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Observabilty singularities and observer design: dual immersion approach

K.A.A. Langueh, O. Datcu, J-P Barbot, G. Zheng, K. Busawon*

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Abstract

It is well-known that, for nonlinear systems, the observability is often only a local property and depends on the input. Moreover, it is often required that the observer be of the same dimension as the original system. A direct consequence of this requirement is that it enlarges the set of observability singularities. If, on one hand, it is impossible to observe the state variables that are structurally unobservable, it is, however, possible to overcome the observability singularities introduced by the constraints on the observer design. In this paper, we propose a novel dual immersion method which allows to reduce the set of observability singularities. In addition, a step by step design of a high order sliding mode observer based on the proposed dual immersion approach is presented. Finally, a thorough analysis and discussion on the simulation results with respect to a non-autonomous system is given.

1 Introduction

It is a well-known that, unlike the linear case, the observability for nonlinear systems is, in general, not only a local property [13] but also depends on the input of the system [11]. A direct implication of these is that, in the nonlinear case, one has to determine the so-called 'observability singularity set' if proper observability analysis has to be carried out. Roughly speaking, the observability singularity set is the set of points in the state space whereby the observability matrix is not of full rank.

On the other hand, owing to the fact that it is easier to design an observer for linear systems, the natural approach to designing nonlinear observers, adopted by several researchers [15, 25, 22, 14, 12, 1, 2], consist in transforming the original system into

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i) either a linear one plus a nonlinear part having some special structures

ii) or a linear one plus a nonlinear part depending only on the input and the output so the observer has linear error dynamics.

Unfortunately, even if the observer design is done using these approaches, the problem introduced by the observability singularity set stays omnipresent and is not overcomed. In this paper, we propose a solution, based on the immersion technique, to tackle this problem under very weak conditions.

It is important to mention that, since the work of [21, 20], the immersion technique was extensively used in the observer design context. However, it is generally used to recover the linearity property by diffeomorphism and output injection [3, 4, 24, 23]. In the majority of the mentioned papers, immersion was realized by adding a dynamic by means of output integration [3]. The stability of such extra dynamics can be problematic and an elegant solution to deal with this issue is proposed in [23]. However, the problem of observability singularity was not treated. In this work, a dual immersion technique using only extra differentiations is proposed to by pass the observability singularity issue. This method is close to the one proposed in [2, 5] in a completely different context. Moreover, for stability reasons, we chose to employ exponentially stable dynamics instead of constant dynamics. It is shown that the proposed approach is realisable thanks to the finite time differentiator as, for example, the one proposed in [6, 7] (but for these methods, the delay appears due to the data acquisition frame) or High Order Sliding Mode (HOSM) [18, 9].

The paper is organized as follows: in the next section some observability and observability singularity definitions are presented and the problem statement is explained. In Section 3, the dual immersion method is presented. After that, some recalls on HOSM differentiator are given in Section 4. In Section 5, a simulation example is given in order to highlight the fact that the proposed method can be extended to nonautonomous system. The paper ends with some conclusions and perspectives.

2 Some recalls and problem statement

Consider the following autonomous system:

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^m$ and the vector fields f and h are assumed to be C^∞. It is also assumed that the outputs are independent for $\forall x \in \mathbb{R}^n$.

Notation: For $1 \leq i \leq m$, we denote by ρ_i the observability index [15] of the output function h_i at $x_0 \in \mathbb{R}^n$.

It is worth noting that, if the smallest order of output derivatives are considered, then the choice of m-tuples $(\rho_1, \cdots, \rho_m)$ is not unique.

Now suppose that around a certain point x_0 the system (1) is observable, i.e. $\sum_{i=1}^{m} \rho_i = n$, and rank $d\mathcal{O}_{|x_0} = n$ where

$$\mathcal{O}(n) = \begin{pmatrix} h_1 \\ \vdots \\ L_f^{\rho_1 - 1} h_1 \\ \vdots \\ h_m \\ \vdots \\ L_f^{\rho_m - 1} h_m \end{pmatrix} \tag{2}$$

is the observability map and

$$d\mathcal{O}(n)_{|x_0} = \begin{pmatrix} dh_1 \\ \vdots \\ dL_f^{\rho_1 - 1} h_1 \\ \vdots \\ dh_m \\ \vdots \\ dL_f^{\rho_m - 1} h_m \end{pmatrix}_{|x_0} \tag{3}$$

is the observability matrix with $L_f^i h = \frac{\partial L_f^{i-1} h}{\partial x} f$ the usual Lie derivative and $dL_f^i h = \left(\frac{\partial L_f^i h}{\partial x_1}, \frac{\partial L_f^i h}{\partial x_2}, ..., \frac{\partial L_f^i h}{\partial x_n} \right)$ its corresponding 1-form.

Since $d\mathcal{O}(n)_{|x_0}$ is only of full rank n around x_0, this implies that there might exist some $\bar{x} \in \mathbb{R}^n$ such that rank $d\mathcal{O}(n)_{|\bar{x}} < n$. Due to this fact, we define the following observability singularity set:

$$\mathcal{S}_n = \{x \in \mathbb{R}^n : \text{rank } \{d\mathcal{O}(n)_{|x}\} < n\} \tag{4}$$

In this case, for the purpose of removing the singularities in $d\mathcal{O}(n)_{|x}$ defined in (3), it is necessary to increase the dimension of (3) by involving more derivatives of the output. The following example illustrates the procedure involved.

Example 1 *Let us consider the following simple system:*

$$\begin{aligned} \dot{x}_1 &= x_2 + x_2^2 \\ \dot{x}_2 &= -x_2^3 + 1 \\ \dot{x}_3 &= x_2 - x_2^3 \end{aligned} \tag{5}$$

with $y_1 = x_1$ and $y_2 = x_3$.

It can be seen that, if we choose the corresponding observability indices as $(\rho_1 = 2, \rho_2 = 1)$, then

$$d\mathcal{O}(3)_{|x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + 2x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we obtain the following observability singularity set:

$$\mathcal{S}_3 = \{x \in \mathbb{R}^3 : x_2 = -0.5\}$$

On the other hand, if the observability indices were chosen as $(\rho_1 = 1, \rho_2 = 2)$*, then*

$$d\mathcal{O}(3)_{|x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1-3x_2^2 & 0 \end{pmatrix}$$

and the observability singularity set is now given by:

$$\mathcal{S}_3 = \{x \in \mathbb{R}^3 : x_2 = \frac{\pm 1}{\sqrt{3}}\}$$

It is clear that both observability matrices contain singularities, but they are not the same. In order to overcome those singularities, one can compute further derivatives of the output. Indeed, consider the observability indices $(\rho_1 = 3, \rho_2 = 1)$*, then we obtain*

$$d\mathcal{O}(4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+2x_2 & 0 \\ 0 & -8x_2^3 - 3x_2^2 + 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathcal{S}_4 = \emptyset$*, i.e. there is no longer any observability singularity. These highlight the fact that the choice of the observability indices is crucial in designing an observer for a nonlinear system. As a result, we obtain the following immersion from* $\mathbb{R}^3$ *to* $\mathbb{R}^4$*.*

$$z = \phi(x) = \begin{pmatrix} x_1 \\ x_2 + x_2^2 \\ -2x_2^4 - x_2^3 + 2x_2 + 1 \\ x_3 \end{pmatrix}$$

The previous example shows that, even if the observability matrix $d\mathcal{O}(n)$ contains singularities, i.e. its rank is not equal to n for some $x \in \mathbb{R}^n$, it is still possible to obtain a higher dimensional map, $\mathcal{O}(n+k)$, that will not contain any singularity. This map can then be regarded as an immersion. It may, therefore, be interesting to design an observer of dimension greater than n for the system; which is equivalent to using a state space representation or order greater than n. More precisely, this immersion is obtained by finding the smallest integer $k \in \mathcal{Z}^+$ such that

$$\text{rank}\{d\mathcal{O}(n+k)\} = n, \forall x \in \mathbb{R}^n \tag{6}$$

where

$$\mathcal{O}(n+k) = \begin{pmatrix} h_1 \\ \vdots \\ L_f^{\rho_1+k_1} h_1 \\ \vdots \\ h_m \\ \vdots \\ L_f^{\rho_m+k_m} h_m \end{pmatrix} \tag{7}$$

with $\sum_{i=1}^{m} \rho_i = n$ and $\sum_{i=1}^{m} k_i = k$. Then, the immersion can be defined as $z = \mathcal{O}(n+k)$ yielding a higher dimensional transformed system

$$\dot{z} = \frac{\partial \mathcal{O}(n+k)}{\partial x} f(x)$$

and which can be rewritten as follows:

$$\begin{pmatrix} \dot{z}_{1,1} \\ \vdots \\ \dot{z}_{\rho_1+k_1} \\ \vdots \\ \dot{z}_{m,1} \\ \vdots \\ z_{\rho_m+k_m} \end{pmatrix} = \begin{pmatrix} z_{1,2} \\ \vdots \\ L_f^{\rho_1+k_1} h_1 \\ \vdots \\ z_{m,2} \\ \vdots \\ L_f^{\rho_m+k_m} h_m \end{pmatrix} \tag{8}$$

For (8) a traditional high-gain observer [11] (or sliding mode observer [18]) can be easily designed to estimate z. One can then obtain an estimation of x as follows

$$\hat{x} = \underset{\hat{x} \in \mathbb{R}^n}{\arg\min} ||\hat{z} - \mathcal{O}(n+k)(\hat{x})||$$

where $\hat{z}$ is the estimate of z.

It is important to note that, for the above optimization task, the Jacobian of $\mathcal{O}(n+k)$ is a non-square matrix of dimension$(n+k)$ by n. This will inevitably cause difficulties when applying numerical methods to solve the optimization problem. Obviously, some additional techniques can be used even if the jacobian matrix of $\mathcal{O}(n+k)$ is not square, However, the optimisation task would be much more easier if one can find a full rank square Jacobian matrix.

By taking into account these previous remarks, we are going to propose, for a given $\mathcal{O}(n)$ defined in (3) containing the singularities, a constructive way to deduce, not only a simple immersion, but a global diffeomorphism $\mathcal{O}(n+k)$ as defined in (7) by increasing the state space dimension arbitrarily in the original coordinates. In the literature, this technique is called immersion and several authors have used this immersion technique in order to obtain a specific normal form [23]. Most often, the immersion is obtained by output integration. In this paper, however, the immersion will be derived by a dual method which will be presented in the next section.

Remark 1 *The above problem statement can be extended to general dynamical systems, including nonautonomous one. For instance, consider the following system:*

$$\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x) \end{aligned} \tag{9}$$

where $u \in C^\infty$ is the input. Then, as it was pointed out in [11], the observability of (9) also depends on the input (which has, in turn led, to the concept of universal input). In this case, instead of Lie derivative it is necessary to use the Lie Bäcklund derivative [8][1]. For system (9), we can define the following observability matrix

$$d\mathcal{O}(n) := \begin{pmatrix} dh_1 \\ \vdots \\ d\tilde{L}_f^{\rho_1-1}h_1 \\ \vdots \\ dh_m \\ \vdots \\ d\tilde{L}_f^{\rho_m-1}h_m \end{pmatrix}_{|_{x,u,\cdots,u^{(n-1)}}}$$

and its associated singularity observability set, parameterized by the input and its derivatives, as follows:

$$\mathcal{S}_n = \{x \in \mathbb{R}^n : \text{rank}\{d\mathcal{O}(n)_{|_{x,u,\cdots,u^{(n-1)}}}\} < n\}$$

3 Dual immersion technique

3.1 Preliminary results

In this section, for a given observability map $\mathcal{O}(n)$ containing singularities in its associated observability singularity set $\mathcal{S}_n$, we will propose a constructive method to deduce a global diffeomorphism $\mathcal{O}(n+k)$ as defined in (7) by increasing the state dimension (and consequently its associated dynamics).
We start by stating the following trivial result.

Lemma 1 *Consider the observability singularity set $\mathcal{S}_n$ associated to the observability map $\mathcal{O}(n)$ as defined in (2), then*

$$\mathcal{S}_{n+i} \subseteq \mathcal{S}_{n+i-1}$$

fori ≥ 1.

[1]The Lie Backlund is $\tilde{L}_f^i h = \frac{\partial L_f^{i-1}h}{\partial x} f + \frac{\partial L_f^{i-1}h}{\partial u}\dot{u} + \cdots + \frac{\partial L_f^{i-1}h}{\partial u^{(j-2)}} u^{(j-1)}$ where $u^{(j)}$ is the jth derivative of u with respect to time.

Proof. For $i \geq 0$, according to the definition, we have

$$\mathcal{O}(n+i) = \begin{pmatrix} h_1 \\ \vdots \\ L_f^{\rho_1+i_1} h_1 \\ \vdots \\ h_m \\ \vdots \\ L_f^{\rho_m+i_m} h_m \end{pmatrix}$$

with $\sum_{i=1}^{m} \rho_i = n$ and $\sum_{j=1}^{m} i_j = i$. Consequently, there exists an elementary matrix S_i such that

$$\mathcal{O}(n+i) = S_i \begin{bmatrix} \mathcal{O}(n+i-1) \\ L_f^{\rho_1+i_1} h_1 \\ \vdots \\ L_f^{\rho_m+i_m} h_m \end{bmatrix}$$

Hence, we have

$$\text{rank}\{d\mathcal{O}(n+i)\} \geq \text{rank}\{d\mathcal{O}(n+i-1)\}$$

which implies that $\mathcal{S}_{n+i} \subseteq \mathcal{S}_{n+i-1}$. ■

In what follows, we make the following assumption on the global observability in order to find a global diffeomorphism by iteratively differentiating the output.

Assumption 1 *We assume that system (1) is globally observable, i.e. there exists a least positive integer $k \in \mathcal{Z}^+$ such that*

$$\text{rank}\{d\mathcal{O}(n+k)\} = n \tag{10}$$

for all $x \in \mathbb{R}^n$, where $\mathcal{O}(n+k)$ is defined in (7).

Remark 2 *It is well known that the choice of observability index is not unique ([15]), consequently, the construction of $\mathcal{O}(n)$ is not unique. From these, the construction of $\mathcal{O}(n+k)$ depends also of the original choice of the observability index this key point will be investigated in forthcoming paper.*

With the above assumption, we can state the following result.

Lemma 2 *For system (1), if Assumption 1 is satisfied, then there exists a least positive integer $k \in \mathcal{Z}^+$ such that the following inclusion is satisfied:*

$$\emptyset = \mathcal{S}_{n+k} \subseteq \cdots \subseteq \mathcal{S}_{n+1} \subseteq \mathcal{S}_n$$

Moreover, $z = \mathcal{O}(n+k)$ is an immersion.

Proof. The subset inclusions can be obtained directly via Lemma 1. Moreover, due to Assumption 1 on the global observability, there exists a least positive integer $k \in \mathcal{Z}^+$ such that no singularities appear in $d\mathcal{O}(n+k)$, thus we have $\mathcal{S}_{n+k} = \emptyset$, since $\text{rank}\{d\mathcal{O}(n+k)\} = n \ \forall x \in \mathbb{R}^n$. ■

The above results show that one can reduce the observability singularity set till the empty set by simply increasing the number of derivatives of the output. Consequently, one can find a least positive integer $k \in \mathcal{Z}^+$ such that $z = \mathcal{O}(n+k)$ is an immersion, but not necessarily a diffeomorphism. In what follows, we will propose a constructive method to calculate a global diffeomorphism $\phi(x)$ from the deduced immersion, $z = \mathcal{O}(n+k)$, in Lemma 2 via the technique of dual immersion, and whose Jacobian $d\phi(x)$ is square and invertible.

3.2 Dual immersion approach

Consider again system (1), and suppose that we have already found a least positive integer $k \in \mathcal{Z}^+$ such that $z = \mathcal{O}(n+k)$ is an immersion. Then, the following algorithm enables to compute a global diffeomorphism $\phi(x, \xi)$ (where $\xi \in \mathbb{R}^k$ will be defined hereafter).

Dual immersion algorithm

1. Initialization: Set $z_{i,1} = \phi_{i,1}(x) = y_i$ for $1 \leq i \leq m$;
2. Calculate $z_{i,j} = \phi_{i,j}(x) = y_i^{(j-1)}$ until that a singularity of observability appears in the row $dL_f^{(j-1)} h_i$;
3. Define $\phi_{i,j} = y_i^{(j-1)} + \xi_{i,1}$ where $\xi_{i,1}$ is an additional state, with the following dynamics:
$$\dot{\xi}_{i,1} = -\epsilon_{i,1}\xi_{i,1} \tag{11}$$
initialized at $\xi_{i,1} = 0$.
4.
   - If y_i^{j+1} does not exist on $\mathcal{O}(n+k)$, define $\phi_{i,j+1} = y_i^{(j)} - \epsilon_{i,1}\xi_{i,1}$
   - If y_i^{j+1} exists, define $\phi_{i,j+1} = y_i^{(j)} - \epsilon_{i,1}\xi_{i,1} + \xi_{i,2}$, with
$$\dot{\xi}_{i,2} = -\epsilon_{i,2}\xi_{i,2} + \dot{\epsilon}_{i,1}\xi_{i,1} - \epsilon_{i,1}^2\xi_{i,1}$$
5. Repeat the operation until we obtain ϕ_{i,ρ_i+k_i} and this for $i \in \{1, ..., m\}$. The resulting matrix $d\phi$ becomes a square matrix. In addition, if there exists $\epsilon_{i,j}$ sufficiently smooth such that the matrix is regular for all x then the algorithm 3.2 converges.

This algorithm yields the following result:

Theorem 1 *For system (1), if Assumption 1 is satisfied and Algorithm 3.2 converges, then $\phi(x, \xi)$ with $\xi \in \mathbb{R}^k$ is a global diffeomorphism. Moreover, the Jacobian of the generated $\phi(x, \xi)$ via the dual immersion satisfies:*

$$\dim d\bar{\phi}(x, \xi) \in \mathbb{R}^{(n+k)\times(n+k)}$$

and $d\phi(x, \xi)$ is full rank for all $x \in \mathbb{R}^n$.

Proof. The proof follows from the algorithm, and Assumption 1 is a necessary condition for the convergence of algorithm 3.2. ■ Considering that, for the order $r = n + k$, $d\mathcal{O}(r)(h, f)$ generates an empty observability set $\mathcal{S}(r)$ or at least an acceptable one with respect to the dynamics (see [10] for chaotic systems), it is possible to define a new change of coordinates $z = \phi(x, \xi)$ as follows: $z_{i,j} = y_i^{(j-1)}$ until an observability singularity appears in the row $dL_f^{(j-1)}h_i$. When a singularity appears in row $y_i^{(j-1)}$, then $z_{i,j}$ is reassigned to $y_i^{(j-1)} + \xi_{i,1}$ where $\xi_{i,1}$ is an extra state, which satisfies the following dynamics

$$\dot{\xi}_{i,1} = -\epsilon_{i,1}\xi_{i,1} \tag{12}$$

that is initialized at $\xi_{i,1} = 0$.

Remark 3 *From the stability and the initialization of the dynamics (12) it is not necessary to simulate this dynamic in the observer design because it is equivalent to adding zero at $y_i^{(j-1)}$. In [2, 5] the proposed solution consists in adding a constant in order to be able to design a high gain observer in the original coordinate.*

After this step, $z_{i,j+l} = y_i^{(j+l-1)} + \xi_{i,1}^{(l)}$. If it is necessary to repeat the procedure for another row of $d\phi$ with respect to the derivative of y_i, then another state $\xi_{i,2}$ is introduced with the dynamics

$$\dot{\xi}_{i,2} = -\epsilon_{i,2}\xi_{i,2}$$

and with $\xi_{i,2} = 0$ as initial state.

As all $\xi_{i,j}$ are always equal to zero, then nothing is added to the original system. Then, from the dual immersion approach, the observability matrix $\phi|_{x,\xi}$ is a square matrix and the $\epsilon_{i,j}$ are chosen so that $d\phi|_{x,\xi}$ is full rank.

Example 2 *(Example 1 continued) For the system (5), we have obtained, as shown in Example* 1*, that*

$$\mathcal{O}(4) = \left(x_1, x_2 + x_2^2, -2x_2^4 - x_2^3 + 2x_2 + 1, x_3\right)^T$$

is an immersion. By applying the proposed method, one obtains

$$\phi(x, \xi) = \left(x_1, x_2 + x_2^2 + \xi_{1,1}, -2x_2^4 - x_2^3 + 2x_2 + 1, x_3\right)^T$$

with $\dot{\xi} = \dot{\xi}_{1,1} = -\epsilon_{1,1}\xi_{1,1}$, whose Jacobian is equal to:

$$d\bar{\phi}(x, \xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + 2x_2 & 0 & 1 \\ 0 & -8x_2^3 - 3x_2^2 + 2 & 0 & -\epsilon_{1,1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It can be easily checked that it is square and non singular for all $x \in \mathbb{R}^n$ if $\epsilon_{1,1}$ is chosen such that $(1 + 2x_2)\epsilon_{1,1} - 8x_2^3 - 3x_2^2 + 2 \neq 0$.

4 Recalls on high-order sliding-mode

In this paper, the proposed method is based on the so-called real-time exact robust HOSM differentiator [17, 19], which be recalled in the following.

Consider a signal $y(t) \in C^k$ (at least k times differentiable), let us suppose $(y, \cdots, y^{(k)}) = (z_1, \cdots, z_{k+1})$. The HOSM robust differentiator proposed in [18] takes the following form:

$$\begin{aligned}
\dot{\hat{z}}_1 &= -\lambda_0 M^{\frac{1}{k}} |\hat{z}_1 - y|^{\frac{k}{k+1}} sign(\hat{z}_1 - y) + \hat{z}_2 \\
\dot{\hat{z}}_2 &= -\lambda_1 M^{\frac{1}{k-1}} |\hat{z}_2 - v_1|^{\frac{k-1}{k}} sign(\hat{z}_2 - v_1) + \hat{z}_3 \\
&\vdots \\
\dot{\hat{z}}_k &= -\lambda_{k-1} M^{\frac{1}{2}} |\hat{z}_k - v_{k-1}|^{\frac{1}{2}} sign(\hat{z}_k - v_{k-1}) + \hat{z}_{k+1} \\
\dot{\hat{z}}_{k+1} &= -\lambda_k M sign(\hat{z}_{k+1} - v_k)
\end{aligned}$$

where M is chosen to be larger than the k-th derivative of $y(t)$, λ_i are positive design parameters, and the adjustment or tuning of those parameters is described in detail in [17] and [16]. Defining the observation errors as: $e_i = z_i - \hat{z}_i$, then the observation errors dynamics is given by:

$$\begin{aligned}
e_1 &= \hat{z}_1 - y \\
e_2 &= \dot{e}_1 = \lambda_0 M^{\frac{1}{k}} |e_1|^{\frac{k}{k+1}} sign(e_1) \\
&\vdots \\
e_k &= \dot{e}_{k-1} = \lambda_{k-1} M^{\frac{1}{2}} |e_{k-1}|^{\frac{1}{2}} sign(e_{k-1}) \\
e_{k+1} &= \dot{e}_k = \lambda_k M sign(e_k)
\end{aligned}$$

It has been proven in [17] that there exists t_0 such that $\forall t > t_0$ we have

$$e_i = z_i - \hat{z}_i = 0 \quad \text{for} \quad 1 \leq i \leq k+1$$

In the next section, an example, representing a type of observability singularity that appears for example in the induction motor where the outputs are two phase currents, is presented. Moreover, since this system is non autonomous, it highlights the fact that the proposed method can be generalized to some class of non autonomous system.

5 Non autonomous multi-output example.

Let us consider the following non autonomous multi-output system:

$$\begin{aligned}
\dot{x}_1 &= x_2 cos(t) \\
\dot{x}_2 &= -x_3 \\
\dot{x}_3 &= x_2 \\
\dot{x}_4 &= x_2 sin(t)
\end{aligned} \tag{13}$$

where $y_1 = x_1$ and $y_2 = x_4$ are the outputs.

Remark 4 *System (13) is exactly of the form (9) with $u = t$. Moreover, with respect to an induction machine model or other AC three-phase machine model,* $\sin(t)$ *and* $\cos(t)$ *play the same role as* $\cos(\omega t)$, $\cos(\omega t + \frac{2\pi}{3})$ *and* $\cos(\omega t + \frac{4\pi}{3})$ *with ω being the electrical pulsation.*

For system (13), if we take the observability indices as $(\rho_1 = 3, \rho_2 = 1)$, then the observability singularity occurs at $t = \frac{\pi}{2} + k\pi$. Similarly for $(\rho_1 = 1, \rho_2 = 3)$ which yields an observability singularity at $t = k\pi$. Finally, for $(\rho_1 = 2, \rho_2 = 2)$ the observability singularity occurs at $t = k\frac{\pi}{2}$. Nevertheless, if we consider the third derivative for both outputs (i.e. $(\rho_1 = 3, \rho_2 = 3)$), we have:

$$\mathcal{O}(4) \quad = \quad [x_1, x_2 \cos t, -x_2 \sin t - x_3 \cos t, x_4, x_2 \sin t, x_2 \cos t - x_3 \sin t]^T$$

for which $d\mathcal{O}(4)$ is of full column rank; i.e $\text{rank}\{d\mathcal{O}(4)\} = 4$ for all $x \in \mathbb{R}^4$ and $t \geq 0$. Consequently:

$$z = \phi(x, \xi) = \begin{pmatrix} x_1 \\ x_2 \cos t + \xi_{1,1} \\ -x_2 \sin t - x_3 \cos t \\ x_4 \\ x_2 \sin t + \xi_{2,1} \\ x_2 \cos t - x_3 \sin t \end{pmatrix} = \begin{pmatrix} y_1 \\ \dot{y}_1 + \xi_{1,1} \\ \ddot{y}_1 - \epsilon_{1,1} \\ y_2 \\ \dot{y}_2 + \xi_{2,1} \\ \ddot{y}_2 - \epsilon_{2,1} \end{pmatrix}$$

with $\dot{\xi}_{1,1} = -\epsilon_{1,1}\xi_{1,1}$ and $\dot{\xi}_{2,1} = -\epsilon_{2,1}\xi_{2,1}$ both with zero initial conditions. Then, $d\phi(x, \xi)$ is square and equal to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos t & 0 & 0 & 1 & 0 \\ 0 & -\sin t & -\cos t & 0 & -\epsilon_{1,1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sin t & 0 & 0 & 0 & 1 \\ 0 & \cos t & -\sin t & 0 & 0 & -\epsilon_{2,1} \end{bmatrix}$$

In this trivial example no numerical method is needed to compute the inverse, since one has the explicit formula of $\phi^{-1}(z)$, which however might not be possible in more complicated examples. Thus, one can obtain an estimate of x via the relation $(x^T, \xi^T)^T = \phi^{-1}(z)$, where $\hat{z} = z$ is obtained in finite time by means of HOSM differentiation. For simulation, initial conditions are chosen from an uniform distribution in $[0, 1]$, and the observer parameters are chosen as $M = 50$ and $\epsilon_{1,1} = \epsilon_{2,1} = 0$. In Fig. 1 and Fig. 2, it can be seen that the observations of x_2 and x_3 are given after a finite time (around $1s$).

When adding noise in the output, the results are depicted in Fig. 3 and Fig. 4, in which we apply firstly a Butterworth filter (as done for many engineering processes to filter the noise in advance), then design the HOSM observer. The variations in amplitude of the estimates are due to the filtering of the noise.

6 Conclusion

In this work, a dual immersion method was proposed in order to overcome the singularity problems due to observability singularity set. The proposed method is based on an immersion which is, in turn, transformed to diffeomorphism by

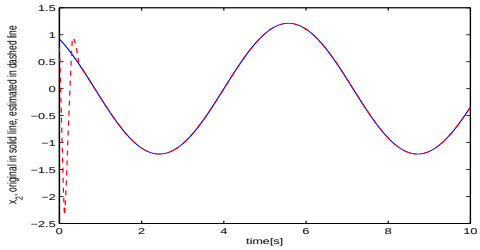

Figure 1: The signal x_2 in blue, and its estimate $\hat{x}_2$ in red.

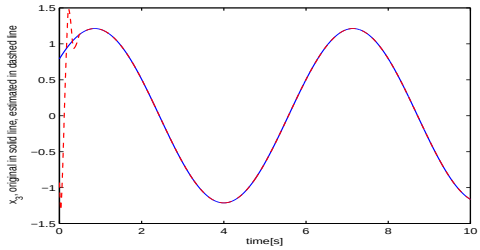

Figure 2: The signal x_3 in blue, and its estimate $\hat{x}_3$ in red.

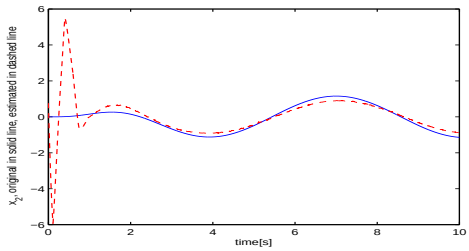

Figure 3: Noisy channel. The signal x_2 in blue, and its estimate $\hat{x}_2$ in red.

adding a stable fictitious dynamics initialized at the equilibrium point to the original system. The inversion of the obtained diffeomorphism is not necessarily a straightforward task. As a result, in a future work, a numerical solution for the inverse of the diffeomorphism will be investigated by taking into account the particular structure of the obtained diffeomorphism. Another promising solution will be to design an observer in the extended original coordinates with the observer design method introduced in [5]. Moreover, an extension of the proposed method to some class of non autonomous system seem to be possible.

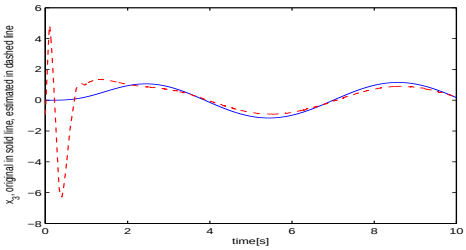

Figure 4: Noisy channel. The signal x_3 in blue, and its estimate $\hat{x}_3$ in red.

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