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Application of Special Nonsmooth Temporal Transformations to Linear and Nonlinear Systems under Discontinuous and Impulsive Excitation

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Abstract. Linear and nonlinear mechanical systems under periodic impulsive excitation are considered. Solutions of the differential equations of motion are represented in a special form which contains a standard pair of nonsmooth periodic functions and possesses a convenient structure. This form is also suitable in the case of excitation with a periodic series of discontinuities of the first kind (a stepwise excitation). The transformations are illustrated in a series of examples. An explicit form of analytical solutions has been obtained for periodic regimes. In the case of parametric impulsive excitation, it is shown that a nonequidistant distribution of the impulses with dipole-like temporal shifts may significantly effect the qualitative characteristics of the response. For example, the sequence of instability zones loses its different subsequences depending on the parameter of the shifts. It is shown that the method's applicability can be extended for nonperiodic regimes by involving the idea of averaging.

Keywords: Pulsed excitation, parametric loading, nonsmooth transformations, nonsmooth dynamics.

1. Introduction

Instantaneous impulses acting on a mechanical system can be taken into account either by imposing the special conditions of matching for the velocities and coordinates of the system at points of the impulses' location or by introducing Dirac's functions into the equations of motion. The first approach, in principal, deals with the differential equations for an unforced system [1]. However, these equations are treated separately in each of the intervals between impulses and, hence, instead of a single system, a whole sequence of systems must be considered. The second method gives a single set of equations over the whole time interval without any conditions of matching imposed on the variables. However, in this case, analysis can be carried out correctly within the framework of the theory of generalized functions (distributions) [2], which requires additional mathematical proofs in nonlinear cases [3, 4] (see this book also for a more complete list of publications regarding systems with impulses). Both of the above-mentioned ways of modeling are fruitfully employed for different quantitative and qualitative considerations of mechanical systems under pulsed excitations [1, 5–9]. A method is described below which enables one, on the one hand, to eliminate the singular terms in the equations and, on the other hand, to obtain solutions in the form of a single analytic expression over the whole time interval. Originally this method was proposed for a class of smooth

vibrating systems. Here, the term ‘smooth’ means that the motion is described by means of smooth (differentiable) functions of time. However, a special feature of the technique is that the smooth solutions to be defined are expressed through a pair of *nonsmooth functions* such as a saw-tooth sine and rectangular cosine. Mathematical properties of the related transformation can be briefly listed as follows [10]:

- an introduction of the nonsmooth (saw-tooth) time transforms the coordinates into elements of two-components hyperbolic numbers algebra¹;
- under certain conditions, the result of the differentiation of the elements remains in the algebra, and the result of integration remains in the algebra;
- a slow timescale can be introduced into the transformation to describe the nonperiodic processes.

Based on these points, the introduction of a nonsmooth independent variable into differential equations of motion can be justified. Different physical and mechanical applications and treatments of saw-tooth temporal transformation were discussed in [11, 12]. It should be noted that the transformation itself has to be viewed as a preliminary step of investigation. It reduces an original system to a boundary-value problem with respect to a new independent variable, i.e., the saw-tooth bounded argument. Then, one can invoke the appropriate method to solve the boundary-value problem accordingly to the physical contents of the model. For example, special successive iterations processes and asymptotic techniques were used for different systems performing weakly and strongly nonlinear periodic motions, and modulated regimes as well [11, 12]. The shooting method was applied to study a complicated set of periodic orbits close to the separatrix [13]. The transformation can also be realized with respect to a spatial independent variable when dealing with different spatially periodic structures [14–16]. As was mentioned above, this paper focuses on a class of dynamical systems under periodic impulsive and discontinuous excitations. A generalized version of the transformation based on the saw-tooth function with an asymmetric slope [17] will also be considered. This enables us to take into account the dipole-like shifts of singularities of the external force and, hence, to involve a more general class of dynamical systems.

2. Some Reasons for Nonsmooth Transformations

This section presents some remarks concerning nonsmooth temporal transformations. Some of the basic relations will be reproduced for future references as well. A theoretical reason for the nonsmooth transformations follows from a simple general observation. Indeed, linear transformation of coordinates may significantly simplify a problem of linear dynamics. Weakly nonlinear transformations of coordinates can be viewed as a basic step in most of the analytical techniques in the theory of quasilinear systems. The theory of solitons gives a great example of a strongly nonlinear theory based on strongly nonlinear transformations of variables. Now one can conclude that dynamical systems with discontinuities can be simplified by means of the appropriate nonsmooth transformations of variables. For instance, in the case of systems with absolutely rigid constraints, special nonsmooth transformations have been proposed in [18]. This realization deals with the *positional variables (coordinates)* of a *vibro-impact*² system and enables one to eliminate the constraints that the vibro-impact

¹ Complex numbers $X + Ye$, where $e^2 = 1$ (see the next section).

² In terms of the related field of the applied mechanics, see V. I. Babitsky, *Theory of Vibroimpact Systems*, Moscow, Nauka, 1978 [in Russian].

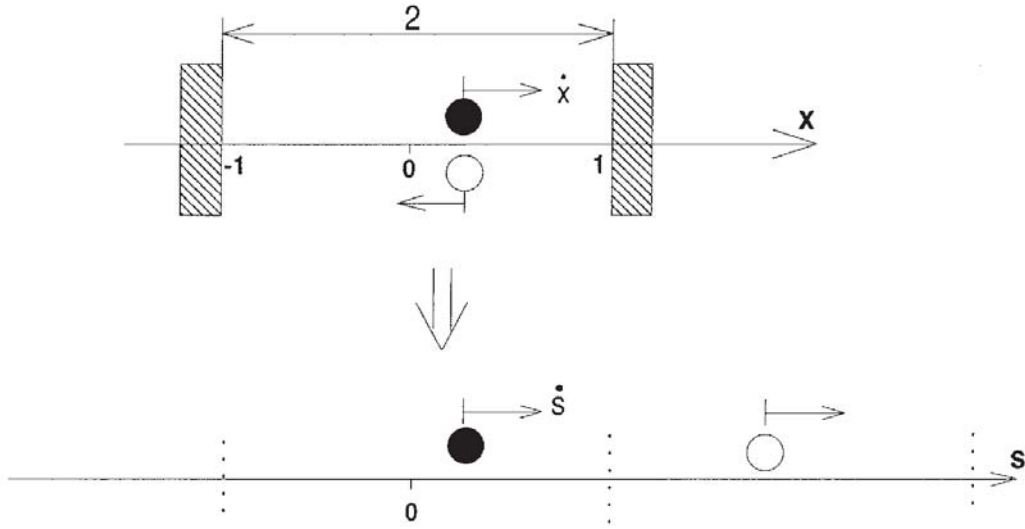


Figure 1. Geometrical treatment of the nonsmooth transformation of a positional variable on the simplest vibro-impact oscillator. In this case, $x \in [-1, 1]$ is the original coordinate of the system, while $s \in (-\infty, \infty)$ is the distance passed by the system. The figure illustrates reflection from the right barrier.

systems must include by definition. The transformations instantaneously change the direction of the coordinate axes at the instance of impact (a geometrical treatment is given in Figure 1). As a result, the transformed system becomes free from its absolutely rigid barriers and the corresponding differential equations do not contain any ‘impact terms’.

Unfortunately, there is no direct way to use nonsmooth transformations of coordinates in the case of the external impacts, due to external force. Another question is how to apply the nonsmooth transformation of coordinates to a broad class of systems which, within a certain meaning, are close to vibro-impact, but do not include any absolutely rigid constrains. A related attempt was made in [19] when considering the localized vibro-impact modes in a linear string with concentrated masses between the two rigid walls. To deal with differential equations of motion over all time domains and under no constraint conditions, the rigid walls were replaced by a special potential field with a power function of a higher odd exponent, x^m . In this case, the classical vibro-impact situation appears when $m \rightarrow \infty$, as the rigid walls limit. Is it possible to construct a transformation of the coordinate $x(t) \rightarrow s(t)$ depending on m as $x = f(s, m)$ and, giving nonsmooth transformation by [18], when $m \rightarrow \infty$? This question has not yet been answered. Probably, it is hardly possible before the solution of the original differential equation is known. However, starting from a study of the asymptotic $m \rightarrow \infty$, it was preliminarily shown in [20] that for certain kind of systems, a nonsmooth transformation of time can be introduced as

$$\tau = \tau(t) : (-\infty, \infty) \ni t \rightarrow \tau \in [-1, 1], \quad (1)$$

$$x(t) = x(\tau(t)) \equiv X(\tau), \quad (2)$$

where $x(t)$ is a periodic function, the period of which is normalized to be equal to four, and τ is a saw-tooth periodic function defined as

$$\tau(t) = \begin{cases} t, & -1 \leq t \leq 1 \\ -t + 2, & 1 \leq t \leq 3 \end{cases} ; \quad \tau(t) \stackrel{\forall t}{=} \tau(t + 4). \quad (3)$$

This function can also be expressed as $\tau(t) = (2/\pi) \arcsin \sin(\pi t/2)$. A possible geometrical treatment of Equation (1) is given in Figure 2. It is important to note that transformation (1)

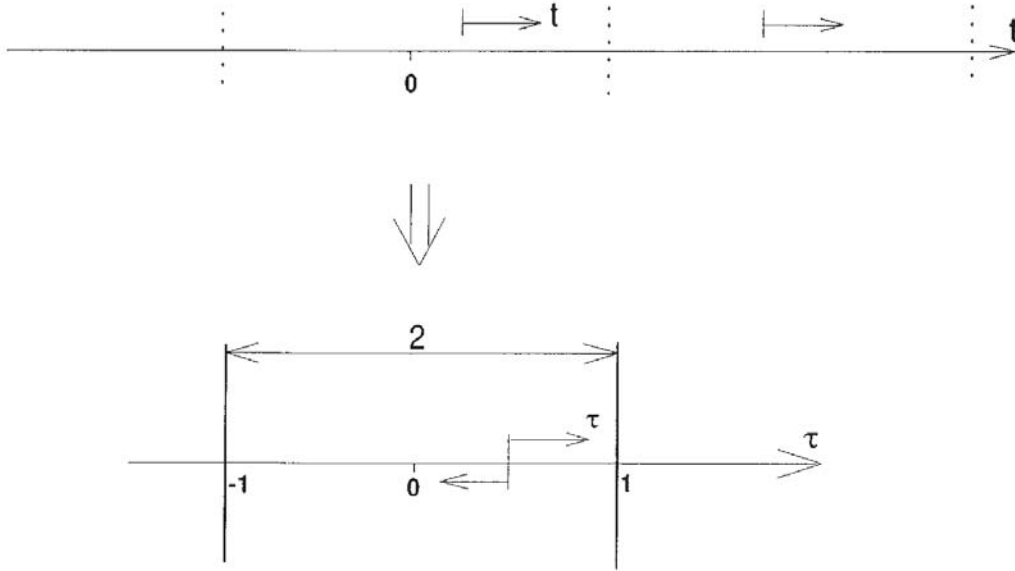


Figure 2. Geometrical treatment of the nonsmooth transformation of time. The new time oscillates in the bounded domain $\tau \in [-1, 1]$ with the constant velocity reflecting at points $\tau = \pm 1$, while the original time is unbounded, $t \in (-\infty, \infty)$.

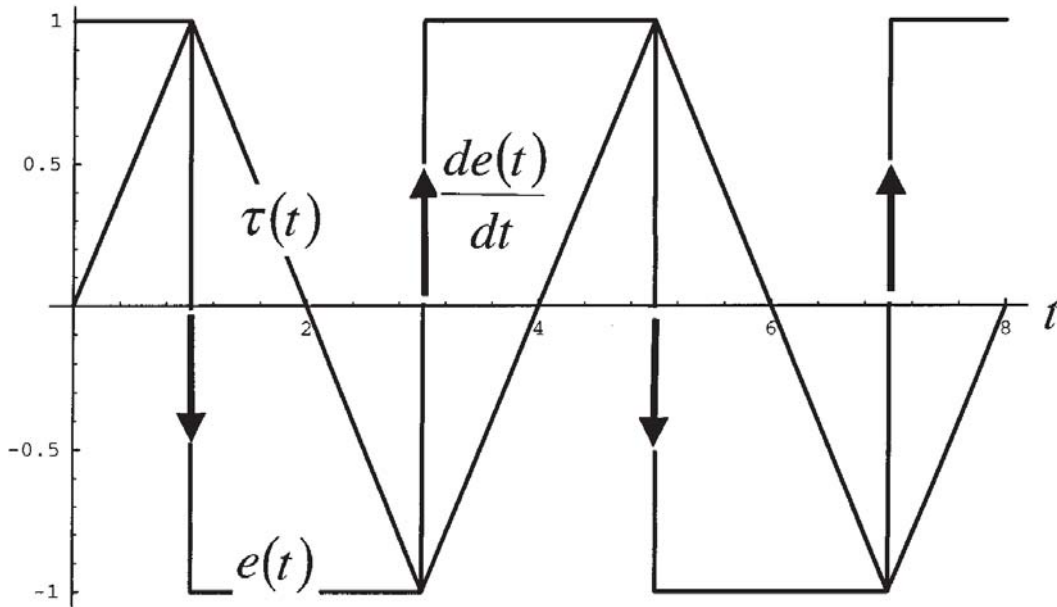


Figure 3. The saw-tooth sine and its first two generalized derivatives.

is not invertible over all periods of motion $T = 4$ and, hence, representation (2) is not always correct. For any periodic motion, a representation has been obtained in a two-component form. It is written in terms of both nonsmooth oscillating time, τ , and its first generalized derivative, $e = d\tau(t)/dt = \dot{\tau}(t)$ (Figure 3), as [10]

$$x(t) = X(\tau) + Y(\tau)e, \quad (4)$$

where the functions X and Y are either known (if the original function, $x(t)$, is known), or supposed to be defined as a solution of the transformed equations.

Expression (4) can be viewed as a special ‘complex’ value with real X and ‘imaginary’ Y components, where the basis ‘imaginary’ element e generates a circling group: $e^2 = 1$,

$e^3 = e$, $e^4 = 1$, \dots . These complex numbers are elements of algebra without division by the basis elements $\{1, e\}$. This remark significantly simplifies all mathematical manipulations with the representation when transforming the differential equations of motion. For example,

$$\begin{aligned}(X + Ye)^2 &= X^2 + Y^2 + 2XYe, \\ (X + Ye)^3 &= X^3 + 3XY^2 + (Y^3 + 3YX^2)e.\end{aligned}$$

Here, the right-hand sides are of the structure $\{\dots\} + \{\dots\}e$, as the element $X + Ye$ itself is. A less trivial equality

$$\exp(X + Ye) = \exp(X)[\cosh(Y) + \sinh(Y)e] \quad (5)$$

will be taken into account below.

Taking into account that $e^2 = 1$, one obtains

$$\frac{1}{X + Ye} = \frac{X}{X^2 - Y^2} - \frac{Y}{X^2 - Y^2}e \quad \text{if } X \neq \pm Y.$$

The last example explains why division is not always possible.

In this paper, we also use a generalized version of representation (4) based on the saw-tooth function with an asymmetric slope:

$$\begin{aligned}\tau(t; \gamma) &= \begin{cases} t/(1 - \gamma), & -1 + \gamma \leq t \leq 1 - \gamma, \\ (-t + 2)/(1 + \gamma), & 1 - \gamma \leq t \leq 3 + \gamma, \end{cases} \\ e(t; \gamma) &= \frac{\partial \tau(t; \gamma)}{\partial t} \quad (\tau(t; \gamma) \stackrel{\forall t}{=} \tau(t + 4; \gamma)),\end{aligned} \quad (6)$$

where $\gamma \in (-1, 1)$ is a parameter characterizing the inclination of the ‘saw’s teeth’ as Figure 4 shows. The parameter of inclination γ is introduced in such a manner that the period $T = 4$ of the function and its amplitude value ($= 1$) do not depend on the inclination.

In this case, the algebraic structure becomes more complicated. The above multiplication rule, $e^2 = 1$, should be replaced with a more complicated one:

$$e^2 = \alpha(\gamma) + \beta(\gamma)e; \quad \alpha = \frac{1}{1 - \gamma^2}, \quad \beta = 2\gamma\alpha. \quad (7)$$

In the symmetric case, one has $\alpha(0) = 1$, $\beta(0) = 0$, and (7) is reduced to the old multiplication rule.

Properties of representation (4) based on the asymmetric saw-tooth function (6) were considered in previous work [17]. Note that the result of differentiation of Equation (4) formally contains a periodic series of singularities at points $\Lambda = \{t : \tau(t; \gamma) = \pm 1\}$. The singularities are discarded by imposing the condition

$$Y|_{t \in \Lambda} = Y|_{\tau = \pm 1} = 0 \quad (8)$$

that can be viewed as a necessary condition for the continuity for $x(t)$. Indeed, the step-wise function $e(t; \gamma)$ has discontinuities at points Λ . It means that the ‘imaginary part’ of representation (4), $Y(\tau)e$, would be discontinuous at Λ , if condition (8) was false.

Now, let us assume that condition (8) is *false* and one has

$$Y|_{\tau = \pm 1} = p \neq 0. \quad (9)$$

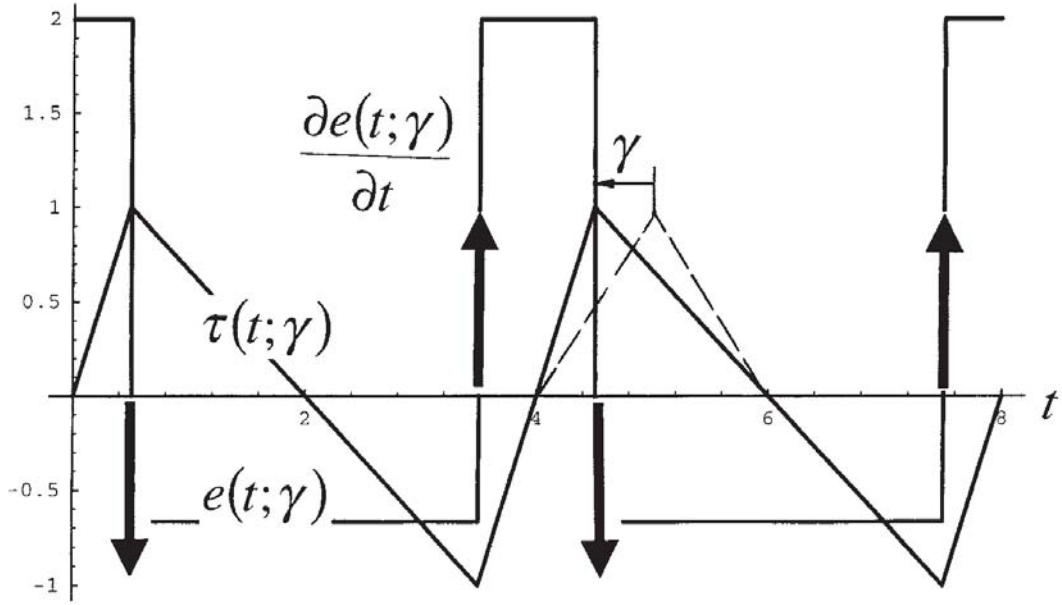


Figure 4. The saw-tooth function with asymmetric slope and its first two generalized derivatives. Parameter γ defines the teeth's inclination; it has been taken as $\gamma = 0.7$.

In this case, function $x(t)$ represented by equality (4) has a periodic series of steps of amplitude $2p/(1 - \gamma^2)$ at points Λ . An idea of this work is to employ these discontinuities in order to describe nonsmooth dynamical processes.

Note that nonsmooth temporal transformation does not necessarily imply any absolutely rigid barriers. *Instead of the coordinates, the parameter of time is transformed by means of the nonsmooth function.* As a result, the coordinates acquire a special algebraic structure with a series of mathematically suitable properties. In the next sections, this kind of transformation will be applied to systems subjected to impulsive and discontinuous excitations. Transformation (1–4) instantaneously changes the direction and scale of time when the external impulses act.

Note that the imaginary basic element e of representation (4) is a periodic piece-wise constant step function of time t . In this connection, we note the paper by Liu [21] where the solutions of quasilinear systems under the action of an impulsive or discontinuous force (with a singularity at the point $t = a$) were presented in the form $x(t) = x_1(t) + x_2(t)H(t - a)$, where $H(t)$ is Heaviside's step function. An analogous form of the solution is used to describe moving discontinuities in the wave theory in [22]. It is known [4, p. 18] that the representation $x(t) = y(t) + vH(t - a)$ ($v = \text{const.}$) transforms the nonlinear system $\dot{x} = f(t, x) + v\delta(t - a)$ into a so-called Carathéodory equation with respect to $y(t)$ (see Appendix 1). The Carathéodory form of equation admits a series of existence theorems. A principal difference is that components $x_1(t)$, $x_2(t)$ of the above representation depend on the original time parameter, whereas equality (4) provides the introduction of the oscillating time bounded as $\tau \in [-1, 1]$.

To this end, note that representations (1), (4) and the representation based on functions (6) as well, stay to transform a system on the manifold of periodic solutions. This may bring doubt regarding the practical meaning of the representations. That is why Sections 5 and 6 show how to extend the applicability of the representation. For example, the idea of averaging can be involved, if inserting slow varying parameters into the transformation. Let $t^\circ = \varepsilon t$ be a

slow time scale, where $0 < \varepsilon \ll 1$. In the case of damped or modulated vibrations, solutions can be sought in the form [10]

$$x = X(\tau, t^\circ) + Y(\tau, t^\circ)e,$$

where $\tau = \tau(\omega t)$, $e = e(\omega t)$ and the constant ω is assumed to be of order 1.

Of course, this representation does not lead to ‘another method’ of the two variables expansion. However, replacing the fast timescale by oscillating argument τ , one brings some special features into the two-variables expansion method. For instance, a system’s fast time oscillation will be described with the boundary-value problem on the standard domain $\tau \in [-1, 1]$ and the slow time (averaged) equations follow from the boundary conditions at points $\tau = \pm 1$ but are not a result of the elimination of the secular terms.

3. Systems with Periodic Pulsed Excitation

We will start from the simple case of a linear system to illustrate an application of the non-smooth temporal transformation with no additional complications which may be brought about by technical details related to a concrete practical problem. Also, the example itself has a definite practical meaning.

3.1. PERIODIC SERIES OF IMPULSES

We will seek periodic solutions of the first-order differential equation

$$\dot{v} + \lambda v = \mu \sum_{k=-\infty}^{\infty} [\delta(t + 1 - 4k) - \delta(t - 1 - 4k)]; \quad \lambda, \mu = \text{const.} \quad (10)$$

For positive λ , the function to be determined, $v(t)$, can be treated as a velocity of a point moving under periodic impulsive force and viscous damping. This equation has a simple form involving the generalized functions. However, the common ways of solving, generally speaking, require first the determination of a class of functions among which the solution should be sought. Then, one can apply either the generalized Fourier series/transforms procedure, or consider the equation between impulses matching pieces of the solution at the points of impulse. The saw-tooth temporal transformation gives us another way to get a closed-form analytical solution.

First, the right-hand side of Equation (10) can be expressed in terms of a generalized derivative of rectangular cosine (see Figure 3). As a result, Equation (10) is rewritten as

$$\dot{v} + \lambda v = \frac{\mu}{2} \dot{e}(t). \quad (11)$$

Representing periodic solutions in the form of (4), $v(t) = X(\tau(t)) + Y(\tau(t))e(t)$, and substituting this into Equation (11), gives

$$(Y' + \lambda X) + \underbrace{(X' + \lambda Y)e + \left(Y - \frac{\mu}{2}\right) \dot{e}(t)} = 0.$$

The first two terms in this expression present an element of structure (4). Both components must be separately equal to zero to provide a complete zero element.³ This will give us a set

³ Note that elements 1 and e are linearly independent on the time domain considered.

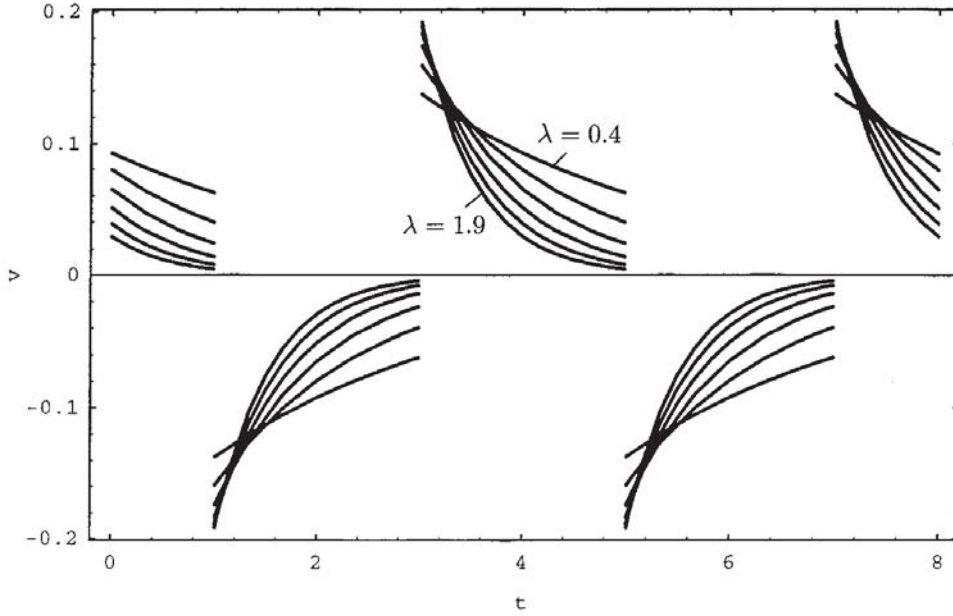


Figure 5. Family of periodic solutions of Equation (11) for $\mu = 0.2$ and $\lambda = 0.4, 0.7, 1.0, 1.3, 1.6, 1.9$.

of two first-order differential equations. The elimination of the last (underlined) term leads to the boundary conditions. As a result, one obtains the boundary-value problem as

$$\begin{aligned} Y' + \lambda X &= 0, & X' + \lambda Y &= 0 \\ Y|_{\tau=\pm 1} &= \frac{\mu}{2}. \end{aligned} \quad (12)$$

So the original differential equation (10) has been transformed into boundary-value problem (12) for two first-order differential equations. The basic advantages, however, is that the transformed system does not include δ -functions any more, and the new independent variable is bounded, $|\tau| \leq 1$.

Solving boundary-value problem (12), one obtains the periodic solution of the original equation as

$$v = X + Ye = \frac{\mu}{2 \cosh \lambda} (-\sinh \lambda \tau + e \cosh \lambda \tau).$$

Taking into account equality (5) and the expression $e^2 = 1$, finally gives

$$v = \frac{\mu}{2 \cosh \lambda} \exp[-\lambda \tau(t)e(t)]e(t). \quad (13)$$

Figure 5 shows dependencies $v(t)$ for different values of λ . Note that the discontinuous solution $v(t)$ is described by a single expression of the closed form (13) using the two standard nonsmooth periodic functions $\tau(t)$ and $e(t)$. These functions admit a clear mechanical treatment as the coordinate and velocity of the vibro-impact oscillator (see Figure 1) under the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$ (see Section 7 for more details).

The boundary-value problem (12) is replaced by a more complicated one, when considering the impulsive excitation with a dipole-like shift, i.e. the right-hand side of Equation (11) is expressed by a derivative of the asymmetric saw-tooth function with a slope γ [17]:

$$\dot{v} + \lambda v = p \frac{\partial e(\omega t; \gamma)}{\partial(\omega t)}$$

$$= \frac{2p}{1-\gamma^2} \sum_{k=-\infty}^{\infty} [\delta(\omega t + 1 - \gamma - 4k) - \delta(\omega t - 1 + \gamma - 4k)]. \quad (14)$$

In this case, representation for a periodic solution is taken as

$$v = X(\tau) + Y(\tau)e; \quad \tau = \tau(\omega t; \gamma), \quad e = e(\omega t; \gamma). \quad (15)$$

Substituting expression (15) into differential equation (14) and collecting separately the ‘real’ and the ‘imaginary’ components and singular terms, one obtains

$$(\omega\alpha Y' + \lambda X) + [\omega(X' + \beta Y') + \lambda Y]e + (\omega Y - p) \frac{\partial e(\omega t; \gamma)}{\partial(\omega t)} = 0.$$

The resulting boundary-value problem is

$$\begin{aligned} \omega(X' + \beta Y') &= -\lambda Y, & \omega\alpha Y' &= -\lambda X, \\ \omega Y|_{\tau=\pm 1} &= p. \end{aligned} \quad (16)$$

Taking the boundary conditions into account, one obtains

$$\begin{aligned} Y &= \frac{p}{\omega} \left[\cosh\left(\gamma \frac{\lambda}{\omega}\right) \frac{\cosh\left(\frac{\lambda}{\omega}\tau\right)}{\cosh\frac{\lambda}{\omega}} + \sinh\left(\gamma \frac{\lambda}{\omega}\right) \frac{\sinh\left(\frac{\lambda}{\omega}\tau\right)}{\sinh\frac{\lambda}{\omega}} \right] \exp\left(\gamma \frac{\lambda}{\omega}\tau\right), \\ X &= -\frac{\omega\alpha}{\lambda} Y'. \end{aligned} \quad (17)$$

The X -component is defined by differentiation according to the second equation in (16).

3.2. HARMONIC OSCILLATOR UNDER PERIODIC IMPULSIVE EXCITATION

We will seek a steady-state periodic solution of the second-order differential equation

$$\ddot{x} + \omega_0^2 x = 2p \sum_{k=-\infty}^{\infty} [\delta(\omega t + 1 - 4k) - \delta(\omega t - 1 - 4k)], \quad (18)$$

where p , ω_0 and ω are constant parameters.

In the first step, Equation (18) is rewritten as

$$\ddot{x} + \omega_0^2 x = p \frac{de(\omega t)}{d(\omega t)}. \quad (19)$$

We seek a periodic solution of the period $T = 4/\omega$ in the form

$$x(t) = X(\tau(\omega t)) + Y(\tau(\omega t))e(\omega t). \quad (20)$$

Substituting this into Equation (19), under the condition of continuity for $x(t)$ (Equation (8)), gives

$$(\omega^2 X'' + \omega_0^2 X) + (\omega^2 Y'' + \omega_0^2 Y)e + (\omega^2 X' - p) \frac{de(\omega t)}{d(\omega t)} = 0.$$

The first two terms in this expression present an element of the algebraic structure considered. Both components must be separately equal to zero to provide a complete zero element. This

will give us a set of two first-order differential equations. Elimination of the last (underlined) term leads to the boundary conditions. As a result, one obtains the boundary-value problem as

$$\begin{aligned} X'' + \left(\frac{\omega_0}{\omega}\right)^2 X &= 0, & Y'' + \left(\frac{\omega_0}{\omega}\right)^2 Y &= 0, \\ X'|_{\tau=\pm 1} &= \frac{P}{\omega^2}, & Y|_{\tau=\pm 1} &= 0. \end{aligned} \quad (21)$$

So the original differential equation (18) has been transformed into boundary-value problem (21) for two second-order differential equations. The basic advantage, however, is that the transformed system does not include δ -functions any more. By solving boundary-value problem (21), one obtains the periodic solution of the original equation as

$$x = X(\tau(\omega t)) = \frac{P}{\omega\omega_0} \frac{\sin\left[\frac{\omega_0}{\omega}\tau(\omega t)\right]}{\cos\frac{\omega_0}{\omega}}, \quad Y \equiv 0. \quad (22)$$

This is a continuous solution for all t , and all possible resonances are given by the roots of the equation $\cos(\omega_0/\omega) = 0$ as

$$\omega = \frac{2}{\pi} \frac{\omega_0}{n}, \quad n = 1, 3, 5, \dots, \quad (23)$$

where the factor $2/\pi$ appeared due to different normalization of the periods for trigonometric and saw-tooth sines.

In the presence of linear viscous damping, boundary-value problem (21) becomes coupled throughout the terms $Y'(\tau)$ and $X'(\tau)$ in the first and second equations, respectively. However, the problem remains resolvable due to the linearity. Let the equation of motion be written as

$$\ddot{x} + 2\zeta\dot{x} + \omega_0^2 x = p \frac{de(\omega t)}{d(\omega t)}, \quad (24)$$

where ζ is the damping factor.

The boundary-value problem for the X and Y components of the steady-state periodic solution has no discontinuous terms and is written as

$$\begin{aligned} X'' + 2\frac{\zeta}{\omega}Y' + \left(\frac{\omega_0}{\omega}\right)^2 X &= 0, & Y'' + 2\frac{\zeta}{\omega}X' + \left(\frac{\omega_0}{\omega}\right)^2 Y &= 0, \\ X'|_{\tau=\pm 1} &= \frac{P}{\omega^2}, & Y|_{\tau=\pm 1} &= 0. \end{aligned} \quad (25)$$

Solving the boundary-value problem in a standard way involving the characteristic equation, one finally obtains

$$\begin{aligned} x = X + Ye &= \frac{P}{\beta\omega^2(\cos^2\beta \cosh^2\alpha + \sin^2\beta \sinh^2\alpha)} \\ &\times [\cosh\alpha \cos\beta \cosh\alpha\tau \sin\beta\tau - \sinh\alpha \sin\beta \sinh\alpha\tau \cos\beta\tau \\ &+ (\sinh\alpha \cos\beta\tau \cosh\alpha\tau \sin\beta - \sinh\alpha\tau \sin\beta\tau \cosh\alpha \cos\beta)e], \end{aligned} \quad (26)$$

where $\tau = \tau(\omega t)$ and $e = e(\omega t)$; $\alpha = \zeta/\omega$ and $\beta = \sqrt{\omega_0^2 - \zeta^2}/\omega$.

Figure 6 illustrates the qualitatively different responses of the system when varying the external excitation frequency. In different proportions, the responses combine properties of

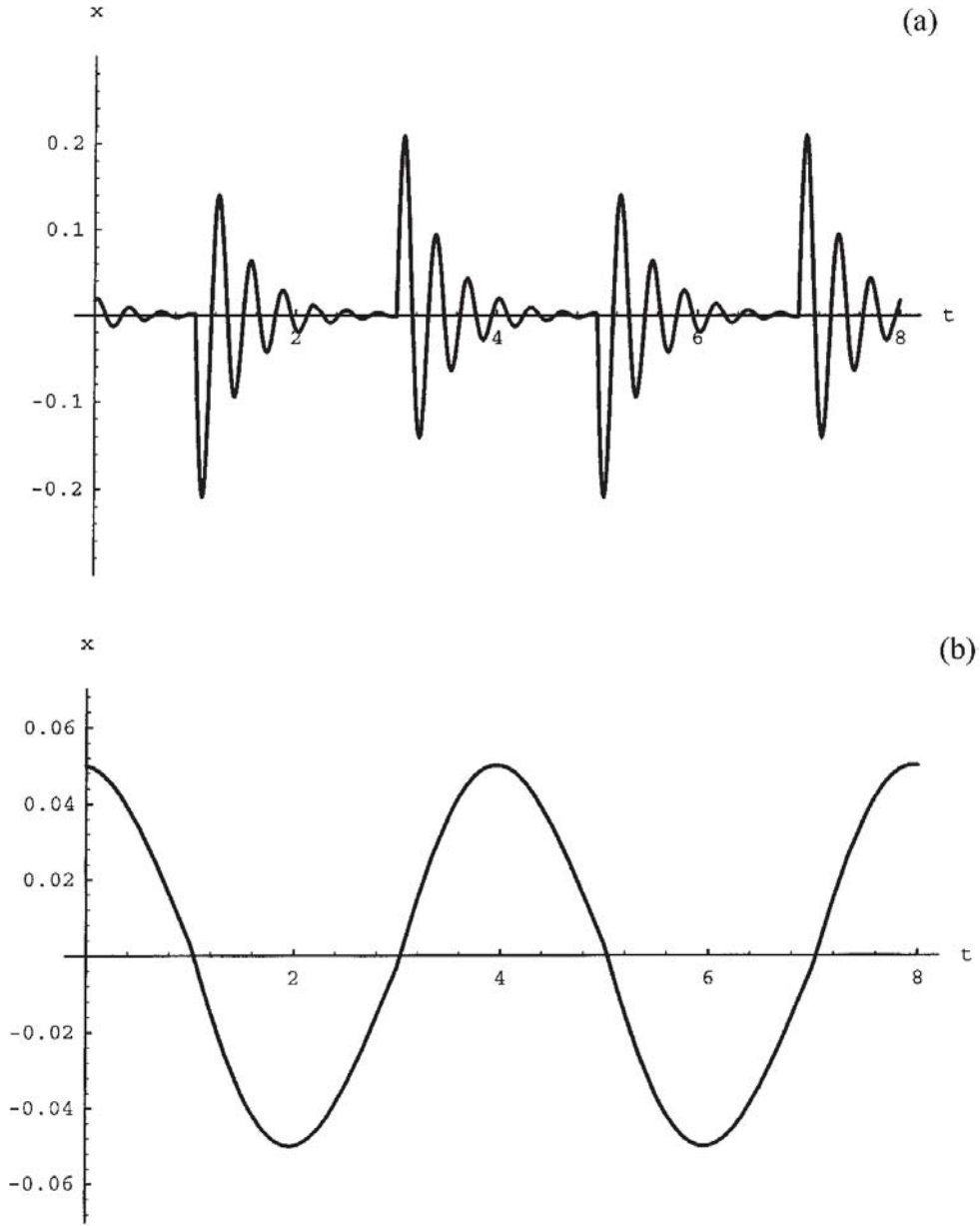


Figure 6. Different responses of the damped harmonic oscillator under the periodic impulsive excitation for $p = 0.1$, $\zeta = 0.5$, $\omega_0 = 4$ and (a) $\omega = 0.2$ (large distance between the impulses); (b) $\omega = (2/\pi)\omega_0 = 2.5465$ (principal resonance).

the harmonic damped and nonsmooth motions generated by the impact loading. For example, when $\omega \gg \omega_0$ and $\omega \gg \zeta$, the system approaches the limit of a free particle under the periodic impulsive force. In this case, the boundary-value problem is reduced to

$$X'' = 0, \quad Y'' = 0; \quad X'|_{\tau=\pm 1} = \frac{P}{\omega^2}, \quad Y|_{\tau=\pm 1} = 0 \quad (27)$$

and its solution gives the saw-tooth time shape of the motion, $x = p\tau(\omega t)/\omega^2$, which is approached by the time history record in Figure 6c.

Now let us consider the N -degrees-of-freedom system

$$M\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{p} \frac{de(\omega t)}{d(\omega t)}, \quad (28)$$

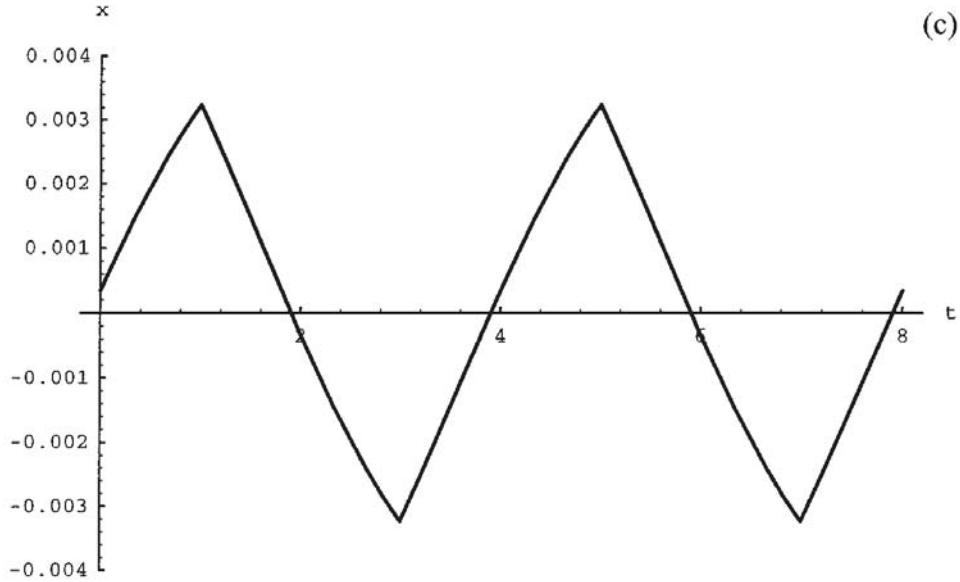


Figure 6. Continued. (c) $\omega = 6$ (small distance between the impulses).

where $\mathbf{y}(t)$ is an N -dimensional vector function, M and K are constant $N \times N$ mass and stiffness matrices, respectively. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ and $\omega_1, \dots, \omega_N$ be the principal coordinate normalized basis and natural frequencies such as the following conditions hold

$$K\mathbf{e}_j = \omega_j^2 M\mathbf{e}_j, \quad \mathbf{e}_k^T M\mathbf{e}_j = \delta_{kj}; \quad k = 1, \dots, N; j = 1, \dots, N.$$

Introducing the principal coordinates $x^j(t)$ as

$$\mathbf{y} = \sum_{j=1}^N x^j(t)\mathbf{e}_j, \quad (29)$$

one obtains a decoupled set of impulsively forced harmonic oscillators

$$\ddot{x}^j + \omega_j^2 x^j = p^j \frac{d\epsilon(\omega t)}{d(\omega t)}, \quad (30)$$

where $p^j = \mathbf{e}_j^T \mathbf{p}$.

Making use of solution (22) for each of the oscillators (30) and taking into account expansion (29), one obtains the steady-state response

$$\mathbf{y} = \sum_{j=1}^N \frac{(\mathbf{e}_j^T \mathbf{p})\mathbf{e}_j}{\omega\omega_j} \frac{\sin\left[\frac{\omega_0}{\omega_j}\tau(\omega_j t)\right]}{\cos\frac{\omega_0}{\omega_j}}. \quad (31)$$

The resonance condition is

$$\omega = \frac{2}{\pi} \frac{\omega_j}{n}; \quad n = 1, 3, 5, \dots; j = 1, \dots, N.$$

These resonances can also be obtained in a regular way, by taking the generalized Fourier expansion of the right-hand side of Equation (30). Note, however, that a convergence of the related Fourier expansions is slow, whereas solutions (22) and (31) are not series and, hence, the convergence problem is of no importance for them.

3.3. NONLINEARIZABLE OSCILLATOR UNDER PERIODIC IMPULSIVE EXCITATION

Let us consider an essentially nonlinear oscillator with a power homogeneous characteristic under the periodic sequence of impulses

$$\ddot{x} + x^m = p \frac{de(\omega t)}{d(\omega t)}, \quad (32)$$

where m is a positive odd number; $p, \omega = \text{const}$.

When $m = 1$, one has the impulsively forced harmonic oscillator considered in Section 3.2. For a large exponent, $m \gg 1$, the system gives a smooth approximation of the simplest vibro-impact oscillator (Figure 1). Free vibration of the system admits an exact analytical solution in terms of complicated-enough special functions [23–25]. The oscillator was of interest to many authors [12, 26–30]. An approximate analytical solution for arbitrary exponent m was obtained in [20] in the power series form with respect to the saw-tooth function of time.

We will seek a periodic ($T = 4/\omega$) solution of Equation (32) in the form

$$x = X(\tau(\omega t)), \quad (33)$$

where the Y -component of the representation is supposed to be zero, based on the final result of the previous subsection. Note that the equation considered is nonlinear and the two components of the transformed system are coupled. However, the system still admits a family of solutions on which $Y \equiv 0$.

Substituting Equation (33) into Equation (32), one obtains

$$(\omega^2 X'' + X^m) + (\omega^2 X' - p) \frac{de(\omega t)}{d(\omega t)} = 0 \quad (34)$$

and the equivalent boundary-value problem

$$\omega^2 X'' + X^m = 0, \quad (35)$$

$$\omega^2 X'|_{\tau=\pm 1} = p, \quad (36)$$

where the boundary condition stays for the elimination of the periodic singular term from the transformed Equation (34). In the case of free vibration ($p = 0$), the boundary condition becomes homogeneous and the parameter ω is treated as oscillator's natural frequency which was previously unknown and should be coupled with a parameter of amplitude.

Due to the symmetry provided by the oddness of m , the system admits a family of solutions where $X(-\tau) \equiv -X(\tau)$. Taking into account this remark, we will seek an odd with respect to τ solution and take care of only one point, $\tau = 1$, of the boundary condition (36).

In this example, the difference between free and forced vibrations is reduced to different values of the right-hand side of the boundary condition (36). A scheme of successive approximation [20] gives an approximate solution of Equation (35) in the power series form

$$X = A \left[\tau - \frac{\eta \tau^{m+2}}{(m+2)} + \frac{m \eta^2 \tau^{2m+3}}{2(m+2)(2m+3)} - \frac{m(m^2+m-1) \eta^3 \tau^{3m+4}}{2(m+2)^2(2m+3)(3m+4)} + \dots \right], \quad (37)$$

where $\tau = \tau(\omega t)$ and A is arbitrary constant, and

$$\eta = \frac{A^{m-1}}{(m+1)\omega^2}.$$

The solution is in power series form. However, it is periodic with respect to the original parameter of time, t .

The substitution of solution (37) into the boundary condition (36) for $\tau = 1$ gives the algebraic equation

$$A \left[1 - \eta + \frac{m\eta^2}{2(m+2)} - \frac{m(m^2+m-1)\eta^3}{2(m+2)^2(2m+3)} + \dots \right] = \frac{p}{\omega^2}. \quad (38)$$

Equation (38) remains for the elimination of the periodic *singular* terms⁴ from the transformed system (34).

If $p = 0$, Equation (38) couples the parameters A and ω associated with the amplitude and frequency of the free vibration. When $p \neq 0$, it gives a nonlinear frequency response for the system under external impulsive excitation (Figure 7).

A point of interest is the limit of the vibro-impact oscillator when $m \rightarrow \infty$ (see Section 1). For a sufficiently large exponent $m = 9$, the phase trajectory's projection on the plane xv and the time history record for the velocity show the discontinuities of the velocity v created by external impulses when the coordinate x reaches its amplitude values (see Figures 8 and 9). In the vibro-impact limit, when $m \rightarrow \infty$ the amplitude values of the coordinate are defined by the rigid barriers appearing at points $x = \pm 1$. It means that both external and internal impacts must act at the same time instances to provide the periodic regime considered.

To this end, note that starting from the boundary-value problem in the form $\omega^2 X'' + f(X) = 0$; $\omega^2 X'|_{\tau=\pm 1} = p$ and using successive approximations, one can consider a more general case of the odd characteristic $f(x)$ represented by Maclaurin's series.

4. Parametric Excitation

In this section, two cases of parametric excitation will be considered based on a relatively simple linear model. The time shapes of a piecewise-constant excitation and an impulsive one will be described by means of the functions $e(\omega t; \gamma)$ and $\partial e(\omega t; \gamma)/\partial(\omega t)$, respectively. Both systems can be considered as particular cases of the more general example represented in [17]. However, the parameters selected in [17] correspond, in principal, to the case of impulsive excitation only (see Section 4.2). Note that both of the systems are piecewise linear and can be solved in a regular manner [31]. These examples have been chosen to illustrate the applicability of the saw-tooth transformation of time to systems under parametric discontinuous excitations. Finally, there are two reason to use the transformation as a preliminary step in analysis. Firstly, the saw-tooth transformation automatically gives the conditions of matching at the points of discontinuity of the excitation. Secondly, the solution is represented as a closed-form expression, and this is an important feature in many cases.

4.1. PIECEWISE-CONSTANT EXCITATION

Let us consider a linear oscillator under the periodic piecewise-constant excitation

$$\ddot{x}(t) + \omega_0^2 [1 + \varepsilon e(\omega t; \gamma)]x = 0, \quad (39)$$

where ω_0 , ω , γ are constants, and the parameter ε is not necessarily small.

⁴ Recall that the quasilinear techniques give the corresponding relations after the elimination of the *secular* terms.

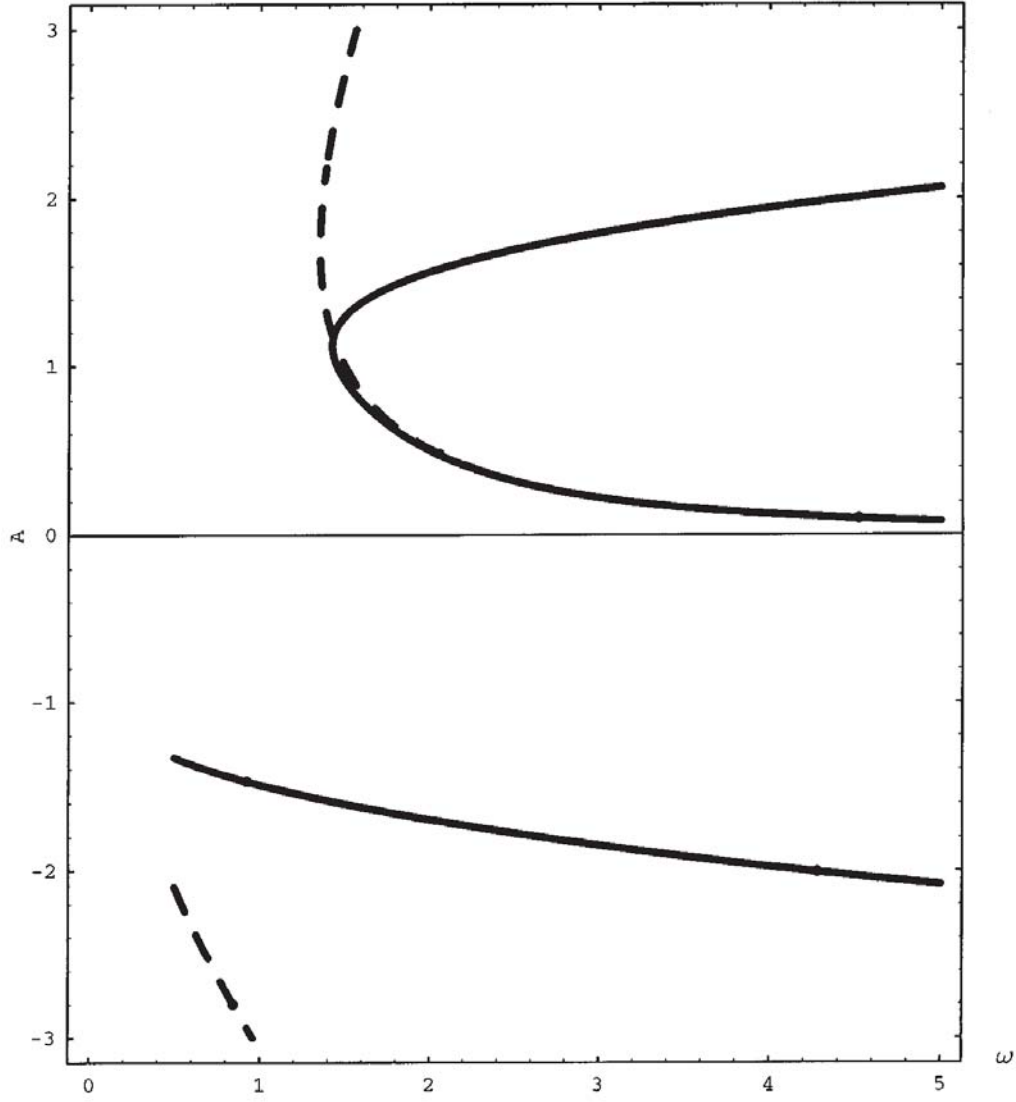


Figure 7. Frequency response of the ‘quasi-impact’ oscillator under periodic impulsive excitation for $p = 2$, $m = 3$ (dashed line) and $m = 9$ (solid line). In standard normalization of the linear vibration theory, the frequency is $(\pi/2)\omega$. The neighborhood of zero frequency is not shown.

We will seek a periodic solution with the period of excitation $T = 4/\omega$ in the form (15). The coordinate of the system $x(t)$ and velocity $\dot{x}(t)$ must be continuous functions of time under the excitation considered. It means that neither velocity $\dot{x}(t)$ nor acceleration $\ddot{x}(t)$ can include any δ -functions. The first derivative of representation (15), due to expression (7), is

$$\dot{x}(t) = \left[\alpha Y' + (X' + \beta Y')e + Y \frac{\partial e(\omega t; \gamma)}{\partial(\omega t)} \right] \omega. \quad (40)$$

The last term consisting of δ -functions can be excluded by imposing the boundary condition for Y -component of the solution as

$$Y|_{\tau=\pm 1} = 0. \quad (41)$$

Under this condition, the second derivative is

$$\ddot{x}(t) = \omega^2[\alpha(X'' + \beta Y'')] + \omega^2[\beta X'' + (\alpha + \beta^2)Y'']e$$

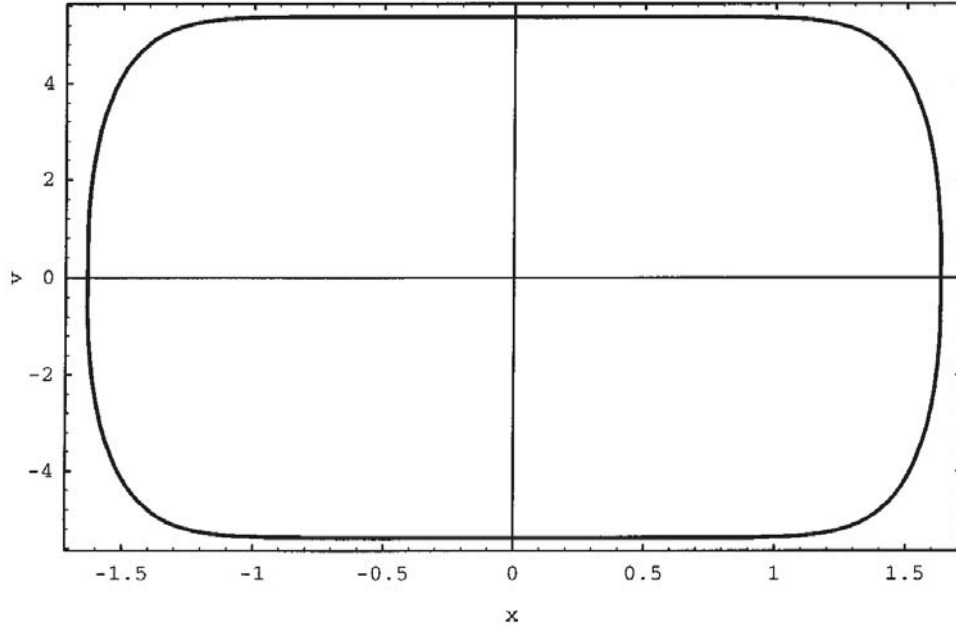


Figure 8. Projection of the impulsively forced ‘quasi-impact’ oscillator’s orbit on the plane ‘coordinate–velocity’, xv . The velocity experiences momentary steps (see also the next figure), when the system’s coordinate reaches its amplitude values. The periodic regime corresponds to the parameters $p = 2$, $m = 9$, $\omega = 3$ and $A = 1.79354$.

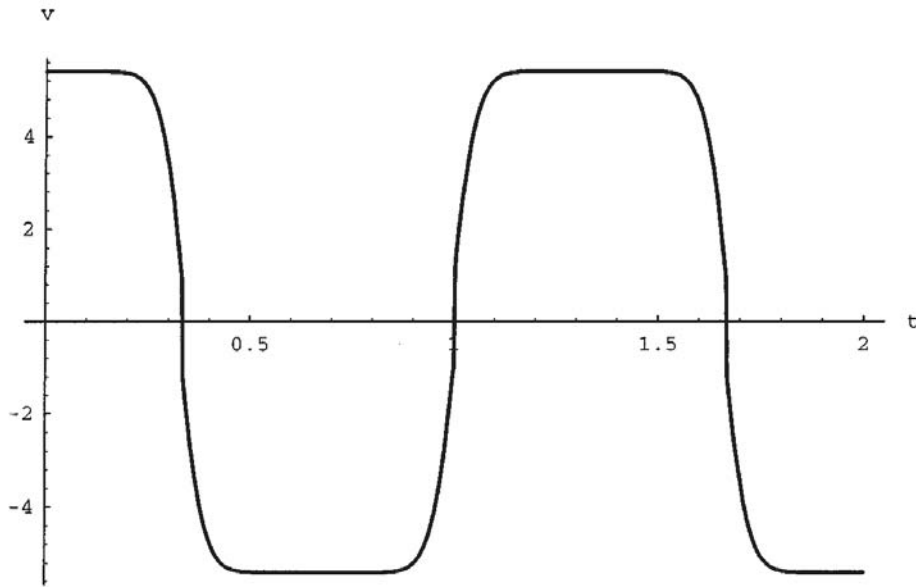


Figure 9. The velocity time record shows sudden vertical ‘steps’ caused by external impulses.

$$\underline{+ \omega^2(X' + \beta Y') \frac{\partial e(\omega t; \gamma)}{\partial(\omega t)},} \quad (42)$$

where the multiplication rule for basic element (7) has been taken into account once again.

Analogously to the previous step, one eliminates the underlined term by means of the conditions

$$(X' + \beta Y')|_{\tau=\pm 1} = 0. \quad (43)$$

Substituting expressions (15) and (42) into equation of motion (39) and taking into account Equation (7) gives an element of the algebraic structure $\{\dots\} + \{\dots\}e$. Equating each of the

two components of the element to zero, one obtains the set of differential equations for $X(\tau)$ and $Y(\tau)$ in matrix form, as

$$\begin{pmatrix} \alpha & \alpha\beta \\ \beta & \alpha + \beta^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}'' + r^2 \begin{pmatrix} 1 & \alpha\varepsilon \\ \varepsilon & 1 + \beta\varepsilon \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0, \quad (44)$$

where $r = \omega_0/\omega$.

We put

$$\begin{pmatrix} X \\ Y \end{pmatrix} = B \begin{pmatrix} 1 \\ \mu \end{pmatrix} \exp(\lambda\tau); \quad B, \mu, \lambda = \text{const.} \quad (45)$$

From the differential equations for X and Y (Equation (44)), subject to the condition $B \neq 0$, we obtain a characteristic equation which has the following two pairs of roots:

$$\lambda^2 = [-(1 - \gamma)\varepsilon - (1 - \gamma)^2]r^2 \equiv \pm k^2,$$

$$\lambda^2 = [(1 + \gamma)\varepsilon - (1 + \gamma)^2]r^2 \equiv \pm l^2.$$

We now consider the case of negative pairs. This means that the coefficient of x in Equation (39) is positive at any time t .

The general solution of a system of equations subjected to this condition will have the form

$$X = B_1 \sin k\tau + B_2 \cos k\tau + B_3 \sin l\tau + B_4 \cos l\tau,$$

$$Y = \mu_1(B_1 \sin k\tau + B_2 \cos k\tau) + \mu_2(B_3 \sin l\tau + B_4 \cos l\tau), \quad (46)$$

$$\mu_1 = -\frac{1}{\alpha} \frac{\alpha k^2 - r^2}{\beta k^2 - \varepsilon r^2}, \quad \mu_2 = -\frac{1}{\alpha} \frac{\alpha l^2 - r^2}{\beta l^2 - \varepsilon r^2}.$$

In view of the homogeneity of the boundary conditions, the requirement that the solution should be nontrivial leads to the relation

$$\begin{aligned} & [\mu_1(1 + \beta\mu_2)l \cos k \sin l - \mu_2(1 + \beta\mu_1)k \cos l \sin k] \\ & \times [\mu_1(1 + \beta\mu_2)l \cos l \sin k - \mu_2(1 + \beta\mu_1)k \cos k \sin l] = 0. \end{aligned}$$

Figure 10 shows a family of curves in which this expression is equal to zero and, hence, periodic solutions exist with related instability zones. Figure 11 shows different time history records when crossing a region covered by two neighboring curves. An increasing of the amplitude in Figures 11b and 11e indicates the parametric instability of the system, when the point (r, ε) is located inside the instability regions shown in Figure 10.

4.2. PARAMETRIC IMPULSIVE EXCITATION

Let us consider the case of a system with a single degree of freedom under the parametric impulsive excitation of the time shape given by a second derivative of the asymmetric sawtooth function, i.e. let us write the equation of motion as (compare with Equation (39))

$$\ddot{x} + \omega_0^2 \left[1 + \varepsilon \frac{\partial e(\omega t; \gamma)}{\partial(\omega t)} \right] x = 0. \quad (47)$$

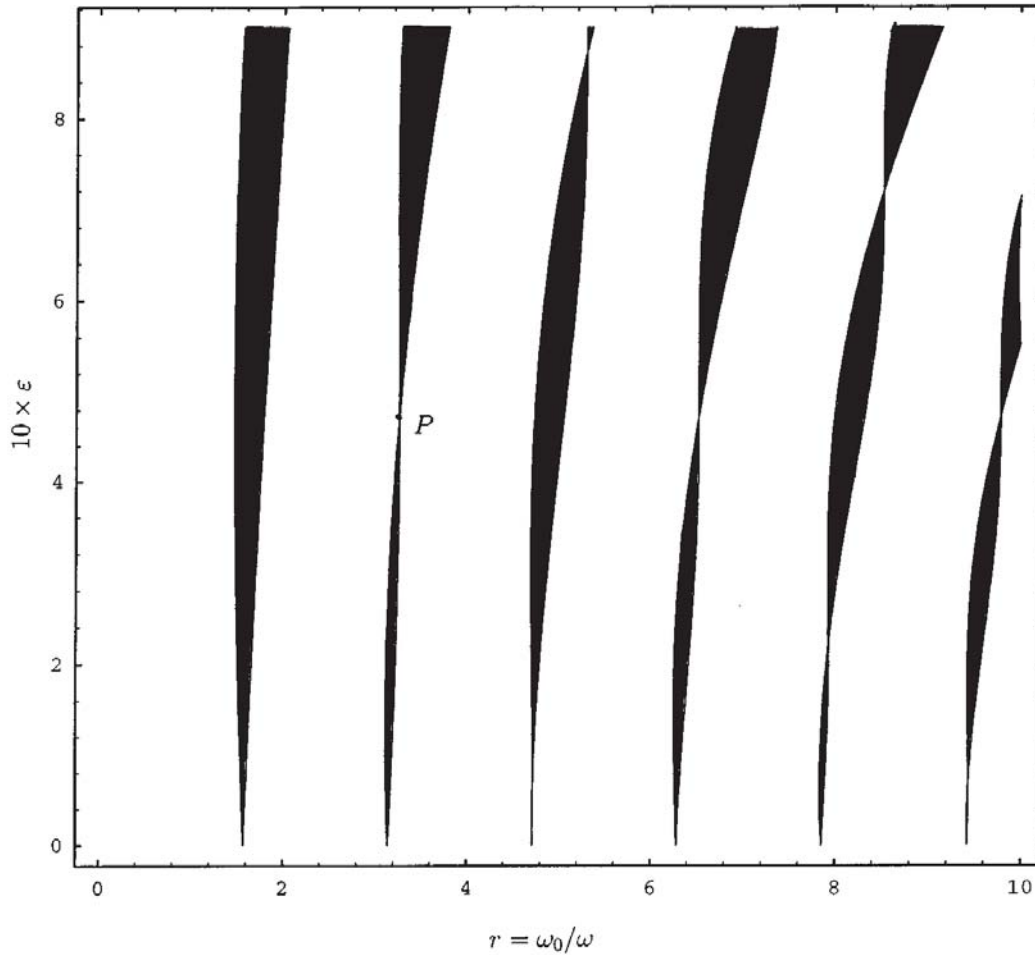


Figure 10. Instability zones of the linear oscillator under the parametric piecewise constant force described by the first derivative of the asymmetric saw-tooth function for the parameter $\gamma = 0.7$. A numerical testing of the self-intersections of the instability zone boundaries is presented by the series of plots in Figure 11.

This case was considered in [17] based on the saw-tooth transformation of time. It was shown that the periodic solutions of the period $T = 4/\omega$ exist when parameters of the system satisfy the relation

$$p^2 = \frac{2r^2(1 - \gamma^2)^2 \sin^2 2r}{\cos 4r - \cos 4\gamma r}, \quad (48)$$

where $r = \omega_0/\omega$ and $p = \epsilon r^2$.

The dependence of p on r for fixed γ has a branched zone-like structure characteristic of similar problems (see Figure 12, where, in view of the symmetry, only the upper half-plane is shown). A curious effect is observed when the parameter γ which characterizes the slope of the ‘saw’ is varied. The question concerns the transverse ‘collapse’ of the sequences of zones which are regularly situated across each of the s zones, where s is an integer which depends on the parameter γ (the form of this relationship will be established later). For instance, when $\gamma = 1/5$, each fifth zone is missing and, when $\gamma = 1/2$, each second zone is missing as Figures 6a and 6b, respectively, show. An explanation of this phenomenon was given in [17].

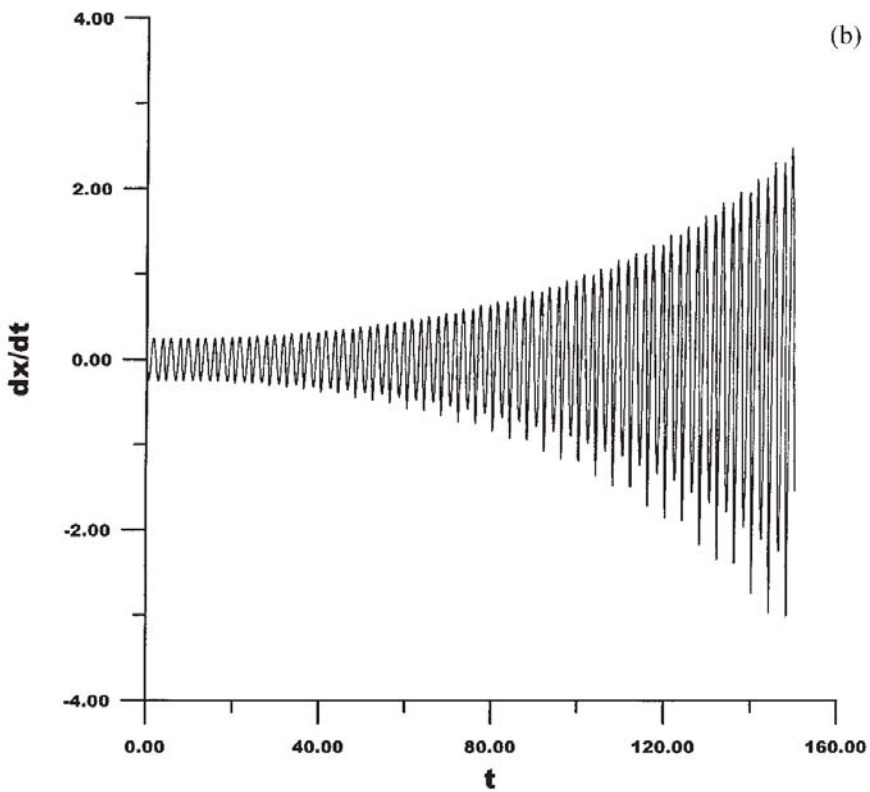
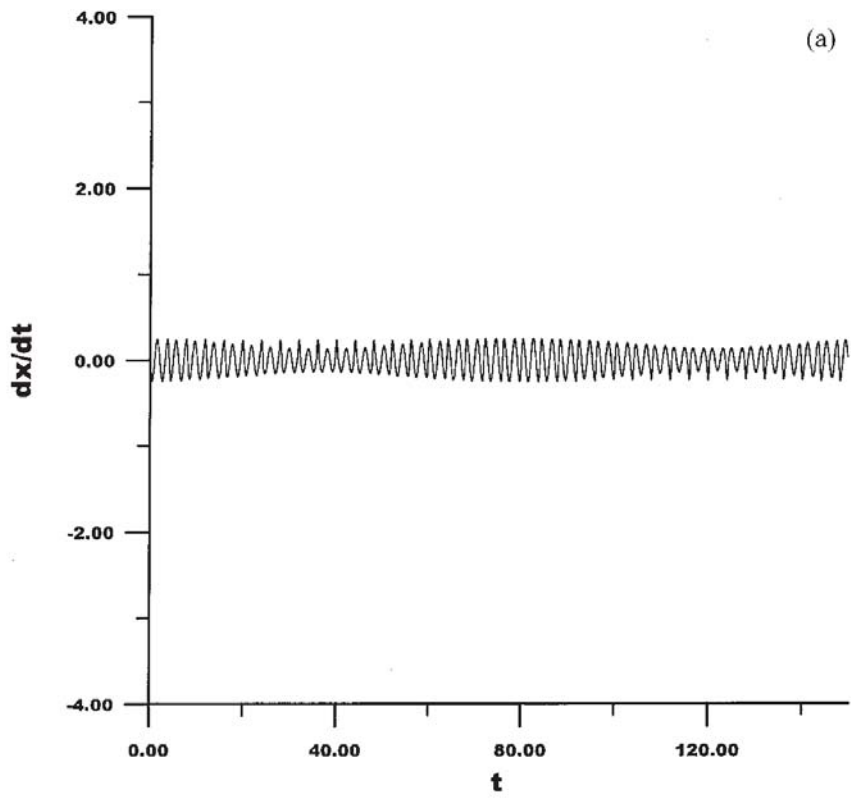


Figure 11. Different time history records when taking the system's parameters gradually around the point of self-intersection of the instability zone (point P in Figure 10). (a) $r = 3.20$, $\varepsilon = 0.42$ (on the left of the instability zone, under point P), (b) $r = 3.24$, $\varepsilon = 0.42$ (inside the instability zone, under point P).

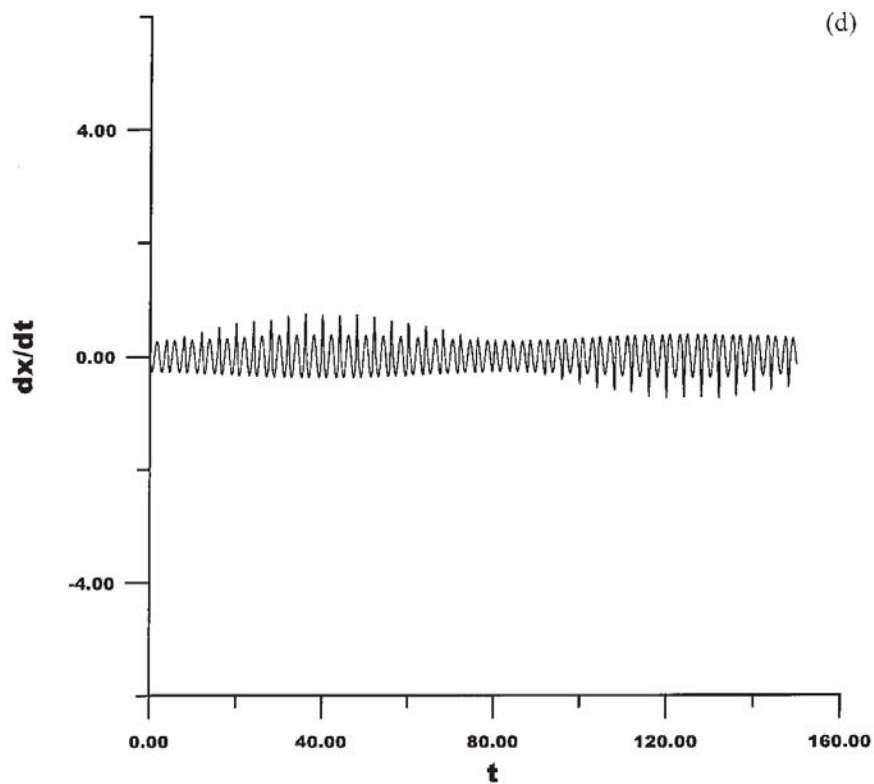
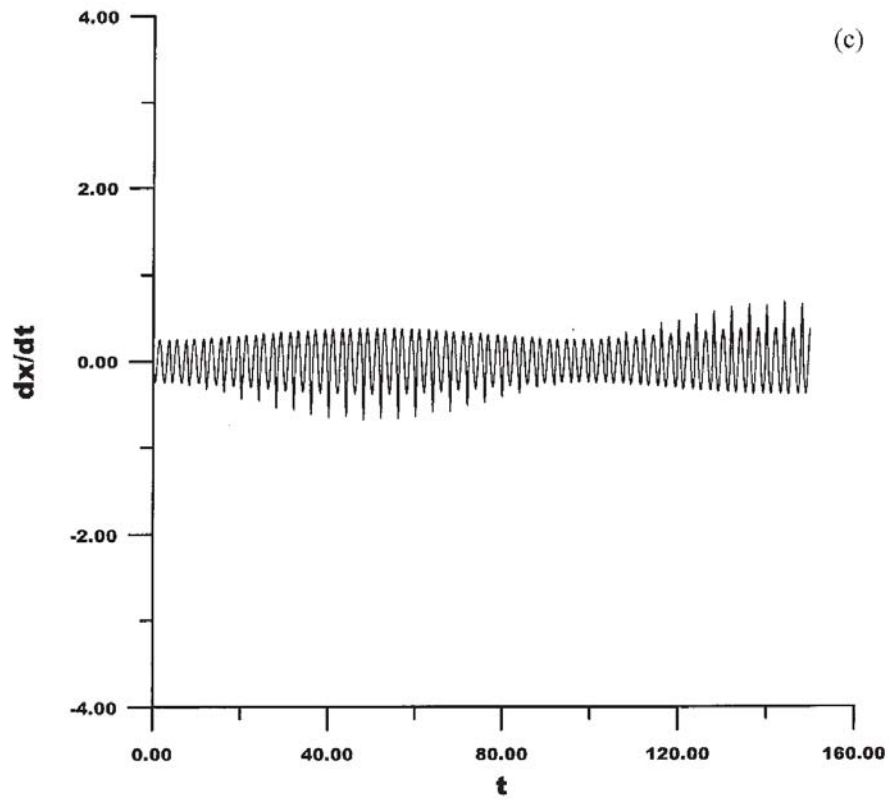


Figure 11. Continued. (c) $r = 3.28$, $\varepsilon = 0.42$ (on the right of the instability zone, under point P), (d) $r = 3.24$, $\varepsilon = 0.52$ (on the left of the instability zone, above point P).

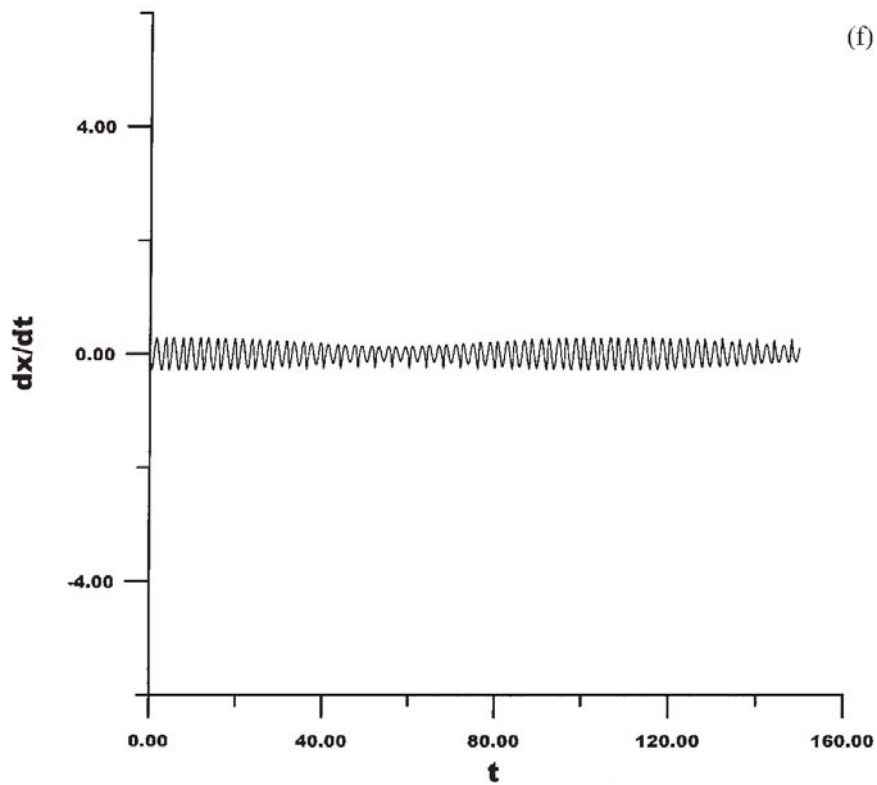
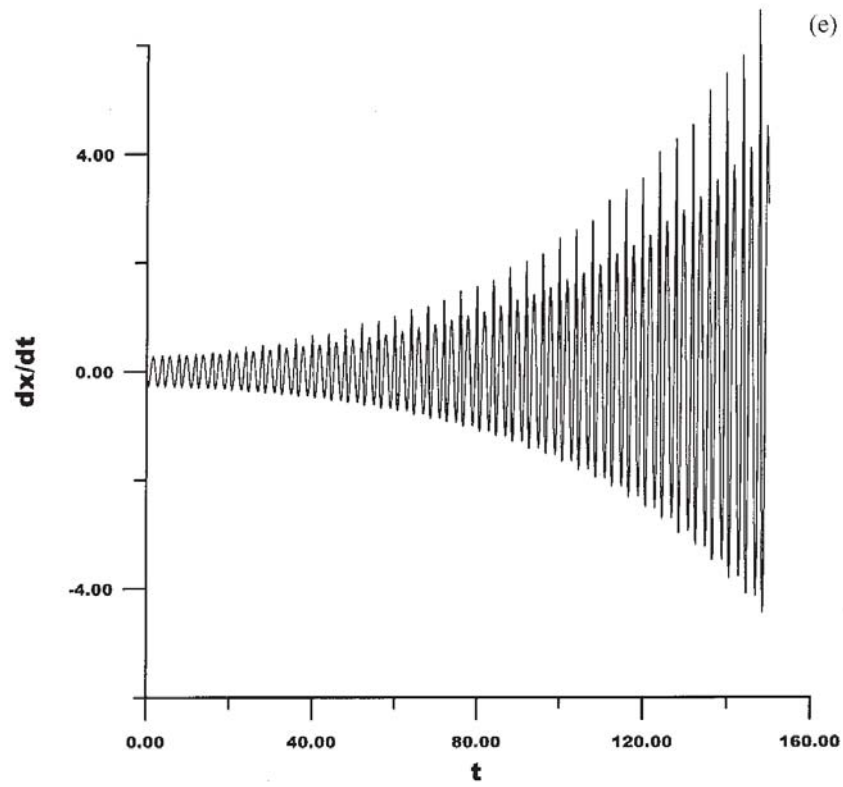


Figure 11. Continued. (e) $r = 3.28$, $\varepsilon = 0.52$ (inside the instability zone, above point P), (f) $r = 3.32$, $\varepsilon = 0.52$ (on the right of the instability zone, above point P).

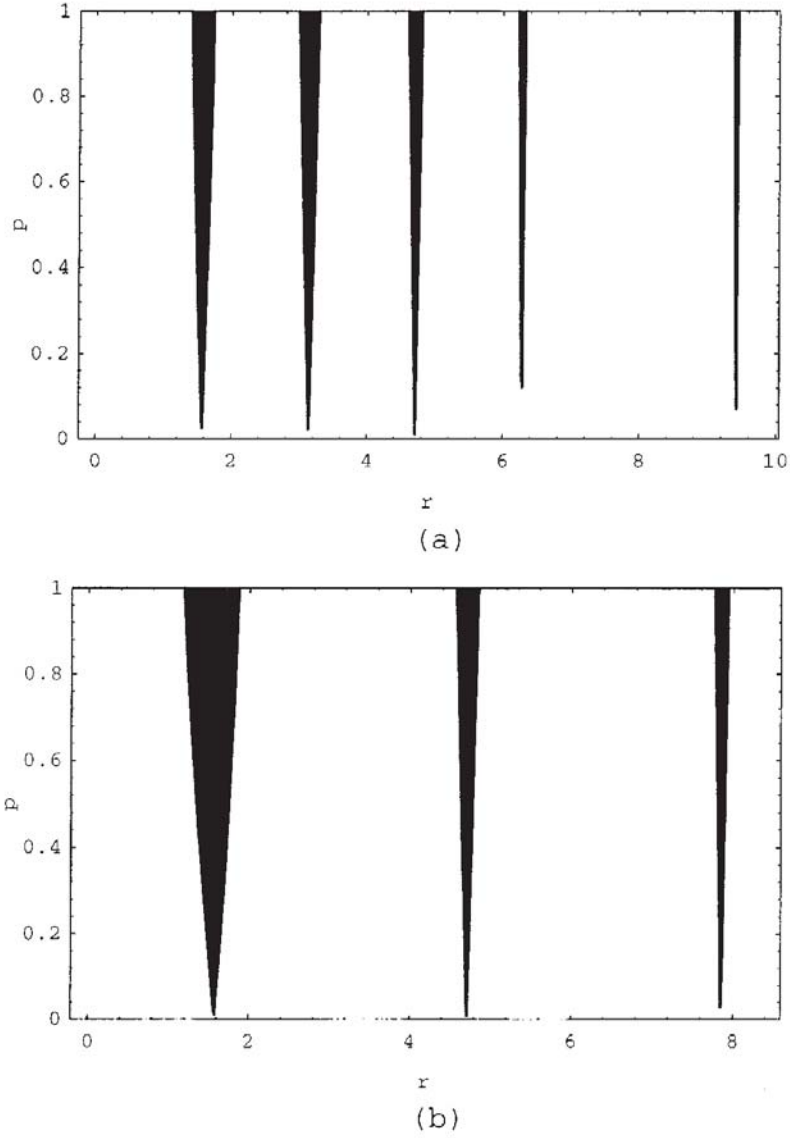


Figure 12. ‘Collapse’ of the instability zones of the harmonic oscillator under the periodic impulsive excitation with the dipole-like shift: (a) $\gamma = 1/5$, each fifth zone is missing, (b) $\gamma = 1/2$, each second zone is missing.

5. On the Use of the Method of Averaging

The above technique is applicable for analyzing periodic solutions, which are only achieved under certain initial conditions. Solutions of the Cauchy problem for an arbitrary initial condition are, generally speaking, nonperiodic and can be obtained by involving the idea of averaging. An appropriate formalization is given by the two-variables expansion technique [32].

Let us consider the first-order differential equation

$$\dot{v} + \varepsilon v^3 = p \frac{de(t)}{dt} \quad (49)$$

under the initial condition

$$v|_{t=0} = v^\circ. \quad (50)$$

This Cauchy problem describes the one-dimensional motion of a material point under a periodic series of impulses, nonlinear damping and no elastic forces. Note that any standard

formulation of the idea of averaging is hard to apply directly to Equation (49), because the right-hand side of the equation is neither smooth nor even a continuous function of time. To proceed correctly, one should specify the class of solutions to be defined, because the equation combines the nonlinearity and singular functions (see Section 1 and Appendix 1). A preliminary analysis shows that a role in the solutions can be played by functions with discontinuities of the first kind at points $\Lambda = \{t : \tau(t) = \pm 1\}$. Solutions of the problem (49), (50) are sought in the form

$$v = X(\tau, t^\circ) + Y(\tau, t^\circ)e; \quad \tau = \tau(t), e = e(t), \quad (51)$$

where the notation $t^\circ = \varepsilon t$ indicates the ‘slow time’ scale. Hence, the solution is assumed to be dependent on two time variables – fast oscillating time τ and slow time t° .

Substituting Equation (51) into Equation (49), we obtain the expression

$$\frac{\partial Y}{\partial \tau} + \varepsilon \left(\frac{\partial X}{\partial t^\circ} + X^3 + 3XY^2 \right) + \left[\frac{\partial X}{\partial \tau} + \varepsilon \left(\frac{\partial Y}{\partial t^\circ} + 3YX^2 + Y^3 \right) \right] e + (Y - p)\dot{e} = 0.$$

This finally leads to the set of partial differential equations

$$\frac{\partial X}{\partial \tau} = -\varepsilon \left(\frac{\partial Y}{\partial t^\circ} + 3YX^2 + Y^3 \right), \quad \frac{\partial Y}{\partial \tau} = -\varepsilon \left(\frac{\partial X}{\partial t^\circ} + X^3 + 3XY^2 \right) \quad (52)$$

with the boundary condition

$$Y|_{\tau=\pm 1} = p. \quad (53)$$

The asymptotic ($\varepsilon \rightarrow 0$) solution of problem (52), (53) can be found using the standard procedure of the method of two-variables expansion in a power series form, as

$$X = \sum_{i=0}^{\infty} \varepsilon^i X_i(\tau, t^\circ), \quad Y = \sum_{i=0}^{\infty} \varepsilon^i Y_i(\tau, t^\circ). \quad (54)$$

Substituting Equation (54) into Equations (52), (53) and matching coefficients of the same powers of ε , gives a zero-order problem

$$\varepsilon^0 : \frac{\partial X_0}{\partial \tau} = 0, \quad \frac{\partial Y_0}{\partial \tau} = 0; \quad Y_0|_{\tau=\pm 1} = p.$$

The solution satisfying the boundary condition is

$$X_0 = A_0(t^\circ), \quad Y_0 = p, \quad (55)$$

where A_0 is an arbitrary function of the slow time.

In the first order of ε , taking into account Equation (55) gives

$$\varepsilon^1 : \frac{\partial X_1}{\partial \tau} = -(3pA_0^2 + p^3), \quad \frac{\partial Y_1}{\partial \tau} = -\left(\frac{dA_0}{dt^\circ} + A_0^3 + 3A_0p^2 \right); \quad Y_1|_{\tau=\pm 1} = 0.$$

Integrating the equations gives

$$X_1 = -(3pA_0^2 + p^3)\tau + A_1(t^\circ), \quad Y_1 = -\left(\frac{dA_0}{dt^\circ} + A_0^3 + 3A_0p^2 \right)\tau + C_1(t^\circ). \quad (56)$$

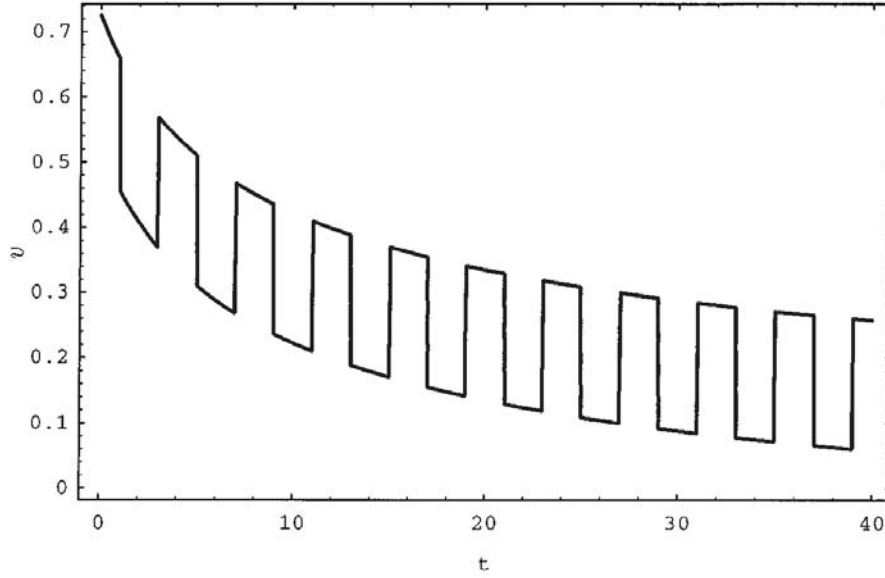


Figure 13. The closed-form analytical approach for the velocity of a nonlinearly damped material point of the periodic impulsive excitation.

Boundary conditions for Y_1 give $C_1(t^\circ) \equiv 0$ and the leading-order averaged differential equation for $A_0(t^\circ)$ is

$$\frac{dA_0}{dt^\circ} + A_0^3 + 3A_0p^2 = 0. \quad (57)$$

So, in the first order of ε , one has $Y_1 \equiv 0$, and the function $A_1(t^\circ)$ in the expression for X_1 remains unknown.

Equation (57) admits an elementary integration which gives a solution in the following explicit form

$$A_0 = \left[-\frac{1}{3p^2} + \text{const} \cdot \exp(6p^2t^\circ) \right]^{-1/2}.$$

Finally, a leading-order asymptotic solution of the Cauchy problem can be written as (Figure 13)

$$v = \pm \left\{ -\frac{1}{3p^2} + \left[\frac{1}{3p^2} + (x^\circ - p)^2 \right] \exp(6p^2\varepsilon t) \right\}^{-1/2} + pe(t) + O(\varepsilon), \quad (58)$$

where the plus or minus signs are taken as $v^\circ - p > 0$ or $v^\circ - p < 0$, respectively.

The first term in expression (58) is a transient response of the system, and it tends to zero when the time tends to infinity. It can be seen that the impulsive external excitation strongly affects the time scale of the transient response, because that timescale is multiplied by the impulse parameter squared, p^2 .

6. Weakly Nonlinear Resonances in the Duffing Oscillator under the Periodic Impulsive Force

Let us consider the Duffing oscillator under the periodic impulsive excitation

$$\ddot{x} + \omega_0^2 x = -\varepsilon \left[x^3 - p \frac{de(\omega t)}{d(\omega t)} \right] \quad (59)$$

in the neighborhood of linear resonance condition (23). Introducing the detuning parameter σ into the differential equation (59) as

$$\omega_0^2 = \left(\frac{n\pi}{2} \omega \right)^2 + \varepsilon \sigma \quad (60)$$

gives

$$\ddot{x} + \left(\frac{n\pi}{2} \omega \right)^2 x = -\varepsilon \left[\sigma x + x^3 - p \frac{de(\omega t)}{d(\omega t)} \right]. \quad (61)$$

We shall seek the solution in the form (51), where $\tau = \tau(\omega t)$, $e = e(\omega t)$. Assuming the continuity condition for $x(t)$,

$$Y|_{\tau=\pm 1} = 0 \quad (62)$$

gives the first derivative of Equation (51) as

$$\dot{x} = \omega \frac{\partial Y}{\partial \tau} + \varepsilon \frac{\partial X}{\partial t^0} + \left(\omega \frac{\partial X}{\partial \tau} + \varepsilon \frac{\partial Y}{\partial t^0} \right) e. \quad (63)$$

Substituting Equations (51) and (63) into differential equation (61) gives

$$\omega^2 \frac{\partial X}{\partial \tau} \Big|_{\tau=\pm 1} = \varepsilon p, \quad (64)$$

$$\omega^2 \frac{\partial^2 X}{\partial \tau^2} + 2\varepsilon \omega \frac{\partial^2 Y}{\partial \tau \partial t^0} + \varepsilon^2 \frac{\partial^2 X}{\partial t^{02}} + \left(\frac{n\pi}{2} \omega \right)^2 X = -\varepsilon (\sigma X + X^3 + 3Y^2 X), \quad (65)$$

$$\omega^2 \frac{\partial^2 Y}{\partial \tau^2} + 2\varepsilon \omega \frac{\partial^2 X}{\partial \tau \partial t^0} + \varepsilon^2 \frac{\partial^2 Y}{\partial t^{02}} + \left(\frac{n\pi}{2} \omega \right)^2 Y = -\varepsilon (\sigma Y + Y^3 + 3X^2 Y). \quad (66)$$

As a result of the nonsmooth transformation of the argument, the boundary-value problem (62), (64–66) does not include any singular functions, and one can correctly use the averaging technique.

We seek an asymptotic solution in a power series form as

$$X = X_0(\tau, t^\circ) + \varepsilon X_1(\tau, t^\circ) + \varepsilon^2 X_2(\tau, t^\circ) + \dots,$$

$$Y = Y_0(\tau, t^\circ) + \varepsilon Y_1(\tau, t^\circ) + \varepsilon^2 Y_2(\tau, t^\circ) + \dots.$$

Substituting this into Equations (62) and (64–66), and matching coefficients of the same degrees of ε , gives a series of boundary value problems with respect to the X and Y components of the solution.

In the first step, one has the homogenous boundary value problem

$$\begin{aligned} \frac{\partial^2 X_0}{\partial \tau^2} + \left(\frac{n\pi}{2}\right)^2 X_0 &= 0, & \frac{\partial X_0}{\partial \tau} \Big|_{\tau=\pm 1} &= 0, \\ \frac{\partial^2 Y_0}{\partial \tau^2} + \left(\frac{n\pi}{2}\right)^2 Y_0 &= 0, & Y_0|_{\tau=\pm 1} &= 0 \end{aligned} \quad (67)$$

and its nonzero solution is written as

$$X_0 = A_0(t^\circ) \sin \frac{n\pi}{2} \tau, \quad Y_0 = D_0(t^\circ) \cos \frac{n\pi}{2} \tau, \quad (68)$$

where $A_0(t^\circ)$ and $D_0(t^\circ)$ are arbitrary functions.

Collecting all terms of order ε gives the boundary-value problem as

$$\begin{aligned} \omega^2 \left[\frac{\partial^2 X_1}{\partial \tau^2} + \left(\frac{n\pi}{2}\right)^2 X_1 \right] &= - \left(2\omega \frac{\partial^2 Y_0}{\partial \tau \partial t^0} + \sigma X_0 + X_0^3 + 3Y_0^2 X_0 \right), \\ \omega^2 \frac{\partial X_1}{\partial \tau} \Big|_{\tau=\pm 1} &= p, \\ \omega^2 \left[\frac{\partial^2 Y_1}{\partial \tau^2} + \left(\frac{n\pi}{2}\right)^2 Y_1 \right] &= - \left(2\omega \frac{\partial^2 X_0}{\partial \tau \partial t^0} + \sigma Y_0 + Y_0^3 + 3X_0^2 Y_0 \right), \\ Y_1|_{\tau=\pm 1} &= 0. \end{aligned} \quad (69)$$

The right-hand sides of Equation (69) depend implicitly on the slow time throughout the still unknown functions $A_0(t^\circ)$, $D_0(t^\circ)$, and its derivatives $A'_0(t^\circ)$ and $D'_0(t^\circ)$.

The solution of the differential equations (69) can be represented as

$$\begin{aligned} X_1 &= A_1(t^0) \sin \frac{n\pi}{2} \tau + B_1(t^0) \cos \frac{n\pi}{2} \tau \\ &+ \frac{1}{\omega} \left[D'_0 + \frac{1}{n\pi\omega} \left(\sigma A_0 + \frac{3}{4} A_0^3 + \frac{3}{4} A_0 D_0^2 \right) \right] \tau \cos \frac{n\pi}{2} \tau \\ &+ \frac{2}{n\pi\omega} \left[D'_0 + \frac{1}{n\pi\omega} \left(\sigma A_0 - \frac{9}{16} A_0^3 - \frac{21}{16} A_0 D_0^2 \right) \right] \sin \frac{n\pi}{2} \tau \\ &- \frac{1}{8n^2\pi^2\omega^2} (A_0^3 - 3A_0 D_0^2) \sin \frac{3n\pi}{2} \tau, \end{aligned}$$

$$\begin{aligned} Y_1 &= C_1(t^0) \sin \frac{n\pi}{2} \tau + D_1(t^0) \cos \frac{n\pi}{2} \tau \\ &+ \frac{1}{\omega} \left[A'_0 - \frac{1}{n\pi\omega} \left(\sigma D_0 + \frac{3}{4} A_0^2 D_0 + \frac{3}{4} D_0^3 \right) \right] \tau \sin \frac{n\pi}{2} \tau \\ &+ \frac{1}{4n^2\pi^2\omega^2} (D_0^3 - 3D_0 A_0^2) \sin n\pi \tau \sin \frac{n\pi}{2} \tau. \end{aligned}$$

The boundary condition for X_1 (see remark at the end of Section 2) gives

$$B_1(t^0) \equiv 0$$

and

$$D'_0 = \frac{1}{n\pi\omega} \left[2(-1)^{(n-1)/2} p + \sigma A_0 + \frac{3}{4} A_0(A_0^2 + D_0^2) \right]. \quad (70)$$

The boundary condition for Y_1 gives

$$C_1(t^\circ) \equiv 0$$

and

$$A'_0 = -\frac{1}{n\pi\omega} \left[\sigma D_0 + \frac{3}{4} D_0(A_0^2 + D_0^2) \right]. \quad (71)$$

In this step, functions $A_1(t^\circ)$ and $D_1(t^\circ)$ remain unknown. Equations (70) and (71) allow us to determine the slow functions $A_0(t^\circ)$ and $D_0(t^\circ)$ and to complete the previous step. These equations can be investigated in a regular manner for the nonlinear vibration theory, introducing polar coordinates.

In the case when no external impulses are acting, i.e. $p = 0$, Equations (70) and (71) stay for elimination of the so-called ‘secular terms’ in the solution. If $p \neq 0$, secular terms occur in the solution. These terms, however, are only secular with respect to ‘oscillating time’ τ and, hence, the solution remains periodic with respect to original time, t .

7. Conclusion

It has been shown that the nonsmooth transformation of the argument is applicable to linear and nonlinear dynamical systems under periodic impulsive and discontinuous excitations. A standard pair of nonsmooth periodic functions plays an important role in the proposed method. In the symmetric case, $\gamma = 0$, the pair of functions which is used $\{\tau(t), e(t)\}$ is associated in a natural way with the motion of a free point mass between two fixed arresting devices, that is, with one of the two simplest mechanical oscillators. These systems are a vibro-impact system with a single impact pair and a harmonic oscillator. It is known that the latter generates a pair of trigonometric functions $\{\sin t, \cos t\}$ possessing a number of convenient mathematical properties. From the mechanical point of view, the functions $\{\sin t, \cos t\}$ are associated with a free rotating disc with angular velocity $\omega = 1$. A ‘complicated’ linear regime of an elastic structure (say, a free mass-spring model of N degrees of freedom) usually involves a set of harmonics $\{\sin \omega_k t, \cos \omega_k t\}$, $k = 1, \dots, N$. It means that the regime can be represented by a set of N free rotating discs with angular velocities ω_k , i.e. it appears that the oscillation of an elastic structure is equivalent to the rigid-bodies motion.

In the same manner, the functions $\{\tau(t), e(t)\}$ are associated to the motion of a free rigid body with absolutely rigid constraints. It has been shown that a more complicated nonsmooth (or a quasi nonsmooth) dynamical regime can be represented by this rigid-body motion and its different deformations, according to the power forms, τ^3, τ^5, \dots , or in another simple way. Hence, the introduction the pair $\{\tau(t), e(t)\}$ is not a purely mathematical manipulation. It has a certain physical meaning.

Another essential physical feature of the transformation is that the oscillating time, τ , is introduced in such a manner that it does not affect the metrical properties of the time, namely, the metric remains constant after the transformation: $[d\tau(t)]^2 = dt^2$. This defines a special role of the saw-tooth transformation among other periodic transformations of time. Due to

this property, the transformed differential equations of motion remain within the framework of Newton's formulation (see Appendix 2).

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Appendix 1

Carathéodory Equation

We will present some remarks concerning nonlinear differential equations with distributions involved as summands. We will also compare the two different representation for solutions based on nonsmooth functions.

Let us consider first-order (nonlinear) differential equation, $\dot{x} = f(t, x)$, where x is a scalar or a vector, and the right-hand side satisfies the Carathéodory conditions [4]. Namely, in the domain D of the (t, x) space, the function $f(t, x)$ is defined and continuous in x for almost all t ; the function $f(t, x)$ is measurable in t for each x ; $|f(t, x)| \leq m(t)$, and the function $m(t)$ is summable (at each finite interval if t is not bounded in D).

The conditions listed above mean that the restriction taken for the right-hand side's dependence on t is less strong than the classical existence theorem requires, i.e. the function $f(t, x)$ is not necessarily continuous in t . It becomes possible if the right-hand side of the equivalent integral equation, $x(t) - x(t_0) = \int_{t_0}^t f(\xi, x(\xi)) d\xi$ is understood in the sense of Lebesgue. Under the Carathéodory conditions, a series of existence theorems holds [4].

Let the right-hand side of the equation additively involve a δ -impulse and the equation is written as

$$\dot{x} = f(t, x) + v\delta(t - a), \quad (72)$$

where a and v are constant parameters (v is a scalar or a vector).

Many mechanical and physical problems can be represented in the form (72).

EXAMPLE 1. A parametrically excited Duffing oscillator subjected to external impact at moment $t = a$ is described with the equation

$$\ddot{u} + \omega_0^2(1 + \varepsilon \cos 2t)u - \beta u^3 = 2p\delta(t - a). \quad (73)$$

Equation (73) is equivalent to Equation (72) when

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(t, x) = \begin{bmatrix} v \\ \beta u^3 - \omega_0^2(1 + \varepsilon \cos 2t)u \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 2p \end{bmatrix}.$$

In this case, the variable v is the velocity.

EXAMPLE 2. After a special scaling of the variables, equation

$$\frac{d^4 W}{d\xi^4} + \gamma(\xi)W = q\delta(t - a) \quad (74)$$

describes a beam's centerline, $W = W(\xi)$, resting on an elastic foundation of variable stiffness, $\gamma(\xi)$ and loaded by a transverse force localized at point $\xi = a$. Equation (74) can be represented in the form of (72) when $t \equiv \xi$,

$$x = \begin{bmatrix} W \\ \frac{dW}{d\xi} \\ \frac{d^2W}{d\xi^2} \\ \frac{d^3W}{d\xi^3} \end{bmatrix}, \quad f(\xi, x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma(\xi) & 0 & 0 & 0 \end{bmatrix} x \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ q \end{bmatrix}.$$

The right-hand side of Equation (72) no longer satisfies the Carathéodory conditions. However, a change of variables

$$x(t) \longrightarrow y(t) : x(t) = y(t) + vH(t - a) \quad (75)$$

brings the differential equation back into the form

$$\dot{y} = f(t, y + vH(t - a)) \equiv \bar{f}(t, y), \quad (76)$$

where the Dirac function has been eliminated due to equality $\dot{H}(t - a) = \delta(t - a)$ and, as a result, the right-hand side, $\bar{f}(t, y)$, satisfies the Carathéodory conditions.

Note that due to $H(t - a)$, the right-hand side of the transformed equation is generally piecewise continuous in t , even if the function $f(t, x)$ at the right-hand side of the original equation (72) is continuous.

'Single-Tooth' Transformation of Time

Let us transform Equation (72) by making use of a nonsmooth argument (a 'single-tooth' version of the saw-tooth transformation of time). Let the new argument be

$$|t - a| = s = s(t). \quad (77)$$

We also introduce the old notation for the first-time derivative of the nonsmooth argument as $\dot{s}(t) = d|t - a|/d(t - a) = e(t)$. One has $e(t) = -1$ for $t - a < 0$ and $e(t) = 1$ for $t - a > 0$ and hence, $e^2 = 1$ at least for almost all t .

LEMMA 1. *An arbitrary function $x(t)$ can be represented in form*

$$x(t) = X(s) + Y(s)e. \quad (78)$$

Proof. A transformation inverted to (77) is expressed as

$$t = a + se. \quad (79)$$

Substituting Equation (79) into Equation (78) and taking $e = -1$ and $e = 1$, one obtains

$$x(a - s) = X(s) - Y(s),$$

$$x(a + s) = X(s) + Y(s)$$

and, hence,

$$\begin{aligned} X(s) &= \frac{1}{2}[x(a+s) + x(a-s)], \\ Y(s) &= \frac{1}{2}[x(a+s) - x(a-s)]. \end{aligned} \quad (80)$$

So when $X(s)$ and $Y(s)$ are selected as Equation (80), the identity (78) holds for any $x(t)$.

Let a solution of Equation (72) be represented in the form of (78). Substituting Equations (78) and (79) into Equation (72) and taking into account the expression $\dot{e}(t) = 2\delta(t-a)$, one obtains

$$Y' + R_f(s, X, Y) + [X' + I_f(s, X, Y)]e + (2Y - v)\delta(t-a) = 0, \quad (81)$$

where the expressions

$$\begin{aligned} R_f &= \frac{1}{2}[f(a+s, X+Y) + f(a-s, X-Y)], \\ I_f &= \frac{1}{2}[f(a+s, X+Y) - f(a-s, X-Y)] \end{aligned}$$

have been obtained analogously to Equation (80).

Equality (81) gives a set of equations and a boundary condition

$$\begin{aligned} X' + I_f(s, X, Y) &= 0, \\ Y' + R_f(s, X, Y) &= 0, \\ Y|_{s=0} &= \frac{v}{2}. \end{aligned} \quad (82)$$

As a result of the transformation, the Dirac function has been eliminated from the system. Equations (82) compared with Equation (76) have a more complicated form. However, one should pay attention, for its smoothness properties may become better. In fact, the step function $H(t-a)$ has not been involved in Equation (82). Another (sometimes suitable) point is that the new time variable s is half-limited, $0 \leq s < \infty$.

Appendix 2

Let us show that the saw-tooth time $\tau(t)$ is associated with the group properties of the oscillator

$$\frac{d^2x}{dt^2} + f(x) = 0 \quad (83)$$

and, hence, plays a special role among all possible periodic transformations of time.

In this particular case, a one-component version of the saw-tooth transformation of time can be realized. Alternatively, one may introduce an oscillating time parameter, say g , in a more general manner as $g = g(t)$ and $x = X(g(t))$, where $g(t)$ is a periodic function, not necessarily saw-tooth. One could consider trigonometric sine, elliptic sine, and other specially

selected functions depending on the problem investigated. As a result, one obtains a new system in the form

$$\left(\frac{dg}{dt}\right)^2 \frac{d^2 X}{dg^2} + \frac{d^2 g}{dt^2} \frac{dX}{dg} + f(X) = 0. \quad (84)$$

This transformation becomes physically reasonable if the variable coefficients appear to be functions of the new time parameter g over all periods of the oscillation. For example, let $g = \sin t$. In this case, the transformation $t \rightarrow g, x \rightarrow X$ is completed as

$$(1 - g^2) \frac{d^2 X}{dg^2} - g \frac{dX}{dg} + f(X) = 0. \quad (85)$$

In this example, one undesirable result of the transformation is that the system has lost its original Newton form given by Equation (83) and, hence, the dynamical properties of the system cannot be treated any more in a regular manner.

To preserve Newton's form of the system, one should set

$$\left(\frac{dg}{dt}\right)^2 = 1 \quad \text{and} \quad \frac{d^2 g}{dt^2} \frac{dX}{dg} = 0 \quad (86)$$

in Equation (84). The first equality is satisfied by $g = \alpha + t$ and $g = \beta - t$, where α and β are arbitrary constants. It means that the function $g(t)$, to be periodic, must consist of different pieces of linear functions with positive and negative slopes of the same absolute value, i.e. it has a piecewise linear saw-tooth form. The simplest (monofrequency) case of the saw-tooth function is given by $\tau(t)$ (Equation (3)), as Figure 3 shows. In this case, the second equality of Equation (86) holds for all t under the condition $(dX/d\tau)|_{\tau=\pm 1} = 0$, and the new equation has the same form as the original one: $(d^2 X/d\tau^2) + f(X) = 0$.

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