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# Computing Attracting Ellipsoids for Nonlinear Systems using an Interval Lyapunov Equation

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## 1 Introduction

The stability analysis of a continuous-time dynamical system:

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad (1)$$

is a fundamental question in control theory. This paper focuses on the simple but important case of *exponential stability*. If  $x^*$  is an exponentially stable equilibrium, all the trajectories starting inside a neighborhood  $\mathcal{A}(x^*)$  of  $x^*$  are known to converge to  $x^*$  faster than an exponential decay, that is:

$$\forall x_0 \in \mathcal{A}(x^*), \exists t_0, \forall t \geq t_0, \quad \|x(t) - x^*\| \leq \alpha \|x_0 - x^*\| e^{-\beta t} \quad (2)$$

for some nonnegative constants  $\alpha$  and  $\beta$ . The value of  $\beta$  characterizes the *strength* of attraction: the greater  $\beta$ , the more attractive  $x^*$ .

Proving exponential stability is a classical result. The point  $x^*$  is an exponentially stable equilibrium if  $f(x^*) = 0$  and all the eigenvalues of the jacobian matrix  $Df(x^*)$  has strictly negative real parts. These eigenvalues also characterize the attraction strength as it is easy to derive from their real parts a bound for  $\beta$  in (2). However, they give no information about the size of the basin of attraction  $\mathcal{A}(x^*)$ . In fact, this basin can be arbitrarily small for arbitrarily strongly attracting equilibria. Consider, for example, the system

$$\dot{x} = c^3 x^3 - cx. \quad (3)$$

The linearization at the equilibrium  $x^* = 0$  is  $\dot{x} = -cx$  and the basin of attraction is  $\mathcal{A}(x^*) = (-\frac{1}{c}, \frac{1}{c})$ . Thus, the greater  $c$ , the more attracting 0 but the smaller the basin of attraction. Inversely, for the system

$$\dot{x} = \frac{x^3}{c} - cx. \quad (4)$$

with the same linearization as before, we have  $\mathcal{A}(x^*) = (-c, c)$  so the more attracting 0, the larger the basin.

It is therefore valuable to estimate the basin of attraction for a stable equilibrium in addition of assessing the stability based on the linearization. However, estimating the actual shape of the basin is too complicated. One could proceed, for instance, statistically by simulating trajectories from a sample of initial conditions, but the result is not reliable, the computational cost is high and blows up with the dimension  $n$ . In this paper, we only build a subset of  $\mathcal{A}(x^*)$  but the algorithms are fast (in polynomial time) and reliable, in the sense that the computed subset is proven to be included in the basin.

Our paper is based on Lyapunov theory, which is the usual framework for addressing this kind of problems. A Lyapunov function is a real-valued function of the state space that can be interpreted, locally around  $x^*$ , as the energy of the system. It must decrease with time for all trajectories converging to the equilibrium, where the energy must be minimal. Then, all the level sets of this

energy function are inside the basin of attraction. Formally, a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lyapunov function if

$$V(x^*) = 0 \tag{5}$$

and there exists an open neighborhood  $\mathcal{N}$  of  $x^*$  such that

$$\forall x \in \mathcal{N} \setminus \{x^*\} \begin{cases} V(x) > 0 \\ \dot{V}(x) < 0 \text{ with } \dot{V}(x) := \nabla V(x)^T f(x). \end{cases} \tag{6}$$

Building a Lyapunov function is actually simple in the case of an exponentially stable point. One just has to consider the linearized system:

$$\dot{x}(t) = Df(x^*) \times x(t), \tag{8}$$

and solve the corresponding *Lyapunov equation*:

$$Df(x^*)^T P + P Df(x^*) = -I. \tag{9}$$

where the positive definite matrix  $P \in \mathbb{R}^{n \times n}$  is the unknown. Then, the quadratic function

$$V(x) = (x - x^*)^T P (x - x^*) \tag{10}$$

is a Lyapunov function. The remaining difficulty is to find a set  $\mathcal{N}$  where (7) is true since, by construction, (6) is already true for any neighborhood.

Interval analysis seems promising in this context. We can indeed fix the set  $\mathcal{N}$  to an arbitrary box enclosing  $x^*$  and try to prove (6) using interval arithmetic. However, two issues make this approach inapplicable:

- (i) Proving that  $\dot{V}$  is strictly negative excepted at one point where it is zero is not in the scope of interval methods, subject to rounding errors and which can only prove generic properties.
- (ii) The equilibrium  $x^*$  cannot be computed exactly, but is instead enclosed inside a tiny box  $[x^*]$ . Therefore, the Lyapunov function  $V$  and its attracting level-sets are uncertain.

We propose two algorithms for building such a set in a guaranteed way, taking into account the uncertainty on  $x^*$  and rounding errors. Both take as input an initial ellipsoid and return a smaller ellipsoid. The first algorithm uses only first-order derivatives of  $f$  but requires to fix heuristically the size of the input ellipsoid until it succeeds (the algorithm is proven to succeed for sufficiently small input size). The second algorithm always succeeds, for any input size, but resorts to second-order derivatives.

## 2 Background

### 2.1 Notations

All along this paper we consider a differentiable mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and denote  $Df$  the jacobian of  $f$ . For our second-order algorithm, we require  $f$  to be twice differentiable. Need to change from  $f$  twice differentiable to  $f \in \mathcal{C}^2$ . and denote  $D^2f$  the tensor of second-order derivatives. No need to define  $D^2f$  the tensor... Not used in the article.. We assume  $x^*$  is an equilibrium point, that is,  $f(x^*) = 0$ , and denote  $\mathcal{A}(x^*)$  the basin of attraction of  $x^*$ .

An application  $V : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is positive definite (PD) if

$$\forall x \in \Omega, V(x) \geq 0 \quad \text{and} \quad V(x) = 0 \iff x = 0, \tag{11}$$

and negative definite (ND) if the inequality is reversed. A matrix  $P$  is positive (resp. negative) definite if the application  $x \mapsto x^T P x$  is positive (resp. negative) definite. We use the abbreviations SPD (resp. SND) for symmetric positive (resp. negative) definite matrices.

Throughout this paper, we consider a SDP matrix  $P \in \mathbb{R}^{n \times n}$ , which, in practice, is obtained by considering the *linearized system*

$$\dot{x}(t) = \text{Df}(\hat{x}) \times x(t), \quad (12)$$

where  $\hat{x}$  stands for an approximation of  $x^*$ , and solving the corresponding Lyapunov equation:

$$\text{Df}(\hat{x})^T P + P \text{Df}(\hat{x}) = -I. \quad (13)$$

We use the  $P$ -norm, induced by  $P$ :

$$\|x\|_P := \sqrt{x^T P x} \quad (14)$$

Add definition  $A \in \mathbb{R}^{n \times n}$ ,  $\|A\|_\infty$ ,  $\|A\|_2$  and  $\|A\|_P$  and denote  $\mathbb{E}(x, r)$  the following ellipsoid of center  $x$  and radius  $r$ :

$$\mathbb{E}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\|_P \leq r\}. \quad (15)$$

Moreover,  $\bar{\lambda}$  and  $\underline{\lambda}$  denote respectively strictly positive upper and lower bounds on the eigenvalues of  $P$ , i.e.,  $0 < \underline{\lambda} \leq \lambda_{\min}(P) \leq \lambda_{\max}(P) \leq \bar{\lambda}$ .

Finally, we note  $V|_\Omega$  the restriction of a function  $V$  to a set  $\Omega$ .

## 2.2 Interval Arithmetic

An interval matrix  $[Q]$  is ND if  $\forall Q \in [Q]$ ,  $Q$  is ND. Add definition  $[A] \in \mathbb{IR}^{n \times n}$ ,  $\|[A]\|_\infty$ ,  $\|[A]\|_2$  and  $\|[A]\|_P$

## 3 Related Works

### 3.1 Using Sum of Square Decomposition

When the function  $f$  is polynomial<sup>1</sup>, a symbolic-numeric approach can be applied. In particular, [7] shows how a polynomial Lyapunov function can be automatically built, using sum of square (SOS) decomposition. They do not consider basins of attraction, but since they accept systems with inequalities, their main theorem can be easily adapted to fit our need. More precisely, the following result holds as a corollary:

*Given a SPD matrix  $P$  and  $r > 0$ , if there exists a polynomial  $p(x)$  such that*

$$p \geq 0 \quad \wedge \quad -P f(x) + p(x) \left( \|x - x^*\|_P - r \right) \geq 0 \quad (16)$$

*then  $\mathbb{E}(x^*, r) \subseteq \mathcal{A}(x^*)$ .*

If we now replace each positivity condition by the more strict condition of being SOS, finding such a polynomial  $p(x)$  becomes a tractable problem that can be solved using LMI solvers.

However, this approach is not applicable to the whole class of nonlinear systems. It is also not robust to rounding errors and turns out to be slower than ours, as our experiments will show.

<sup>1</sup>More generally, the approach applies if  $f$  can be transformed to a polynomial by introducing auxiliary variables subject to polynomial constraints.

## 4 $\Omega$ -Lyapunov Functions

We present in this section preliminar concepts and results that will serve further as a basis for the computation of the basin of attraction. More precisely, we give a test for a quadratic function to be a so-called  $\Omega$ -Lyapunov function. In this section, we also assume, with no loss of generality, that the equilibrium is the origin ( $x^* = 0$ ).

Our test will involve the *slope matrix*, defined as follows:

**Definition 1** (Slope Matrix).

$$\forall x \in \mathbb{R}^n, \text{ we call slope matrix the matrix } S(x) := \int_0^1 Df(tx) dt. \quad (17)$$

The fundamental theorem of analysis then gives

$$\forall x \in \mathbb{R}^n \quad f(x) = f(0) + S(x)x = S(x)x. \quad (18)$$

$$Df([x]) \supseteq S([x])$$

**Definition 2.**

Given a neighborhood  $\Omega \subseteq \mathbb{R}^n$  of 0, a differentiable mapping  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a  $\Omega$ -Lyapunov function if

$$V|_{\Omega} \text{ is PD and } \dot{V}|_{\Omega} \text{ is ND}, \quad (19)$$

where

$$\dot{V}(x) := \nabla V(x)^T f(x). \quad (20)$$

**Proposition 1.**

Let  $\Omega$  be a 0-stared set 0-stared to convex?.  $\forall x \in \Omega$ , define  $Q(x)$  as follows:

$$Q(x) := S(x)^T P + P S(x). \quad (21)$$

If  $Q(\Omega)$  contains only ND matrices, then the application  $V : x \mapsto x^T P x$  is an  $\Omega$ -Lyapunov function.

*Proof.*

First,  $V|_{\Omega}$  is PD since  $P$  is PD by hypothesis. Second,

$$\forall x \in \Omega, \quad \dot{V}(x) = \nabla V(x)^T f(x) = \nabla V(x)^T S(x)x \quad (\text{using (18)}) \quad (22)$$

$$= 2x^T P S(x)x \quad (\text{because } \nabla V(x) = 2x^T P) \quad (23)$$

$$= x^T (S(x)^T P + P S(x))x \quad (\text{because } x^T A x = \frac{1}{2} x^T (A^T + A)x) \quad (24)$$

$$= x^T Q(x)x \quad (25)$$

and since  $Q(x)$  is ND by hypothesis,  $\dot{V}|_{\Omega}$  is ND. Hence,  $V$  is a  $\Omega$ -Lyapunov function.  $\square$

## 5 A First-Order Ellipse Computation

We now consider again the general case of an equilibrium  $x^*$ , which can be any point. The point  $\hat{x}$  represents an approximation of  $x^*$ . Although  $P$  can be chosen arbitrarily, the following corollary works if  $P$  is an approximate solution of (13).

We first give a condition on the radius  $\hat{r}$  of an ellipse centered on  $\hat{x}$  for being included in an ellipse centered on  $x^*$  and for containing  $x^*$ .

**Lemma 1.** Let  $\hat{r} > 0$ ,  $\hat{x} \in \mathbb{R}^n$  and  $\bar{\Delta} \geq \|x^* - \hat{x}\|_P$ . Then

- (i)  $\mathbb{E}(\hat{x}, \hat{r}) \subseteq \mathbb{E}(x^*, \hat{r} + \bar{\Delta})$ .
- (ii)  $\bar{\Delta} \leq \hat{r} \implies x^* \in \mathbb{E}(\hat{x}, \hat{r})$ .

*Proof.*

- (i)  $x \in \mathbb{E}(\hat{x}, \hat{r}) \implies \|x\|_P \leq \hat{r} \implies \|x - x^*\|_P \leq \hat{r} + \bar{\Delta} \implies x \in \mathbb{E}(x^*, \hat{r} + \bar{\Delta}) \implies \|x - x^*\|_P \leq \|x - \hat{x}\|_P + \|\hat{x} - x^*\|_P \leq \hat{r} + \bar{\Delta}$
- (ii)  $\bar{\Delta} \leq \hat{r} \implies \|x^* - \hat{x}\|_P \leq \hat{r} \implies x^* \in \mathbb{E}(\hat{x}, \hat{r})$ . □

We give now our first corollary, the main result of this section, which gives a test for proving that an ellipse is inside the basin of attraction of the equilibrium  $x^*$ . We take now into account the uncertainty on  $x^*$ , by considering that only a box  $[x^*] \in \mathbb{IR}^n$  enclosing  $x^*$  is known. In practice,  $[x^*]$  is a tiny box and  $\hat{x} \approx \text{mid}[x^*]$ .

The proof is based on the famous Lyapunov Why is Lyapunov's theorem here and not in the  $\Omega$ -Lyapunov function since it is used in both first and second order? stability criterion:

**Theorem 1** (Lyapunov). *A system is asymptotically stable in 0 if and only if there exists an  $\Omega$ -Lyapunov function  $V$  for some neighborhood  $\Omega$  of 0. Furthermore, the level sets of  $V$  included in  $\Omega$  are inside the basin of attraction  $\mathcal{A}(0)$ .*

**Corollary 1.** Let  $x^*, \hat{x} \in [x^*] \in \mathbb{IR}^n$  and  $\hat{r} \geq 0$ . Define  $\bar{\Delta} \geq 0$ ,  $[x] \in \mathbb{IR}^n$  and  $[J], [Q] \in \mathbb{IR}^{n \times n}$  such that

$$\bar{\Delta} \geq \sqrt{([x^*] - \hat{x})^T P ([x^*] - \hat{x})}, \quad (26)$$

$$[x] \supseteq [x^*] + \square(\mathbb{E}(0, \hat{r} + \bar{\Delta})), \quad (27)$$

$$[J] \supseteq [Df]([x]), \quad (28)$$

$$[Q] \supseteq [J]^T P + P[J]. \quad (29)$$

If  $[Q]$  is ND and  $\bar{\Delta} \leq \hat{r}$  then  $x^* \in \mathbb{E}(\hat{x}, \hat{r}) \subseteq \mathbb{E}(x^*, \hat{r} + \bar{\Delta}) \subseteq \mathcal{A}(x^*)$ .

*Proof.* We prove the three inclusions from left to right. First, if  $\bar{\Delta} \leq \hat{r}$ , by Lemma 1.(ii),  $x^* \in \mathbb{E}(\hat{x}, \hat{r})$ . Second,  $\mathbb{E}(\hat{x}, \hat{r}) \subseteq \mathbb{E}(x^*, \hat{r} + \bar{\Delta})$  by Lemma 1.(i).

Third, we can consider without loss of generality  $x^* = 0$ , since such a translation preserves derivatives. We have  $[J] \supseteq S([x])$  and therefore  $Q([x]) \subseteq [Q]$ . Hence, by Proposition 1,  $V : u \mapsto u^T P u$  is a  $[x]$ -Lyapunov function. Now,  $\mathbb{E}(x^*, r^*)$  is by definition a level set of  $V$  and (27) implies  $\mathbb{E}(x^*, r^*) \subseteq [x]$ . So, by Theorem 1,  $\mathbb{E}(x^*, r^*)$  is inside the attraction basin of  $x^*$ . Remove the  $r^*$  and use  $\hat{r} + \bar{\Delta}$  no? □

## 6 A Second-Order Ellipse Computation

**Lemma 2.** Let  $\mathcal{M}$  be a path connected set of symmetric nonsingular matrices. Then  $\exists M \in \mathcal{M}$  such that  $M$  is SPD implies  $\forall M \in \mathcal{M}$ ,  $M$  is SPD. As a consequence,  $\|M - I\| < 1$  implies  $M$  is SPD, or equivalently,  $\|M + I\| < 1$  implies  $M$  is SND.

*Proof.* This follows from the continuity of eigenvalues with respect to the matrix entries: Let  $M, M' \in \mathcal{M}$  such that  $M$  is SPD and  $M'$  is not, and consider a continuous path  $\Gamma : [0, 1] \rightarrow \mathcal{M}$  such that  $\Gamma(0) = M$  and  $\Gamma(1) = M'$ . Since all eigenvalues of  $M$  are symmetric then eigenvalue is real, or that the real part does not crosses imaginary axis. of  $\Gamma(0)$  are strictly positive and at least one eigenvalue of  $\Gamma(1)$  is strictly negative, one matrix  $\Gamma(t)$  needs to have a null eigenvalue, hence is singular, which contradicts the nonsingularity hypothesis. Todo: Prove the second statement. □

**Corollary 2.** *Add nu, [J\*] and [Q\*] in requirements Let  $x^*, \hat{x} \in [x^*] \in \mathbb{I}\mathbb{R}^n$  and  $\hat{r} > 0$ . Suppose furthermore that  $0 < \underline{\lambda} \leq \bar{\lambda}$  are lower and upper bounds respectively on the eigenvalues of  $P$ . Define  $\bar{\Delta} \geq 0$  and  $[x] \in \mathbb{I}\mathbb{R}^n$  such that*

$$\bar{\Delta} \geq \sqrt{([x^*] - \hat{x})^T P ([x^*] - \hat{x})}, \quad (30)$$

$$[x] \supseteq [x^*] + \square(\mathbb{E}(0, \hat{r} + \bar{\Delta})), \quad (31)$$

$$[J^*] \supseteq [Df]([x^*]), \quad (32)$$

$$[Q^*] \supseteq [J^*]^T P + P [J^*], \quad (33)$$

$$\nu \geq \sqrt{\frac{n\bar{\lambda}}{\underline{\lambda}}} \| [Q^*] + I \|_{\infty}, \quad (34)$$

Suppose that  $\nu < 1$  Move to conditions? and  $S(x)$  is Lipschitz continuous inside  $[x]$ , entailing  $S(x)^T$  Shouldn't it be  $Df(x)$ ? is Lipschitz continuous inside  $[x]$ , We should say that  $L$  and  $L'$  are constants for the  $P$ -norm with constants  $L$  and  $L'$  Move to intro of corollary? respectively. Finally define

$$\tilde{r} := \frac{1}{(L + L') \frac{\bar{\lambda}}{1-\nu}} - \bar{\Delta}. \quad (35)$$

If  $\min\{\hat{r}, \tilde{r}\} \geq \bar{\Delta}$  then  $x^* \in \mathbb{E}(\hat{x}, \min\{\hat{r}, \tilde{r}\}) \subseteq \mathcal{A}(x^*)$ .

**Remark 1.** *As previously, we expect  $\bar{\Delta} \ll 1$ . Furthermore, since  $Df(\hat{x})^T P + P Df(\hat{x}) \approx -I$  and  $[J^*] \ni Df(\hat{x})$  is tiny, we expect that  $\nu \ll 1$  too.*

*Proof.* Without loss of generality, we can consider  $x^* = 0$  since a translation preserves first and second derivatives. We will prove that  $Q(\mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta}))$  contains only SPD matrices. This entails by Theorem 1 that  $V(x) = (x - x^*)^T P (x - x^*)$  is a Lyapunov function inside this ellipsoid, and therefore that  $\mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta}) \subseteq \mathcal{A}(x^*)$ . Finally, by Lemma 1 we have  $x^* \in \mathbb{E}(\hat{x}, \min\{\hat{r}, \tilde{r}\}) \subseteq \mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta})$ , which will end the proof. First, we prove that  $Q(x^*)$  is SND and that

$$\|Q(x^*)^{-1}\|_P \leq \frac{1}{1-\nu}. \quad (36)$$

We haveq

$$\|Q(x^*) + I\|_P \stackrel{\text{Lemma 4}}{\leq} \sqrt{\frac{n\bar{\lambda}}{\underline{\lambda}}} \|Q(x^*) + I\|_{\infty} \stackrel{(34)}{\leq} \nu < 1, \quad (37)$$

As a consequence,  $Q(x^*)$  is SND by Lemma 2, and the Neumann series  $\sum_{k=0}^{\infty} (Q(x^*) + I)^k$  converges toward  $(I - (Q(x^*) + I))^{-1} = -Q(x^*)^{-1}$ . Furthermore

$$\|Q(x^*)^{-1}\|_P = \left\| \sum_{k=0}^{\infty} (Q(x^*) + I)^k \right\|_P \leq \sum_{k=0}^{\infty} \nu^k = \frac{1}{1-\nu}. \quad (38)$$

We can prove that  $Q(\mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta}))$  contains only SND matrices. Note that  $Q(\mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta}))$  is path connected, as the image of a path connected set by continuous function. Furthermore we proved that  $Q(x^*)$  is SND, therefore it is enough by Lemma 2 to prove that  $Q(\mathbb{E}(x^*, \min\{\hat{r}, \tilde{r}\} + \bar{\Delta}))$  contains only nonsingular matrices. In fact, we prove that  $Q(x^*)^{-1}Q(x)$  is nonsingular using Neumann

series:

$$\|Q(x^*)^{-1}Q(x) - I\|_P = \|Q(x^*)^{-1}(Q(x) - Q(x^*))\|_P \quad (39)$$

$$\leq \|Q(x^*)^{-1}\|_P \|Q(x) - Q(x^*)\|_P \quad (40)$$

$$\leq \frac{1}{1-\nu} \|Q(x) - Q(x^*)\|_P \quad (41)$$

$$= \frac{1}{1-\nu} \|S(x)^T P + PS(x) - S(x^*)^T P - PS(x^*)\|_P \quad (42)$$

$$= \frac{1}{1-\nu} \|(S(x)^T - S(x^*)^T)P + P(S(x) - S(x^*))\|_P \quad (43)$$

$$\leq \frac{\|P\|_P}{1-\nu} (\|S(x)^T - S(x^*)^T\|_P + \|S(x) - S(x^*)\|_P) \quad (44)$$

$$\leq \frac{\|P\|_P}{1-\nu} (L + L') \|x - x^*\|_P. \quad (45)$$

The proof is concluded using Lemme 3.  $\square$

## 7 Algorithmic Considerations

In this section we discuss several issues encountered while implementing and different solutions we have explored. We provide the procedures and equations so as this article to be self-contained.

We first start by tackling issues encountered before applying the corollaries. The use of these does not require an exact knowledge of  $x^*$  but rather rely on an enclosure  $[x^*] \in \mathbb{IR}^n$ . This can be performed using guaranteed algorithms such as Newton interval. For details about such approaches, see [4, 5]. In addition, computations require a point  $\hat{x} \in [x^*]$ . It has usually been chosen as  $\hat{x} \approx \text{mid}[x^*]$ . In practice, we used the approximation  $\hat{J} \approx \text{mid}[Df]([x^*])$ . Thus, the matrix  $P$  was obtained by solving

$$\hat{J}^T P + P \hat{J} = -I. \quad (46)$$

This equation can be solved naively by solving the  $n^2$ -by- $n^2$  system of equations. However, this solution only works for small values of  $n$  since its computational complexity is  $\mathcal{O}(n^6)$  (using a numerical method that can solve a system of equation in  $\mathcal{O}(n^3)$ ). More sophisticated approaches can be used here, such as the Bartels-Stewart algorithm which complexity is  $\mathcal{O}(n^3)$ . For more details, see [2].

Now we discuss the problem of testing positive definiteness of an interval matrix  $[M] \in \mathbb{IR}^{n \times n}$ . In fact, so as to apply both Corollaries 1 and 2, one must be able to test whether  $P$  is indeed PD. Moreover, to use Corollary 1 one must be able to test negative definiteness of matrix  $[Q]$  outputted by equation (29). These are the reasons we implemented and tested three different methods to verify positive definiteness<sup>2</sup>. From the least precise to the most precise, and at the same time from the cheapest to the most expensive: strict diagonal dominance test, Sylvester's criterion and Rohn's criterion. Strict diagonal dominance implies positive definiteness, thus it can be used as quick verification. For interval matrices, strict diagonal dominance is written as:

$$1 \leq i \leq n, \quad \min\{|\underline{m}_{ii}|, |\overline{m}_{ii}|\} > \sum_{j \neq i} \max\{|\underline{m}_{ij}|, |\overline{m}_{ij}|\}. \quad (47)$$

We can see that this test takes  $n^2$  operations, hence it is linear in the size of the matrix. Sylvester's criterion relies on computing the leading minors, which is indeed heavier since it requires usually at least  $n^2$  operations, thus the overall  $\mathcal{O}(n^3)$  for Sylvester's criterion. However, one must note that this test is a necessary and sufficient condition. Henceforth, this test is particularly more precise than the diagonal dominance one, even if the pessimism on the determinant computation might end-up in rejecting PD matrices. But this fact is negligible. Finally, Rohn's criterion is the most precise one. It uses  $2^{n-1}$  tests of positive definiteness on the corners of  $[M]$ . The approach is the following: let

<sup>2</sup>Since  $-[M]$  is PD implies  $[M]$  is ND.



$M_c = \frac{1}{2} (\underline{M} + \overline{M})$ ,  $M_\Delta = \frac{1}{2} (\overline{M} - \underline{M})$ ,  $\mathcal{Z} = \{z \in \mathbb{R}^n \mid |z_i| = 1, 1 \leq i \leq n\}$  and  $T_z$  the matrix whose diagonal components are those of  $z$ . Then  $[M]$  is PD if

$$\forall z \in \mathcal{Z}, M_z = M_c - T_z M_\Delta T_z \quad (48)$$

is PD. In fact, one can show that if  $M_z$  is PD, then  $M_{-z}$  is PD as well, this divides the number of tests by a factor 2. In practice, Rohn's procedure for testing is quite heavier since one must then verify that each  $M_z$  is PD as well, using Sylvester's criterion for instance. In practice, for random matrices, one will get better results using either Sylvester's or Rohn's criterion. However, by construction, the interval matrix  $[Q]$  obtained by Corollary 1 is close to  $-I_n$ . Thus the diagonal dominance test is in fact not as less precise. This will be highlighted in Section ??.

In the following paragraphs, one must be able to compute a guaranteed enclosure of a given interval matrix  $[M] \in \mathbb{IR}^{n \times n}$ . For self-containment again, we provide a short description on how to proceed. It is taken from [6]. Using a standard linear algebra tool, compute the approximation  $\tilde{N} \approx (\text{mid}[M])^{-1}$ . Let the scalar  $\beta = \|\tilde{N} \times [M] - I_n\|_\infty$ , vectors  $u$  and  $w \in \mathbb{R}^n$ ,  $u$  having all components equal to 1 and  $w_i = \|\tilde{N}_{:,i}\|_\infty$ ,  $\tilde{N}_{:,i}$  being the  $i$ -th column of  $\tilde{N}$ . If  $\beta < 1$ , then we have the following enclosure of the inverse of  $[M]$

$$[M^{-1}] \subseteq \tilde{N} + \frac{\beta}{1-\beta} [-u^T w, u^T w] \quad (49)$$

Replace  $P^{-1}$  by  $[P^{-1}]$  and according notations.

Corollaries 1 and 2 require to compute an enclosure of an ellipsoid. Let  $P \in \mathbb{R}^{n \times n}$  be a PD matrix,  $\mathbb{E}(x, r)$  the ellipsoid of characteristic matrix  $P$ , of center  $x \in \mathbb{R}^n$  and radius  $r$ . In [3] one can find a method to compute the hull of  $\mathbb{E}(x, r)$ . The procedure is the following: one first needs to compute  $\tilde{x}$ ,  $d \in \mathbb{R}^n$  and  $\delta \in \mathbb{R}$  defined by

$$\begin{cases} \tilde{x} = -P^{-1}x & (50) \\ d_i = \sqrt{(P^{-1})_{ii}} & (51) \\ \delta = \sqrt{r + x^T P^{-1}x}. & (52) \end{cases}$$

Then Theorem 1.7 in [3] provides a hull  $[x] \in \mathbb{IR}^n$  for the ellipsoid, given by

$$\square \mathbb{E}(x, r) = [\tilde{x} - \delta d, \tilde{x} + \delta d]. \quad (53)$$

In Corollary 1 we considered the radius  $\hat{r}$  to be given. In practice, one gets this value from an overheard algorithm, for instance, a dichotomy process can be used to build an enclosure of the largest  $\hat{r}$  for which the test passes.

We continue by addressing the issue of computing guaranteed bounds  $\underline{\lambda}$  and  $\overline{\lambda}$  on the eigenvalues of a given symmetric interval matrix  $[M] \in \mathbb{IR}^{n \times n}$ . This is required in order to apply Corollary 2. We suggest here an interval adaptation of Gerschgorin's Circles Theorem Reference?. First, define the radius of the  $i$ -th Gerschgorin's circle,  $1 \leq i \leq n$ , by

$$R_i = \sum_{j \neq i} \max\{|\underline{m}_{ij}|, |\overline{m}_{ij}|\}. \quad (54)$$

Then, an enclosure of the  $i$ -th Gerschgorin's circles  $\forall M \in [M]$  is given by

$$[g_i] = [\underline{a}_{ii} - R_i, \overline{a}_{ii} + R_i]. \quad (55)$$

Finally, Gerschgorin's Theorem Reference? entails that

$$\sigma([M]) \subseteq \bigcup_i [g_i] \subseteq [\min_i \{\underline{a}_{ii} - R_i\}, \max_i \{\overline{a}_{ii} + R_i\}] =: [g], \quad (56)$$

where  $\sigma([M])$  is the set of all eigenvalues  $\forall M \in [M]$ . One must note that the enclosure relies on the  $R_i$ 's which can be arbitrarily big, even for diagonal dominant matrices. However, using the eigenvalue decomposition  $M = U^T D U$ , where  $D$  is the diagonal matrix of eigenvalues and  $U$  an orthonormal

eigenvectors basis, can lead to drastic improvements. Suppose you can compute an approximation  $\tilde{U}$  of  $U$ , with a standard linear algebra tool. Let  $[\tilde{U}^{-1}]$  be a guaranteed enclosure of its inverse. Since  $\tilde{U}$  is orthogonal, one can compute  $[\tilde{D}] = \tilde{U}M[\tilde{U}^{-1}]$ , which is an interval matrix close to be diagonal, and with same eigenvalues as  $M$  since  $\tilde{U}$  is nonsingular. Now, the values of the  $\tilde{R}_i$ 's computed on  $[\tilde{D}]$  are really small compared to the initial  $R_i$ 's. The previous procedure will again provide an enclosure  $[\tilde{g}]$ . As expected,  $[\tilde{g}]$  will be tighter than  $[g]$ , which results in tighter bounds  $\underline{\lambda}$  and  $\bar{\lambda}$ .

We close this section by discussing an optimization problem that can be run once one has obtained a radius  $\hat{r}$  from Corollary 1 or 2. This optimization problem aims to find the largest attractive ellipsoid that can be built using  $V$ .

$$\min r \text{ s.t. } V(x) \geq \hat{r}^2 \wedge V(x) = r^2 \wedge \dot{V}(x) = 0, \quad (57)$$

where  $V$  is the quadratic Lyapunov function obtained from the corollaries and  $\dot{V}$  its orbital derivative, i.e.  $V(x) = (x - [x^*])^T P (x - [x^*])$  and  $\dot{V}(x) = (x - [x^*])^T P f(x)$ . Another way of expressing this problem can be done by relaxing the two last constraints:

$$\min r \text{ s.t. } V(x) \geq \hat{r}^2 \wedge V(x) \leq r^2 \wedge \dot{V}(x) \geq 0. \quad (58)$$

One must note that several parameters impact particularly the execution time for solving this optimization problem. First the radius. Obviously, there are categories of initial  $\hat{r}$ , if it is  $\varepsilon$ -comparable, the algorithm will require a large amount of time in its initial steps, which would dramatically affect the overall execution time. Second, the dimension. Usually, such problem is solved by bisecting boxes on the boundary, which is reflected exponentially in the execution time.

## 8 Numerical Example

In this section we illustrate the different algorithms on an example with numerical values. The algorithms have been implemented in C++ using the Ibex [1] library, implementing interval arithmetic routines and interval Newton method.

The system we study is the following:

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= x - (1 - x^2)y \end{cases} \quad (59)$$

It is a modified version of the well-known Van Der Pol equation, such that the origin is a stable equilibrium and the limit cycle is repulsive.

We will suppose that the interval Newton method provides a box  $[x^*] = [-0.03, 0.01] \times [-0.01, 0.03]$ . We will use its center as  $\hat{x}$ , that is  $\hat{x} = (-0.02, 0.02)$ .

Solving Lyapunov's linear equation at  $\hat{x}$  leads to

$$P = \begin{pmatrix} 1.50025 & -0.50050 \\ -0.50050 & 1.00100 \end{pmatrix}. \quad (60)$$

This matrix is positive definite, hence we can continue.

### 8.1 First order condition

We will here discuss the first order algorithm, hence verify that conditions of Corollary 1 are met. We won't simulate the whole dichotomy process, but instead, we will select a radius and discuss the different steps of the test algorithm.

Let  $\hat{r} = 0.45$ . This leads to  $[\Delta] = [0.00000, 0.03743]$  and hence  $\hat{r} + \bar{\Delta} = 0.48742$ . One can verify that  $[x] = [-0.46595, 0.44595] \times [-0.54376, 0.56376]$  contains  $[x^*] + \square(\mathbb{E}(0, \hat{r} + \bar{\Delta}))$ . The Jacobian

evaluation of  $f$  on  $[x]$  is then

$$[J] = \begin{pmatrix} [0.00000, 0.00000] & [-1.00000, -1.00000] \\ [0.47463, 1.50673] & [-1.00000, -0.78289] \end{pmatrix}, \quad (61)$$

and thus we obtain

$$[Q] = \begin{pmatrix} [-1.50763, -0.47492] & [-0.63355, 0.50793] \\ [-0.63355, 0.50793] & [-1.00090, -0.56636] \end{pmatrix}. \quad (62)$$

One can verify that  $[Q]$  is unfortunately not diagonally strictly dominant, hence a straightforward test would eventually fail. However, using a more sophisticated test, such as the one described by Rohn, would succeed in proving that  $[Q]$  is negative definite.  $\bar{\Delta} \leq \hat{r}$  is trivially proven to hold.

Thus condition to apply Corollary 1 are met, hence the ellipse of center  $\hat{x}$  and radius  $\hat{r}$  is included in the basin of attraction of  $x^*$ .

The result is shown on Figure ???. The light gray part is the basin of attraction, i.e.  $\mathcal{A}(0)$ , whereas the dark gray part is outside the basin of attraction. The blue box corresponds to  $[x^*]$ , the dotted red box corresponds to  $[x]$ , and the dashed red ellipse to  $\mathbb{E}(\hat{x}, \hat{r})$ .

## A $P$ -norms

Consider the eigenvalue decomposition  $P = C^T C$  with  $C = D^{\frac{1}{2}} U$ ,  $D$  being the diagonal matrix of positive eigenvalues and  $U$  an orthonormal eigenvector basis.

**Lemma 3.**  $\|A\|_P = \|CAC^{-1}\|_2$  and  $\|P\|_P \leq \bar{\lambda}$ .

*Proof.* For the first equality,

$$\|A\|_P^2 := \max_{\|u\|_P^2=1} \|Au\|_P^2 = \max_{u^T C^T C u=1} u^T A^T C^T C A u \quad (63)$$

$$= \max_{v^T v=1} v^T C^{-T} A^T C^T C A C^{-1} v =: \|CAC^{-1}\|_2^2. \quad (64)$$

For the inequality:  $\|P\|_P = \|CC^T CC^{-1}\|_2 = \|CC^T\|_2 = \|D^{\frac{1}{2}} U U^T D^{\frac{1}{2}}\|_2 = \|D\|_2 \leq \bar{\lambda}$ .  $\square$

**Lemma 4.**  $\|x\|_\infty \leq \sqrt{\frac{1}{\lambda}} \|x\|_P$  and  $\|A\|_P \leq \sqrt{\frac{n\bar{\lambda}}{\lambda}} \|A\|_\infty$ .

*Proof.* First, upper bounds on  $\|C\|_2$  and  $\|C^{-1}\|_2$  are computed:

$$\|C\|_2^2 := \lambda_{\max}(C^T C) = \lambda_{\max}(P) \leq \bar{\lambda}. \quad (65)$$

$$\|C^{-1}\|_2^2 := \lambda_{\max}(C^{-T} C^{-1}) = \lambda_{\max}(D^{-\frac{1}{2}} U U^T D^{-\frac{1}{2}}) = \lambda_{\max}(D^{-1}) \leq \frac{1}{\lambda}. \quad (66)$$

Now, for the first statement, using  $\|x\|_P^2 := x^T P x = (Cx)^T (Cx) = \|Cx\|_2^2$ ,

$$\|x\|_\infty^2 \leq \|x\|_2^2 = \|C^{-1} C x\|_2^2 \leq \|C^{-1}\|_2^2 \|x\|_P^2 \leq \frac{1}{\lambda} \|x\|_P^2. \quad (67)$$

By Lemma 3,  $\|A\|_P \leq \|C\|_2 \|C^{-1}\|_2 \|A\|_2 \leq \sqrt{n} \|C\|_2 \|C^{-1}\|_2 \|A\|_\infty$ . The statement is finally proved using (65) and (66).  $\square$

## B A Lipschitz constant of $S(x)$ for the $P$ -norm

A Lipschitz constant for  $S(x) := \int_0^1 Df(\tau x) d\tau$  and the  $P$ -norm is computed using bounds on the second derivatives of  $f$ . First, the Lipschitz constant of  $S(x)$  is related to the one of  $Df(x)$ .

**Proposition 2.** *Given an arbitrary operator norm and an arbitrary vector norm, if  $Df$  is  $L$ -Lipschitz continuous on the 0-stared set  $\Omega$  for these norms then  $S$  is  $\frac{L}{2}$ -Lipschitz continuous on  $\Omega$  for these norms.*

*Proof.* Consider arbitrary  $x, y \in \Omega$ . Then by direct computations:

$$\|S(x) - S(y)\| = \left\| \int_0^1 Df(\tau x) - Df(\tau y) d\tau \right\| \leq \int_0^1 \|Df(\tau x) - Df(\tau y)\| d\tau \quad (68)$$

$$\leq \int_0^1 L \|\tau x - \tau y\| d\tau = \frac{1}{2} L \|x - y\|. \quad (69)$$

The Lipschitz continuity can be used for the second inequality because  $\tau x, \tau y \in \Omega$  for all  $\tau \in [0, 1]$ , since  $\Omega$  is 0-stared. For the last equality, just note that  $\int_0^1 \tau d\tau = \frac{1}{2}$ .  $\square$

Second, a Lipschitz constant for  $Df$  and the  $\infty$ -norm is computed using bounds on the second derivatives of  $f$ . This second derivative based Lipschitz constant is classical, but its proof provided here for completeness.

**Lemma 5.** Let  $\Omega \subset \mathbb{R}^n$  be convex,  $g : \Omega \rightarrow \mathbb{R}$  be differentiable and  $\bar{g}_k \geq \max_{x \in \Omega} |\frac{\partial g}{\partial x_k}(x)|$ . Then for all  $x, y \in \Omega$ ,

$$|g(x) - g(y)| \leq \left( \sum_k \bar{g}_k \right) \|x - y\|_\infty. \quad (70)$$

*Proof.* Since  $\Omega$  is convex,  $x + \xi(y - x) \in \Omega$  for all  $\xi \in [0, 1]$ . Hence, the mean-value theorem applies and there exists  $\xi \in [0, 1]$  such that  $g(x) - g(y) = Dg(u)(x - y)$  with  $u = x + \xi(y - x) \in \Omega$ . Therefore,

$$|g(x) - g(y)| = \left| \sum_k \frac{\partial g}{\partial x_k}(u) (x_k - y_k) \right| \leq \sum_k \left| \frac{\partial g}{\partial x_k}(u) \right| |x_k - y_k| \leq \left( \sum_k \left| \frac{\partial g}{\partial x_k}(u) \right| \right) \|x - y\|_\infty. \quad (71)$$

Finally,  $u \in \Omega$  entails  $|\frac{\partial g}{\partial x_k}(u)| \leq \bar{g}_k$ .  $\square$

**Proposition 3.** Let  $\Omega \subseteq \mathbb{R}^n$  be convex,  $f : \Omega \rightarrow \mathbb{R}^n$  be twice differentiable, and  $\bar{f}_{ijk} \geq \max_{x \in \Omega} |\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x)|$ . Then,  $Df$  is  $L_\infty$  Lipschitz continuous for the operator and vector  $\infty$ -norms inside  $\Omega$  with

$$L_\infty := \max_i \sum_{jk} \bar{f}_{ijk}. \quad (72)$$

*Proof.* For some arbitrary  $x, y \in \Omega$ ,

$$\|Df(x) - Df(y)\|_\infty = \max_i \sum_j \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right|. \quad (73)$$

Using Lemma 5 with  $g(x) = \frac{\partial f_i}{\partial x_j}(x)$ , so  $\frac{\partial g}{\partial x_k}(x) = \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x)$  and  $\bar{g}_k := \bar{f}_{ijk}$  satisfies the hypothesis of the lemma, we obtain

$$\|Df(x) - Df(y)\|_\infty \leq \max_i \sum_{jk} \bar{f}_{ijk} \|x - y\|_\infty = \left( \max_i \sum_{jk} \bar{f}_{ijk} \right) \|x - y\|_\infty. \quad (74)$$

$\square$

The constants  $\bar{f}_{ijk}$  can be computed by evaluating the second derivatives of  $f$  using interval arithmetic. Finally, a Lipschitz constant for  $Df$  and the  $P$ -norm is computed from a Lipschitz constant for the  $\infty$ -norm.

**Proposition 4.** Suppose that  $Df$  is  $L_\infty$ -Lipschitz continuous inside the convex  $\Omega \subseteq \mathbb{R}^n$  for the vector and matrix  $\infty$ -norms, then it is  $L_P$ -Lipschitz continuous inside  $\Omega$  for the vector and matrix  $P$ -norms with

$$L_P := L_\infty \frac{\sqrt{n\bar{\lambda}}}{\lambda}. \quad (75)$$

*Proof.* For arbitrary  $x, y \in \Omega$ ,

$$\|Df(x) - Df(y)\|_P \leq \sqrt{\frac{n\bar{\lambda}}{\lambda}} \|Df(x) - Df(y)\|_\infty \leq L_\infty \sqrt{\frac{n\bar{\lambda}}{\lambda}} \|x - y\|_\infty \leq L_\infty \frac{\sqrt{n\bar{\lambda}}}{\lambda} \|x - y\|_P. \quad (76)$$

The first and last inequalities result of Lemma 4, the second from the Lipschitz continuity for  $\infty$ -norms.  $\square$

Finally,  $S(x)$  is proved to be  $L$ -Lipschitz continuous inside  $\Omega$  for the  $P$ -norm with

$$L := \frac{1}{2} \left( \max_i \sum_{jk} \bar{f}_{ijk} \right) \frac{\sqrt{n\bar{\lambda}}}{\lambda}. \quad (77)$$

Similarly, one can prove that  $S(x)^T$  is  $L'$ -Lipschitz continuous inside  $\Omega$  for the  $P$ -norm with

$$L' := \frac{1}{2} \left( \max_j \sum_{ik} \bar{f}_{ijk} \right) \frac{\sqrt{n\bar{\lambda}}}{\lambda}. \quad (78)$$