Admissible subsets and Littelmann paths in affine Kazhdan-Lusztig theory

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Abstract

The center of an extended affine Hecke algebra is known to be isomorphic to the ring of symmetric functions associated to the underlying finite Weyl group W_0. The set of Weyl characters s_λ forms a basis of the center and Lusztig showed in [11] that these characters act as translations on the Kazhdan-Lusztig basis element C_{w_0} where w_0 is the longest element of W_0, that is we have $C_{w_0}\mathsf{s}_\lambda = C_{w_0 t_\lambda}$. As a consequence, the coefficients that appear when decomposing $C_{w_0 t_\lambda}\mathsf{s}_\tau$ in the Kazhdan-Lusztig basis are tensor multiplicities of the Lie algebra with Weyl group W_0. The aim of this paper is to explain how admissible subsets and Littelmann paths, which are models to compute such multiplicities, naturally appear when working out this decomposition.

1 Introduction

Let W_e be an extended affine Weyl group with underlying finite Weyl group W_0. Then $W_e = W_0 \ltimes P$ where P denotes the set of weights associated to W_0. Let $\mathcal{H}$ be the generic affine Hecke algebra of W_e defined over $\mathcal{A}$ the ring of Laurent polynomials with one indeterminate q and let $\{C_w \mid w \in W_e\}$ be the Kazhdan-Lusztig basis of $\mathcal{H}$. The center of the affine Hecke algebra $\mathcal{H}$ associated to W_e is known to be isomorphic to the ring of symmetric functions $\mathcal{A}[P]^{W_0}$. The set of Weyl characters $\{\mathsf{s}_\lambda \mid \lambda \in P^+\}$ forms a basis of $\mathcal{A}[P]^{W_0}$ and we have $C_{w_0}\mathsf{s}_\lambda = C_{w_0 t_\lambda}$ where t_λ denotes the translation by $\lambda \in P^+$ in W_e; see [11, 13] and the references therein.

Denote by $V(\tau)$ the irreducible highest weight module of weight $\tau \in P^+$ for the simple Lie algebra over $\mathbf{C}$ with Weyl group W_0 and weight lattice P. Then the character of $V(\lambda)$ is s_λ and for all $\tau, \lambda \in P^+$ we have $\mathsf{s}_\lambda \mathsf{s}_\tau = \sum m^\mu_{\lambda,\tau}\mathsf{s}_\mu$ where $m^\mu_{\tau,\lambda}$ is the multiplicity of $V(\mu)$ in the tensor product $V(\lambda) \otimes V(\tau)$. Computing the multiplicities $m^\mu_{\tau,\lambda}$ is one of the most basic question in representation theory of simple Lie algebras over $\mathbf{C}$. Littelmann showed [9] that such multiplicities can be determined by counting certain kind of paths in the weight lattice P constrained to stay in the fundamental chamber. Later on, Lenart and Postnikov [7, 8] showed that these multiplicities can be determined using admissible subsets associated to a fix reduced expression of $t_\tau \in W_e$. In [7] they used a geometric approach based on equivariant K-theory while in [8] their approach is more axiomatic and is based on the fact that their model satisfies a set of axioms, introduced by Stembridge in [16], that encode the combinatorics of Weyl characters. The model of Lenart and Postnikov can be viewed as a discrete counterpart of Littelmann paths model and they explicitly constructed a bijection between admissible subsets and Lakshmibai-Seshadri paths (which are certain kind of Littelmann paths).

In the extended affine Hecke algebra, we must have

$$C_{w_0 t_\lambda}\mathsf{s}_\tau = C_{w_0}\mathsf{s}_\lambda\mathsf{s}_\tau = \sum m^\mu_{\lambda,\tau} C_{w_0 t_\mu}.$$

The aim of this paper is to explain how admissible subsets (and thus Littelmann paths) naturally appear when decomposing $C_{w_0 t_\lambda}\mathsf{s}_\tau$ in the Kazhdan-Lusztig basis. Ultimately, this will be a consequence of a multiplication formula for two standard basis elements in the extended affine Hecke algebra [2, proof of Proposition 5.1]. More precisely, we will, for all $\lambda, \tau \in P^+$

1. construct elements $\mathsf{h}_\tau \in T_{t_\tau} + \sum_{y<t_\tau} \mathsf{q}^{-1}\mathbf{Z}[\mathsf{q}^{-1}]T_y$ such that $C_{w_0}\mathsf{h}_\tau = C_{w_0 t_\tau} = C_{w_0}\mathsf{s}_\tau$;
2. show that $C_{w_0 t_\lambda}\mathsf{h}_\tau$ is a $\mathbf{Z}$-linear combination of Kazhdan-Lusztig basis elements;
3. show that determining the expansion in (2) is equivalent to finding terms of maximal degree in products of the form $T_{w_0 t_\lambda v}T_{t_\tau}$ $(v \in W_0)$ expressed in the standard basis;
4. show that the maximal terms in (3) are indexed by admissible subsets J associated to a reduced expression of t_τ as defined by Lenart and Postnikov.

The paper is organised as follows. In Section 2, we introduce all the needed material on (extended) affine Weyl groups. In Section 3, we present Kazhdan-Lusztig theory for affine Hecke algebras with unequal parameters and we describe the center of these algebras. In Section 4 we prove (1)–(3) above: this will essentially be a consequence of results on the lowest two-sided cells in [5]. We will prove Statement (4) in Section 5. In Section 6, following [8], we study the connections between our work and the results of Lenart and Postnikov and we describe the bijection between admissible subsets and Lakshmibai-Seshadri paths.

2 Affine Weyl groups

Let V be an Euclidean space with scalar product $(\cdot,\cdot)$. We denote by V^* the dual of V and by $\langle\ ,\ \rangle$: $V \times V^* \longrightarrow \mathbf{R}$ the canonical pairing. Let Φ be a root system and let $\Phi^\vee$ be the dual root system. If $\alpha \in \Phi$ then $\alpha^\vee \in \Phi^\vee$ is defined by $\langle x, \alpha^\vee\rangle = 2(x,\alpha)/(\alpha,\alpha)$. We fix a set of positive roots Φ^+ and a simple system $\Delta = \{\alpha_1, \dots, \alpha_N\}$ such that $\Delta \subset \Phi^+$.

2.1 Geometric presentation of an affine Weyl group

We denote by $H_{\alpha,n}$ the hyperplane defined by the equation $\langle x, \alpha^\vee\rangle = n$ and by $\mathcal{F}$ the collection of all such hyperplanes. We will say that an hyperplane H is of direction $\alpha \in \Phi^+$ if there exists a pair $(\alpha, n) \in \Phi^+ \times \mathbf{Z}$ such that $H = H_{\alpha,n}$. We then write $\overline{H} = \alpha$. For any subset $F \subset \mathcal{F}$ we set $\overline{F} := \{\overline{H} \mid H \in F\} \subset \Phi^+$. Playing with notations yields $\overline{\{H\}} = \{\overline{H}\}$.

Let W_a be the group generated by the set of orthogonal reflections $s_{\alpha,n}$ with respect to $H_{\alpha,n}$ where $\alpha \in \Phi$ and $n \in \mathbf{Z}$. The group W_a is an affine Weyl group of type $\Phi^\vee$ and it is isomorphic to $W_0 \ltimes Q$ where Q is the lattice generated by Φ. For $\lambda \in Q$, we will denote by t_λ the translation by λ in W_a. It is well known that W_a is generated by the set $S := \{s_{\alpha_i,0} \mid \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$ where $\tilde{\alpha}^\vee$ is the highest root of $\Phi^\vee$. We will simply write s_{α_i} for $s_{\alpha_i,0}$ where $1 \leq i \leq N$ and s_{α_0} for $s_{\tilde{\alpha},1}$. Let W_0 be the stabiliser of 0 in W_a and w_0 be the longest element of W_0. Clearly $W_0 = \langle S_0\rangle$ where $S_0 = \{s_{\alpha_1}, \dots, s_{\alpha_N}\}$. We will denote by id the identity element in W_a.

The set of alcoves, denoted $\mathrm{Alc}(\mathcal{F})$, is the set of connected components of $V \backslash \mathcal{F}$. The fundamental alcove A_0 is defined by

$$\begin{aligned} A_0 &= \{x \in V \mid 0 < \langle x, \alpha^\vee\rangle < 1,\ \forall \alpha \in \Phi^+\} \\ &= \{x \in V \mid 0 < \langle x, \alpha^\vee\rangle < 1,\ \forall \alpha \in \Delta\}. \end{aligned}$$

The set of hyperplanes bounding A_0 is equal to $\{H_{\alpha,0} \mid \alpha \in \Delta\} \cup \{H_{\tilde{\alpha},1}\}$: these are called the walls of A_0. A face of A_0 (that is a codimension 1 facet) is said to be of type s_{α_i} if it is contained in the hyperplane $H_{\alpha_i,0}$ and of type s_{α_0} if it is contained in the hyperplane $H_{\tilde{\alpha},1}$.

The group W_a acts simply transitively on the set of alcoves and we can extend the definition of walls and faces to all alcoves. Two alcoves A and A' are then said to be s-adjacent where $s \in S$ if they share a face f of type s and we write $A \sim_s A'$. In other words, A and A' are s-adjacent if there exists $w \in W_a$ such that wf is the face of type s of A_0. We will denote by A_y the alcove yA_0.

Remark 2.1. It is important to notice that the alcoves A_w and A_{ws} are s-adjacent for all $w \in W_a$ and all $s \in S$. Therefore, any expression $\vec{w} = s_1 \dots s_n$ ($s_i \in S$) of $w \in W_a$ defines a sequence of adjacent alcoves starting at the fundamental alcove A_0 and finishing at the alcove A_w:

$$e \sim_{s_1} A_{s_1} \sim_{s_2} A_{s_1 s_2} \sim_{s_3} \dots \sim_{s_n} A_{s_1 \dots s_n} = A_w.$$

Further, there exists a unique pair $(\alpha, k) \in \Phi^+ \times \mathbf{Z}$ such that the hyperplane $H_{\alpha,k}$ separates the alcoves A_w and A_{ws} and we have $s_{\alpha,k} A_w = A_{ws}$.

Any hyperplane $H_{\alpha,n}$ where $\alpha \in \Phi^+$ divides the space V into two half spaces

$$H^+_{\alpha,n} = \{\lambda \in V \mid \langle \lambda, \alpha^\vee\rangle > n\} \quad \text{and} \quad H^-_{\alpha,n} = \{\lambda \in V \mid \langle \lambda, \alpha^\vee\rangle < n\}.$$

We say that an hyperplane $H \in \mathcal{F}$ separates the alcoves A and B if and only if $A \in H^\epsilon$ and $B \in H^{-\epsilon}$ where $\epsilon = \pm$. Given two alcoves $A, B \in \mathrm{Alc}(\mathcal{F})$, we set

$$H(A,B) = \{H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B\}.$$

For all $A \in \mathrm{Alc}(\mathcal{F})$, $\alpha \in \Phi$ and $n \in \mathbf{Z}$, we write $n < A[\alpha]$ (respectively $n > A[\alpha]$) if and only if for all $\lambda \in A$ we have $n < \langle \lambda, \alpha^\vee\rangle$ (respectively $n > \langle \lambda, \alpha^\vee\rangle$). For two alcoves A and A', we write $A[\alpha] < A'[\alpha]$ if and only if $\langle \lambda, \alpha^\vee \rangle < \langle \mu, \alpha^\vee \rangle$ for all $(\lambda, \mu) \in A \times A'$. For all alcoves A and all roots $\alpha \in \Phi$, there exists a unique $n \in \mathbf{Z}$ such that $n < A[\alpha] < n+1$.

The following proposition gathers some well known results about the length function and the action of W_a on the set of alcoves. A standard reference for these results is [6].

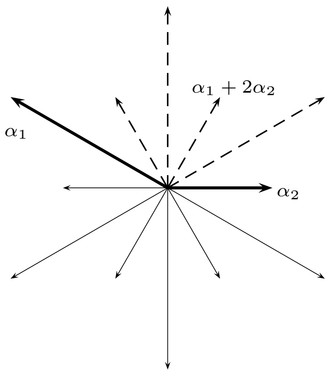
Proposition 2.2. *Let $w \in W_a$. We have*

1. $\ell(w) = |H(A_0, A_w)|$
2. *Let $s \in S$ and let $H_{\alpha,n}$ where $\alpha \in \Phi^+$ be the unique hyperplane separating w and ws. We have $ws < w$ if and only one of the following statement holds:*
   (a) $A_w \in H^+_{\alpha,n}$, $A_{ws} \in H^-_{\alpha,n}$ *and* $n > 0$,
   (b) $A_w \in H^-_{\alpha,n}$, $A_{ws} \in H^+_{\alpha,n}$ *and* $n \leq 0$.
3. *We have* $\overline{H(A_0, A_v)} = \{\alpha \in \Phi^+ \mid v^{-1}\alpha \in \Phi^-\}$ *for all* $v \in W_0$.

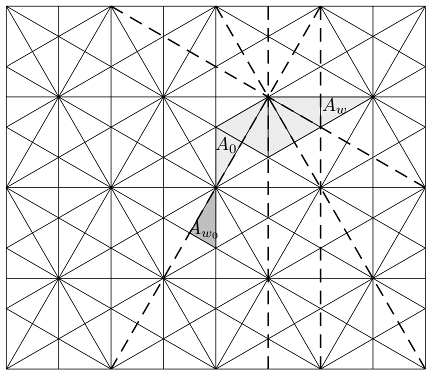
Example 2.3. Let Φ be a root system of type G_2 and let $\Delta = \{\alpha_1, \alpha_2\}$ be a simple system in Φ such that α_2 is the short root so that

$$\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.$$

Then $\Phi^\vee$ is also of type G_2 and the coroot of $\alpha_1 + 2\alpha_2$ is $2\alpha_1^\vee + 3\alpha_2^\vee$ which is the highest root of $\Phi^\vee$. In Figure 1a, we represent the root system Φ. In Figure 1b, we represent the alcoves A_0, A_{w_0} and the alcove A_w where $w = s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_0}$.



(a) Root system Φ



(b) Action on A_0

Fig. 1: Roots, hyperplanes and alcoves in type G_2

The set of hyperplanes that separates A_0 and A_w is

$$H(A_0, A_w) = \{H_{\alpha_2,2}, H_{\alpha_1+2\alpha_2,2}, H_{\alpha_1+3\alpha_2,1}, H_{\alpha_2,1}, H_{\alpha_1,0}, H_{\alpha_0,1}\}$$

and these are represented with dashed lines. Note that the cardinal of $H(A_0, wA_0)$ is indeed the length of w. The alcoves of the sequence

$$(A_0, A_{s_0}, A_{s_0s_1}, A_{s_0s_1s_2}, A_{s_0s_1s_2s_1}, A_{s_0s_1s_2s_1s_2}, A_w)$$

are colored in light gray. Finally we have

$$s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_0}A_0 = s_{\alpha_2,2}s_{\alpha_1+2\alpha_2,2}s_{\alpha_1+3\alpha_2,1}s_{\alpha_2,1}s_{\alpha_1,0}s_{\alpha_0,1}A_0.$$

2.2 Weight functions and special points

Let L be a positive weight function on W_a, that is a function $L : W_a \longrightarrow \mathbf{N}$ such that $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$. To determine a weight function, it is enough to give its values on the conjugacy classes of generators of S. From now on, we fix such a positive weight function L on W_a.

Let H be an hyperplane in $\mathcal{F}$. We say that H is of weight $L(s)$ if it contains a face of type $s \in S$. This is well-defined since if H contains a face of type s and s' then s and s' are conjugate and $L(s) = L(s')$ see [2, Lemma 2.1]. We denote the weight of an hyperplane H by $L(H)$. If $\alpha \in \Phi$, we set $L(\alpha) = \max_{\overline{H}=\alpha} L(H)$. For any $\lambda \in V$ we set

$$L(\lambda) = \sum_{H, \lambda \in H} L(H).$$

Let $\nu = \max_{\lambda \in V} L(\lambda)$. We call λ an L-weight if $L(\lambda) = \nu$ and we denote by P the set of L-weights. Further we denote by P^+ the set of dominant L-weights that is $P^+ = \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$ and by P^- the set of anti-dominant weights, that is $P^- = -P^+$. Without loss of generality, we will always assume that $0 \in V$ is an L-weight. The action of the longest element $w_0 \in W_0$ on the set of weights is an involution that sends P^+ onto P^- and we set $\lambda^* = w_0\lambda$ for all $\lambda \in P$.

2.3 Extended affine Weyl groups

The extended affine Weyl group is defined by $W_e = W_0 \ltimes P$; it acts naturally on the set of alcoves $\mathrm{Alc}(\mathcal{F})$ but the action is no longer faithful. If we denote by Π the stabiliser of A_0 in W_e then we have $W_e = \Pi \ltimes W_a$. Note that Π permutes the weight that belong to the closure of A_0. Further the group Π is isomorphic to P/Q, hence it is abelian and its action on W_a is given by an automorphism of the Dynkin diagram; see Planches I–IX in [1]. We denote t_λ where $\lambda \in P$ the translation by λ in W_e.

An extended alcove is a pair (A, μ) where $A \in \mathrm{Alc}(\mathcal{F})$ and μ is a vertex of A which lies in P. We denote by $\mathrm{Alc}_e(\mathcal{F})$ the set of extended alcoves. The group W_e acts naturally on $\mathrm{Alc}_e(\mathcal{F})$ and the action is faithfull and transitive. Indeed, if $(A, \mu) \in \mathrm{Alc}_e(\mathcal{F})$, there exists $w \in W_a$ such that $wA = A_0$ and $w\mu = \mu'$ where $\mu' \in \overline{A_0}$. If we let $\pi \in \Pi$ be such that $\pi\mu' = 0$ we obtain $\pi w(A, \mu) = (A_0, 0)$ as required. To simplify the notation, we will simply write A_0 for $(A_0, 0)$ and for all $w \in W_e$ we set $A_w = wA_0$.

All the notions and notations for alcoves in $\mathrm{Alc}(\mathcal{F})$ can be extended to $\mathrm{Alc}_e(\mathcal{F})$. We just omit the part with the weight when needed. For instance if $A' = (A, \lambda) \in \mathrm{Alc}_e(\mathcal{F})$, we write $A'[\alpha] < 0$ to mean $A[\alpha] < 0$. The length function, the weight function, the Bruhat order all naturally extend to W_e by setting $\ell(aw) = \ell(w)$, $L(aw) = L(w)$ and $aw < a'w'$ if and only if $a = a'$ and $w < w'$ where $a, a' \in \Pi$ and $w, w' \in W_a$.

2.4 Quarter of vertex λ

The quarters of vertex $\lambda \in V$ are the connected components of

$$V \backslash \bigcup_{H \in \mathcal{F}, \lambda \in H} H.$$

Given $\lambda \in P$ and $v \in W_0$, we denote by $\mathcal{C}_{\lambda,v}$ the quarter of vertex λ which contains $t_\lambda v$. When we consider a quarter with vertex 0 we will omit the 0 in the notation. The set of Weyl chambers is then $\{\mathcal{C}_w \mid w \in W_0\}$ and the fundamental Weyl chamber is $\mathcal{C}_{\mathsf{id}}$. We have

$$\begin{aligned} \mathcal{C}_{\mathsf{id}} &:= \{x \in V \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi^+\} \quad \text{and} \\ \mathcal{C}_{w_0} &:= \{x \in V \mid \langle x, \alpha^\vee \rangle < 0 \text{ for all } \alpha \in \Phi^+\}. \end{aligned}$$

Let X_0 be the set of right coset representatives of minimal length of W_0 in W_e. Then X_0 is the set of $x \in W_e$ that satisfies $\ell(w_0 x) = \ell(w_0) + \ell(x)$ and

$$x \in X_0 \Longleftrightarrow A_x \in \mathcal{C}_{\mathsf{id}} \Longleftrightarrow A_x[\alpha] > 0 \text{ for all } \alpha \in \Delta.$$

Any element w of W_e can be uniquely written under the form $w = vx$ where $v \in W_0$ and $x \in X_0$. For all $x \in X_0$, $\lambda \in P$ and $v \in W_0$, the alcove $t_\lambda vx$ lies in $\mathcal{C}_{\lambda,v}$.

For $\lambda \in V$ and $\alpha \in \Phi^+$ we set $\lambda_\alpha = \langle \lambda, \alpha^\vee \rangle$. Let $\mathcal{C}$ be a quarter of vertex $\lambda \in P$ and fix $\alpha \in \Phi^+$. We have either

$$\{\langle x, \alpha^\vee \rangle \mid x \in \mathcal{C}\} =]\lambda_\alpha, +\infty[\quad \text{or} \quad \{\langle x, \alpha^\vee \rangle \mid x \in \mathcal{C}\} =]-\infty, \lambda_\alpha[.$$

In the first case we say that $\mathcal{C}$ is oriented toward $+\infty$ in the direction α and we write $\mathcal{C}[\alpha] = +\infty$. In the second case we say that $\mathcal{C}$ is oriented toward $-\infty$ in the direction α and we write $\mathcal{C}[\alpha] = -\infty$.

Lemma 2.4. *Let $\lambda \in P$ and $v \in W_0$. We have for all $\alpha \in \Phi^+$:*

$$\mathcal{C}_{\lambda,v}[\alpha] = \begin{cases} +\infty & \text{if } v^{-1}\alpha \in \Phi^+, \\ -\infty & \text{if } v^{-1}\alpha \in \Phi^-. \end{cases}$$

Proof. Let $x \in \mathcal{C}_{\lambda,v}$. There exist $x_0 \in \mathcal{C}_v$ and $y_0 \in \mathcal{C}_{\mathsf{id}}$ such that $x = x_0 + \lambda$ and $x_0 = vy_0$. We have

$$\langle x, \alpha^\vee \rangle = \langle x_0 + \lambda, \alpha^\vee \rangle = \langle vy_0, \alpha^\vee \rangle + \lambda_\alpha = \langle y_0, (v^{-1}\alpha)^\vee \rangle + \lambda_\alpha.$$

Since $y_0 \in \mathcal{C}_{\mathsf{id}}$ we have $\langle y_0, (v^{-1}\alpha)^\vee \rangle \geq 0$ if and only if $v^{-1}\alpha \in \Phi^+$ as required. ∎

3 Affine Hecke algebra with unequal parameters

3.1 Affine Hecke algebra

Let $\mathcal{A} = \mathbf{C}[\mathsf{q}, \mathsf{q}^{-1}]$ where q is an indeterminate. The Iwahori-Hecke algebra $\mathcal{H}$ associated to W_e is the free $\mathcal{A}$-module with basis $(T_w)_{w\in W_e}$ and relations given by

$$T_uT_v = T_{uv} \text{ whenever } \ell(uv) = \ell(u) + \ell(v)$$

and

$$(T_s - \mathsf{q}^{L(s)})(T_s + \mathsf{q}^{-L(s)}) = 0 \text{ if } s \in S.$$

From this relation, we easily find that for all $s \in S$ and all $w \in W$, we have

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + \xi_s T_w & \text{if } \ell(sw) < \ell(w) \end{cases} \quad \text{where } \xi_s = \mathsf{q}^{L(s)} - \mathsf{q}^{-L(s)}.$$

The basis $(T_w)_{w\in W_e}$ is called the standard basis. We write $f_{x,y,z}$ for the structure constants with respect to this basis:

$$T_xT_y = \sum_{z\in W_e} f_{x,y,z}T_z.$$

The elements $f_{x,y,z}$ are polynomials in $\{\xi_s \mid s \in S\}$ with positive coefficients. The degree of $f_{x,y,z}$ will be denoted $\deg(f_{x,y,z})$ and is the highest power of q that appears in $f_{x,y,z}$.

3.2 Multiplication of the standard basis

In this section, we present a result of [4] on a bound on the degree of the polynomials $f_{x,y,z}$. Recall that for two alcoves $A, B \in \mathrm{Alc}_e(\mathcal{F})$, we have set $H(A, B) = \{H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B\}$. Then for $x, y \in W_e$ we set

$$\mathsf{H}_{x,y} = H(A_0, A_x) \cap H(A_x, A_{xy}) \quad \text{and} \quad c_{x,y}(\alpha) := \max_{\substack{H\in \mathsf{H}_{x,y} \\ \overline{H}=\alpha}} L_H.$$

Then according to [4, Theorem 2.4] we have:

Theorem 3.1. *The degrees of the polynomials $f_{x,y,z}$ are bounded by $\sum_{\alpha\in\overline{\mathsf{H}_{x,y}}} c_{x,y}(\alpha)$.*

We obtain the following corollary [5, Proposition 5.3] which will be crucial in the following section.

Corollary 3.2. *Let $x \in X_0^{-1}$, $v \in W_0$ and $y \in X_0$. We have*

$$\deg(f_{xv,y,z}) \leq L(w_0) - L(v).$$

3.3 Kazhdan-Lusztig basis

Let $^-$ be the ring involution of $\mathcal{A}$ which takes q to q^{-1}. This involution can be extended to a ring involution of $\mathcal{H}$ via the formula

$$\overline{\sum_{w\in W_e} a_wT_w} = \sum_{w\in W_e} \overline{a}_w T^{-1}_{w^{-1}} \quad (a_w \in \mathcal{A}).$$

We set

$$\begin{array}{rlcl} \mathcal{A}_{<0} = & \mathsf{q}^{-1}\mathbf{Z}[\mathsf{q}^{-1}] & , & \mathcal{H}_{<0} = \bigoplus_{w\in W_e} \mathcal{A}_{<0}T_w \\ \mathcal{A}_{\leq 0} = & \mathbf{Z}[\mathsf{q}^{-1}] & \text{and} & \mathcal{H}_{\leq 0} = \bigoplus_{w\in W_e} \mathcal{A}_{\leq 0}T_w. \end{array}$$

For each $w \in W_e$ there exists a unique element $C_w \in \mathcal{H}$ (see [12, Theorem 5.2]) such that (1) $\overline{C}_w = C_w$ and (2) $C_w \equiv T_w \mod \mathcal{H}_{<0}$. For any $w \in W_e$ we set

$$C_w = \sum_{y\in W_e} P_{y,w}T_y \quad \text{where } P_{y,w} \in \mathcal{A}_{<0}.$$

The coefficients $P_{y,w}$ are called the Kazhdan-Lusztig polynomials. It is well known ([12, §5.3]) that $P_{y,w} = 0$ whenever $y \not\leq w$ and that $P_{w,w} = 1$. It follows that $(C_w)_{w\in W_e}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$ known as the Kazhdan-Lusztig basis. According to [12, Theorem 6.6] we have

$$\forall w \in W_e, \forall s \in S, P_{x,y} = \mathsf{q}^{-L(s)}P_{xs,y} \text{ whenever } x < xs \text{ and } ys < y.$$

Remark 3.3. Using Corollary 3.2 and the definition of the Kazhdan-Lusztig basis, one can show that the element $C_{w_0t_\lambda}T_y$ lies in $\mathcal{H}_{\leq 0}$ for all $\lambda \in P^+$ and $y \in X_0$; see Proposition 5.3 in [5]. Indeed we have

$$\begin{aligned}
C_{w_0t_\lambda}T_y &= \sum_{x \leq w_0t_\lambda} P_{x,w_0t_\lambda}T_xT_y \\
&= \sum_{x_0\in X_0, v\in W_0} P_{x_0^{-1}v,t_{\lambda^*}w_0}T_{x_0^{-1}v}T_y \\
&= \sum_{x_0^{-1}v\leq w_0t_\lambda} \mathsf{q}^{L(v)-L(w_0)}P_{x_0^{-1}w_0,t_{\lambda^*}w_0}T_{x_0^{-1}v}T_y
\end{aligned}$$

The result follows since $\deg(f_{x_0^{-1}v,y,z}) \leq L(w_0) - L(v)$ for all $z \in W_e$.

3.4 The center of the affine Hecke algebra

In this section we follow the presentation of Nelsen and Ram [13] and we refer to it and the references therein for details and proofs. We start by introducing another presentation of the affine Hecke algebra which is more convenient to describe its center $Z(\mathcal{H})$. For each $\lambda \in P$, we set $e^\lambda := T_{t_\mu}T_{t_\nu}^{-1}$ where $\mu, \nu \in P^+$ are such that $\mu - \nu = \lambda$. This can be shown to be independent of the choice of μ and ν. Then $\mathcal{H}$ is generated by the sets $\{T_s \mid s \in S_0\}$ and $\{e^\lambda \mid \lambda \in P\}$ and we have the relations

$$e^\lambda e^\mu = e^{\lambda+\mu} = e^\mu e^\lambda \quad \text{and} \quad e^\lambda T_{s_i} = T_s e^{s_i\lambda} + \xi_s\frac{e^\lambda - e^{s_i\lambda}}{1-e^{-\alpha_i}}.$$

Let $\mathcal{A}[P]$ be the subalgebra of $\mathcal{H}$ generated by $\{e^\lambda \mid \lambda \in P\}$. Then W_0 acts naturally on $\mathcal{A}[P]$ via $w \cdot e^\lambda = e^{w\lambda}$.

Theorem 3.4. *The sets $(e^\lambda T_w)_{\lambda\in P, w\in W_0}$ and $(T_we^\lambda)_{w\in W_0,\lambda\in P}$ are $\mathcal{A}$-basis of $\mathcal{H}$ and the center of $\mathcal{H}$ is*

$$\mathcal{Z}(\mathcal{H}) = \{f \in \mathcal{A}[P] \mid w\cdot f = f \text{ for all } w \in W_0\}.$$

We define the Weyl characters s_λ as follows

$$a_\lambda := \sum_{w\in W_0}(-1)^{\ell(w)}e^{w\lambda} \quad \text{and} \quad \mathsf{s}_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{where } \rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

Then $(\mathsf{s}_\lambda)_{\lambda\in P^+}$ form a basis of $\mathcal{Z}(\mathcal{H})$. According to [13, Theorem 2.9], the element s_λ acts as translation on the Kazhdan-Lusztig element C_{w_0}.

Theorem 3.5. *We have $\mathsf{s}_\lambda C_{w_0} = C_{w_0}\mathsf{s}_\lambda = C_{w_0t_\lambda}$ for all $\lambda \in P^+$ and*

$$C_{w_0t_\lambda}\mathsf{s}_\tau = C_{w_0}\mathsf{s}_\lambda\mathsf{s}_\tau = \sum_\mu m^\mu_{\lambda,\tau}C_{w_0t_\mu}.$$

4 Decomposition into the Kazhdan-Lusztig basis

The aim of this section is to show that in order to determine the decomposition of $C_{w_0t_\lambda}\mathsf{s}_\tau$ in the Kazhdan-Lusztig basis, it is enough to determine the terms of maximal degree in the products $T_{w_0t_\lambda v}T_{t_\tau}$ where $v \in W_0$. This will be done in 3 steps:

1. we construct some special elements h_τ such that $C_{w_0t_\lambda}\mathsf{s}_\tau = C_{w_0t_\lambda}\mathsf{h}_\tau$ and $\mathsf{h}_\tau \in T_{t_\tau} + \bigoplus_{y<t_\tau, y\in X_0}\mathcal{A}_{<0}T_y$,
2. we show that $C_{w_0t_\lambda}\mathsf{h}_\tau \equiv \sum a_xT_x \mod \mathcal{H}_{<0}$ where $a_x \in \mathbf{Z}$,
3. we show that $\overline{C_{w_0t_\lambda}\mathsf{h}_\tau} = C_{w_0t_\lambda}\mathsf{h}_\tau$ and therefore $C_{w_0t_\lambda}\mathsf{h}_\tau = \sum a_xC_x$.

Let $x \in X_0$ and set $\mathsf{h}_x := \sum_{x'\in X_0}P_{w_0x',w_0x}T_{x'}$. Following [18, Lemma 2.5] we get

$$\begin{aligned}
C_{w_0x} &= \sum_{y\in W_e}P_{y,w_0x}T_y \\
&= \sum_{x'\in X_0,u\in W_0}P_{ux',w_0x}T_{ux'} \\
&= \sum_{x'\in X_0,u\in W_0}\mathsf{q}^{L(u)-L(w_0)}P_{w_0x',w_0x}T_uT_{x'} \\
&= \left(\sum_{u\in W_0}\mathsf{q}^{L(u)-L(w_0)}T_u\right)\sum_{x'\in X_0}P_{w_0x',w_0x}T_{x'} \\
&= C_{w_0}\mathsf{h}_x.
\end{aligned}$$

When $\tau \in P^+$, we know that $t_\tau \in X_0$ and we will write h_τ instead of h_{t_τ}. For all $\tau, \lambda \in P^+$ we have

$$C_{w_0}\mathsf{s}_\tau = C_{w_0}\mathsf{h}_\tau \quad \text{and} \quad C_{w_0 t_\lambda}\mathsf{s}_\tau = \mathsf{s}_\lambda C_{w_0}\mathsf{s}_\tau = \mathsf{s}_\lambda C_{w_0}\mathsf{h}_\tau = C_{w_0 t_\lambda}\mathsf{h}_\tau.$$

This concludes part (1) since the elements h_τ have the required form. Similarly, we can construct elements g_x for all $x \in X_0^{-1}$ such that $\mathsf{g}_x C_{w_0} = C_{xw_0}$. Setting $\mathsf{g}_{\mathsf{t}_\tau} = \mathsf{g}_\tau$ $(\tau \in P^-)$ we have $C_{w_0}\mathsf{h}_\tau = C_{w_0 t_\tau} = C_{t_{\tau^*} w_0} = \mathsf{g}_{\tau^*} C_{w_0}$.

Let $\tau, \lambda \in P^+$. Using Remark 3.3 and the fact that $P_{w_0 x, w_0 t_\tau} \in \mathcal{A}_{<0}$ whenever $x < t_\tau$, we get

$$C_{w_0 t_\lambda}\mathsf{h}_\tau = C_{w_0 t_\lambda} T_{t_\tau} + \sum_{x < t_\tau, x \in X_0} \underbrace{P_{w_0 x, w_0 t_\tau} C_{w_0 t_\lambda} T_x}_{\in \mathcal{H}_{<0}} \equiv C_{w_0 t_\lambda} T_{t_\tau} \mod \mathcal{H}_{<0}$$

and

$$C_{w_0 t_\lambda} T_{t_\tau} = T_{w_0 t_\lambda} T_{t_\tau} + \sum_{y < w_0 t_\lambda} P_{y, w_0 t_\lambda} T_y T_{t_\tau}.$$

Let $y \in W_e$ and $(y_r, y_0) \in X_0^{-1} \times W_0$ be such $y = y_r y_0$. On the one hand $P_{y, w_0 t_\lambda} = \mathsf{q}^{L(y_0) - L(w_0)} P_{y_r w_0, t_{\lambda^*} w_0}$ and on the other hand, by Corollary 3.2, the maximal degree that can appear in $T_{y_r y_0} T_{t_\tau}$ is $L(w_0) - L(y_0)$. Therefore if $y_r w_0 < t_{\lambda^*} w_0$ we get that $P_{y, w_0 t_\lambda} T_y T_{t_\tau} \in \mathcal{H}_{<0}$ and

$$C_{w_0 t_\lambda} T_{t_\tau} \equiv \sum_{y_0 \in W_0} \mathsf{q}^{L(y_0) - L(w_0)} T_{t^*_\lambda y_0} T_{t_\tau} \equiv \sum_{v \in W_0} \mathsf{q}^{-L(v)} T_{t^*_\lambda w_0 v} T_{t_\tau} \mod \mathcal{H}_{<0}.$$

Finally

$$C_{w_0 t_\lambda}\mathsf{h}_\tau \equiv \sum_{v \in W_0} \underbrace{\mathsf{q}^{-L(v)} T_{w_0 t_\lambda v} T_{t_\tau}}_{\in \mathcal{H}_{\leq 0}} \mod \mathcal{H}_{<0} \tag{1}$$

as claimed in statement (2).

We now prove Statement (3).

Lemma 4.1. *For all $\lambda, \tau \in P^+$, the elements $C_{w_0 t_\lambda}\mathsf{h}_\tau$ are stable under the $^-$-involution.*

Proof. First we have $C_{w_0 t_\lambda}\mathsf{h}_\tau = C_{w_0}\mathsf{h}_\lambda \mathsf{h}_\tau = \mathsf{g}_{\lambda^*}\mathsf{g}_{\tau^*} C_{w_0}$ so that $C_{w_0 t_\lambda}\mathsf{h}_\tau$ lies in the intersection of $\mathcal{H}C_{w_0}$ and $C_{w_0}\mathcal{H}$. Next

$$\mathcal{H}C_{w_0} = \langle C_{xw_0} \mid x \in X_0^{-1} \rangle_{\mathcal{A}} \quad \text{and} \quad \mathcal{H}C_{w_0} = \langle C_{w_0 x} \mid x \in X_0 \rangle_{\mathcal{A}}$$

from where we get

$$\mathcal{H}C_{w_0} \cap C_{w_0}\mathcal{H} = \langle C_{w_0 t_\nu} \mid \nu \in P^+ \rangle_{\mathcal{A}}$$

since $X_0^{-1} w_0 \cap w_0 X_0 = \{w_0 t_\nu \mid \nu \in P^+\}$. Therefore there exist $b_\nu \in \mathcal{A}$ such that $C_{w_0 t_\lambda}\mathsf{h}_\tau = \sum_{\nu \in P^+} b_\nu C_{w_0 t_\nu}$. At this stage, in order to show that $\overline{C_{w_0 t_\lambda}\mathsf{h}_\tau} = C_{w_0 t_\lambda}\mathsf{h}_\tau$ it is now enough to show that $b_\nu \in \mathbf{Z}$ since $\overline{C_{w_0 t_\nu}} = C_{w_0 t_\nu}$. The following argument is inspired by [17, Proof of Theorem 6.2]. We have

$$\begin{aligned} h_{w_0, w_0, w_0} C_{w_0 t_\lambda}\mathsf{h}_\tau &= C_{t_{\lambda^*} w_0} C_{w_0}\mathsf{h}_\tau \\ &= C_{t_{\lambda^*} w_0} C_{w_0 t_\tau} \\ &= \sum_{z \in W_e} h_{t_{\lambda^*} w_0, w_0 t_\tau, z} C_z \\ &= \sum_{\nu \in P^+} b_\nu h_{w_0, w_0, w_0} C_{w_0 t_\nu} \end{aligned}$$

where $h_{x,y,z} \in \mathcal{A}$ denote the structure constants with respect to the Kazhdan-Lusztig basis. According to [12, §13.4], the degree of $h_{x,y,z}$ is bounded by $L(w_0)$ for all $x, y, z \in W_e$. Therefore $\deg(h_{\tau^{-1} w_0, w_0 t_\tau, z}) \leq L(w_0)$ and since $\deg(h_{w_0, w_0, w_0}) = L(w_0)$, this forces $b_\nu \in \mathbf{Z}$ as required. ☐

Statement (3) now follows using the following lemma.

Lemma 4.2. *Let $h \in \mathcal{H}$ be such that $\bar{h} = h$ and $h \equiv \sum a_x T_x \mod \mathcal{H}_{<0}$ where $a_x \in \mathbf{Z}$. Then $h = \sum a_x C_x$.*

Proof. We know [12, §5.2.(e)] that if $h' \in \mathcal{H}_{<0}$ satisfies $\bar{h}' = h$ then $h' = 0$. The lemma is an easy consequence of this result setting $h' = h - \sum a_x C_x$. ☐

As a consequence, in order to determine the decomposition of $C_{w_0t_\lambda}\mathsf{h}_\tau$ in the Kazhdan-Lusztig basis, we need to determine which product $\mathsf{q}^{-L(v)}T_{w_0t_\lambda v}T_{t_\tau}$ can actually give rise to a non zero term modulo $\mathcal{H}_{<0}$. In other words, we need to determine which terms in the decomposition of $T_{w_0t_\lambda v}T_{t_\tau}$ in the standard basis has a coefficient of (maximal) degree $L(v)$.

The remainder of this section is devoted to set up the notation in order to study the product $T_{w_0t_\lambda v}T_{t_\tau}$. Let $x, y \in W_e$ and let $\vec{y} = s_1 \dots s_n a$ be a reduced expression of y where $a \in \Pi$ and $s_i \in S$ for all i. Let $J = \{i_1, \dots, i_p\}$ be a subset of $\{1, \dots, n\}$. For all $1 \leq \ell, k < n$, we set

$$\vec{y}^J = \left(\prod_{r=1, r\notin J}^{n} s_r\right) a, \quad \vec{y}^J[\ell,k] = \prod_{r=\ell, r\notin J}^{k} s_r \quad \text{and} \quad \vec{y}^J[\ell,n] = \left(\prod_{r=\ell, r\notin J}^{n} s_r\right) a.$$

When J is empty we will simply write $\vec{y}[\ell,k]$ instead of $\vec{y}^{\varnothing}[\ell,k]$ for the product $s_\ell \dots s_k$ and $\vec{y}[\ell,n]$ instead of $\vec{y}^{\varnothing}[\ell,n]$. Finally we denote by $p_J(x;\vec{y})$ the sequence of alcoves $(A_{x\vec{y}^J[1,k]})_{1\leq k\leq n}$. We see that

- there can be repetitions in this sequence (see below);
- any two consecutive alcoves are either equal or adjacent.

We can therefore represent $p_J(x;\vec{y})$ by a path going through the sequence of alcoves $(A_{x\vec{y}^J[1,k]})_{1\leq k\leq n}$ and which folds on the s_i-face of $A_{x\vec{y}^J[1,i-1]}$ for all $i \in J$ (and hence goes twice through the alcove $A_{x\vec{y}^J[1,i-1]} = A_{x\vec{y}^J[1,i]}$). We will say that the path $p_J(x;\vec{y})$ is included in a certain subset of V if all the alcoves that appear in $p_J(x;\vec{y})$ lie in this subset.

Let $\mathfrak{I}_{x,\vec{y}}$ the set of all subsets $\{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$ such that $1 \leq i_1 < \dots < i_p \leq n$ and

$$x\vec{y}^J[1, i_\ell - 1]s_{i_\ell} < x\vec{y}^J[1, i_\ell - 1] \text{ for all } \ell \in \{1, \dots, p\}.$$

For $J = \{i_1, \dots, i_p\}$ in $\mathfrak{I}_{x,\vec{y}}$, we set $\xi_J = \prod_{k=1}^p \xi_{s_{i_k}}$ so that we have [2, Proof of Proposition 5.1]

$$T_xT_y = \sum_{J\in\mathfrak{I}_{x,\vec{y}}} \xi_J T_{x\vec{y}^J}.$$

Let $H_{\alpha,n}$ ($\alpha \in \Phi^+$) be the hyperplane that separates $A_{x\vec{y}^J[1,i_\ell-1]}$ and $A_{x\vec{y}^J[1,i_\ell-1]s_{i_{i_\ell}}}$. By definition of $\mathfrak{I}_{x,\vec{y}}$ we have $x\vec{y}^J[1,i_\ell-1]s_{i_\ell} < x\vec{y}^J[1,i_\ell-1]$ and therefore

- $A_{x\vec{y}^J[1,i_\ell-1]} \in H^-_{\alpha,n}$ and $A_{x\vec{y}^J[1,i_\ell-1]s_{i_\ell}} \in H^+_{\alpha,n}$ if $n \leq 0$;
- $A_{x\vec{y}^J[1,i_\ell-1]} \in H^+_{\alpha,n}$ and $A_{x\vec{y}^J[1,i_\ell-1]s_{i_\ell}} \in H^-_{\alpha,n}$ if $n > 0$.

We fix a reduced expression $\vec{t_\tau}$ of t_τ and we set

$$\mathfrak{I}_{\lambda,v,\tau} = \mathfrak{I}_{t^*_\lambda w_0 v, \vec{t_\tau}} \quad \text{and} \quad \mathfrak{I}^{\max}_{\lambda,v,\tau} = \{J \in \mathfrak{I}_{\lambda,v,\tau} \mid \deg(\xi_J) = L(v)\}$$

so that according to (1) and the fact that the leading term of ξ_J is $\mathsf{q}^{L(v)}$ we have

$$C_{w_0t_\lambda}\mathsf{h}_\tau \equiv T_{w_0t_{\lambda+\tau}} + \sum_{v\in W_0\setminus\{\mathsf{id}\}} \mathsf{q}^{-L(v)}T_{w_0t_\lambda v}T_{t_\tau} \equiv \sum_{v\in W_0}\sum_{J\in\mathfrak{I}^{\max}_{\lambda,v,\tau}} T_{w_0t_\lambda v t^J_\tau} \mod \mathcal{H}_{<0}.$$

and

$$C_{w_0t_\lambda}\mathsf{h}_\tau = \sum_{v\in W_0}\sum_{J\in\mathfrak{I}^{\max}_{\lambda,v,\tau}} C_{w_0t_\lambda v t^J_\tau}.$$

5 Description of $\mathfrak{I}^{\max}_{\lambda,v,\tau}$ in terms of admissible subsets

Once and for all in this section, we fix $\tau \in P^+$ and a reduced expression $\vec{t_\tau} = s_1 \dots s_n a$ where $a \in \Pi$ and $s_i \in S$. Let $(\beta_1, \dots, \beta_n) \in (\Phi^+)^n$ and $(N_1, \dots, N_k) \in \mathbf{N}^n$ be such that the unique hyperplane separating $A_{s_1\dots s_{k-1}}$ and $A_{s_1\dots s_k}$ is H_{β_k,N_k}. Following [8], we now introduce the concept of admissible subsets.

Definition 5.1. A subset $J = \{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$ will be called an admissible subset if

$$\mathsf{id} < s_{\beta_{i_p}} < s_{\beta_{i_p}}s_{\beta_{i_{p-1}}} < \dots < s_{\beta_{i_p}}s_{\beta_{i_{p-1}}}\dots s_{\beta_{i_1}}$$

is a saturated chain in the Bruhat order on W_0. We set $v_J := s_{\beta_{i_p}}s_{\beta_{i_{p-1}}}\dots s_{\beta_{i_1}}$.

Saying that the chain is saturated in the Bruhat order on W_0 is equivalent to say that $\ell(s_{\beta_{i_p}} \dots s_{\beta_{i_k}}) = p-k+1$ for all $1 \leq k \leq p$. We note that if $\{i_1, \dots, i_p\}$ is an admissible subset then so is $\{i_\ell, \dots, i_p\}$ for all $\ell \leq p$ and we denote this subset by $J_{\ell-1}$ so that $J_0 = J$ and $J_p = \varnothing$. Then we have $v_{J_{\ell-1}} = s_{\beta_{i_p}} \dots s_{\beta_{i_\ell}}$.

Example 5.2. Let W be of type $\tilde{G}_2$ as in Example 2.3 and let $\tau = 2\alpha_1 + 3\alpha_2 \in P^+$. We fix the following reduced expression

$$\vec{t_\tau} = s_{\alpha_0} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_0} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}.$$

The sequence of roots $(\beta_1, \dots, \beta_{10})$ associated to $\vec{t_\tau}$ is

$$(\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_1).$$

Following [8, Example 10.12], we know that there are 14 admissible subsets and we describe these sets in the table below. In the column saturated chains, we only put the extremal element and one can recover the full chain by adding to the chain all the elements above in the same column: for instance, the saturated chain associated to the admissible subset $\{3, 9, 10\}$ is $\mathsf{id} < s_{\alpha_1} < s_{\alpha_1} s_{\alpha_1+\alpha_2} < s_{\alpha_1} s_{\alpha_1+\alpha_2} s_{2\alpha_1+3\alpha_2}$.

Saturated chains	Reduced expression	admissible subset
1	1	$\emptyset$
s_{α_1}	s_{α_1}	$\{10\}$
$s_{\alpha_1} s_{\alpha_1+\alpha_2}$	$s_{\alpha_2} s_{\alpha_1}$	$\{9, 10\}, \{5, 10\}, \{2, 10\}$
$s_{\alpha_1} s_{\alpha_1+\alpha_2} s_{2\alpha_1+3\alpha_2}$	$s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$	$\{8, 9, 10\}, \{3, 9, 10\}, \{3, 5, 10\}$
$s_{\alpha_1} s_{\alpha_1+\alpha_2} s_{2\alpha_1+3\alpha_2} s_{\alpha_1+2\alpha_2}$	$s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$	$\{7, 8, 9, 10\}, \{4, 8, 9, 10\}, \{1, 8, 9, 10\}, \{1, 3, 9, 10\}, \{1, 3, 5, 10\}$
$s_{\alpha_1} s_{\alpha_1+\alpha_2} s_{2\alpha_1+3\alpha_2} s_{\alpha_1+2\alpha_2} s_{\alpha_1+3\alpha_2}$	$s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$	$\{6, 7, 8, 9, 10\}$

Remark 5.3. Our definition of admissible subset is slightly different than the one in [8] where they work with reduced expressions of $t_{-\tau}$. The connection between those two definitions will be made clear in the next section.

Recall the definition of $L(\beta)$ $(\beta \in \Phi^+)$ and $\mathcal{C}_v$ $(v \in W_0)$ in Section 2.2 and 2.4 respectively.

Definition 5.4. Let $J = \{i_1, \dots, i_p\}$ be an admissible subset. We say that J is

1. λ-dominant for $\lambda \in P^+$ if the $p_J(t_\lambda v; \vec{t_\tau}) \subset \mathcal{C}_{\mathsf{id}}$,
2. maximal if $L(s_{i_\ell}) = L(\beta_{i_\ell})$ for all $1 \leq \ell \leq p$.

Recall that we always assume that 0 is an L-weight so that $L(v_J) = \sum_{k=1}^{p} L(\beta_{i_k})$ for all J. In particular, if $J \in \mathfrak{I}^{\max}_{\lambda,v,\tau}$ then J must be maximal in order to satisfy $\deg(\xi_J) = L(v_J)$.

We are now ready to state the main result of this paper. Recall the notations introduced at the end of the previous section.

Theorem 5.5. *Let $\lambda \in P^+$ and $v \in W_0$.*

1. *If J is admissible, λ-dominant and maximal then $J \in \mathfrak{I}^{\max}_{\lambda,v_J,\tau}$.*
2. *If $J \in \mathfrak{I}^{\max}_{\lambda,v,\tau}$ then $v = v_J$, J is admissible, λ-dominant and maximal.*

The rest of this section is devoted to the proof of this theorem.

Proposition 5.6. *Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \dots, n\}$.*

1. *The hyperplane separating the alcoves $t_\lambda v t_\tau[1,k-1]A_0$ and $t_\lambda v t_\tau[1,k]A_0$ is $H_k := H_{v\beta_k, N_k + \langle \lambda, v\beta_k{}^\vee \rangle}$.*
2. *We have $t_\lambda v t_\tau[1,k-1]A_0 = t_{\lambda'} v' t_\tau[1,k]A_0$ where $v' = v s_{\beta_k}$ and $\lambda' = s_{H_k}\lambda$ where s_{H_k} denotes the affine reflection with respect to H_k.*

Proof. The hyperplane separating $vt_\tau[1,k-1]A_0$ and $vt_\tau[1,k]A_0$ is $vH_{\beta_k,N_k} = H_{v\beta_k,N_k}$. Hence the hyperplane $H_k = H_{v\beta_k, N_k + \langle \lambda, v\beta_k^\vee \rangle}$ separates the two alcoves $t_\lambda v t_\tau[1,k-1]A_0$ and $t_\lambda v t_\tau[1,k]A_0$. We have

$$\begin{aligned} t_\lambda v t_\tau[1,k-1]A_0 &= t_\lambda v t_\tau[1,k] s_k A_0 \\ &= s_{H_k} t_\lambda v t_\tau[1,k]A_0 \\ &= t_{s_{H_k}\lambda} s_{v\beta_k} v t_\tau[1,k]A_0 \\ &= t_{s_{H_k}\lambda} v s_{\beta_k} t_\tau[1,k]A_0. \end{aligned}$$

∎

Let $(\lambda, v) \in P \times W_0$ and $J = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$. For all $0 \leq \ell \leq p$ we define the elements v_ℓ, λ_ℓ and J_ℓ by

- $v_0 = v$ and $v_\ell = v_{\ell-1} s_{\beta_{i_\ell}}$;
- $\lambda_0 = \lambda$ and $\lambda_\ell = s_{H_\ell} \lambda_{\ell-1}$ where $H_\ell := H_{v_{\ell-1}\beta_{i_\ell}, N_{i_\ell} + \langle \lambda, v_{i_{\ell-1}} \beta_{i_\ell}{}^\vee \rangle}$;
- $J_\ell := \{i_{\ell+1}, \dots, i_p\}$.

Note that the path $p_{J_\ell}(t_{\lambda_\ell} v_\ell; \vec{t_\tau})$ folds exactly $p - \ell$ times and the first fold occurs at position $i_{\ell+1}$ on the hyperplane $H_{\ell+1}$. By a straightforward induction using Proposition 5.6, we see that we have for all $1 \leq \ell \leq p$

$$t_\lambda v t^J_\tau[1, i_\ell - 1] A_0 = t_{\lambda_\ell} v s_{\beta_{i_1}} \dots s_{\beta_{i_\ell}} t_\tau[1, i_\ell] A_0 = t_{\lambda_\ell} v_\ell t_\tau[1, i_\ell] A_0.$$

In the case where J is an admissible subset and $v = v_J$ we have $t_\lambda v t^J_\tau[1, i_\ell - 1] A_0 = t_{\lambda_\ell} v_{J_\ell} t_\tau[1, i_\ell] A_0$. Further since $J_p = \varnothing$, the path $p_{J_p}(t_\lambda v; \vec{t_\tau})$ does not fold.

Example 5.7. Let W be of type $\tilde{G}_2$ as in Example 2.3 and 5.2. Let $\tau = 2\alpha_1 + 3\alpha_2 \in P^+$ and fix the reduced expression $t_\tau = s_{\alpha_0} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_0} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$. Let $J := \{3, 5, 10\}$, $\lambda \in P$ and $v = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} \in W_0$. In Figure 2 we describe the paths $p_{J_\ell}(t_{\lambda_\ell} v_\ell; \vec{t_\tau})$ for $0 \leq \ell \leq 3$ and the sequence $(\lambda_0, \dots, \lambda_3)$ obtained in the procedure above. We write ⓛ for the alcove $A_{t_{\lambda_\ell} v_\ell}$ (so that the path $p_{J_\ell}(t_{\lambda_\ell} v_\ell; \vec{t_\tau})$ starts at the alcove ⓛ) and the light gray alcoves represent $A_{t_{\lambda_\ell}}$.

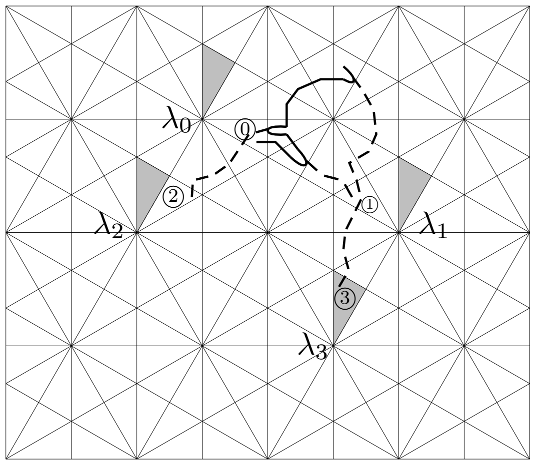


Fig. 2: Sequences associated to the set $J = \{3, 5, 10\}$.

We are now ready to prove the first part of Theorem 5.5.

Proposition 5.8. *Let $\lambda \in P^+$. If J is admissible, λ-dominant and maximal then $J \in \mathfrak{I}^{\max}_{\lambda, v_J, t_\tau}$.*

Proof. In order to prove that $J = \{i_1, \dots, i_p\} \in J \in \mathfrak{I}^{\max}_{\lambda, v_J, t_\tau}$, since J is maximal, we need to show that

$$w_0 t_\lambda v_J t^J_\tau[1, i_\ell - 1] s_{i_\ell} < w_0 t_\lambda v_J t^J_\tau[1, i_\ell - 1] \text{ for all } 1 \leq \ell \leq p.$$

We have

1. $w_0 t_\lambda v_J t^J_\tau[1, i_\ell - 1] = w_0 t_{\lambda_{\ell-1}} v_{J_{\ell-1}} t_\tau[1, i_\ell - 1]$;
2. $v_{J_{\ell-1}} = s_{\beta_{i_p}} \dots s_{\beta_{i_\ell}}$ so that $v_{J_{\ell-1}} \beta_{i_\ell} \in \Phi^-$ and $w_0 v_{J_{\ell-1}} \beta_{i_\ell} \in \Phi^+$;
3. the hyperplane separating

$$w_0 t_{\lambda_{\ell-1}} v_{J_{\ell-1}} t_\tau[1, i_\ell - 1] A_0 \quad \text{and} \quad w_0 t_{\lambda_{\ell-1}} v_{J_{\ell-1}} t_\tau[1, i_\ell - 1] s_{i_\ell} A_0$$

is equal to $H_{w_0 v_{J_{\ell-1}} \beta_{i_\ell}, m}$ where $m < 0$ since J is λ-dominant.

We have $v^{-1}_{J_{\ell-1}} w_0 w_0 v_{J_{\ell-1}} \beta_{i_\ell} = \beta_{i_\ell} \in \Phi^+$ which implies that the quarter $\mathcal{C}_{\lambda^*_{\ell-1}, w_0 v_{J_{\ell-1}}}$ is oriented toward $+\infty$ in the direction $w_0 v_{J_{\ell-1}} \beta_{i_\ell}$. It follows that

$$w_0 t_{\lambda_{\ell-1}} v_{J_{\ell-1}} t_\tau[1, i_\ell - 1] A_0 \in H^-_{w_0 v_{J_{\ell-1}} \beta_{i_\ell}, m} \quad \text{and}$$
$$w_0 t_{\lambda_{\ell-1}} v_{J_{\ell-1}} t_\tau[1, i_\ell - 1] s_{i_\ell} A_0 \in H^+_{w_0 v_{J_{\ell-1}} \beta_{i_\ell}, m}$$

hence the result by Proposition 2.2 since $m < 0$. ∎

We now focus on the second part. We start by proving some technical lemmas. Recall the definitions of $A[\alpha] < n$ and $A[\alpha] < A'[\alpha]$ at the end of Section 2.1 and of $\mathsf{H}_{x,y}$ for $x, y \in W_e$ in Section 3.2.

Lemma 5.9. *Let $\lambda \in P^+$ and $v \in W_0$. We have $\overline{\mathsf{H}_{t_{\lambda^*}w_0v,t_\tau}} \subset \{\delta \in \Phi^+ \mid v^{-1}w_0\delta \in \Phi^+\}$.*

Proof. Let $\delta \in \Phi^+$ be such that

$$H_{\delta,N} \in \mathsf{H}_{t_{\lambda^*}w_0v,t_\tau} = H(A_0, A_{t_{\lambda^*}w_0v}) \cap H(A_{t_{\lambda^*}w_0v}, A_{t_{\lambda^*}w_0vt_\tau}).$$

First, since $\lambda \in P^+$ we have $A_{t_{\lambda^*}w_0v}[\delta] < 1$. Next since $H_{\delta,N} \in H(A_0, A_{t_{\lambda^*}w_0v})$, it follows that $A_{t_{\lambda^*}w_0v}[\delta] < N \leq 0$. If the quarter $\mathcal{C}_{\lambda^*,w_0v}$ is oriented towards $-\infty$ in the direction δ then $N > A_{t_{\lambda^*}w_0v}[\delta] \geq A_{t_{\lambda^*}w_0vt_\tau}[\delta]$ but in this case we cannot have $H_{\delta,N} \in H(A_{t_{\lambda^*}w_0v}, A_{t_{\lambda^*}w_0vt_\tau})$. This shows that the quarter $\mathcal{C}_{\lambda^*,w_0v}$ is oriented towards $+\infty$ in the direction δ and thus $v^{-1}w_0\delta \in \Phi^+$ as required. $\square$

We remark that if $\lambda \in P^+$ and $A_{t_{\lambda^*}w_0v} \notin \mathcal{C}_{w_0}$ then $A_{t_\lambda v} \notin \mathcal{C}_{\mathsf{id}}$ and there exists a simple root $\alpha_i \in \Phi^+$ such that $v^{-1}\alpha_i \in \Phi^-$ and $-1 < A_{t_\lambda v}[\alpha_i] < 0$. Equivalently $-w_0\alpha_i \in \Phi^+$ and $0 < A_{w_0t_\lambda v}[-w_0\alpha_i] < 1$. This implies, considering the positive root $-w_0\alpha_i \in \Phi^+$, that the inclusion

$$\overline{\mathsf{H}_{t_{\lambda^*}w_0v,t_\tau}} \subset \{\delta \in \Phi^+ \mid v^{-1}w_0\delta \in \Phi^+\}$$

is strict and $\mathfrak{I}^{\max}_{\lambda,v,\tau}$ has to be empty by Theorem 3.1.

The next lemma generalises the idea above and gives some restrictions for the set $\mathfrak{I}^{\max}_{\lambda,v,\tau}$ to be non-empty.

Lemma 5.10. *Let $\lambda \in P$ and $v \in W_0$. Let $k \in \mathbf{N}$ be such that $A_{t_{\lambda^*}w_0vt_\tau[1,k]} \in \mathcal{C}_{w_0}$ and let $J = \{i_1, \ldots, i_p\} \in \mathfrak{I}_{\lambda,v,\tau}$ be such that $i_1 > k$. We have*

1. $\overline{\mathsf{H}_{t_{\lambda^*}w_0vt_\tau[1,k],t_\tau[k+1,n]}} \subset \{\delta \in \Phi^+ \mid v^{-1}w_0\delta \in \Phi^+\}$*;*
2. *if $A_{t_{\lambda^*}w_0vt_\tau[1,i_1-1]} \notin \mathcal{C}_{w_0}$ then $\xi_J = \displaystyle\sum_{k=1}^{p} L(s_{i_k}) < L(v)$.*

Proof. Let $\delta \in \Phi^+$ be such that $H_{\delta,N} \in \mathsf{H}_{t_{\lambda^*}w_0vt_\tau[1,k],t_\tau[k+1,n]}$. Assume first that the quarter $\mathcal{C}_{\lambda^*,w_0v}$ is oriented towards $-\infty$ in the direction δ. Then we must have

$$A_{t_{\lambda^*}w_0v}[\delta] \geq \underbrace{A_{t_{\lambda^*}w_0vt_\tau[1,k]}[\delta]}_{<0} \geq A_{t_{\lambda^*}w_0vt_\tau[1,k+1]}[\delta] \geq A_{t_{\lambda^*}w_0vt_\tau}[\delta].$$

But $H_{\delta,N} \in H(A_0, A_{t_{\lambda^*}w_0vt_\tau[1,k]})$ implies that $0 \geq N > A_{t_{\lambda^*}w_0vt_\tau[1,k]}[\delta]$. Therefore in this case we cannot have $H_{\delta,N} \in H(A_{t_{\lambda^*}w_0vt_\tau[1,k+1]}, A_{t_{\lambda^*}w_0vt_\tau})$. This shows that the quarter $\mathcal{C}_{\lambda^*,w_0v}$ has to be oriented towards $+\infty$ in the direction δ that is $v^{-1}w_0\delta \in \Phi^+$.

We prove (2). Since $J \in \mathfrak{I}_{\lambda,v,\tau}$, there is a term of degree $\sum_{k=1}^p L(s_{i_k})$ that appear in the product $T_{t^*_\lambda w_0v_J}T_{t_\tau}$. Further, for all k such that $k < i_1$, there is also a term of degree $\sum_{k=1}^p L(s_{i_k})$ in product $T_{t^*_\lambda w_0v_Jt_\tau[1,k]}T_{t_\tau[k+1,n]}$. Let $k_0 < i_1$ be the index such that $A_{t_{\lambda^*}w_0vt_\tau[1,k_0-1]} \in \mathcal{C}_{w_0}$ and $A_{t_{\lambda^*}w_0vt_\tau[1,k_0]} \notin \mathcal{C}_{w_0}$ and let $\alpha \in \Delta$ be the direction of the hyperplane that separates those two alcoves. The quarter $\mathcal{C}_{\lambda^*,w_0v}$ has to be oriented toward $+\infty$ in the direction α so that $v^{-1}w_0\alpha \in \Phi^+$. Next

$$\overline{\mathsf{H}_{t_{\lambda^*}w_0vt_\tau[1,k_0],t_\tau[k_0+1,n]}} = \underbrace{\overline{\mathsf{H}_{t_{\lambda^*}w_0vt_\tau[1,k_0-1],t_\tau[k_0,n]}}}_{\subset\{\delta\in\Phi^+ \mid v^{-1}w_0\delta\in\Phi^+\}} - \{\alpha\}.$$

Theorem 3.1 now implies that the maximal degree that can appear in the product $T_{t^*_\lambda w_0v_Jt_\tau[1,k_0]}T_{t_\tau[k_0+1,n]}$ is strictly less than $L(v)$. But there is a term of degree $\sum_{k=1}^p L(s_{i_k})$, hence the result. $\square$

Lemma 5.11. *Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \ldots, n\}$ such that*

$$t_{\lambda^*}w_0vt_\tau[1,k-1]s_k < t_{\lambda^*}w_0vt_\tau[1,k-1] \quad \text{and} \quad A_{t_{\lambda^*}w_0vt_\tau[1,k-1]} \in \mathcal{C}_{w_0}.$$

Then $w_0v\beta_k \in \Phi^+$. In particular $w_0vs_{\beta_k} > w_0v$ and $vs_{\beta_k} < v$.

Proof. The hyperplane H that separates $A_{t_{\lambda^*}w_0vt_\tau[1,k-1]}$ and $A_{t_{\lambda^*}w_0vt_\tau[1,k-1]s_k}$ is $H = H_{w_0v\beta_k,N_k+\langle\lambda^*,w_0v\beta_k^\vee\rangle}$. Since $A_{t_{\lambda^*}w_0vt_\tau[1,k-1]} \in \mathcal{C}_{w_0}$, we must have

$$A_{t_{\lambda^*}w_0vt_\tau[1,k-1]} \in H^- \quad \text{and} \quad A_{t_{\lambda^*}w_0vt_\tau[1,k-1]s_k} \in H^+,$$

which means that the quarter $\mathcal{C}_{\lambda^*,w_0v}$ has to be oriented toward $+\infty$ in the direction $|w_0v\beta_k| \in \Phi^+$, where $|w_0v\beta_k|$ denotes the unique positive root colinear to $w_0v\beta_k$. According to Lemma 2.4, since $w_0v^{-1}w_0v\beta_k = \beta_k \in \Phi^+$, we have $w_0v\beta_k \in \Phi^+$. $\square$

We are now ready to prove the second statement of Theorem 5.5. Let $\lambda \in P^+$, $v \in W_0$ and assume that $J = \{i_1, \dots, i_p\} \in \mathfrak{I}^{\max}_{\lambda,v,\tau}$. We need to show that J is admissible, λ-dominant and that $v = v_J$.

By maximality of J, we have $\sum_{k=1}^{p} L(s_{i_k}) = L(v)$ and the second statement of Lemma 5.10 implies

$$A_{t_{\lambda^*} w_0 v t_\tau[1,i_1-1]} = A_{t^*_{\lambda_1} w_0 v_1 t_\tau[1,i_1]} \in \mathcal{C}_{w_0}.$$

Note also that $J_1 \in \mathfrak{I}_{\lambda_1,v_1,t_\tau}$. Applying Lemma 5.10 to J_1 and $i_2 > i_1$ we get

$$\overline{\mathsf{H}_{t^*_{\lambda_1} w_0 v_1 t_\tau[1,i_1], t_\tau[i_1+1,n]}} \subset \{\delta \in \Phi^+ \mid v_1^{-1} w_0 \delta \in \Phi^+\}.$$

Then Theorem 3.1 implies that

$$\sum_{k=2}^{n} L(s_{i_k}) \leq L(v_1).$$

But $v_1 = v s_{\beta_{i_1}}$ which is strictly less than v by Lemma 5.11. From there we see that $L(v_1) \leq L(v) - L(\beta_1) = L(v) - L(s_{i_1})$ the last equality coming from the fact that J is maximal. Putting everything together, we get

$$L(v) - L(s_{i_1}) = \sum_{k=2}^{n} L(s_{i_k}) \leq L(v_1) \leq L(v) - L(s_{i_1}).$$

It follows that we have equality throughout. Further, $\deg(\xi_{J_1}) = L(v_1)$ also implies that $A_{t^*_{\lambda_1} w_0 v_1 t_\tau[1,i_2-1]} \in \mathcal{C}_{w_0}$ by Lemma 5.10. At this stage, we have

$$J_2 \in \mathfrak{I}_{\lambda_2,v_2,t_\tau} \quad \text{and} \quad A_{t^*_{\lambda_1} w_0 v_1 t_\tau[1,i_2-1]} = A_{t^*_{\lambda_2} w_0 v_2 t_\tau[1,i_2]} \in \mathcal{C}_{w_0}.$$

We can therefore argue by induction to show that for all $\ell \in \{1, \dots, p\}$ we have

$$A_{t^*_{\lambda_\ell} w_0 v_\ell t_\tau[1,i_\ell]} \in \mathcal{C}_{w_0} \quad \text{and} \quad L(v_\ell) = L(v) - \sum_{k=1}^{\ell} L(s_{i_k}).$$

In particular, we have $v_p = \mathsf{id}$. But $v_p = v s_{\beta_{i_1}} \dots s_{\beta_{i_p}}$ so $v = s_{\beta_{i_p}} \dots s_{\beta_{i_1}} = v_J$.

We have also seen that for all $1 \leq \ell \leq p-1$

$$\underbrace{A_{t^*_{\lambda_\ell} w_0 v_\ell t_\tau[1,i_\ell]}}_{=A_{t^*_\lambda w_0 v t^J_\tau[1,i_\ell]}} \in \mathcal{C}_{w_0} \quad \text{and} \quad \underbrace{A_{t^*_{\lambda_\ell} w_0 v_\ell t_\tau[1,i_{\ell+1}-1]}}_{=A_{t^*_\lambda w_0 v t^J_\tau[1,i_{\ell+1}-1]}} \in \mathcal{C}_{w_0}$$

hence showing that for all $k \leq i_p$ we have $A_{t^*_\lambda w_0 v t^J_\tau[1,k]} \in \mathcal{C}_{w_0}$. For the last values of k we simply note that

$$A_{t^*_\lambda w_0 v t^J_\tau[1,i_p]} = A_{t^*_{\lambda_p} w_0 t_\tau[1,i_p]} \in \mathcal{C}_{w_0} \text{ (since } v_p = \mathsf{id})$$

and since $\tau \in P^+$ we must have $A_{t^*_{\lambda_p} w_0 t_\tau[1,k]} \in \mathcal{C}_{w_0}$ for all $k \geq i_p$. It follows that J is λ-dominant.

It remains to show that J is admissible. Applying the proof above in the case where L is the usual length function ℓ shows that $\ell(v) = p$. We have proved that

$$\mathsf{id} < v s_{\beta_{i_1}} \dots s_{\beta_{i_p}} < \dots < v s_{\beta_{i_1}} s_{\beta_{i_2}} < v s_{\beta_{i_1}} < v$$

which implies that

$$\mathsf{id} < s_{\beta_{i_p}} < s_{\beta_{i_p}} s_{\beta_{i_{p-1}}} < \dots < s_{\beta_{i_p}} \dots s_{\beta_{i_2}} s_{\beta_{i_1}}.$$

This sequence is of length $p+1$ hence it has to be saturated. This completes the proof of Theorem 5.5.

Corollary 5.12. *Let $\lambda \in P^+$ and $\tau \in P^+$. We have*

$$C_{w_0 t_\lambda} \mathsf{s}_\tau = C_{w_0 t_\lambda} \mathsf{h}_\tau = \sum_J C_{w_0 t_\lambda v_J t^J_\tau}$$

where the sum is taken over all λ-dominant maximal admissible subset J.

Remark 5.13. If we are working in the case of equal parameters or when W is not of type $\tilde{C}_n$ or $\tilde{A}_1$, we can remove the condition of maximality since all parallel hyperplanes have same weights and therefore all admissible subsets are maximal.

Any element in W_e can be uniquely written under the form $t_{\mathrm{wt}(w)}\theta(w)$ where $\mathrm{wt}(w) \in P$ is called the weight of w and $\theta(w) \in W_0$. In this section, we have shown that $\theta(v_J t_\tau^J) = \mathsf{id}$ so that if we set $\mu(J) := \mathrm{wt}(v_J t_\tau^J)$, Corollary 5.12 now reads

$$C_{w_0 t_\lambda} \mathsf{s}_\tau = C_{w_0 t_\lambda} \mathsf{h}_\tau = \sum_J C_{w_0 t_{\lambda+\mu(J)}}$$

where the sum is taken over all λ-dominant maximal admissible subset J.

Remark 5.14. Gaussent and Littelmann [3] have shown that the Littelmann path model is equivalent to the gallery model and they expressed $m^\mu_{\lambda,\tau}$ as the number of certain galleries. In [15], Schwer has computed the structure constants of the MacDonald spherical functions $P_\lambda(\mathsf{q}^{-1})$ using galleries. The functions $P_\lambda(\mathsf{q}^{-1})$ interpolate the Weyl characters and the monomial functions as we have $P_\lambda(0) = \mathsf{s}_\lambda$ and $P_\lambda(1) = m_\lambda$. Schwer then obtained a combinatorial description of $m^\mu_{\lambda,\tau}$ by describing which of the coefficients in the expansion $P_\lambda(\mathsf{q}^{-1})P_\tau(\mathsf{q}^{-1})$ survives the specialisation $\mathsf{q}^{-1} = 0$. The result of Schwer is indeed equivalent to our result but its proof is quite different : it uses the link between the combinatorics of positive folded galleries of Gaussent and Littelmann and the action of the Bernstein basis on the periodic Hecke module of Lusztig [10]. Another approach to computing structure constants of MacDonald spherical functions can be found in [14] where Parkinson gives a formula in terms of intersection cardinalities in affine buildings.

6 Admissible subsets and Littelmann paths

In this section we compare our result with the result of Lenart and Postnikov and then explain following [8, §9] how admissible subsets give rise to Lakshmibai-Seshadri paths (which are special kind of Littelmann's paths).

6.1 Lenart and Postnikov result

In [7, 8], Lenart and Postnikov introduced the notion of admissible subset in order to compute the product $\mathsf{s}_\tau \mathsf{s}_\mu$ where $\lambda, \mu \in P^+$, see Theorem 6.3. In this section we explain the connection between our result. We start by recalling their definition of admissible subset. To distinguish between those two definitions, we will say that a subset is LP-admissible if it is admissible in the sense of [8, Definition 6.1].

Let $t_\tau = s_1 \ldots s_n a$ be a fixed reduced expression and denote by H_{β_i, N_i} the hyperplane separating the alcoves $A_{s_1\ldots s_{i-1}}$ and $A_{s_1\ldots s_{i-1}s_i}$. Let $s'_1 \ldots s'_n a^{-1}$ be the reduced expression of $t_{-\tau}$ obtained by inverting the reduced expression above and moving a^{-1} to the left. Then, the hyperplane separating $A_{s'_1\ldots s'_{i-1}}$ and $A_{s'_1\ldots s'_{i-1}s'_i}$ is $H_{\beta'_i, N'_i}$ where $\beta'_i = \beta_{n-i+1}$ and $N'_i = N_{n-i+1} - \langle \tau, \beta^\vee_{n-i+1}\rangle$. A set $J := \{j_1, \ldots, j_p\}$ is said to be LP-admissible if the sequence

$$\mathsf{id} < s_{\beta'_{i_1}} < s_{\beta'_{i_1}} s_{\beta'_{i_2}} < \ldots < s_{\beta'_{i_1}} s_{\beta'_{i_2}} \ldots s_{\beta'_{i_p}}$$

is a saturated sequence in the Bruhat order in W_0.

Remark 6.1.

1. The involution $i \mapsto n-i+1$ of $\{1, \ldots, n\}$ induces a bijection $J \mapsto J^\dagger$ between admissible subsets in the sense of Definition 5.1 and admissible subsets in the sense of Lenart and Postnikov.
2. In [8, Definition 5.2], the weight of an LP-admissible subset is defined to be $\mathrm{wt}(t^{J^\dagger}_{-\tau})$. The equality $t^{J^\dagger}_{-\tau}(t^J_\tau)A_0 = A_0$ implies that $\mathrm{wt}(t^{J^\dagger}_{-\tau}) = -\mathrm{wt}(v_J t^J_\tau) = -\mu(J)$ where the function μ is defined at the end of the previous section. We set $\mu(J^\dagger) = \mu(J)$.

Before being able to state the result of Lenart and Postnikov, we need to introduce some more data associated to the LP-admissible subset $J^\dagger$.

Let $H_{-\gamma_i, \ell_i}$ where $\gamma_i \in \Phi^+$ be the hyperplane that separates the alcoves $A_{t^{J^\dagger}_{-\tau}[1,i]}$ and $A_{t^{J^\dagger}_{-\tau}[1,i+1]}$ for all $1 \leq i \leq n$. Let α be a simple root and $\ell^\alpha_\infty = \langle \mu(J), \alpha^\vee\rangle$. Following [8], we set

$$\begin{aligned} I(J^\dagger, \alpha) &= \{i \mid \gamma_i = \alpha\} \\ L(J^\dagger, \alpha) &= \big\{\ell_i \mid i \in I(J^\dagger, \alpha)\big\} \cup \{\ell^\alpha_\infty\} \\ M(J^\dagger, \alpha) &= \max L(J^\dagger, \alpha). \end{aligned}$$

Example 6.2. We keep the setting of Example 5.2. Consider the admissible subset $J := \{3, 5, 10\}$ and the corresponding LP-admissible subset $J^\dagger := \{1, 6, 8\}$. We have $\mu(J) = \alpha_2$ so that $\ell^{\alpha_2}_\infty = 2$ and $\ell^{\alpha_1}_\infty = -1$. Next we compute

$$I(J^\dagger, \alpha_2) = \{2, 6, 10\} \quad \text{and} \quad I(J^\dagger, \alpha_1) := \{1, 8\}$$

so that

$$L(J^\dagger,\alpha_2) = \{0,1,2\} \quad \text{and} \quad L(J^\dagger,\alpha_1) := \{0,-1\}.$$

Finally we obtain $M(J^\dagger,\alpha_2) = 2$ and $M(J^\dagger,\alpha_1) = 0$. In the figure below, we have drawn the path $p_J(\mathsf{id}; \vec{t}^{J^\dagger}_{-\tau})$. The set $I(J^\dagger,\alpha_i)$ indicates when the path is crossing an hyperplane of direction α_i and the integers $M(J^\dagger,\alpha_i)$ are such that the whole path lies in $H^+_{\alpha_i,-M(J^\dagger,\alpha_i)}$. Further, these are the minimal integers with this property.

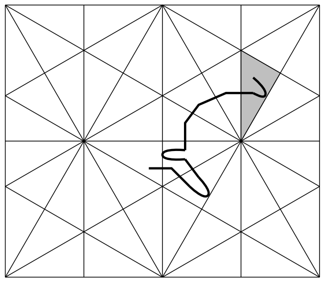


Fig. 3: The path $p_J(\mathsf{id}; \vec{t}^{J^\dagger}_{-\tau})$

We are now ready to state Lenart and Postnikov result.

Theorem 6.3 ([8], Corollary 8.3)**.** *In a simple Lie algebra with Weyl group W_0, we have for $\lambda,\tau \in P^+$*

$$\mathsf{s}_\tau \mathsf{s}_\lambda = \sum_{J^\dagger} \mathsf{s}_{\lambda+\mu(J^\dagger)} \tag{2}$$

where the sum is over all LP-admissible subsets $J^\dagger$ satisfying $\langle \lambda + \mu(J^\dagger), \alpha^\vee\rangle \geq M(J^\dagger,\alpha)$ for all simple root α.

We need to check that the condition on $J^\dagger$ above is equivalent to J being λ-dominant. It will be a consequence of the following three observations:

1. the path $p_J(A_{t_\lambda}; \vec{t}^{J^\dagger}_{-\tau})$ lies in the half plane $H^+_{\alpha,\langle\lambda,\alpha^\vee\rangle - M(J^\dagger,\alpha)}$ and the integer $\langle\lambda,\alpha^\vee\rangle - M(J^\dagger,\alpha)$ is maximal with this property;
2. J is λ-dominant if and only if the path $p_J(t_\lambda v_J; \vec{t_\tau})$ lies in $\mathcal{C}_{\mathsf{id}}$;
3. $p_J(t_\lambda v_J; \vec{t_\tau})$ is obtained from $p_J(A_{t_\lambda}; \vec{t}^{J^\dagger}_{-\tau})$ by translating all the alcoves by $\mu(J)$ and reversing the order.

We have

$$\begin{aligned}
p_J(t_\lambda v_J; \vec{t_\tau}) \subset \mathcal{C}_{\mathsf{id}} &\iff p_J(t_\lambda v_J; \vec{t_\tau}) \subset H^+_{\alpha,0} \text{ for all simple root } \alpha \\
&\iff \left(p_J(A_{t_\lambda}; \vec{t}^{J^\dagger}_{-\tau})\right) t_{\mu(J)} \subset H^+_{\alpha,0} \text{ for all simple root } \alpha \\
&\iff p_J(A_{t_\lambda}; \vec{t}^{J^\dagger}_{-\tau}) \subset H^+_{\alpha,-\langle\mu(J),\alpha^\vee\rangle} \text{ for all simple root } \alpha \\
&\iff \langle\lambda,\alpha^\vee\rangle - M(J^\dagger,\alpha) \geq -\langle\mu(J),\alpha^\vee\rangle \text{ for all simple root } \alpha \\
&\iff \langle\lambda+\mu(J),\alpha^\vee\rangle \geq M(J^\dagger,\alpha) \text{ for all simple root } \alpha.
\end{aligned}$$

This theorem is equivalent to Theorem 5.5 in the equal parameter case when there is no condition on the maximality of J.

6.2 Lakshmibai-Seshadri paths

We explain following [8, §9], how to construct an Lakshmibai-Seshadri path (LS paths for short) from an admissible subset J. There is no new result in this section but this construction exhibits a strong link between LS paths and our approach in the proof of Theorem 5.5: compare for instance Figure 2 and Figure 4.

A rational W_a-path of shape $\mu \in P$ is a pair of sequences $(\underline{v},\underline{a})$ such that $\underline{v} = (v_0,v_1,\ldots,v_r)$ is a strictly decreasing chain (in the Bruhat order) of minimal length left coset representatives of $\mathrm{stab}_\mu(W_0)$ and $\underline{a} = (a_0,a_1,\ldots,a_{r+1})$ is an increasing sequence of rational numbers such that $a_0 = 0$ and $a_{r+1} = 1$.

We identify π with the path $\pi : [0,1] \longrightarrow V$ defined by

$$\pi(t) := \sum_{i=1}^{j-1}(a_i - a_{i-1})v_i\mu + (t - a_{j-1})v_j\mu \text{ for } a_{j-1} \leq t \leq a_j.$$

The weight of π is equal to $\pi(1)$.

A rational W_a-path of shape μ is called a Lakshmibai-Seshadri path if there exists a sequence of positive roots $(\delta_1, \ldots, \delta_r)$ such that

$$v_0 > v_1 = v_0 s_{\delta_1} > v_2 = v_0 s_{\delta_1} s_{\delta_2} > \ldots > v_r = v_0 s_{\delta_1} \ldots s_{\delta_r},$$

$\ell(v_i) = \ell(v_{i-1}) - 1$ and $a_i\langle v_i\mu, \delta_i^\vee\rangle \in \mathbf{Z}$.

Remark 6.4. Our definition of LS paths is slightly different from the one of Littelmann in [9] but nearly obviously equivalent. One only needs to notice that if $a = a_q = a_{q+1} = \ldots = a_{q+r}$ then the chain $(v_q, \ldots, v_{q+r})$ is an a-chain as defined by Littelmann.

For the rest of this section we will need to work with a specific reduced expression of t_τ. This choice is important in the proof of the third statement of Theorem 6.5; see [8, §9]. Fix a total order on the set of simple roots $\{\alpha_1, \ldots, \alpha_N\}$ and write $\{\omega_1, \ldots, \omega_N\}$ for the corresponding fundamental weights. We define the map

$$\begin{array}{cccc} \mathbf{h}: & H(A_0, A_{t_\tau}) & \longrightarrow & \mathbf{R}^{N+1} \\ & H_{\delta,k} & \longmapsto & \frac{1}{\langle \tau, \delta^\vee\rangle}\left(k, \langle\omega_1, \delta^\vee\rangle, \ldots, \langle\omega_N, \delta^\vee\rangle\right). \end{array}$$

The lexicographic order on $\mathbf{R}^{N+1}$ induces a total order on the set $H(A_0, A_\tau)$. Let

$$H(A_0, A_\tau) = \{H_{\beta_1,N_1}, \ldots, H_{\beta_n,N_n}\}$$

be such that $H_{\beta_i,N_i} < H_{\beta_{i+1},N_{i+1}}$ for all i. Then there exists a reduced expression of t_τ of the form $s_1 \ldots s_n a$ where $a \in \Pi$, $s_i \in S$ and such that H_{β_i,N_i} is the hyperplane separating $A_{s_1\ldots s_{i-1}}$ and $A_{s_1\ldots s_i}$.

Let $J = \{i_1, \ldots, i_p\}$ be an admissible subset as in Definition 5.1 and consider the pair $(0, v_J) \in P \times W_0$. Let $(\lambda_0, \ldots, \lambda_p)$, $(v_0, \ldots, v_p)$ and $(J_0, \ldots, J_p)$ be the sequences associated to $(0, v_J)$ constructed in Section 5. Since J is admissible we have $v_\ell = s_{\beta_{i_p}} \ldots s_{\beta_{i_{\ell+1}}} = v_{J_\ell}$. We set

- $\tau_\ell = v_{J_\ell}\tau$ for $0 \leq \ell \leq p$,
- $\pi_\ell : [0,1] \longrightarrow V$ to be the straight path defined by $t \longmapsto \mu_\ell + t \cdot \tau_\ell$,
- $a_\ell = \dfrac{N_{i_\ell}}{\langle \tau, \beta_{i_\ell}^\vee\rangle}$ for $1 \leq \ell \leq p$.

Finally, we set $\underline{v} = (v_{J_0}, \ldots, v_{J_p})$ and $\underline{a} = (a_0, \ldots, a_{p+1})$ where $a_0 = 0$ and $a_{p+1} = 1$. By construction, the path $\pi = (\underline{v}, \underline{a})$ coincide with the path π_ℓ for all $a_\ell \leq t \leq a_{\ell+1}$. Then according to [8, §9] one can show the following result where once again, we assume that we are in the equal parameter case. The choice of our specific reduced expression for t_τ plays a crucial role in the proof of Statement (3). Recall the definition of $\mu(J)$ at the end of Section 5.

Theorem 6.5. *Let J be an admissible subset, $\lambda \in P^+$ and $\pi = (\underline{v_J}, \underline{a})$ be the W_a-rational path defined above. We have*

1. *π is a LS-path of shape τ and weight $\mu(J)$.*
2. *The set of LS-paths of shape τ is in bijection with the set of admissible subsets.*
3. *π is λ-dominant (i.e. $\lambda + \pi(t) \in \overline{\mathcal{C}_{id}}$ for all $t \in [0,1]$) if and only if the path $p_J(t_\lambda v_J; t_\tau) \subset \mathcal{C}_{id}$.*

As a direct consequence of this theorem and Theorem 5.5, we obtain for $\lambda, \tau \in P^+$

$$C_{w_0 t_\lambda}\mathsf{h}_\tau = \sum_\pi C_{w_0 t_{\lambda+\pi(1)}}$$

where the sum is over all λ-dominant LS-paths of shape τ.

Example 6.6. Let W be of type G_2 and keep the notations of the previous examples. We compute the different elements needed in the construction of the path π_J where $J = \{3, 5, 10\}$. Note that most of these have already been constructed in Example 5.7.

ℓ	J_ℓ	β_{i_ℓ}	v_{J_ℓ}	τ_ℓ	N_{i_ℓ}	a_ℓ
0	$\{3, 5, 10\}$		$s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$	$-\alpha_1$		0
1	$\{5, 10\}$	δ_4	$s_{\alpha_2} s_{\alpha_1}$	α_1	1	1/2
2	$\{10\}$	δ_5	σ_{α_1}	$\alpha_1 + 3\alpha_2$	2	2/3
3	$\emptyset$	δ_6	id	$2\alpha_1 + 3\alpha_2$	1	1

The LS-path π_J associated to J is represented by a thick line in Figure 4. The paths π_1, π_2 and π_3 are represented by dashed lines. We see that that

* π_J follows the direction $-\alpha_1$ for a time $1/2$;
* π_J follows the direction α_1 for a time $1/6$;
* π_J follows the direction $\alpha_1 + 3\alpha_2$ for a time $1/3$.

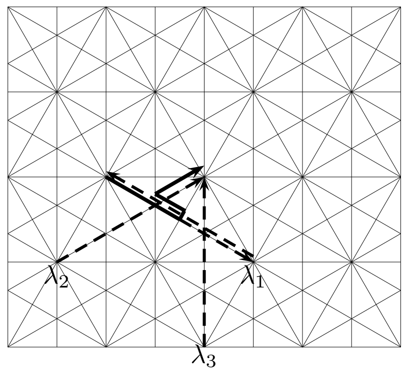


Fig. 4: LS path associated to the set $J = \{3, 5, 10\}$.

When doing the above procedure for all admissible subsets J (described in Example 5.2) we obtain the paths described in Figure 5.

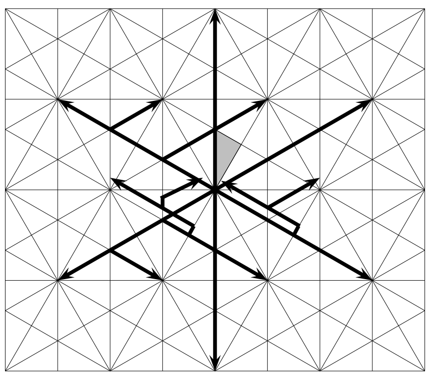


Fig. 5: LS paths in type G_2.

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