

Topological alterations of 3D digital images under rigid transformations

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Résumé

Rigid transformations in \mathbb{R}^n are known to preserve the shape, and are often applied to digital images. However, digitized rigid transformations, defined as digital functions from \mathbb{Z}^n to \mathbb{Z}^n do not preserve shapes in general; indeed, they are almost never bijective and thus alter the topology. In order to understand the causes of such topological alterations, we first study the possible loss of voxel information and modification of voxel adjacencies induced by applications of digitized rigid transformations to 3D digital images. We then show that even very simple structured images such as digital half-spaces may not preserve their topology under these transformations. This signifies that a simple extension of the two-dimensional solution for topology preservation cannot be made in three dimensions.

Mots clé : digital topology, rigid transformation, digital image

1. Introduction

Three-dimensional digital images can be viewed as finite subsets of \mathbb{Z}^3 . In the context of applications of rigid transformations to digital images, they are defined as digital functions $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$. In contrast with rigid transformations defined as $\mathbb{R}^n \rightarrow \mathbb{R}^n$, their digitized analogues do not preserve distances and angles in general; indeed, they are almost never bijective and thus alter both geometry and topology.

Recently, 2D discrete rigid transformations were studied by Ngo *et al.* [NKPT13], with their impact on topological properties [NKPT14], which finally led to establishing sufficient conditions for topology preservation under digitized rigid transformations [NPKT14]. The purpose of this study is to investigate whether the conditions for topology preservation established by Ngo *et al.* [NPKT14] in the case of two dimensions can be extended to three dimensions. In particular, we study how fundamental problems induced by digitization differ between two- and three-dimensional transformations. In order to answer these questions, we will identify the following three main problems: the possible loss of information during digitized rigid transformations, the alteration of distances between a pair of adjacent points, and their impact on topological properties. Our study will show that the main two problems, which are common in both dimensions, have different impacts on topological properties, and the link between two- and three-dimensional conditions for topology

preservation under digitized rigid transformations cannot be straightforwardly established. Indeed, our main result shows that some simple topological properties of images cannot be preserved under digitized rigid transformations even for “planar” surfaces, which have one of the simplest geometrical shapes.

2. Rigid transformations

Rigid transformations in \mathbb{R}^3 are bijective isometry maps [GS07] which preserve distances and angles between every pair and triple of points respectively. The set of rigid transformations includes rotations, translations, reflections, and combinations of these. Hereafter, only rotations, translations and their compositions will be considered. Reflections are excluded due to application demand. It may be noted that reflections are easily obtained by composing a \mathbb{Z}^n -preserving reflection with a rotation; hence the same issues hold for reflections. In \mathbb{R}^3 , a rigid transformation is defined as a function:

$$\begin{cases} \mathcal{U} : \mathbb{R}^3 & \rightarrow \mathbb{R}^3 \\ \mathbf{x} & \mapsto \mathbf{R}\mathbf{x} + \mathbf{t} \end{cases} \quad (1)$$

where \mathbf{t} denotes a translation vector and \mathbf{R} is a rotation matrix. Note also that the inverse function $\mathcal{T} = \mathcal{U}^{-1}$ is also a rigid transformation.

According to (1), in general, $\mathcal{U}(\mathbb{Z}^3) \not\subseteq \mathbb{Z}^3$. Therefore, rigid transformations defined as maps $\mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ have to be treated with care. The most common solution is to apply rigid transformations together with digitization operator $\mathcal{D} : \mathbb{R}^3 \rightarrow \mathbb{Z}^3$.

Then digitized rigid transformations are defined as:

$$U = \mathcal{D} \circ \mathcal{U}|_{\mathbb{Z}^3} \quad (2)$$

and

$$T = \mathcal{D} \circ \mathcal{T}|_{\mathbb{Z}^3} = \mathcal{D} \circ (\mathcal{U}^{-1})|_{\mathbb{Z}^3} \quad (3)$$

which are the digitized versions of \mathcal{U} and \mathcal{T} , respectively. Due to behavior of \mathcal{D} that maps $\mathbb{R}^3 \rightarrow \mathbb{Z}^3$, digitized rigid transformations are not bijective in general.

3. Voxel statuses

According to (2), it may happen that none, or more than one point of finite subset of \mathbb{Z}^3 will enter into the same voxel after transformation of image. To analyze such situations, the status of voxels after digitized rigid transformation can be defined as follows. Let us consider the voxel whose center is $\mathbf{y} \in \mathbb{Z}^3$. For a given digitized rigid transformation U , the set of points whose transformation is \mathbf{y} is given by $M(\mathbf{y}) = \{\mathbf{x} \in \mathbb{Z}^3 \mid U(\mathbf{x}) = \mathbf{y}\}$. If we consider $|M(\mathbf{y})|$ as the status of voxel whose center is \mathbf{y} , it can be proven that only five statuses are possible, namely $|M(\mathbf{y})| \in \{0, 1, 2, 3, 4\}$. This indicates that digitized rigid transformations can induce a large loss of information; in addition, five voxel statuses exist in 3D, instead of three statuses in 2D, where $|M(\mathbf{y})| \in \{0, 1, 2\}$ [NR05]. Moreover, two or more voxels such that their status is zero can be *6-adjacent*, while the analogous case in 2D, namely *4-adjacent* pixels of $|M(\mathbf{y})| = 0$, is impossible [NR05].

4. Distance alterations

In general, digitized rigid transformations do not preserve distances in both 2D and 3D. The most critical case for 2D is an *8-adjacent* between which the Euclidean distance $d_e = \sqrt{2}$; after digitized transformations, it can become $\sqrt{5}$ at most (in [NKPT14] authors incorrectly gave 2 as a maximum). In 3D, both *18-* and *26-adjacencies* provide critical cases – the set of possible distances between an adjacent pair of points contains relatively higher values, such as $\sqrt{6}$ and 3, respectively. More precisely, considering two points \mathbf{p}, \mathbf{q} then, we have

$$d_e(\mathbf{p}, \mathbf{q}) = 1 \Rightarrow d_e(\mathbf{p}', \mathbf{q}') \in \{0, 1, \sqrt{2}, \sqrt{3}\}$$

$$d_e(\mathbf{p}, \mathbf{q}) = \sqrt{2} \Rightarrow d_e(\mathbf{p}', \mathbf{q}') \in \{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}\}$$

$$d_e(\mathbf{p}, \mathbf{q}) = \sqrt{3} \Rightarrow d_e(\mathbf{p}', \mathbf{q}') \in \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{8}, 3\}$$

where $\mathbf{p}' = U(\mathbf{p})$ and $\mathbf{q}' = U(\mathbf{q})$. Such modifications of distances induced by digitized transformations are potential causes of topological issues like merging or splitting of connected components.

5. Topological issues

The main result of this study shows differences between digitized rigid transformation defined in \mathbb{Z}^2 and \mathbb{Z}^3 , that implies that the solution for topology preservation under digitized rigid transformations established by Ngo *et al.* [NPKT14] cannot be simply extended from 2D to 3D. Indeed, as noted by Figure 1 which presents \mathbf{x} and its *6-neighborhood* under rigid transformation, alterations of topological properties are possible even for simple digital shapes, such as discrete

planes [KBSS08]. This makes difficult to extend the definition of regular images, which allows, in 2D, to preserve their topology [NPKT14]. Our result adds to the set of known problems in digital topology, which are difficult to overcome when increasing the space dimension from two to three, such as homotopy thinning [PCB08] and digitization [SLS07].

6. Conclusion

In this study, the first step to understand digitized rigid transformations in \mathbb{Z}^3 was taken, and the essential topological problems and their causes were highlighted. Indeed, in order to propose efficient methods for topology reparation after digitized rigid transformations, it is crucial to understand how the adjacency structure of digital images changes under digitized rigid transformations first, in particular, the relations between these changes and the statuses of voxels or between the induced distances. It has to be mentioned that, due to the lack of combinatorial and mathematical tools, such as those developed for 2D digitized transformations [NR05], [NKPT13], a future study should start with developing an extension of these tools to 3D. It is also important to mention that preservation of geometrical properties of digital images under digitized rigid transformations is still an open problem, which deserves a better understanding in both 2D and 3D.

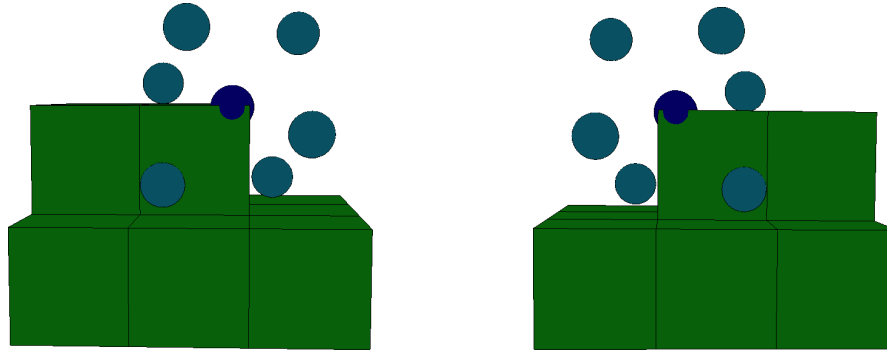


Figure 1: In green, a $3 \times 3 \times 3$ part of initial image. The spheres represent selected points of transformed space after application of T . The darkened blue sphere corresponds to the point which was mapped onto border voxel. Bright spheres correspond to its 6-adjacent neighbors. Even if the dark blue point is a part of the digital plane after a transformation, its neighboring points are not.

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