# An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems

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# An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems

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*A method is proposed for analyzing the steady-state response of nonlinear dynamic systems. The method iterates to obtain the discrete Fourier transform of the system response, returning to the time domain at each iteration to take advantage of the ease in evaluating nonlinearities there—rather than analytically describing the nonlinear terms in the frequency domain. The updated estimates of the nonlinear terms are transformed back into the frequency domain in order to continue iterating on the frequency spectrum of the steady-state response. The method is demonstrated by solving a problem with friction damping in which the excitation has multiple discrete frequencies.*

## 1 Introduction

A variety of methods are presently available for calculating the response of nonlinear dynamic systems, (see, for example, Iwan (1974) for analytical approaches, and Press et al. (1986) for numerical approaches). The method introduced in this paper offers advantages over most other methods in its generality, efficiency, accuracy, and ease-of-use for calculating the steady-state response of forced nonlinear systems. The method involves an iterative scheme that alternates from the frequency to the time domain in order to take advantage of the convenience in performing certain tasks in each domain. Consequently, this *Alternating Frequency/Time* (AFT) method easily provides either time trajectory or frequency spectrum solutions.

The purpose of this paper is to introduce the AFT method and illustrate its utility with a friction damping problem: the response of a single degree-of-freedom (sdf) system with an elastic/perfectly-plastic friction damper subject to three discrete frequencies of excitation is calculated.

## 2 Alternative Spectral Methods

One of the most popular methods for approximating the frequency response of nonlinear systems is known as the *Harmonic Balance* (HB) method in the vibration literature (e.g., Nayfeh and Mook, 1979), and as the *Describing Function* (DF) approach in the control literature (e.g., Gelb and Vander Velde, 1968). This is essentially a spectral method in which the solution is assumed to have one dominant frequency component—or at most two.[1] In situations where it can be applied, and trusted to give accurate solutions (where a single harmonic description of the nonlinearity is adequate), the HB method is by far the most computationally efficient method for obtaining steady-state solutions to nonlinear dynamics problems. It has been used extensively to study friction damping (e.g., Griffin, 1980; Sinha and Griffin, 1983 and 1984; Dowell, 1983; and Jezequel, 1983). Generally, however, this method is limited to harmonically excited systems and it becomes increasingly inaccurate as higher harmonics in the expansion of the nonlinear force become significant. A fair amount of work is also required to expand the nonlinear force, although DF's for a large number of nonlinearities have been tabulated (Gelb and Vander Velde, 1968, Appendix B).

An *Incremental Harmonic Balance* (IHB) method which accommodates multiple harmonic components in the description of the nonlinear force was introduced by Lau et al. in 1982. The method was originally applied to continuous (cubic) nonlinearities (Lau et al., 1982, 1983), but Pierre et al. (1985) extended the method to analyze the rigid/perfectly-plastic friction damper problem in which the nonlinear force is discontinuous. The IHB method appears to handle strong nonlinearities well, and provides accurate solutions over a much broader range than the HB method since it accommodates multiple harmonics. The method is ideally suited to parametric studies since it can step from a known state of vibration to a neighboring state which corresponds to an incremental change in one of the governing parameters of the

[1] The DF approach has been extended to allow two input frequencies in the case of odd, memoryless (nonhysteretic) nonlinearities (Gelb and Vander Velde, 1968, Chapter 5 and Appendix D), but the nonlinear force and the system response are described only in terms of the two input frequencies.

system. However, there are also significant disadvantages to the IHB method.

Computationally the IHB method is no more efficient than numerical integration, at least for the discontinuous friction damping problem (cf., Pierre et al., 1985, "Conclusions"), and the computational effort grows geometrically with the number of harmonics considered. As currently implemented, the frequencies are integer multiples of the excitation frequency, which will hide the subharmonic resonances that may occur in nonlinear systems. Another disadvantage of the IHB method is the amount of analytical work required. Functions in the equation are expanded in a first order Taylor series (which may be difficult to obtain for discontinuous functions) and substituted into the original equation to obtain a linear "incrementalized" equation. The incrementalized equation is solved by the Galerkin method, which requires further expansions and integrations. Finally, although the approach is general, the expansions are specialized, so each new dynamic system requires its own formulation and implementation.

Other numerical methods, using techniques similar to those in the AFT method, describe functions in both the frequency and time domains. In computational fluid dynamics, "Spectral" and "Pseudo-Spectral" methods (cf., Zang and Hussaini, 1985) use Fourier transform techniques to expand functions into (trigonometric or Chebyshev) polynomials. This permits derivatives and other operations on those functions to be performed with higher order accuracy than finite difference methods achieve for similar effort. Cesari (1963) appears to be the first to obtain steady-state solutions to nonlinear dynamic equations by using a trigonometric polynomial expansion, which is equivalent to a truncated Fourier series or a discrete Fourier transform. Urabe (1965) expands Cesari's results and (Urabe and Reiter, 1966) shows how the Fourier coefficients may be computed at each iteration with a discrete Fourier transform, once the nonlinear functions have been evaluated in the time domain. Urabe identifies this approach as an application of the Galerkin method. Ling and Wu (1987) use a fast Fourier transform (FFT) to implement Urabe's approach, calling it the "Fast Galerkin" (FG) method. Although there are fundamental similarities in the concepts underlying the FG and AFT methods, there are differences in how the methods are formulated and implemented. These differences are discussed later.

## 3 An Alternating Frequency/Time (AFT) Method

**3.1 Fundamental Concepts.** Frequency domain methods have long been used to analyze linear systems because the frequency content of a signal is often of greater interest than the time history and because convolution – a convenient solution method – is more efficiently accomplished in the frequency domain. In general, nonlinear forces are nonlinear functions of the past and present states of the system. Since they are not linear functions of the states or explicit functions of time, they cannot be transformed directly into the frequency domain. This makes nonlinear systems difficult to analyze in the frequency domain, as evidenced by the complicated expansions required in the frequency domain methods discussed previously. Nonlinear forces are easy to evaluate in the time domain, however, since the state history of the system is readily available. This is evidenced by the ease of implementation and greater complexity allowed by numerical integration. The AFT method employs the advantages of both domains by performing convolution (as well as filtering and convergence testing) in the frequency domain and by switching to the time domain to evaluate the nonlinear forces as functions of the state trajectory. The cost involved in this technique is the time required to switch back and forth between the frequency and time domains. A fast Fourier transform[2] algorithm makes this cost reasonable, especially in obtaining steady-state solutions to periodically excited systems.

Consider a nonlinear dynamic system in which the terms containing nonlinearities are grouped on the right hand side of the equation with the external forcing term. The equation of motion for this system may be represented by:

$$L\{x(t)\} = f(t) + N_f(x,\dot{x}) \tag{1}$$

where $L$ is the differential operator representing the linear terms, $f(t)$ is the external forcing term, $N_f(\cdot)$ represents the nonlinear force terms, and $x(t)$ is the desired system response. These are vector functions for multi-degree-of-freedom (mdf) systems, or scalar functions for sdf systems. Taking the Fourier transform (FT) of both sides yields:

$$\Lambda(\omega)\cdot X(\omega) = F(\omega) + \eta(\omega, X(\omega)) \tag{2}$$

where $\Lambda(\omega)$, the transform of $L$, is a (matrix) function of the frequency, $\omega$, $F(\omega)$ is the transform of $f(t)$, $\eta(\omega,X(\omega))$ is the transform of $N_f(\cdot)$, and $X(\omega)$ is the transform of $x(t)$. In the transform domain the nonlinear forces depend only on $\omega$ and $X(\omega)$ because derivatives of $x(t)$ transform into linearly scaled forms of $X(\omega)$:[3]

$$\begin{aligned} x(t) &\Leftrightarrow X(\omega) \\ \dot{x}(t) &\Leftrightarrow -j\omega X(\omega) \\ \ddot{x}(t) &\Leftrightarrow -\omega^2 X(\omega) \end{aligned} \tag{3}$$

("$\Leftrightarrow$" denotes a Fourier Transform pair).

Since an FFT will be used to compute and represent the frequency dependent functions, (2) is required to hold at the discrete frequencies, $\omega_k$, introduced by the FFT. Thus, (2) becomes a system of nonlinear algebraic equations[4]:

$$\Lambda(\omega)\cdot X(\omega) = F(\omega) + \eta(\omega, X(\omega)) \tag{4}$$

where

$$\omega = \{\omega_k\} \tag{5}$$

and

$$\omega_k = \frac{2\pi k}{\Delta T \cdot N} \qquad k = 0, \ldots, \frac{N}{2}. \tag{6}$$

Here, $\Delta T$ is the sampling period and $N$ is the number of samples in the FFT of each time domain function. In general, $\Lambda$ is a matrix whose components are functions of the independent variable ($\omega_k$), while $X$, $F$, and $\eta$ are vector functions of $\omega_k$. For example, at a given frequency, say $\omega_4$, there will be $X_1(\omega_4), \ldots, X_m(\omega_4)$ if there are $m$ degrees-of-freedom. Therefore, the length of $X(\omega)$ is the product of the number of frequencies in the FFT ($N/2+1$) times the number of degrees-of-freedom ($m$). To minimize the bandwidth of $L$, the equations should be ordered so that $X(\omega)$ appears as $\{X_1(\omega_0), X_2(\omega_0), \ldots, X_m(\omega_0), X_1(\omega_1), \ldots, X_m(\omega_1), \ldots, X_1(\omega_{N/2}), \ldots X_m(\omega_{N/2})\}$.

Various iterative methods are available for solving nonlinear algebraic equations, like (4). One problem facing any method is evaluation of the nonlinear term $\eta(\omega, X(\omega))$. To add it directly to $F(\omega)$ this term must be known explicitly in terms of $\omega$, as $\eta(\omega)$, but this form is difficult to determine in

[2] "Fast Fourier transform" refers to a class of algorithms used to calculate the discrete Fourier transform (DFT – cf., Jackson, 1986, Chapter 7; and Oppenheim and Schafer, 1975, Chapter 3). The notation "FFT" will be used here to refer to the DFT as well as its implementation. The authors use an FFT subroutine from Press, et al. (1986).

[3] The Fourier Transform definition used here is $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$ and the inverse transform is $x(t) = 1/2\pi \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega$. Analogous definitions are employed in the FFT.

[4] The "$\omega$" notation in (4) is for conceptual purposes. For digital implementation (4) is cast in stricter vector notation – $\Lambda(k)\cdot X(k) = F(k) + \eta(k)$ – where the $k$th component of each vector represents a sampled version of the corresponding continuous function at the frequency $\omega_k$ (ignoring scale factors and errors).

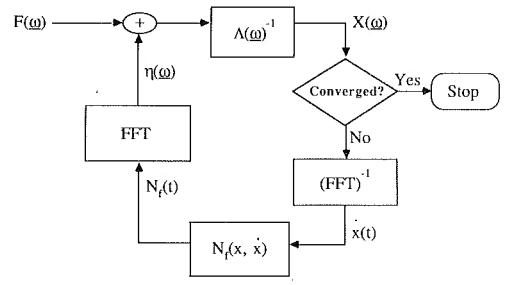


**Fig. 1 Direct iteration implementation of the AFT method**

the frequency domain. However, if $N_f(\cdot)$ can be expressed as an explicit function of time, $N_f(t)$, then it can be transformed to provide $\eta(\omega)$. To find $N_f(t)$, the most recent estimate of $X(\omega)$ is inverse transformed to provide an estimate of $x(t)$. Equation (3) – or the analogous FFT relationships – may be used to obtain other state variables so that $N_f(\cdot)$ may be evaluated as $N_f(t)$, and transformed to $\eta(\omega)$. For nonlinearities that involve products of time domain functions this avoids frequency domain convolution. For history-dependent nonlinearities, this procedure allows the nonlinearity to be evaluated directly from the state history as long as the time period used in the analysis is longer than the "term of memory" of the nonlinearity. With short term memory nonlinearities, values of the nonlinear terms at the beginning of the period may be obtained from periodicity requirements.

Another problem for nonlinear equation solvers is obtaining an initial estimate of the solution. An initial estimate (designated $X^0(\omega)$) may be obtained from the linear system, ignoring $\eta(\cdot)$, or from a known solution corresponding to a closely related state. For example, consider a sdf system with $f(t) = A\cos(\omega_e t)$ where a parametric study of the effect of $A$ on the system response is desired. Once the solution, $X(\omega)$, is obtained for one value of $A$, it may be used as the initial estimate for a neighboring case with a slightly different value of $A$.

**3.2 Implementation Considerations.** The authors presently use two methods to solve (4). The first method, direct iteration, is an iterative implementation of (4):

$$\Lambda(\omega)\cdot X^{(i)}(\omega) = F(\omega) + \eta(\omega, X^{(i-1}(\omega)). \tag{7}$$

$\Lambda(\omega)$[5] and $F(\omega)$ are fixed by the system and the external forcing function, respectively, once the $\omega_k$'s are selected. New generations of $X^{(i)}(\omega)$ may be found by evaluating $\eta(\cdot)$, as described previously, adding it to $F(\omega)$, and multiplying by $\Lambda^{-1}$. Since $\Lambda$ is a constant matrix representing the linear part of the system, it may be inverted once and saved. A flow chart of the direct iteration approach is shown in Fig. 1.

The second approach used by the authors for solving (4) is the Newton-Raphson method (cf., Stoer and Bulirsch, 1980), with a difference quotient approximation of the Jacobian. For the example problem in this paper, Newton-Raphson is more robust than direct iteration (it converges at times that direct iteration fails) and it converges in fewer iterations. However, when both methods converge, computation time is greater for Newton-Raphson than for direct iteration because of the effort required to recompute the Jacobian at each iteration. Newton-Raphson is also more difficult to implement. In some cases Newton-Raphson fails to converge, so more robust methods (cf., Wolfe, 1959; Broyden, 1969; Stoer and Bulirsch, 1980; Shacham, 1986) may be required if a solution is desired for all regions of the parameter space.

[5]Professor Alok Sinha of Pennsylvania State University has pointed out that in the absence of damping $\Lambda$ may be singular. This may be remedied by adding a linear damping term to both sides of any equation at fault. The damping term may be lumped with the nonlinear term(s) on the right-hand side of an equation.

Equation (4) represents $m(1+N/2)$ equations (for $m$ degrees-of-freedom) which for most methods is unwieldy, except for small values of $N$. The computational effort of the direct iteration method, $O[mN(m+\log(N)]$ multiplications per iteration, is much less affected by "$N$" than Newton-Raphson, which has $O[(mN)^2\cdot\log(N)]$ multiplications per iteration. In the approach used here, rather than limiting $N$, which would greatly reduce the number of frequencies considered in the solution, the direct iteration approach is used first to estimate which frequency components are significant. Then, if Newton-Raphson is employed, equation (4) is implemented only for those values of $k$ for which $X^{(i)}(\omega_k)$ is a significant component in the last estimate of the solution vector obtained from the direct iteration method, $\{X^{(i)}(\omega_o), \ldots, X^{(i)}(\omega_{N/2})\}$. This is discussed further in Section 5. This approach, by solving only the significant frequency components, keeps the number of equations and unknowns at a minimum while still allowing a full spectrum of frequencies to be considered in the solution.

There are two types of error that may corrupt FFT values. *Aliasing* error arises when the FT is nonzero for frequencies above the maximum frequency considered by the FFT ($\omega_{\max} = \pi/\Delta T$). Power at the higher frequencies is aliased or folded back and added to the FFT values for $\omega < \pi/\Delta T$. *Leakage* error occurs when the frequency values of the FFT ($k\cdot\Delta\omega$, or $\omega_k$) do not coincide precisely with each frequency in the FT that has nonzero power. Power at a frequency not represented in the FFT is leaked to frequencies that are represented. Leakage can only be eliminated if the sampled time signal is periodic.

For the following sample problem aliasing is insignificant (as verified by taking smaller time steps), and both the excitation on the system and the system response are periodic, so no leakage will occur. Most nonlinear forces are periodic when subject to a periodic input, so the assumption of a periodic response to a periodic excitation is typically valid. Discussions of the DFT, FFT algorithms, aliasing, and leakage may be found in Jackson (1986), Oppenheim and Schafer (1975), and Press et al. (1986).

## 4 Sample Problem

In many mechanical applications the most significant source of vibrational damping is provided by friction. For example, friction damping occurs naturally at joints in fabricated structures, friction damping is used to control the earthquake response of buildings, and friction dampers are used in high speed turbomachinery to reduce resonant stresses in blades. An elastic/perfectly-plastic friction model may also be used to analyze the elasto-plastic behavior of materials. Interest in these applications has motivated considerable research into how friction affects the dynamic response of structures (cf., Plunkett's review of friction damping research, 1980).

In the problem considered the nonlinear element consists of a spring in series with a friction constraint (Fig. 2). The friction contact is assumed to exhibit simple Coulomb-type behavior, i.e., when slip occurs the force is constant in magnitude and directed to oppose relative motion. Since the damper force in this model depends on the history of the response, it is easiest to illustrate the damper force by example. In Fig. 2 (*b*), for instance, the damper force exhibits bilinear hysteresis, which occurs when the input to the damper is harmonic. For more complicated motion the damper force could take on any value within a similar hysteresis loop, where the upper and lower limits of the input, which represent the extreme amplitudes of motion, need not be symmetric about zero. In the sample problem the equations have been normalized so that the friction force, $\mu C_f$, is unity and does not appear explicitly.

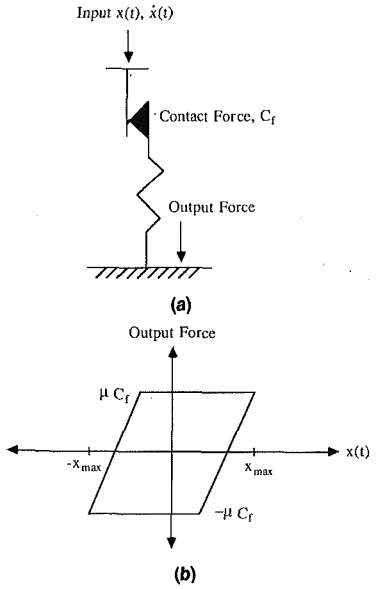


**Fig. 2 Elastic/perfectly-plastic friction damper; (*a*) graphic representation and (*b*) sample output for harmonic input**

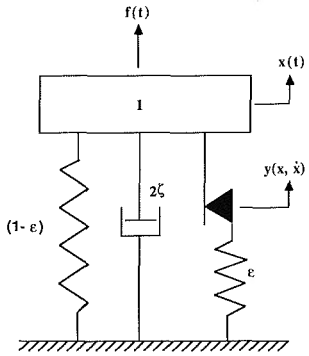


**Fig. 3 Elastic/perfectly-plastic damper and system**

The system used to illustrate the AFT method has only one degree-of-freedom. From the formulation in Section 3.1 it is clear that the method easily extends to multiple degrees-of-freedom (mdf). The computational savings of the AFT method over numerical integration should be even more dramatic for mdf systems since the number of frequencies to consider depends only on the excitation and the nonlinearities, not on the number of degrees-of-freedom.

**4.1 An Elastic/Perfectly-Plastic Friction Damper.** The sample problem was previously unsolved, although it can be solved easily by numerical integration. It would be either difficult or impossible to solve by other methods. The accuracy of the solution is judged by comparison to numerical integration. The comparison is made in the time domain to show the accuracy of the AFT method in describing both the amplitude and phase of the steady-state response.

Multiple discrete frequencies of excitation arise in high speed turbomachinery applications because as the turbine blades rotate they pass through a nonuniform, circumferentially distributed pressure field. This introduces excitation frequencies that are integer multiples of the rotation speed of the turbomachine. It is also frequently necessary to model friction dampers as having compliance, rather than being perfectly rigid. These physical considerations motivate the sdf model with an elastic/perfectly-plastic friction damper subjected to three discrete frequencies of excitation (Fig. 3). The nondimensionalized equation of motion for this system is:

$$\ddot{x}+2\zeta\dot{x}+(1-\epsilon)x=f(t)-\epsilon y(x,\dot{x}) \tag{8}$$

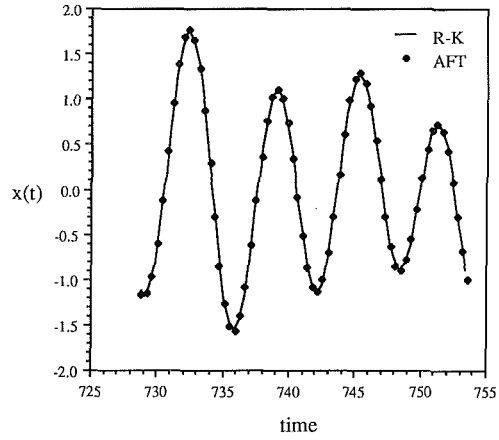


**Fig. 4 Amplitude and phase comparison between AFT method and fourth-order Runge-Kutta solutions for three frequency excitation**

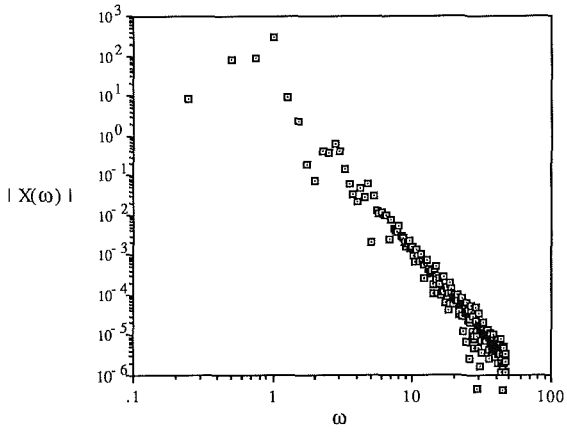


**Fig. 5 Discrete Fourier transform of response for three frequency excitation**

where $f(t)=0.75\ \sin\ (0.5\ t)+0.5\ \sin(0.75t)+0.375\ \sin(t)$, $\zeta$, the fraction of critical viscous damping, is 0.01 (1 percent), and $\epsilon$, a measure of the relative spring stiffnesses, is 0.25. The distance the mass must displace for slip of the friction damper to occur is unity. The nonlinear force on the mass is equal to the stiffness, $\epsilon$, of the elastic friction damper times the displacement of the friction contact point, $y(\bullet)$. The contact slips when the magnitude of the friction force equals $\epsilon$; the resulting force resists the motion. The specific excitation was chosen to provide an interesting response.

One period of the steady-state response of this system is shown in Fig. 4. The AFT and numerical integration solutions compare almost perfectly in both amplitude and phase. Since slip occurs at an amplitude of unity, there is considerable participation of the friction damper in this response. The frequency spectrum of the response (FFT) is shown in Fig. 5. This problem is used in the following section to make accuracy and efficiency comparisons between the AFT method and fourth-order Runge-Kutta integration.

**4.2 Convergence, Error, and Computational Efficiency.** In the absence of aliasing and leakage the FFT of a function is equivalent to a truncated Fourier series representation. Therefore, the AFT method should exhibit the convergence and error characteristics of Fourier series. Given a periodic excitation, with period $\Pi$, and assuming the time sampling is over one period, the maximum frequency in the FFT representation is:

$$\omega_{max}=\frac{\pi}{\Pi}N. \tag{9}$$

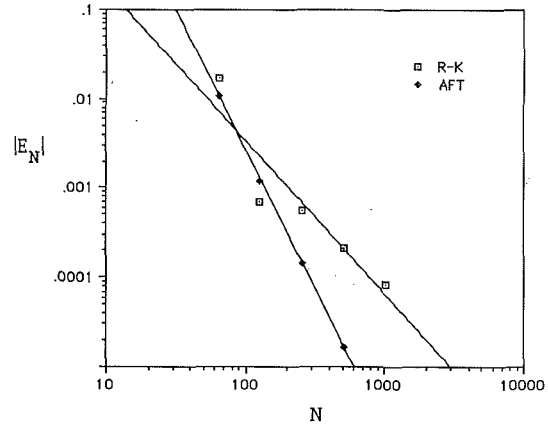


**Fig. 6 Error convergence comparison between AFT and fourth-order R-K methods**

Since the highest frequency is proportional to the number of samples per period, $N$, a continuous function which exhibits $\omega^{-2}$ convergence, like $\Lambda(\omega)^{-1}$ for second order systems, will exhibit $N^{-2}$ convergence in its FFT components. For the sample problem, the nonlinear force on the system, $N_f(t)$, is continuous, but has a discontinuous derivative, so $\eta(\omega)$ will also converge $O(N^{-2})$. Therefore, $X(\omega)$ is expected to converge $O(N^{-4})$, which is confirmed by Fig. 5. ($F(\omega)$ has infinite order convergence so it does not affect the convergence of $X(\omega)$.)

In order to compare convergence of the time domain errors, as $N$ increases, for the AFT and Runge-Kutta methods the error is assumed to be of the form:

$$E_N = x_N - x_{ex} = aN^c \tag{10}$$

where $x_N$ is the value of $x(t)$ at a specified time using $N$ sample points, $x_{ex}$ is an extrapolated "exact" answer, $E_N$ is the estimated error, $N$ is the number of sample points per period, "$a$" is a constant, and "$c$" is the constant that gives the order of convergence of the error. Using three values of $N$ gives three equations that allow the parameters $x_{ex}$, $a$, and $c$ to be evaluated. The consistency of the error convergence is checked with two of the original three values of $N$, plus and an additional, fourth value. These permit another calculation of the parameters in (10).

For the AFT method, the $O(N^{-4})$ convergence of the FFT should produce similar convergence in the time domain error. This is the case if the FFT components are known precisely. However, changing $N$ affects truncation and aliasing errors, so for different values of $N$ the FFT components for the same frequency vary slightly. Consequently, additional errors appear in the time domain. Using $N$ = 64, 128, and 256, in single precision, the AFT time domain error for equation (10) was found to be $O(N^{-3.1})$. Since the sampling period, $\Delta T$, is $\Pi/N$, this is equivalent to $O(\Delta T^{3.1})$ convergence in the error. The error was consistent when estimated with $N$= 128, 256 and 512.

When the functions in a problem are infinitely differentiable a fourth-order Runge-Kutta (R-K) integration has error $O(\Delta T^4)$, or $O(N^{-4})$. When a function has discontinuous derivatives, like $\epsilon y(t)$ (the nonlinear force in problem 4.1), that error estimate is invalid. Since the R-K method steps progressively forward in time, accumulated roundoff error can also dominate truncation error. In fact, for this problem, no consistent error convergence (using (10)) was observed for R-K in single precision, with $N$= 64, 128, 256, 512, and 1024. Using the same values of $N$ in double precision R-K simulations, the error was $O(\Delta T^{1.6})$, or $O(N^{-1.6})$. The error results for both R-K and AFT methods are plotted in Fig. 6.

It is impossible to make a general comparison of computational efficiency between the AFT and R-K methods. The efficiencies of both depend strongly on system parameters. Depending on damping, for example, the system may require any number of periods of oscillation before steady-state is reached. Another concern for R-K is that with complicated wave forms, like the solution to this problem (Fig. 4), it is very difficult to determine when steady-state has been achieved. The AFT method tests for convergence in the frequency domain where it is easy to discern. However, for a sample comparison, the following test was performed. This sample problem – a sdf system with three input frequencies and an elastic/perfectly-plastic friction damper – was analyzed through the 30th period of oscillation. The 30th period was selected as representative of an average number of cycles to reach steady-state. In order to achieve the same accuracy (i.e., similar error in Fig. 6) the AFT method was implemented in single precision with 256 points per period, and the R-K method was implemented in double precision with 512 points per period. In this case the AFT method took 21.7 seconds[6] and Runge-Kutta took 389 seconds on a Macintosh II, a factor of 18 difference.

## 5 Features of the AFT Method

Certain fundamental concepts underlying the AFT method are also employed in the FG (Fast Galerkin) method introduced by Ling and Wu (1987). Ling and Wu implement Urabe's Galerkin method with an FFT, making the method computationally efficient. In Urabe's approach the system of nonlinear equations is put into state variable form, a solution is asssumed in the form of a truncated Fourier series, and the coefficients of the Fourier series are solved for iteratively – with nonlinear terms evaluated in the time domain. Since the coefficients of the Fourier series are equivalent to the components of the DFT, the AFT and FG methods are similar in this respect. There are, however, differences in how the methods formulate and implement solutions of equations. These differences highlight some of the features of AFT method.

The first difference between the AFT and Galerkin methods lies in the form of the equations to be solved. The AFT method permits the equations to be of any order, while Urabe's formulation, which Ling and Wu repeat, assumes the equations are in first order state variable form. The AFT method only finds the linearly independent functions, thereby reducing the number of equations to be solved.

Another feature of the AFT formulation is the general manner in which it handles nonlinearities. In the problems considered by Urabe, as well as Ling and Wu, the nonlinearities are expressed analytically. In many applications, such as the elastic-plastic damper problem, the nonlinearities are hysteretic and analytical expressions may be difficult or impossible to obtain. In the AFT approach the user is provided with the full state history in order to compute the nonlinear forces in a user supplied subprogram, in a manner similar to that used in standard time integration codes.

Perhaps the most significant computational advantage of the AFT method is its technique for reducing the number of equations. As discussed previously, increasing the number of frequencies included in the solution can significantly increase

[6] It would also be possible to solve this problem using Runge-Kutta time integration in conjunction with a two point boundary value formulation. In this approach the displacement and velocity of the mass at the beginning of the period are unknowns whose values must be determined so that the solution satisfies the periodicity requirements. Such an approach requires an iterative method for solving the resulting nonlinear constraint equations (for a discussion of this approach see, for example, Press et al., 1986, Chap. 16). This procedure is less standard than time integration and, consequently, was not investigated here. However, it could be significantly more efficient than time integration and more competitive with the computational efficiency of the AFT method. Again, the outcome of such a comparison would depend on the specific physical system considered, as well as on the complexity of the frequency content of the excitation.

the computational effort, but in many problems it may be necessary to consider both low and high frequencies. The AFT implementation permits the user to include a relatively large number of frequencies and to set a "cutoff" fraction. While using direct iteration (which is computationally efficient), the algorithm scans all the frequency components to estimate which components lie above the cutoff fraction relative to the largest component. These components are considered "significant." If it becomes necessary to employ an intensive equation solver, like Newton-Raphson, only the equations corresponding to the significant components are included. This approach also makes it easy for the user to examine tradeoffs between higher accuracy (low cutoff) and lower computation time (high cutoff).

## 6 Concluding Remarks

In the absence of aliasing, leakage, and roundoff errors the AFT method converges uniformly to the exact answer as $N \to \infty$. Even when it cannot be eliminated, means are available for limiting the error so that good approximate solutions may be obtained. For example, in many problems the power in a signal decays asymptotically to zero as the frequency increases, so aliasing cannot be eliminated (without analog filtering). However, it is still possible to obtain exact solutions to within the precision of the computer by selecting $\Delta T$ so that power at frequencies exceeding $(\pi/\Delta T)$ cannot numerically affect the solution. In the sample problem the error in the AFT method dropped off much more quickly than the error in numerical integration.

Solutions generated by the AFT method are conveniently presented in either the frequency or time domain, or both. The AFT method also lends itself to parametric and sensitivity studies because the solution for one set of parameters supplies a good estimate of the solution for a "neighboring" set of parameters, which aids in convergence of the iterations.

Relative to other frequency domain methods, the AFT method is amenable to more general, complicated nonlinearities and excitations. By switching to the time domain to evaluate the nonlinear forces, the AFT method can use the same description of the nonlinear forces used for numerical integration. The only constraint on the excitation is that its Fourier transform can be approximated using the FFT approach.

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