta_\lambda(\Phi(T))) = 2h(\langle Ax, x \rangle).$$

If $\alpha_1 = \alpha_2 = 0$, we have $T = \beta e_1 \otimes e_2$, thus $T^2 = 0$ and $\Delta_\lambda(T) = 0$, by Lemma 2.4. Hence $\Delta_\lambda(\Phi(T)) = \Phi(\Delta_\lambda(T)) = 0$, since Φ commutes with Δ_λ . Consequently,

$$\text{tr}(T) = 2 \langle Ax, x \rangle = 0 = \text{tr}(\Delta_\lambda(\Phi(T))),$$

and, in this case, (3.14) is satisfied.

Now suppose that $(\alpha_1, \alpha_2) \neq (0, 0)$.

First, note that

$$T \circ (e_1 \otimes e_1) = \frac{1}{2}(e_1 \otimes (\bar{\beta}e_2 + 2\bar{\alpha}_1 e_1)) = \frac{1}{2}(e_1 \otimes v),$$

with $v = \bar{\beta}e_2 + 2\bar{\alpha}_1 e_1$ and $\langle e_1, v \rangle = 2\alpha_1$. From Proposition 2.1, it follows that

$$\Delta_\lambda(T \circ (e_1 \otimes e_1)) = \frac{1}{2}\Delta_\lambda((e_1 \otimes v)) = \alpha_1 \frac{1}{\|v\|^2}(v \otimes v).$$

By (1.3), we get

$$\begin{aligned} h(\alpha_1)\Phi\left(\frac{1}{\|v\|^2}(v \otimes v)\right) &= \Phi(\Delta_\lambda(T \circ (e_1 \otimes e_1))) \\ &= \Delta_\lambda(\Phi(T) \circ \Phi(e_1 \otimes e_1)). \end{aligned}$$

We have also

$$T \circ (e_2 \otimes e_2) = \frac{1}{2}(\beta e_1 + 2\alpha_2 e_2) \otimes e_2.$$

Hence

$$\Delta_\lambda(T \circ (e_2 \otimes e_2)) = \alpha_2 e_2 \otimes e_2.$$

Again (1.3) implies that

$$\begin{aligned} h(\alpha_2)\Phi(e_2 \otimes e_2) &= \Phi(\Delta_\lambda(T \circ (e_2 \otimes e_2))) \\ &= \Delta_\lambda(\Phi(T) \circ \Phi(e_2 \otimes e_2)) \end{aligned}$$

Now, observe that $T \circ (e_1 \otimes e_1 + e_2 \otimes e_2) = T$. We apply again (1.3), then

$$\begin{aligned} \Delta_\lambda(\Phi(T)) &= \Phi(\Delta_\lambda(T \circ (e_1 \otimes e_1 + e_2 \otimes e_2))) \\ &= \Delta_\lambda(\Phi(T) \circ \Phi(e_1 \otimes e_1 + e_2 \otimes e_2)) \\ &= \Delta_\lambda(\Phi(T) \circ \Phi(e_1 \otimes e_1) + \Phi(T) \circ \Phi(e_2 \otimes e_2)). \end{aligned}$$

Since $\Phi(\frac{1}{\|v\|^2}(v \otimes v))$ and $\Phi(e_2 \otimes e_2)$ are orthogonal projections, then

$$\begin{aligned} tr(\Delta_\lambda(\Phi(T))) &= tr(\Delta_\lambda(\Phi(T) \circ \Phi(e_1 \otimes e_1) + \Phi(T) \circ \Phi(e_2 \otimes e_2))) \\ &= tr(\Phi(T) \circ \Phi(e_1 \otimes e_1) + \Phi(T) \circ \Phi(e_2 \otimes e_2)) \\ &= tr(\Phi(T) \circ \Phi(e_1 \otimes e_1)) + tr(\Phi(T) \circ \Phi(e_2 \otimes e_2)) \\ &= tr(h(\alpha_1)\Phi(\frac{1}{\|v\|^2}(v \otimes v)) + tr(h(\alpha_2)\Phi(e_2 \otimes e_2)) \\ &= h(\alpha_1) + h(\alpha_2) \\ &= h(\alpha_1 + \alpha_2) = h(tr(T)) \\ &= 2h(\langle Ax, x \rangle). \end{aligned}$$

Now, we show that

$$(3.15) \quad tr(\Delta_\lambda(\Phi(T))) = 2 \langle \Phi(A)y, y \rangle .$$

By (1.3) and $h(2) = 2$, we get

$$\begin{aligned} \Delta_\lambda(\Phi(T)) &= \Delta_\lambda(\Phi(2A \circ (x \otimes x))) = \Delta_\lambda(h(2)\Phi(A \circ (x \otimes x))) \\ &= h(2)\Phi(\Delta_\lambda(A \circ (x \otimes x))) \\ &= 2\Delta_\lambda(\Phi(A) \circ \Phi(x \otimes x)) \\ &= 2\Delta_\lambda(\Phi(A) \circ (y \otimes y)). \end{aligned}$$

Thus

$$\begin{aligned} tr(\Delta_\lambda(\Phi(T))) &= 2tr(\Delta_\lambda(\Phi(A) \circ (y \otimes y))) = 2tr(\Phi(A) \circ (y \otimes y)) \\ &= 2 \langle \Phi(A)y, y \rangle . \end{aligned}$$

From (3.14) and (3.15), it follows that

$$\langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle),$$

for all $A \in \mathcal{B}(H)$ and for all $x \in H$ and $y \in H$ such $\Phi(x \otimes x) = y \otimes y$, which completes the proof of (3.13).

Step. 2 The function h is continuous.

Let \mathcal{E} be a bounded subset in \mathbb{C} and $A \in \mathcal{B}(H)$ such that $\mathcal{E} \subset W(A)$, where $W(A)$ is the numerical range of A . By (3.13),

$$h(\mathcal{E}) \subset h(W(A)) = W(\Phi(A))$$

Since $W(\Phi(A))$ is bounded, h is bounded on the bounded subset, which implies that h is continuous, since it is additive (see Corollary 3.1). We then have, using Proposition 3.2 (ii) and Corollary 3.1, that h is an automorphism

continuous over the complex field \mathbb{C} . It follows that h is the identity or the complex conjugation map (see, for example, [12]).

Step. 3 Φ is linear or anti-linear.

Let $y \in K$ and $x \in H$ be unital vectors such that $y \otimes y = \Phi(x \otimes x)$. Let $\alpha \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$ be arbitrary. Using (3.13), we get

$$\begin{aligned} \langle \Phi(A + B)y, y \rangle &= h(\langle (A + B)x, x \rangle) \\ &= h(\langle Ax, x \rangle + \langle Bx, x \rangle) \\ &= h(\langle Ax, x \rangle) + h(\langle Bx, x \rangle) \\ &= \langle \Phi(A)y, y \rangle + \langle \Phi(B)y, y \rangle \\ &= \langle (\Phi(A) + \Phi(B))y, y \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \Phi(\alpha A)y, y \rangle &= h(\langle \alpha Ax, x \rangle) \\ &= h(\alpha)h(\langle Ax, x \rangle) \\ &= h(\alpha) \langle \Phi(A)y, y \rangle. \end{aligned}$$

Hence, we immediately obtain the following

$$\forall A, B \in \mathcal{B}(H), \forall \alpha \in \mathbb{C}, \quad \Phi(A + B) = \Phi(A) + \Phi(B) \text{ and } \Phi(\alpha A) = h(\alpha)\Phi(A).$$

Therefore Φ is linear or anti-linear, since h is the identity or the complex conjugation.

Step. 4 $\Phi(A) = UAU^*$ every $A \in \mathcal{B}(H)$, for some unitary operator $U \in \mathcal{B}(H, K)$. By Step.3, we have Φ or Φ^* is linear, where Φ^* is defined by $\Phi^*(A) = (\Phi(A))^*$. From Theorem 3.1 (i), Φ commute with Δ_λ . Now, by [5, Theorem 1], there exists a unitary operator $V : H \rightarrow K$, such that Φ take one of the following forms

$$(3.16) \quad \Phi(A) = VAV^* \text{ for all } A \in \mathcal{B}(H),$$

either

$$(3.17) \quad \Phi(A) = VA^*V^* \text{ for all } A \in \mathcal{B}(H).$$

In order to complete the proof we have to show Φ can not take the form (3.17).

Seeking a contradiction, suppose that (3.17) holds. Then for every $A \in \mathcal{B}(H)$,

$$\begin{aligned} \Delta_\lambda(A^*) &= \Delta_\lambda(V^*\Phi(A)V) \\ &= V^*\Delta_\lambda(\Phi(A))V \\ &= V^*\Phi(\Delta_\lambda(A))V \\ &= (\Delta_\lambda(A))^*. \end{aligned}$$

Therefore

$$(3.18) \quad \Delta_\lambda(A^*) = (\Delta_\lambda(A))^*, \quad \text{for every } A \in \mathcal{B}(H).$$

Now, let us consider $A = x \otimes x'$ with x, x' are unit, independent and non-orthogonal vectors in H . Then $A^* = x' \otimes x$ and by Proposition 2.1, we have

$$(\Delta_\lambda(A))^* = \langle x', x \rangle (x' \otimes x') \quad \text{and} \quad \Delta_\lambda(A^*) = \langle x', x \rangle (x \otimes x),$$

which contradicts (3.18). So if we take $U = V$, Φ gets the form

$$\Phi(A) = UAU^*, \quad \text{for all } A \in \mathcal{B}(H).$$

This completes the proof of our main theorem. \square

4. STAR JORDAN PRODUCT COMMUTING MAPS

We end this paper by characterizing the bijective maps $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, which verify

$$(4.1) \quad \Delta_\lambda(\Phi(A) \circ (\Phi(B))^*) = \Phi(\Delta_\lambda(A \circ B^*)) \quad \text{for all } A, B \in \mathcal{B}(H).$$

Theorem 4.1. Let H and K two complex Hilbert space, such that H is of dimension greater than 2. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a bijective map. Then Φ satisfies (4.1), if and only if, there exists an unitary operator $U : H \rightarrow K$ such that

$$\Phi(A) = UAU^* \quad \text{for all } A \in \mathcal{B}(H).$$

$\ddot{u}_{\frac{1}{2}}$

Remark 4.1. Based on the arguments and the methods developed in the proof of Theorem 1.1, it is not difficult to proof Theorem 4.1. For example, If $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is a bijective map, satisfying (4.1), then

- (i) $\Phi(0) = 0$.,
- (ii) $\Phi(I) = I$.,
- (iii) $\Delta_\lambda(\Phi(A)) = \Phi(\Delta_\lambda(A))$ and $\Delta_\lambda((\Phi(A))^*) = \Phi(\Delta_\lambda(A^*))$ for all $A \in \mathcal{B}(H)$,
- (iv) Φ preserve the set of quasi-normal operator in both directions,
- (v) Φ preserves the set of self adjoint operators in both directions.

Proof. (i) Since Φ is onto, then there exists $B \in \mathcal{B}(H)$ such that $\Phi(B) = 0$. Hence, by (4.1) $\Phi(0) = \Delta_\lambda(\Phi(0) \circ \Phi(B)^*) = 0$.

(ii) Let us denote $T := \Phi(I)$. First we show that T is one-to-one. Let $y \in K$ such that $Ty = 0$. Now since Φ is onto, there exists $B \in \mathcal{B}(H)$ such that $\Phi(B) = y \otimes y$. By (4.1), we have $\Delta_\lambda(T \circ y \otimes y) = y \otimes y$. Thus $y \otimes y = 0$ and it follows that $y = 0$.

Again from (4.1) it follows that $T \circ T^* = T$. Hence T is self adjoint and $T^2 = T \circ T^* = T$. Thus $T = I$ since T is one-to-one.

(iii) is immediate, and (iv) is deduced directly from (iii).

(v) Let B be a self adjoint operator, and $S := \Phi(B)$. By (iii) S is quasi-normal operator, and by (4.1) with $A = I$ we get, that $\Delta(S^*) = S$. By Lemma 2.3, $S = S^*$ and thus Φ preserves the self adjoint operators. \square

The rest of the proof of Theorem 4.1 is similar to that of Theorem 1.1. We do not give details of those arguments.

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