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ON A VARIANT OF SCHANUEL CONJECTURE FOR THE CARLITZ EXPONENTIAL

F. PELLARIN

ABSTRACT. We introduce and discuss a variant of Schanuel conjecture in the framework of the Carlitz exponential function over Tate algebras and allied functions. Another purpose of the present paper is to widen the horizons of possible investigations in transcendence and algebraic independence in positive characteristic.

1. INTRODUCTION

Schanuel's conjecture is an unproven statement predicting the behavior of the intersections with algebraic sub-varieties defined over \mathbb{Q} of a certain analytic subvariety of $\mathbb{G}_a^n(\mathbb{C}) \times \mathbb{G}_m^n(\mathbb{C})$ of dimension n , built on the graph of the classical exponential function. Somehow, the conjecture expects that these intersections behave in the simplest possible way:

Conjecture 1 (Schanuel). *Let u_1, \dots, u_n be complex numbers which are linearly independent over \mathbb{Q} . Then the transcendence degree over \mathbb{Q} of the subfield of \mathbb{C} :*

$$\mathbb{Q}(u_1, \dots, u_n, e^{u_1}, \dots, e^{u_n})$$

is $\geq n$.

This conjecture first appeared in print in Lang's book [17]. It is surprising to see such a syntactically simple statement governing an intricate constellation of results of independence of classical mathematical constants. We mention that the *Lindemann-Weierstrass Theorem* confirms Schanuel's conjecture in the case of u_1, \dots, u_n algebraic numbers, and can be seen as a particular case of Siegel-Shidlovski Theorem for the values of Siegel E -functions at algebraic numbers (see Beukers' [8]). Baker's Theorem on linear forms of logarithms of algebraic numbers, on the other hand, partially clarifies the case in which e^{u_1}, \dots, e^{u_n} are algebraic numbers. Outside these two cases, u_1, \dots, u_n algebraic or e^{u_1}, \dots, e^{u_n} algebraic, fragmentary information is available. Among those, a corollary of Nesterenko's Theorem asserts that the numbers

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π and e^π are algebraically independent over \mathbb{Q} . We do not give precise references for these statements. Instead, we refer to the survey of Waldschmidt [26] and its detailed bibliography. The paper of Scanlon [23] gives a nice introduction of related topics.

As a final, maybe less known and more recent consequence of Schanuel's conjecture, we also mention that in [19, Theorem 1.6], Marker deduces from the Schanuel conjecture that every non-zero exponential polynomial $P(x, e^x) \in \mathbb{Q}[x, e^x]$ such that $P(X, Y)$ depends on both X and Y , has infinitely many algebraically independent roots in \mathbb{C} . In particular, one deduces, from Schanuel's conjecture, that any finite set of distinct such fixed points are algebraically independent over \mathbb{Q} .

In the present paper, we shall discuss variants of another conjecture, very similar to Schanuel's conjecture in aspect, but in the framework of function fields of positive characteristic. Although at first sight, our interest in this framework may seem rather artificial (why looking at an analogue conjecture while we already have so many interesting open problems in the framework of the classical Schanuel conjecture?), we hope that the reader, after having looked into the present paper, will be finally convinced that some new structures emerge in this framework which would have remained hidden in the classical setting, more related to the theory of difference algebra and fields, perhaps giving a new view on the original Schanuel conjecture itself.

This text grew up from the tentative of the author to understand, in the viewpoint of the theory of transcendence and algebraic independence, the meaning of the fact that certain L -values introduced in [22] also behave as "Stark-Anderson units" and are thus sent to a polynomial by the Carlitz exponential function, extended over Tate algebras (see [3, 5, 6]).

1.1. Some background. We set $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$, where \mathbb{F}_q is the finite field with q elements and θ is an indeterminate. In all the following, we also denote by p the characteristic of \mathbb{F}_q so that $q = p^e$ for an integer $e > 0$. We denote by K_∞ the local field completion of K at the infinity place of K ; then K_∞ can be identified with the field of formal Laurent series $\mathbb{F}_q((1/\theta))$, with valuation v_∞ normalized by $v_\infty(\theta) = -1$. Let K_∞^{ac} be an algebraic closure of K_∞ . Then we denote by \mathbb{C}_∞ the completion of K_∞^{ac} for the unique extension of v_∞ . Note that \mathbb{C}_∞ is a K_∞^{ac} -vector space of infinite dimension, and that K_∞^{ac} is of infinite dimension over K_∞ . The field \mathbb{C}_∞ carries a unique extension

$$\mathbb{C}_\infty \xrightarrow{v_\infty} \mathbb{Q} \cup \{\infty\}$$

of the valuation v_∞ . Sometimes, we will also use the associated norm $|\cdot| = q^{-v_\infty(\cdot)}$. Just as the ring \mathbb{Z} is discrete and co-compact in \mathbb{R} , we have that the \mathbb{F}_q -algebra A is

discrete and co-compact in K_∞ ; note also that the infinity place is the only place of K with this property.

1.1.1. *The Carlitz module.* Let $\iota : A \rightarrow B$ be a commutative A -algebra. Then B is equipped with the \mathbb{F}_q -algebra endomorphism $\tau : B \rightarrow B$ which sends $b \in B$ to $b^q \in B$ (the raising to the power q is relative to the algebra structure of B and is \mathbb{F}_q -linear). Let $B\{\tau\}$ be the skew ring of finite sums $\sum_{i \geq 0} b_i \tau^i$ with $b_i \in B$ for all i , and product defined by the rule $\tau b = \tau(b)\tau$ for $b \in B$; the ring $B\{\tau\}$ acts on B , as well as on any B -algebra, by *evaluation*. If $f = \sum_i \tau^i \in B\{\tau\}$, then for all $b \in B$, we define the evaluation of f at b as:

$$f(b) := \sum_i f_i \tau^i(b) = \sum_i f_i b^{q^i}.$$

The *Carlitz module* $C(B)$ over B is by definition the A -module whose underlying \mathbb{F}_q -vector space is B , in which the multiplication by $\theta \in A$, sufficient to define the entire A -module structure, is given by the evaluation of the skew polynomial $\iota(\theta) + \tau$. We shall write $C_a(b)$ for the multiplication of an element b of B by $a \in A$. For example, we have

$$C_\theta(b) = \iota(\theta)b + \tau(b), \quad b \in B.$$

In particular, \mathbb{C}_∞ is an A -algebra with ι the identity map, and the above construction gives rise to the A -module $C(\mathbb{C}_\infty)$.

1.1.2. *The Carlitz exponential.* The Carlitz exponential

$$\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$$

allows to *analytically uniformize* $C(\mathbb{C}_\infty)$. In what concerns the foundation theory of this function, we are going to follow Goss' treatise [15] which can be consulted by any interested reader. We first remark, to define this function, that there is, available in the theory over A , an analogue of the sequence of numbers $n!$, defined as follows:

$$d_n = \prod_a a,$$

where the product runs over the monic polynomials a of A of degree n (this is, more properly, an analogue of the number $q^n!$). We set, for $z \in \mathbb{C}_\infty$,

$$\exp_C(z) := \sum_{n \geq 0} d_n^{-1} \tau^n(z) \in \mathbb{C}_\infty.$$

One verifies that $v_\infty(d_n) = nq^n$, from which we deduce that \exp_C is an entire (hence necessarily surjective by simple considerations of Newton polygons), \mathbb{F}_q -linear map

$\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$. Since it is entire, its kernel Λ , an \mathbb{F}_q -vector space, determines \exp_C uniquely, and we have the convergent Weierstrass product expansion

$$(1) \quad \exp_C(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}_\infty.$$

This function \exp_C in fact is the unique entire \mathbb{F}_q -linear map $\mathbb{C}_\infty \xrightarrow{F} \mathbb{C}_\infty$ which satisfies $F' = 1$, and which induces an exact sequence of A -modules:

$$0 \rightarrow \Lambda \rightarrow \mathbb{C}_\infty \rightarrow C(\mathbb{C}_\infty) \rightarrow 0,$$

where $\Lambda \subset \mathbb{C}_\infty$ is an A -sub-module of \mathbb{C}_∞ . The study of the Newton polygon of \exp_C implies that Λ is of rank one. In particular, there exists an element $\tilde{\pi} \in \mathbb{C}_\infty^\times$ such that $\Lambda = \tilde{\pi}A$; it is defined up to multiplication by an element of $\mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$. Note that for all $a \in A$ and $z \in \mathbb{C}_\infty$,

$$C_a(\exp_C(z)) = \exp_C(az),$$

hence providing an analytic isomorphism of A -modules

$$C(\mathbb{C}_\infty) \cong \frac{\mathbb{C}_\infty}{\tilde{\pi}A}.$$

The element $\tilde{\pi} \in \mathbb{C}_\infty$ can be constructed explicitly by limit processes in several ways. For instance, we recall from [15] that $\tilde{\pi}$ is the value in \mathbb{C}_∞ of a convergent infinite product

$$(2) \quad \tilde{\pi} := -(-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty,$$

uniquely defined up to the choice of a root $(-\theta)^{\frac{1}{q-1}}$. It has been proved in a variety of ways (see [21] for a survey) that $\tilde{\pi}$ is moreover transcendental over K ; the first proof of which was obtained by Wade in [25].

2. A CARLITZIAN ANALOGUE OF SCHANUEL'S CONJECTURE

The following conjecture is due to Laurent Denis (see [13]).

Conjecture 2 (Denis). *Let u_1, \dots, u_n be elements of \mathbb{C}_∞ which are A -linearly independent. Then the transcendence degree over K of the sub-field of \mathbb{C}_∞*

$$K(u_1, \dots, u_n, \exp_C(u_1), \dots, \exp_C(u_n))$$

is $\geq n$.

The case in which u_1, \dots, u_n are algebraic over K has been solved by Thiery in [24] and can be viewed as the analogue of Lindemann-Weierstrass theorem for the Carlitz exponential. Note that the analogue for the Carlitz exponential of the Theorem of Hermite-Lindemann already appears in Wade [25] (see also [21, Theorem 2.5]). In the arithmetic theory of function fields in positive characteristic, what it is usually called the "analogue of Hermite-Lindemann Theorem" is a corollary of a very general transcendence result by Yu in [27] (see also [21, Theorem 2.2]). Years later, Papanikolas solved, in [20], the case in which $\exp_C(u_1), \dots, \exp_C(u_n)$ are algebraic over K by using the "criterion of linear independence of Anderson-Brownawell-Papanikolas". Denis also obtained [14], at the same time as Papanikolas, a partial result in this direction, by using a characteristic p variant of "Mahler's method". Additionally, in [13, Corollaire 2 (a)], Denis proved that $\tilde{\pi}$ and $\tilde{e} := \exp_C(1)$ are algebraically independent over K if $q \geq 3$ (see [22] for an overview of these results).

2.1. Digression: strengthening. We mention, briefly, a natural way to reinforce Conjecture 2, but this has to be considered as a digression, since the nature of the statements we are primarily interested in, is different. Denis, in [13], proposes a strengthening of Conjecture 2 in order to give an interpretation of a multitude of results of algebraic independence he was obtaining, not only for special constants related to the Carlitz exponential, but also for their derivatives in the variable θ . Although not central for our further investigations, these strengthenings tell us that in the present settings (over \mathbb{C}_∞), there are more apparent structures that there seems to be over \mathbb{C} , and this allows us to propose more elaborate statements.

Denote by $\exp_C^{(i)} \in K[[z]]$ the i -th higher derivative of $\exp_C \in K[[z]]$ with respect to θ (note indeed that the formal series \exp_C can also be viewed as a double formal series in $\mathbb{F}_q[[\theta^{-1}, z]]$); it is an \mathbb{F}_q -linear entire function. We recall that p is the prime dividing q , that is, the characteristic of \mathbb{F}_p .

Conjecture 3 (Denis). *Let $u_1, \dots, u_n \in \mathbb{C}_\infty$ be elements which are A -linearly independent. Then the transcendence degree over K of the sub-field of \mathbb{C}_∞*

$$K(u_1, \dots, u_n, \exp_C(u_1), \dots, \exp_C(u_n), \dots, \exp_C^{(p-1)}(u_1), \dots, \exp_C^{(p-1)}(u_n))$$

is $\geq pn$.

In fact, the original statement in [13] was for $u_1, \dots, u_n \in K_\infty^{ac}$. However, we do not see any reason for which Conjecture 3 should fail for $u_i \in \mathbb{C}_\infty \setminus K_\infty^{ac}$ for some i . This conjecture obviously implies Conjecture 2.

3. THE BASIC SETTINGS FOR OUR OPERATOR-THEORETIC CONJECTURE

In the following, we will keep focusing on statements similar to Conjecture 2. We want to formulate - this is the aim of the present paper - an *operator-theoretic conjecture* (see Conjecture 18 below) implying Conjecture 2. In order to do so, we first need to introduce certain difference Banach algebras. In §3.2, we will extend the Carlitz exponential to these algebras. This will give rise to several notions of independence, generalizing algebraic independence over K essential for our statement, and studied in §3.4. The statement of our conjecture will appear in §3.5 and we will give examples of application in §4.

We shall start with the algebra $\mathbb{C}_\infty[t_1, \dots, t_s]$ for variables t_1, \dots, t_s . If $q = p^e$ for an integer $e > 0$, then we set $\tau = \mu^e$ which is $\mathbb{F}_q[t_1, \dots, t_s]$ -linear. It is customary at this point, to take, when it is well defined, the completion of the above \mathbb{C}_∞ -algebra for the unique extension of the valuation v_∞ which is trivial over $\mathbb{F}_q[t_1, \dots, t_s]$, giving rise to a *Tate algebra*. Observe that if $s = 0$, then we just have the field \mathbb{C}_∞ . Let

$$\mathbb{C}_\infty[t_1, \dots, t_s] \xrightarrow{v_\infty} \mathbb{Q} \cup \{\infty\}$$

be the unique extension of the valuation v_∞ over \mathbb{C}_∞ which is trivial over $\mathbb{F}_q[t_1, \dots, t_s]$. The completion of $\mathbb{C}_\infty[t_1, \dots, t_s]$ with respect to this valuation is the standard *s-dimensional Tate algebra* denoted by \mathbb{T}_s in all the following. It is also called, in several papers, the *free s-dimensional affinoid algebra over \mathbb{C}_∞* . It is well known that \mathbb{T}_s is a ring which is Noetherian, factorial, of Krull dimension s (see [9] for the general theory of these algebras). It is isomorphic to the \mathbb{C}_∞ -algebra of the formal series

$$f = \sum_{i_1, \dots, i_s \geq 0} f_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s} \in \mathbb{C}_\infty[[t_1, \dots, t_s]]$$

which satisfy

$$\lim_{\min\{i_1, \dots, i_s\} \rightarrow \infty} f_{i_1, \dots, i_s} = 0.$$

Thus, we have, for f a formal series of \mathbb{T}_s expanded as above, and non-zero, that

$$v_\infty(f) = \inf_{i_1, \dots, i_s} v_\infty(f_{i_1, \dots, i_s}) = \min_{i_1, \dots, i_s} v_\infty(f_{i_1, \dots, i_s}).$$

We also set, for convenience,

$$\|\cdot\| := q^{-v_\infty(\cdot)}$$

and $0 = \|0\| = q^{-\infty}$. In all the following, we are going to view the algebras \mathbb{T}_s as one embedded in the other, so that

$$\mathbb{C}_\infty = \mathbb{T}_0 \subset \mathbb{T}_1 \subset \cdots \subset \mathbb{T}_s \subset \cdots .$$

It is easy to see that, for all $i > 0$,

$$\mathbb{T}_i = \left\{ f = \sum_{j \geq 0} f_j t_i^j; f_j \in \mathbb{T}_{i-1}, v_\infty(f_j) \rightarrow \infty \right\},$$

where v_∞ denotes here the Gauss valuation extending the valuation of \mathbb{C}_∞ over \mathbb{T}_{i-1} .

Then the map $\mathbb{C}_\infty \xrightarrow{\mu} \mathbb{C}_\infty$, $x \mapsto \mu(x) = x^p$ extends, uniquely, to a continuous, open $\mathbb{F}_p[t_1, \dots, t_s]$ -linear automorphism of \mathbb{T}_s (for all s) such that, for all $f \in \mathbb{T}_s$, we have that

$$v_\infty(\mu(f)) = p v_\infty(f).$$

It is also easy to prove that the subring $\mathbb{T}_s^{\mu=1} = \{f \in \mathbb{T}_s; \mu(f) = f\}$ is equal to $\mathbb{F}_p[t_1, \dots, t_s]$ (all these properties are proved in detail in [5]).

3.1. Some complete difference fields. It is suitable to also have, at hand, complete difference fields containing \mathbb{T}_s , not just difference algebras. Let L be any commutative field. We denote by $L\langle\langle\mathbb{Q}\rangle\rangle_\infty$ the set of formal series

$$\sum_{i \in \mathcal{I}} c_i \theta^{-i}, \quad c_i \in L,$$

where $\mathcal{I} \subset \mathbb{Q}$ is a *well-ordered* subset, that is, any non-empty subset has a minimum element. Then with the natural valuation v_∞ trivial over L and such that $v_\infty(\theta) = -1$, it is well known that $L\langle\langle\mathbb{Q}\rangle\rangle_\infty$ is a valued field which is complete and has residual field L (it is in fact henselian). Moreover, it has no proper immediate extensions and, if L is algebraically closed, then $L\langle\langle\mathbb{Q}\rangle\rangle_\infty$ is algebraically closed. Since the residue field of \mathbb{C}_∞ , \mathbb{F}_q^{ac} , is algebraically closed, there is an isometric map of \mathbb{C}_∞ in $\mathbb{F}_q^{ac}\langle\langle\mathbb{Q}\rangle\rangle_\infty$, and therefore, if $L = \mathbb{F}_q^{ac}(t_1, \dots, t_s)$, then $\mathbb{C}_\infty(t_1, \dots, t_s)$ maps isometrically in

$$\mathbb{K}_s := \mathbb{F}_q^{ac}(t_1, \dots, t_s)\langle\langle\mathbb{Q}\rangle\rangle_\infty.$$

The complete field \mathbb{K}_s has thus residue field $\mathbb{F}_q^{ac}(t_1, \dots, t_s)$ and valuation group \mathbb{Q} . In particular, there is an isometric map from the completion of the fraction field of \mathbb{T}_s in \mathbb{K}_s .

The map $\mu : \mathbb{K}_s \rightarrow \mathbb{K}_s$ which sends an element $x = \sum_{i \in \mathcal{I}} c_i \theta^{-i} \in \mathbb{K}_s$ to the element $\mu(x) = \sum_{i \in \mathcal{I}} \mu(c_i) \theta^{-pi}$ extends the previously introduced $\mathbb{F}_p[t_1, \dots, t_s]$ -linear automorphism of \mathbb{T}_s (seen as embedded in \mathbb{K}_s as indicated above) is a continuous, open $\mathbb{F}_q(t_1, \dots, t_s)$ -linear automorphism. Further, we have the identity

$$\mathbb{K}_s^{\tau=1} = \{f \in \mathbb{K}_s : \mu(f) = f\} = \mathbb{F}_p(t_1, \dots, t_s)$$

which follows directly from the definition of the field \mathbb{K}_s .

3.2. Extensions of the Carlitz exponential function. We briefly recall some basic facts about the Carlitz exponential over Tate algebras, from [5]. We can construct a continuous, open $\mathbb{F}_q[t_1, \dots, t_s]$ -linear endomorphism

$$\mathbb{T}_s \xrightarrow{\exp_C} \mathbb{T}_s,$$

by setting

$$\exp_C(f) = \sum_{i \geq 0} d_i^{-1} \tau^i(f) = \sum_{i \geq 0} d_i^{-1} \mu^{ei}(f), \quad f \in \mathbb{T}_s.$$

Note that the restriction of \exp_C to the subring $\mathbb{T}_0 = \mathbb{C}_\infty \cong \mathbb{C}_\infty.1 \subset \mathbb{T}_s$ returns the Carlitz exponential function defined in §1.1.2. Following the arguments of §1.1.1, we endow \mathbb{T}_s with the structure of an $A[t_1, \dots, t_s]$ -module $C(\mathbb{T}_s)$ in the following way. The underlying $\mathbb{F}_q[t_1, \dots, t_s]$ -module is just that of \mathbb{T}_s . Moreover, the multiplication by $\theta \in A$ is given by $C_\theta = \theta + \tau$ and all this produces a structure of $A[t_1, \dots, t_s]$ -module in an unique way.

Example. To give a concrete example of how this $A[t_1, \dots, t_s]$ -module structure works, let us suppose that $s = 1$. In this case, we more simply write $t = t_1$ and $\mathbb{T} = \mathbb{T}_1$. Let us consider $f = t_1 - \theta = t - \theta$, which belongs to $A[t] \subset \mathbb{T}$. Then $\tau(f) = t - \theta^q$ and

$$C_\theta(f) = t(\theta + 1) - (\theta^2 + \theta^q).$$

The following result is proved in [4].

Proposition 4. *The map \exp_C induces an exact sequence of $A[t_1, \dots, t_s]$ -modules:*

$$(3) \quad 0 \rightarrow \tilde{\pi}A[t_1, \dots, t_s] \rightarrow \mathbb{T}_s \xrightarrow{\exp_C} C(\mathbb{T}_s) \rightarrow 0.$$

Remark 5. For $s = 0$, we have seen that \exp_C defines an entire function $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$. In particular, \exp_C , as an entire function, is uniquely defined by the divisor of its zeroes and has the Weierstrass product expansion (1) with $\Lambda = \tilde{\pi}A$. However, for $s > 0$, the extension of \exp_C to the Tate algebra \mathbb{T}_s that we have defined above, is no longer entire. For example, it has no Weierstrass product expansion over \mathbb{T}_s in contrast with the case $s = 0$.

More generally, we have the following.

Proposition 6. *The Carlitz exponential \exp_C gives rise to an exact sequence of $\mathbb{F}_q(t_1, \dots, t_s)[\theta]$ -modules:*

$$0 \rightarrow \tilde{\pi}\mathbb{F}_q(t_1, \dots, t_s)[\theta] \rightarrow \mathbb{K}_s \xrightarrow{\exp_C} C(\mathbb{K}_s) \rightarrow 0.$$

Observe that the $\mathbb{F}_q(t_1, \dots, t_s)[\theta]$ -module $C(\mathbb{K}_s)$ is well defined; the multiplication of $f \in \mathbb{K}_s$ by θ for this module structure is $C_\theta(f) = \theta f + \tau(f) = \theta f + \mu^e(f)$, and all the operators are extended $\mathbb{F}_q(t_1, \dots, t_s)$ -linearly. Restricted over the image of \mathbb{T}_s in \mathbb{K}_s , this exact sequence gives back the exact sequence (3).

Proof of Proposition 6. Since \mathbb{K}_s is complete, \exp_C is well defined. It is easy to see, writing down explicit generalized series of \mathbb{K}_s , that if $g \in \mathbb{K}_s$, then there exists a solution $f \in \mathbb{K}_s$ of the equation $C_\theta(f) = \theta f + \tau(f) = g$. This means that $C(\mathbb{K}_s)$ is θ -divisible and we can construct a continuous section of \exp_C ; this implies that \exp_C is surjective. Now, it is immediate that $\mathbb{F}_q(t_1, \dots, t_s)[\theta]\tilde{\pi} \subset \text{Ker}(\exp_C)$. But then the Newton polygon of $\sum_{i \geq 0} d_i^{-1} Z^{q^i} \in K_\infty[[Z]]$, the $\mathbb{F}_q(t_1, \dots, t_s)$ -linearity of our map and the fact that it gives rise of an isometric $\mathbb{F}_q(t_1, \dots, t_s)$ -linear isomorphism of the disk $\{f \in \mathbb{K}_s : v_\infty(f) > -\frac{q}{q-1}\}$ tells us that the above is precisely the kernel of \exp_C . \square

3.3. A first statement. We set

$$K_s = \mathbb{F}_p(\theta, t_1, \dots, t_s).$$

This field will play the role of the field $K_0 = \mathbb{F}_p(\theta)$ (note that $K = K_0$ if and only if $q = p$; we hope that the fact that the notations K and K_s are so similar will not confuse the reader). In the settings of §3.2, it is natural to state the following conjecture, which generalizes Denis' conjecture 2:

Conjecture 7. *Let u_1, \dots, u_n be elements of \mathbb{K}_s which are $A[t_1, \dots, t_s]$ -linearly independent. Then the transcendence degree over K_s of the sub-field of \mathbb{K}_s*

$$K_s(u_1, \dots, u_n, \exp_C(u_1), \dots, \exp_C(u_n))$$

is $\geq n$.

We now explain why the above statement does not look so interesting; we set $s = 1$. The *Anderson-Thakur function* can be defined as the element

$$\omega = \exp_C \left(\frac{\tilde{\pi}}{\theta - t} \right) \in \mathbb{T}^\times.$$

It is the generator of the $\mathbb{F}_q[t]$ -module $\text{Ker}(C_{\theta-t}) \cap \mathbb{T}$, free of rank one (see [4]). Since

$$(4) \quad \tau(\omega) = (t - \theta)\omega$$

(this is equivalent to saying that $\omega \in \text{Ker}(C_{\theta-t})$), we also deduce the next Proposition in quite an elementary way (see [4] for more details).

Proposition 8. *The following properties hold:*

(1) *We have the product expansion*

$$\omega = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}} \right)^{-1},$$

convergent in \mathbb{T}^\times .

- (2) ω , as an element of \mathbb{T} , extends to a meromorphic function over \mathbb{C}_∞ and has, as unique singularities, simple poles at the points $t = \theta, \theta^q, \theta^{q^2}, \dots$. The residues can be explicitly computed. In particular, we have $\text{Res}_{t=\theta}(\omega) = -\tilde{\pi}$.
- (3) The function $1/\omega$ extends to an entire function $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ with unique zeros located at the poles of ω .

Since ω has infinitely many poles, it is transcendental over $K(t)$ and it is easy to see that $\tilde{\pi}, \omega$ are algebraically independent over $K(t)$, but the conjecture 7 in the case $n = 1$ and $s = 1$ only implies that one among the two elements $\tilde{\pi}$ and ω is transcendental.

In fact, ω is transcendental over $K(t)$ but is also "nearly algebraic"; it satisfies the difference equation (4) which is a kind of algebraic relation involving τ , even though this is not an algebraic morphism. What is missing to Conjecture 7 is the sensitiveness to such difference equations, and this is what we want to view now.

3.4. A suitable notion of independence. Before presenting our operator-theoretic generalization of Denis-Schanuel's conjecture 3 and of Conjecture 7 that will take place in the valued field \mathbb{K}_s . We need to describe the kind of relations which are relevant for it.

We recall Conjectures 1, 3 and 7. In the simplest portrait, we have a complete environment field (\mathbb{C} in the first conjecture, \mathbb{C}_∞ in the second conjecture and \mathbb{K}_s in the third) a discrete subfield of coefficients (\mathbb{Q} in the first conjecture, K in the second conjecture and K_s in the third). Moreover, we have notions of linear relation (over \mathbb{Z} , A and $A[t_1, \dots, t_s]$) and of independence (over the fraction fields of these rings). Our statement will take place in the environment field \mathbb{K}_s , concerns the extension of the Carlitz exponential discussed in the previous section, and will vaguely sound as follows, with a (momentarily) non-specified discrete subfield of coefficients $L_s \subset \mathbb{K}_s$ and a non-specified notion of dependence over L_s :

Conjecture (Prototype) *Let f_1, \dots, f_n be elements of \mathbb{K}_s which are $\mathbb{F}_q(t_1, \dots, t_s)[\theta]$ -linearly independent. Then, among the $2n$ elements f_1, \dots, f_n and $\exp_C(f_1), \dots, \exp_C(f_n)$, n are "independent" over L_s .*

Of course, since our conjecture is tailored to imply Conjectures 3 and 7, we want that our relations over L_s extend algebraic dependence over $K = \mathbb{F}_q(\theta)$, which is equivalent to algebraic dependence over $\mathbb{F}_p(\theta)$. Now, we are going to suppose that $s \geq 1$. In our μ -difference settings, the difference field extension (\mathbb{K}_s, μ) of the difference field (K_s, μ) , there are several noticeable classical notions of independence over K_s but we will see soon that unfortunately, they do not seem to be suitable for our purposes.

3.4.1. *Transformatal independence.* The first notion we want to discuss is that of *transformatal independence* over K_s (read Levin, [18, §2.2]); as announced, it will turn out to be inappropriate.

Let L be a μ -difference subfield of \mathbb{K}_s containing K_s .

Definition 9. Elements $f_1, \dots, f_n \in \mathbb{K}_s$ are *transformatally μ -independent* over L if the elements

$$f_1, \mu(f_1), \dots, f_2, \mu(f_2), \dots, \dots, f_n, \mu(f_n), \dots$$

are algebraically independent over L ; the notions of *transformatal μ -dependence*, *transformatal μ -algebraicity* and *transformatal μ -transcendence* can be defined accordingly (see [18]). This also leads to a notion of *transformatal μ -independence degree*. Let f_1, \dots, f_n be elements of \mathbb{K}_s and let us denote by:

$$\mathcal{L} = L(f_1, \dots, f_n)_\mu$$

the smallest μ -difference subfield of \mathbb{K}_s containing L and f_1, \dots, f_n . Then the transformatal μ -independence degree of \mathcal{L} over L

$$\text{transf deg}_L(\mathcal{L})$$

is by definition the minimal cardinality of a transformatally μ -independent subset of \mathcal{L} over L . If f_1, \dots, f_n are transformatally μ -independent over L , then

$$\text{transf deg}_L(\mathcal{L}) = n.$$

The notions above are not tailored for the statement we want to produce. To illustrate this, we first recall that the generalization of Conjecture 2 (see Conjecture 18) that we have in mind is of arithmetic nature, and is similar to [7, Hypothesis (Σ) p. 252], which is equivalent to the Schanuel conjecture itself (see *ibid.*, Theorem 2). There, a discrete base subfield is required (the field \mathbb{Q}).

Now, observe that since $c^p = \mu(c)$ for $c \in \mathbb{C}_\infty$, every element of \mathbb{C}_∞ is transformatally μ -algebraic over K , independently of the fact that it is algebraic over K or not. Worse, for all $c \in \mathbb{C}_\infty$ and for all $f \in K_s$, the element $g = cf \in \mathbb{C}_\infty(t_1, \dots, t_s)$ is transformatally μ -algebraic. Indeed, $\mu(f)g^p = f^p\mu(g)$. We deduce from [18, Theorem 4.1.2] that every element of $\mathbb{C}_\infty(t_1, \dots, t_s)$ is transformatally μ -algebraic, no matter if the coefficients of the rational fractions are algebraic or not. In particular, the subfield generated by the elements $x \in \mathbb{K}_s$ which are transformatally μ -algebraic over K_s is not countable.

For statements of arithmetic nature such as Schanuel's Conjecture ⁽¹⁾, it is natural to have a countable base field; this means that the notion of transformatal μ -independence of Levin is not suitable in our framework. We now make an attempt to give a more refined notion of relation.

¹Observe that we are not considering here, statements such as [7, (SP)].

3.4.2. *Analytically critical μ -polynomials.* It is likely that the obstruction above is limited to the mere existence of the vanishing of the \mathbb{C}_∞ -linear map

$$x \mapsto x^p - \mu(x), \quad x \in \mathbb{K}_s$$

over \mathbb{C}_∞ , which is not identically zero on \mathbb{K}_s . Since we presently do not have a proof that this property is true, we now introduce the notion of analytically critical μ -polynomial. We choose a finite set of symbols

$$\underline{X} = (X_1, \dots, X_n)$$

and we introduce infinitely many indeterminates

$$\mu^0(X_i), \mu^1(X_i), \mu^2(X_i), \dots, \quad i = 1, \dots, n.$$

To simplify the notation and to make it more natural, we shall write $X_i = \mu^0(X_i)$ and $\mu(X_i) = \mu^1(X_i)$. We introduce the ring in infinitely many indeterminates

$$L[\underline{X}]_\mu = L[X_i, \mu(X_i), \mu^2(X_i), \dots : i = 1, \dots, n],$$

that we turn into a μ -difference ring by setting

$$\mu(\mu^j(X_i)) = \mu^{j+1}(X_i).$$

The elements of $L[\underline{X}]_\mu$ are called the μ -polynomials in the symbols \underline{X} with coefficients in L . Any element $P \in L[\underline{X}]_\mu$ can be written in a unique way as

$$(5) \quad P(\underline{X}) = \sum_{\underline{i} \in \mathbb{N}^{(k+1)n}} c_{\underline{i}} X_1^{i_{1,0}} \mu(X_1)^{i_{1,1}} \dots \mu^k(X_1)^{i_{1,k}} \dots X_n^{i_{n,0}} \mu(X_n)^{i_{n,1}} \dots \mu^k(X_n)^{i_{n,k}}, \quad c_{\underline{i}} \in L,$$

and the smallest $k \in \mathbb{N}$ for which the expansion (5) holds is called the *depth* of P . Let P be in $L[\underline{X}]_\mu$ as in (5) and let $\underline{f} = (f_1, \dots, f_n)$ be an element of \mathbb{K}_s^n . Then the *evaluation of P at \underline{f}* is the element of \mathbb{K}_s :

$$P(\underline{f}) := \sum_{\underline{i} \in \mathbb{N}^{(k+1)n}} c_{\underline{i}} f_1^{i_{1,0}} \mu(f_1)^{i_{1,1}} \dots \mu^k(f_1)^{i_{1,k}} \dots f_n^{i_{n,0}} \mu(f_n)^{i_{n,1}} \dots \mu^k(f_n)^{i_{n,k}}.$$

If $P(\underline{f}) = 0$, we say that f is a *zero* or a *root* of P . We choose a μ -difference subfield L' of \mathbb{K}_s containing the coefficients of a μ -polynomial P . The μ -variety of zeroes of P in L' is the subset of $\underline{f} \in L'^n$ such that $P(\underline{f}) = 0$. It is denoted by $Z_{L'}(P)$ or more simply, by $Z(P)$ when $L' = \mathbb{K}_s$. We will not need this in the sequel, but we mention that, through the notion of μ -closed subset, it is possible to endow $Z_{L'}(P)$ and more generally, μ -varieties defined for example by the simultaneous vanishing of a collection of μ -polynomials, with a topology which is finer than the Zariski topology.

Definition 10. We set $D := D(0, 1) = \{z \in \mathbb{C}_\infty : v_\infty(z) \geq 0\}$. A *critical map* F of D^n in \mathbb{K}_s^n is the datum of an n -tuple of elements $\underline{F} = (F_1, \dots, F_n)$ with

$$F_i = F_i(t_1, \dots, t_s, z_1, \dots, z_n) = \sum_{\underline{k} \geq \underline{0}} F_{i, \underline{k}} z_1^{k_1} \dots z_n^{k_n} \in \mathbb{K}_s \widehat{\otimes}_{\mathbb{C}_\infty} \mathbb{T}_n, F_{i, \underline{k}} \in \mathbb{K}_s, i = 1, \dots, n$$

(the copy of \mathbb{T}_n above is a \mathbb{C}_∞ -algebra of functions in the independent variables z_1, \dots, z_n and $\underline{k} \geq \underline{0}$ means $k_i \geq 0$ for all i) such that the Jacobian

$$J_{\underline{z}}(\underline{F}) = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \dots & \frac{\partial F_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial z_1} & \dots & \frac{\partial F_n}{\partial z_n} \end{pmatrix}$$

has maximal rank in D^n . Since F , as a function of z_1, \dots, z_n is then locally injective everywhere in D^n , the image of a critical map can be seen as a copy of the polydisk D^n in \mathbb{K}_s^n . Let P be a non-zero element of $\mathbb{K}_s[\underline{X}]_\mu$. We say that it is *analytically critical* if there exists a critical map

$$D^n \rightarrow Z(P) \subset \mathbb{K}_s^n.$$

3.4.3. Examples of analytically critical μ -polynomials. Any polynomial $P \in \mathbb{K}_s[\underline{X}]_\mu$ which is multi-homogeneous for the unique multi-graduation which assigns to $\mu^i(X_j)$ the degree p^i is analytically critical, as soon as it has a root $\underline{f} \in (\mathbb{K}_s \setminus \{0\})^n$. Indeed, if $\underline{f} = (f_1, \dots, f_n) \in \mathbb{K}_s^n$ is a root, then also $(c_1 f_1, \dots, c_n f_n)$ for all $c_1, \dots, c_n \in \mathbb{C}_\infty$ is a root, due to the fact that $c^p = \mu(c)$ if $c \in \mathbb{C}_\infty$.

There are analytically critical μ -polynomials which are not homogeneous for the above graduations; we shall see them in the case $n = 1$ ($X = X_1$) but the example is easily generalizable to $\underline{X} = (X_1, \dots, X_n)$ with $n > 1$. We denote by \mathfrak{A} the abelian group $(\mathbb{K}_s[X]_\mu, +)$ with the structure of a non-commutative \mathbb{K}_s -algebra given by the composition \circ as a multiplication. Let $B_1, B_2 \in \mathfrak{A}$ be such that $B_1(0) = 0$ and $B_2(\mathbb{K}_s) \supset D$. Then $B_1 \circ P \circ B_2 \in \mathfrak{A}$ is analytically critical, as the reader can easily see.

Remark 11. We expect that there are no other ways to construct analytically critical μ -polynomials. It is plausible that, in the case $n = 1$, any μ -polynomial P such that $Z(P)$ is discrete ⁽²⁾ is analytically non-critical. Observe that the set of zeroes of a non-constant polynomial in one variable is discrete (indeed it is finite). Note also that the set of zeroes of an analytically critical polynomial (case $n = 1$) is obviously not discrete.

²That is, such that any convergent subsequence is ultimately constant.

3.4.4. *Examples of analytically non-critical μ -polynomials.* We start with the next Lemma.

Lemma 12. *Any non-zero polynomial $P \in \mathbb{K}_s[\underline{X}]$ is analytically non-critical.*

Proof. It is easy to see, by the Jacobian criterion, that the components of any critical map are algebraically independent over \mathbb{K}_s . \square

A homogeneous linear μ -difference equation of order one with coefficients in \mathbb{K}_s is analytically non-critical. Indeed, either this equation has no non-zero solutions, or the set of solutions is a $\mathbb{K}_s^{\mu=1}$ -vector space of dimension one. But $\mathbb{K}_s^{\mu=1} = \mathbb{F}_p(t_1, \dots, t_s)$, although possibly infinite (case $s > 0$), is nevertheless discrete in \mathbb{K}_s . The same holds for linear μ -difference equations of any order. More generally, the next definition provides a larger class of analytically non-critical μ -polynomials.

Definition 13. Let $P \in \mathbb{K}_s[\underline{X}]_\mu$ be as in (5). We say that it is *tame* if $c_{\underline{i}} \neq 0$ implies that, writing $\underline{i} = (i_{l,j})_{\substack{0 \leq l \leq n \\ 0 \leq j \leq k}}$, we have that $0 \leq i_{l,j} < p$ for all l, j .

In other, more loosely words, a tame μ -polynomial is one such that no exponent exceeds, or is equal to p .

Lemma 14. *Every non-zero tame μ -polynomial is analytically non-critical.*

Proof. We suppose that $\underline{X} = (X_1, \dots, X_n)$. Let $P \in \mathbb{K}_s[\underline{X}]_\mu$ be a non-zero tame μ -polynomial with μ -expansion as in (5). Let us suppose by contradiction that it is analytically critical. Then there is a critical map

$$D^n \xrightarrow{F=(F_1, \dots, F_n)} Z(P).$$

Since the Tate algebra \mathbb{T}_n in the variables z_1, \dots, z_n is stable for the partial derivatives $\frac{\partial}{\partial z_i}$ for all i , the map $\underline{z} \mapsto P(\underline{F}(\underline{z}))$ is differentiable in each variable z_i (in the polydisk D^n). By the fact that

$$\frac{\partial}{\partial z_j}(\mu^l(X_k)^i(\underline{F}(\underline{z}))) = 0, \quad j = 1, \dots, n, \quad i, l > 0$$

and by the usual chain rule, we have that

$$\underline{0} = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right) P(\underline{F}(\underline{z})) = \left(\frac{\partial P}{\partial z_1}(\underline{F}(\underline{z})), \dots, \frac{\partial P}{\partial z_n}(\underline{F}(\underline{z})) \right) \cdot J_{\underline{z}}(\underline{F}).$$

Over D^n , $J_{\underline{z}}(\underline{F})$ is invertible and we deduce that

$$\frac{\partial P}{\partial z_1}(\underline{F}(\underline{z})) = \dots = \frac{\partial P}{\partial z_n}(\underline{F}(\underline{z})) = 0.$$

Let k be the depth of P . We can assume, without loss of generality, that there exists i with $1 \leq i \leq n$ such that $P \neq \mu(Q)$ for $Q \in \mathbb{K}_s[\underline{X}]_\mu$. Observe that $0 < \deg_{X_i}(P) < p$.

Then we have proved that there exists i such that $\frac{\partial P}{\partial X_i} \neq 0$, but $\frac{\partial P}{\partial X_i}(F(\underline{z})) = 0$ for $\underline{z} \in D^n$. We have constructed a non-zero tame μ -polynomial \tilde{P} of depth k , of degree in X_i strictly less than that of P , and which is analytically critical. Iterating this process (i.e. replacing P with \tilde{P} various times) we are thus led to a non-zero tame μ -polynomial P which also is analytically critical, of zero degree in X_i . There exists $Q \in \mathbb{K}_s[\underline{X}]_\mu$ tame of depth $< k$ and analytically critical, and there exists $m > 0$ with $\mu^m(Q) = P$. This descent on degrees in X_i and depth brings us to a non-zero analytically critical constant (of course tame) polynomial, which is impossible. \square

Remark 15. Note that a polynomial $P \in \mathbb{K}_s[\underline{X}]$, that is, a μ -polynomial of depth zero, needs not to be tame. Also, the product of two tame μ -polynomials needs not to be tame; however, there exists a natural \mathbb{K}_s -algebra structure over the \mathbb{K}_s -vector space of tame μ -polynomials in $\mathbb{K}_s[\underline{X}]_\mu$ which makes it isomorphic to $\mathbb{K}_s[\underline{X}]$. See the appendix §5.3 with the details of the construction (it will not be used in this paper). Note also that the maps $\mathbb{C}_\infty^n \rightarrow \mathbb{K}_s$ that tame polynomials define, are in natural \mathbb{K}_s -linear isomorphism with the polynomial maps $\mathbb{C}_\infty^n \rightarrow \mathbb{K}_s$ induced by polynomials of $\mathbb{K}_s[\underline{X}]$.

Definition 16. Let L be a μ -difference subfield of \mathbb{K}_s containing K_s . Let f_1, \dots, f_n be elements of \mathbb{K}_s . We say that they are *non-critically μ -independent over L* if for all $P \in L[\underline{X}]_\mu$ analytically non-critical (hence non-zero), $P(\underline{f}) \neq 0$, where $\underline{f} = (f_1, \dots, f_n)$. We say that f_1, \dots, f_n are *tamely μ -independent over L* if the only tame μ -polynomial $P \in L[\underline{X}]_\mu$ such that $P(\underline{f}) = 0$ is the zero polynomial.

If f_1, \dots, f_n are tamely μ -dependent, then they are non-critically μ -dependent. Also, if f_1, \dots, f_n are non-critically μ -dependent, then they are transformally μ -dependent in the sense of Levin [18]. But the reverse implications are all false. We collect a few more properties in the following proposition.

Proposition 17. *Let f_1, \dots, f_n be elements of \mathbb{K}_s . We have:*

- (1) *If f_1, \dots, f_n are non-critically independent over L , then they are algebraically independent over L .*
- (2) *If f_1, \dots, f_n are in \mathbb{C}_∞ and are algebraically independent over K , then they are non-critically independent over K_s .*

Note that there exist algebraic elements $f \in \mathbb{K}_s$ which are transcendental over K_s but non-critically μ -dependent over K_s ; examples are provided by certain torsion points of the Carlitz module, see §9.

Proof of Proposition 17. (1). It follows from Lemma 12. (2). We suppose to have elements $f_1, \dots, f_n \in \mathbb{C}_\infty$ and an analytically non-critical μ -polynomial $P \in K_s[\underline{X}]_\mu$ with $P(\underline{f}) = 0$ and $\underline{f} = (f_1, \dots, f_n)$. This implies, in particular, that the map

$\mathbb{C}_\infty^n \rightarrow \mathbb{K}_s$ induced by P is not identically zero. This map is further equal, by the fact that μ and $x \mapsto x^p$ agree on \mathbb{C}_∞ , to a polynomial map induced by a polynomial $Q \in K_s[\underline{X}]$; since the map is not identically zero by hypothesis, Q is non-zero. Since K^s is dense in D^s , there exists an element $\underline{x} = (x_1, \dots, x_s) \in D^s \cap K^s$ such that the evaluation of the coefficients of Q , which are rational functions of K_s , is well defined and does not map to $\{0\}$. This yields a non-trivial algebraic dependence relation of f_1, \dots, f_n over K . \square

3.5. A refinement of Denis-Schanuel's conjecture. We fix an integer $s \geq 0$ and we set $q = p^e$ with $e > 0$. We recall that $K_s = \mathbb{F}_p(\theta, t_1, \dots, t_s) \subset \mathbb{K}_s$. We denote by L_s the smallest μ -difference subfield of \mathbb{K}_s containing K_s and all the $f \in \mathbb{K}_s$ which are non-critically μ -algebraic over K_s ; note that, by construction, it is discrete. The exponential function $\exp_C : \mathbb{K}_s \rightarrow C(\mathbb{K}_s)$ is defined by

$$\exp_C = \sum_{i \geq 0} d_i^{-1} \tau^i = \sum_{i \geq 0} d_i^{-1} \mu^{ei}$$

as in §3.2. We can now state our conjecture.

Conjecture 18 (Operator-theoretic generalization of Denis-Schanuel's Conjecture). *Let f_1, \dots, f_n be element of \mathbb{K}_s and let us write $g_i = \exp_C(f_i) \in \mathbb{K}_s$ for all $i = 1, \dots, n$. If f_1, \dots, f_n are $\mathbb{F}_q(t_1, \dots, t_s)[\theta]$ -linearly independent, then n among the elements*

$$f_1, \dots, f_n, g_1, \dots, g_n$$

are non-critically μ -independent over L_s .

We see that in the case $s = 0$ or if f_1, \dots, f_n all belong to \mathbb{C}_∞ , then this conjecture reduces to Conjecture 2 by (2) of Proposition 17. Also, for any s , Conjecture 18 implies Conjecture 2 and Conjecture 7. We do not know if, reciprocally, Conjecture 2 implies Conjecture 18 in analogy with the hypothesis (Σ) of [7], which is equivalent to the Schanuel conjecture; this looks unlikely.

4. SOME EXAMPLES

We are going to give some examples of consequences of our Conjecture 3.5. We first review basic properties of certain special functions that are used in our examples.

4.1. Torsion of the Carlitz exponential. Our Carlitz exponential $\exp_C : \mathbb{K}_s \rightarrow C(\mathbb{K}_s)$ has quite a rich torsion structure that we review here (see [5] for more properties). Let $a \in \mathbb{F}_q(t_1, \dots, t_s)[\theta]$ be of degree $d \geq 0$ in θ . Then for all $j = 0, \dots, \deg_\theta(a) - 1$, $\frac{\tilde{\pi}\theta^j}{a} \in \mathbb{K}_s$. In particular $\exp_C\left(\frac{\tilde{\pi}\theta^j}{a}\right) \in \mathbb{K}_s$ and one sees that

$$C_a\left(\exp_C\left(\frac{\tilde{\pi}\theta^j}{a}\right)\right) = 0, \quad j = 0, \dots, \deg_\theta(a) - 1.$$

In fact, the elements $\exp_C \left(\frac{\tilde{\pi}\theta^j}{a} \right) \in \mathbb{K}_s$ constitute an $\mathbb{F}_q(t_1, \dots, t_s)$ -basis of the submodule $\text{Ker}(C_a) \subset C(\mathbb{K}_s)$, which is of dimension d . Every torsion point satisfies a non-trivial linear τ -difference equation with coefficients in $\mathbb{F}_q(t_1, \dots, t_s)[\theta]$, it satisfies a non-trivial linear μ -difference equation with coefficients in K_s and we have that

$$\text{transf deg}_{K_s} K_s \left(\bigcup_{\substack{a \in \mathbb{F}_q(t_1, \dots, t_s)[\theta] \\ a \neq 0}} \text{Ker}(C_a) \right)_{\mu} = 0.$$

If moreover a is a polynomial of $\mathbb{F}_q[t_1, \dots, t_s, \theta]$ which is monic in θ , then $a^{-1} \in \mathbb{T}_s$ and we have $\exp_C \left(\frac{\tilde{\pi}\theta^j}{a} \right) \in \mathbb{T}_s$ for all j . We see that, for $j = 0, \dots, d-1$, the above elements span the rank d free $\mathbb{F}_q[t_1, \dots, t_s]$ -module $\text{Ker}(C_a) \cap \mathbb{T}_s$.

The simplest case (with $s = 1$) is given by the Anderson-Thakur function and comes with the choice of $a = \theta - t$. Note that it is also non-critically μ -algebraic; indeed, from (4), we deduce that the μ -polynomial $\mu^e(X) - (t - \theta)X = C_{\theta-t}(X) \in K_s[X]_{\mu}$ vanishes at ω and is tame, hence analytically non-critical.

4.2. Zeta values. The so-called *Carlitz zeta values* are defined as follows, for $n \geq 1$ an integer, where A^+ denotes the multiplicative monoid of monic polynomials of A , and the product runs over the irreducible polynomials $P \in A^+$:

$$\zeta_A(n) = \sum_{a \in A^+} a^{-n} = \prod_P \left(1 - \frac{1}{P^n} \right)^{-1} \in 1 + \theta^{-1}\mathbb{F}_q[[\theta^{-1}]].$$

Carlitz essentially proved in [10] that

$$(6) \quad \exp_C(\zeta_A(1)) = 1.$$

Conjectures 2 and 18 immediately imply that $\zeta_A(1)$ is transcendental over K and indeed, this follows from the transcendence theory of the Carlitz module in a variety of ways. First of all, this can be viewed as a consequence of the analogue of Hermite-Lindemann Theorem that can be found in [25]. Also, this follows from the Theorem of Papanikolas [20], see also Denis [14] and the survey [21] by the author, not to mention other proofs, with more diophantine, or automatic flavor. More generally, Chieh-Yu Chang and Jing Yu have proved in [12] the following result.

Theorem 19 (Chieh-Yu Chang & Jing Yu). *The element $\tilde{\pi}$ and all the zeta-values $\zeta_A(n)$ for all $n \geq 1$ not divisible by p and $q - 1$, are algebraically independent over K .*

Note that the elements of the above theorem generate the K -algebra

$$K(\tilde{\pi}, \zeta_A(1), \zeta_A(2), \dots).$$

Indeed, on one side, we have the “trivial sum-shuffle relations”

$$(7) \quad \zeta_A(pn) = \zeta_A(n)^p = \mu(\zeta_A(n)),$$

and on the other hand, we have the result of Carlitz

$$(8) \quad \zeta_A(k(q-1)) \in K^\times \tilde{\pi}^{k(q-1)}$$

for all $k \geq 1$.

4.2.1. *Zeta values in Tate algebras.* We recall from [5] the construction of *Carlitz zeta values in \mathbb{T}_s* . They are defined as follows, for $n \geq 1$ and an integer $s \geq 0$:

$$\zeta_A(n; s) = \sum_{a \in A^+} a^{-n} a(t_1) \cdots a(t_s) = \prod_P \left(1 - \frac{P(t_1) \cdots P(t_s)}{P^n} \right)^{-1}.$$

The above product converges in

$$1 + \theta^{-1} \mathbb{F}_q(t_1, \dots, t_s)[[\theta^{-1}]] \subset \mathbb{T}_s^\times.$$

The classical Carlitz zeta values are a special case of our construction with $s = 0$; in our notations, $\zeta_A(n) = \zeta_A(n; 0)$. In terms of the variables t_1, \dots, t_s , these series define entire functions $\mathbb{C}_\infty^s \rightarrow \mathbb{C}_\infty$ (see [3]). Therefore, evaluation at $t_i = \theta^{q^{k_i}}$, $i = 1, \dots, s$ and $k_i \in \mathbb{Z}$ makes sense and, for $n > 0$, we have the identities

$$\zeta_A(n) = \zeta_A(n; 0) = \zeta_A(n + q^{k_1} + \cdots + q^{k_s}; s) \Big|_{t_i = \theta^{q^{k_i}}}.$$

In this respect, we can view these functions as entire *interpolations of Carlitz zeta values*.

4.2.2. *Consequences of Conjecture 18.* We begin by reviewing an important property of the Carlitz zeta values $\zeta_A(1; s)$. In [5], the following generalization of Carlitz’s identity (6) is proved:

Theorem 20. *For $s \geq 0$, we have*

$$\exp_C(\zeta_A(1; s) \omega(t_1) \cdots \omega(t_s)) = P_s(t_1, \dots, t_s) \omega(t_1) \cdots \omega(t_s),$$

where $P_s \in A[t_1, \dots, t_s]$. Moreover, for $s > 1$, we have $P_s = 0$ if and only if $s \equiv 1 \pmod{q-1}$. In this case, we have

$$(9) \quad \zeta_A(1; s) = \frac{\tilde{\pi} B_s}{\omega(t_1) \cdots \omega(t_s)},$$

with $B_s \in A[t_1, \dots, t_s]$.

We deduce:

Theorem 21. *Conjecture 18 implies the truth of the following statement: The element $\tilde{\pi}$ and all the elements $\zeta_A(1; s)$ for $s \not\equiv 1 \pmod{q-1}$ are non-critically independent over K_s .*

In particular, we deduce that the values considered are algebraically independent and also tamely μ -independent over K_s .

Proof of Theorem 21. We choose $s_0 > 0$ and we denote by \mathfrak{S} the set of integers s with $s_0 \geq s \geq 0$ such that $s \not\equiv 1 \pmod{q-1}$; we choose s_0 so that \mathfrak{S} is non-empty. We also denote by ω_s , for $s \in \mathfrak{S}$, the product $\omega(t_1) \cdots \omega(t_s)$, which belongs to \mathbb{T}_{s_0} . Observe that, by the difference equation (4), for all $s \in \mathfrak{S}$,

$$\mu^e(\omega_s) = (t_1 - \theta) \cdots (t_s - \theta)\omega_s,$$

which implies that ω_s is non-critically μ -algebraic over K_s .

For fixed $s \in \mathfrak{S}$, the function ω_s also defines a meromorphic function $\mathbb{C}_\infty^{s_0} \rightarrow \mathbb{C}_\infty$ whose polar divisor D_s is, by Proposition 8, given by the free sum (with multiplicity one) of the affine subsets

$$D_{i,k,s} = \{(x_1, \dots, x_{i-1}, \theta^{q^k}, x_{i+1}, \dots, x_s, x_{s+1}, \dots, x_{s_0}) : x_j \in \mathbb{C}_\infty \forall j\} \subset \mathbb{C}_\infty^{s_0},$$

for $k \geq 0$ and $1 \leq i \leq s$, $s \in \mathfrak{S}$. We note that if $s < s'$ and $s, s' \in \mathfrak{S}$, then $D_s \subset D_{s'}$ and $D_{s'} \setminus D_s$ has infinitely many irreducible components.

We claim that the elements $\tilde{\pi}$ and $\omega_s \zeta_A(1; s)$ with $s \in \mathfrak{S}$ are $\mathbb{F}_q(t_1, \dots, t_{s_0})[\theta]$ -linearly independent. We write $\mathfrak{S} = \{s_1, \dots, s_n\}$. Let us assume by contradiction that there exists a non-trivial linear dependence relation

$$a_0 \tilde{\pi} + a_1 \omega_{s_1} \zeta_A(1; s_1) + \cdots + a_n \omega_{s_n} \zeta_A(1; s_n) = 0, \quad a_i \in A[t_1, \dots, t_{s_0}], s_i \in \mathfrak{S}.$$

For all $\underline{k} := (k_1, \dots, k_{s_0}) \in \mathbb{N}^{s_0}$, the congruence condition on $s \in \mathfrak{S}$ implies that the evaluation $\text{ev}_{\underline{k}}(\zeta_A(1; s))$ of $\zeta_A(1; s)$ at the point $(t_1, \dots, t_s) = (\theta^{q^{k_1}}, \dots, \theta^{q^{k_s}})$ is non-zero (this follows from a well known result of Goss, in [15]). This means that for all j , the function $\omega_{s_j} \zeta_A(1, s_j)$ has polar divisor given by D_{s_j} . Since the locus of the zeros of a_j has only finitely many irreducible components and since the polar divisors of the functions $\omega_{s_j} \zeta_A(1, s_j)$ are embedded one in the other along the total order of \mathfrak{S} induced by $<$ on \mathbb{N} , we deduce a contradiction. This shows that $a_0 = a_1 = \cdots = a_n = 0$ and the elements $\tilde{\pi}$ and $\omega_{s_j} \zeta_A(1, s_j)$ are linearly independent as expected.

We set $f_j = \omega_{s_j} \zeta_A(1, s_j)$, $f_0 = \tilde{\pi}$ and $g_j = \exp_C(f_j)$ for $0 \leq j \leq n$ (note that $g_0 = 0$). By Theorem 20,

$$\exp_C[\omega_s \zeta_A(1; s)] \in \omega_s A[t_1, \dots, t_s], \quad s \geq 0.$$

In particular, g_1, \dots, g_n are non-critically μ -algebraic over K_{s_0} (it is easy to see that they satisfy linear μ -difference equations with coefficients in K_{s_0} , so that they are tamely μ -algebraic). Conjecture 18 then implies that f_1, \dots, f_n are non-critically independent over L_{s_0} . Now, the theorem follows from the fact that $\omega_s \in L_{s_0}$ for all $s \leq s_0$ and that $K_{s_0} \subset L_{s_0}$. \square

The next Proposition goes in the direction of Theorem 21. The statement is so much weaker, but holds unconditionally and is deduced from Chang and Yu's Theorem 19.

Proposition 22. *Assuming that $q = p$, the elements $\tilde{\pi}$ and $\zeta_A(1; s)$ for $0 \leq s \leq p-1$ are tamely independent over K_s .*

Proof. Assume by contradiction that the statement is false. Then there exists a non-trivial relation $P(f_1, \dots, f_p) = 0$ with $P \in K(t_1, \dots, t_s)[\underline{X}]_\mu$ tame and non-zero, with $f_0 = \tilde{\pi}$, $f_1 = \zeta_A(1)$, $f_2 = \zeta_A(1; 1)$, \dots , $f_p = \zeta_A(1; p-1)$. Since the set

$$\{(\theta^{p^{-k_1}}, \dots, \theta^{p^{-k_{p-1}}}); \underline{k} = (k_1, \dots, k_{p-1}) \in \mathbb{N}^{p-1}\}$$

is Zariski-dense, there exists a subset \mathcal{I} of \mathbb{N}^{q-1} of positive density such that the evaluation $\text{ev}_{\underline{k}}(c_i)$ of a coefficient c_i of P at the point

$$(t_1, \dots, t_{p-1}) = (\theta^{p^{-k_1}}, \dots, \theta^{p^{-k_{p-1}}})$$

for all $\underline{k} = (k_1, \dots, k_{p-1}) \in \mathcal{I}$ is non-zero. Further we have, for $m \geq 0$, $n > 0$, $s \leq p-1$ and $k := \max\{k_1, \dots, k_{p-1}\}$,

$$\text{ev}_{\underline{k}}(\mu^m(\zeta_A(n; s))) = \zeta_A(n - p^{-m-k_1} \dots - p^{-m-k_s})^{p^m} = \zeta_A(p^{m+k} n - p^{k-k_1} \dots - p^{k-k_s})^{p^{-k}}.$$

Now, it is easy to see, due to our simple choice of parameters (especially the fact that $s \leq p-1$), that our assumption yields a non-trivial algebraic dependence relation among $\tilde{\pi}$ and the values

$$\zeta_A \left(p^{m+k} - \sum_{i=1}^{s'} p^{k-k_i} \right), \quad m \geq 0, \quad 1 \leq s' \leq p-1,$$

$k := \max\{k_1, \dots, k_{p-1}\}$. First of all, we can choose $\underline{k} = (k_1, \dots, k_{p-1})$ so that, for all i , $k_i > 0$. This implies that for all m and s' , the integer

$$n_{m,s'} := p^{m+k} - \sum_{i=1}^{s'} p^{k-k_i}$$

is positive. Further, for all m, s' , we have that $n_{m,s'} \not\equiv 0 \pmod{p-1}$; this already avoids the existence of a relation of the Euler-Carlitz-type (8). In order to avoid the trivial sum shuffle relations (7) we need to show that $n_{m,s'} = n_{m',s''} p^j$ for $1 \leq s', s'' \leq p-1$ and $j, m, m' \geq 0$ implies $m = m'$, $j = 0$ and $s' = s''$. This is very easy and left to the reader, but note that the condition $s', s'' \leq p-1$ is crucial. The conclusion follows from Chang and Yu's Theorem 19. \square

Remark 23. The above proof does not extend to the case $q \neq p$. The reason is that the identity $\mu^k(\zeta_A(n; s)) = \zeta_A(p^k n; s)$ holds if and only if $e \mid k$ and not for all k .

In a similar vein, we have the next Lemma.

Lemma 24. *Supposing that $p = q > 2$, we have that $f = \sum_{a \in A^+} a(t)^2 a^{-p} \in \mathbb{T}$ is transformally μ -transcendental, hence non-critically μ -transcendental over K_1 .*

Proof. We assume by contradiction that there is a non-trivial relation of algebraic dependence

$$\sum_{\underline{i} \in \mathbb{N}^k} c_{\underline{i}} f^{i_0} \mu(f)^{i_1} \cdots \mu^{k-1}(f)^{i_{k-1}}, \quad c_{\underline{i}} \in K_1.$$

We can suppose, without loss of generality, that all the coefficients $c_{\underline{i}}$ are in $A[t]$ and that they are not all divisible by $t - \theta$. This means that the evaluation at $t = \theta$ of the coefficients $c_{\underline{i}}$ yields a non-zero vector with entries in A . Then observing that the evaluation at $t = \theta$ of $\mu^j(f)$ is equal to the Carlitz zeta value $\zeta_A(p^{j+1} - 2)$, we have thus a non-trivial relation of algebraic dependence with coefficients in A of the Carlitz zeta values $\zeta_A(p^j - 2)$ for $j = 1, \dots, k$. Since $p - 1, p \nmid p^j - 2$ for all $j > 0$, this is again in contradiction with Chang and Yu's Theorem 19. \square

4.2.3. *Two more conjectures.* Additionally, we propose the following, for general q :

Conjecture 25. *Any finite subset of the set whose elements are $\tilde{\pi}, \zeta_A(n; s)$ with $p \nmid n$ and $n \not\equiv s \pmod{q-1}$ is non-critically independent over K_s .*

This conjecture does not seem to be a consequence of Conjecture 18 and is probably also quite a difficult one. In order to present also something which is perhaps provable in the near future, we mention that Conjecture 18 implies, for $n = 1$, the following conjecture:

Conjecture 26. *If $f \in \mathbb{K}_s \setminus \{0\}$, then either f or $\exp_C(f)$ is non-critically transcendental over K_s .*

The author presently does not know if this result can be directly deduced from the analogue of the theorem of Hermite-Lindemann for the Carlitz exponential over \mathbb{C}_∞ of [24].

5. TRANSCENDENCE DEGREE OF DIFFERENCE SUBFIELDS

Let $\underline{f} = (f_1, \dots, f_n)$ be an element of \mathbb{K}_s^n . We suppose that for all i , $g_i = \exp_C(f_i) \in \mathbb{K}_s$ are non-critically μ -algebraic. We address the following question.

Question 27. *Assume that $s > 0$. Is there a procedure allowing to compute the transcendence degree over K_s of the μ -field*

$$K_s(g_1, \dots, g_n)_\mu?$$

What are the possible μ -algebraic dependence relations among the values $\exp_C(f)$ with f varying in \mathbb{K}_s ?

This is likely to be a difficult question, the transcendence degree behaves wildly if $s > 0$ and does not seem to be in transparent (conjectural) relation with the linear forms over $\mathbb{F}_q(t)[\theta]$ satisfied by the various f , as we are going to see. From now on, we are going to assume that $s = 1$ for commodity. Therefore, we are going to write $t = t_1$, $\mathbb{T} = \mathbb{T}_1$ and $\mathbb{K} = \mathbb{K}_1$.

We recall that the \mathbb{C}_∞ -algebra \mathbb{T} is stable under the action of the divided derivatives \mathcal{D}_n , which are, for all $n \geq 0$, the \mathbb{C}_∞ -linear endomorphisms uniquely defined by the rule

$$\mathcal{D}_n t^m = \binom{n}{m} t^{m-n}.$$

Proposition 28 (Analogue of Hölder's theorem for the function ω). *We set $f_i = \frac{\tilde{\pi}}{(\theta-t)^{i+1}}$ for $i \geq 0$. The following properties hold:*

- (1) For all $i \geq 0$, $\exp_C(f_i) = \mathcal{D}_i(\omega)$.
- (2) The functions $\mathcal{D}_i(\omega)$ are a solution of the τ -difference system

$$\tau(X_i) = (t - \theta)X_i + X_{i-1}, \quad i \geq 0, \quad X_{-1} := 0.$$

- (3) The functions $\mathcal{D}_i(\omega)$ are algebraically independent over $K(t)$. In particular, we have, for all $n \geq 0$,

$$\begin{aligned} \text{transf deg}_{K(t)} K(t)(\mathcal{D}_i(\omega) : 0 \leq i \leq n)_\tau &= 0, \\ \text{tr deg}_{K(t)} K(t)(\mathcal{D}_i(\omega) : 0 \leq i \leq n)_\tau &= n + 1. \end{aligned}$$

Proof. The first two properties follow easily from the fact that, over \mathbb{T} ,

$$\exp_C \frac{d}{dt} = \frac{d}{dt} \exp_C,$$

and from the difference equation (4). For (3), since ω is transcendental over $K(t)$ we can argue by induction and suppose that $\omega, \mathcal{D}_1(\omega), \dots, \mathcal{D}_{n-1}(\omega)$ are algebraically independent over $K(t)$. Let us suppose by contradiction that $\mathcal{D}_n(\omega)$ is algebraically dependent of the previous functions; then there exists a non-zero irreducible polynomial P of $K(t)[X_0, \dots, X_n]$ which vanishes at the point determined by the functions $\omega, \dots, \mathcal{D}_n(\omega)$. The property (2) of the proposition implies that, if we denote by P^τ the polynomial of $K(t)[X_0, \dots, X_n]$ obtained by letting τ act on the coefficients, Then $P \mid P^\tau$ and in fact, $P^\tau = (t - \theta)^d P$ with d the total degree of P in its indeterminates, looking at the homogeneous part of highest degree of P . But looking at the monomials of smallest total degree of P , we see that this is impossible. The last property is clear. \square

The above proposition tells us that the linear relations over $\mathbb{F}_q(t)[\theta]$ of $f_1, \dots, f_n \in \mathbb{K}$ do not say much about the transcendence degree over $K(t)$ of the τ -difference field generated by $g_1 = \exp_C(f_1), \dots, g_n = \exp_C(f_n)$.

5.1. Solving linear difference equations in \mathbb{K} . We are going to discuss how the Carlitz exponential \exp_C can be used to solve the equations

$$\tau^{-1}(X) = X + g, \quad g \in \mathbb{K}.$$

Lemma 29. *Let g be an element of \mathbb{K} . Let $v \in \mathbb{K}$ be such that*

$$\exp_C(v) = -\tau(g)(t - \theta)\omega.$$

Then all the solutions $x \in \mathbb{K}$ of the equation

$$\tau^{-1}(X) = X + g$$

are the elements of the set

$$\mathcal{T}(g) := \omega^{-1} \exp_C\left(\frac{v}{\theta - t}\right) + \mathbb{F}_q(t).$$

Proof. All we need to show is that $f_v := \omega^{-1} \exp_C\left(\frac{v}{\theta - t}\right)$ is a solution of our equation. But:

$$\begin{aligned} \tau(f_v) &= \tau\left(\exp_C\left(\frac{v}{\theta - t}\right)\right) \tau(\omega)^{-1} \\ &= \left(C_{\theta - t}\left(\exp_C\left(\frac{v}{\theta - t}\right)\right) - (\theta - t) \exp_C\left(\frac{v}{\theta - t}\right)\right) \tau(\omega)^{-1} \\ &= \left(\exp_C(v) - (\theta - t) \exp_C\left(\frac{v}{\theta - t}\right)\right) ((t - \theta)\omega)^{-1} \\ &= f_v + \frac{\exp_C(v)}{(t - \theta)\omega} \\ &= f_v - \tau(g), \end{aligned}$$

and the equation $\tau(X) = X - \tau(g)$ has the same solutions as our equation. \square

Remark 30. We have that

$$\mathcal{T}(g) = \omega^{-1} \exp_C \frac{1}{\theta - t} \exp_C^{-1}((\theta - t)\omega\tau(g)).$$

We define two towers of field extensions of $K^{ac}(t)$, inductively. The first one is defined by $L_0 = K^{ac}(t)$ and, for all i ,

$$L_i = L_{i-1}(\omega)(f \in \mathbb{K} : \tau^{-1}(f) = f + \omega^l g : l \in \mathbb{Z}, g \in L_{i-1}).$$

For the second one, we set again, $M_0 = K^{ac}(t)$ and then inductively,

$$M_i = M_{i-1}(\cup_{l \in \mathbb{Z}} \exp_C(M_{i-1}\omega^l) \cup \cup_{l' \in \mathbb{Z}} \exp_C^{-1}(M_{i-1}\omega^{l'})).$$

Then writing $L_\infty = \cup_i L_i$ and $M_\infty = \cup_i M_i$, we obtain $L_\infty \subset M_\infty$ as a consequence of Lemma 29. In the next subsection, we are going to notice that the entries of the

rigid analytic trivializations associated to *Chang's multiple polylogarithms* are all in the field L_∞ (as in [11]).

5.2. Polylogarithm t -motives. An important feature of the entire function

$$\Omega := \tau(\omega)^{-1}$$

is highlighted in the paper of Anderson, Brownawell and Papanikolas [2] and in the paper of Papanikolas [20]: Ω is a *rigid analytic trivialization* of the *Carlitz dual t -motive* C . The reader can consult [20, §1.1.2 and §3.4] for the definition and the basic properties of *Anderson (dual) t -motives* (note that the operator τ^{-1} is used therein). The basic properties of the function Ω and of the Carlitz t -motive are discussed in [2, §3.1.2] and [20, §3.4.3, §3.3.4], while the notion of rigid analytic trivializations of t -motives is discussed in [20, §1.1.3 and §3.3].

In essence, the rigid analytic triviality of an object M of the Tannakian category \mathcal{T} over $\mathbb{F}_q(t)$ of *t -motives* (containing the category of Anderson's t -motives and introduced by Papanikolas, see [20, §1.1.6 and §3]) allows one to realize M as the solutions $\Psi \in \mathrm{GL}_r(\mathbb{K})$ (for some r) of a linear τ^{-1} -difference system

$$(10) \quad \tau^{-1}(\Psi) = \Phi\Psi,$$

with $\Phi \in \mathrm{GL}_r(\mathbb{C}_\infty(t))$.

Let Ψ be a rigid analytic trivialization associated to a multiple polylogarithm as in [11], that is, a matrix $\Psi \in \mathrm{GL}_{d+1}(\mathbb{K})$ satisfying the τ^{-1} -difference system (10) with $\Phi \in \mathrm{GL}_{d+1}(K^{ac}(t))$ as in [11, (5.3.3)] with $s_1, s_2, \dots, s_d \in \mathbb{Z}_{>0}$ and $Q_1, \dots, Q_d \in K^{ac}(t)$. We shall prove:

Proposition 31. *The entries of Ψ belong to L_∞ .*

Proof. If we write

$$\Psi = \begin{pmatrix} \Omega^{s_1+\dots+s_d} & 0 & \dots & 0 & 0 \\ x_{1,0} & \Omega^{s_2+\dots+s_d} & \dots & 0 & 0 \\ x_{2,0} & x_{2,1} & \dots & 0 & 0 \\ \dots & \vdots & & \vdots & \vdots \\ x_{d,0} & x_{d,1} & \dots & x_{d,d-1} & 1 \end{pmatrix},$$

solving the system (10) for this choice of Φ amounts to solve the iterated system of τ^{-1} -difference equations

$$\tau^{-1}(x_{i,j}) = \tau^{-1}(Q_i)(t - \theta)^{s_i+\dots+s_d} x_{i-1,j} + (t - \theta)^{s_{i+1}+\dots+s_d} x_{i,j}, \quad 0 \leq j < i \leq d,$$

setting also $x_{i,i} = \Omega^{s_{i+1}+\dots+s_d}$ for $i = 1, \dots, d$. This system is equivalent, by setting

$$y_{i,j} = \frac{x_{i,j}}{\Omega^{s_{i+1}+\dots+s_d}},$$

to the system of equations:

$$\tau^{-1}(y_{i,j}) = \tau^{-1}(Q_i)((t - \theta)\Omega)^{s_i} y_{i-1,j} + y_{i,j}, \quad 0 \leq j < i \leq d,$$

and the result follows. \square

5.2.1. *Example.* Let s_1, s_2 be positive integers. Let us choose $\alpha_1, \alpha_2 \in K^{ac}$. We set:

$$\Phi = \begin{pmatrix} (t - \theta)^{s_1+s_2} & 0 & 0 \\ (t - \theta)^{s_1+s_2} \tau^{-1}(\alpha_1) & (t - \theta)^{s_2} & 0 \\ 0 & (t - \theta)^{s_2} \tau^{-1}(\alpha_2) & 1 \end{pmatrix}.$$

Any matrix

$$\Psi = \begin{pmatrix} \Omega^{s_1+s_2} & 0 & 0 \\ x_1 & \Omega^{s_2} & 0 \\ x_3 & x_2 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{K})$$

such that

$$\tau^{-1}(\Psi) = \Phi\Psi$$

is a rigid analytic trivialization of the t -motive $N \in \mathcal{T}$ (Papanikolas category) associated to the multiple polylogarithm

$$\mathrm{Li}_{s_1, s_2}(\alpha_1, \alpha_2) = \sum_{i_1 > i_2 \geq 0} \frac{\tau^{i_1}(\alpha_1) \tau^{i_2}(\alpha_2)}{l_{i_1}^{s_1} l_{i_2}^{s_2}}$$

with $l_i = (\theta - \theta^q)^{i-1}$ and $l_0 = 1$, (if α_1, α_2 have small norm) and sitting in an exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow 1 \rightarrow 0,$$

with $M = L \otimes C^{\otimes s_2}$, with L sitting in an exact sequence

$$0 \rightarrow C^{\otimes s_1} \rightarrow L \rightarrow 1 \rightarrow 0.$$

We compute the coefficients x_1, x_2, x_3 .

Computation of x_1 . We have $\tau^{-1}(x_1) = (t - \theta)^{s_1+s_2} \tau^{-1}(\alpha_1) \Omega^{s_1+s_2} + (t - \theta)^{s_2} x_1$ so that $\tau(x_1) = \frac{1}{(t - \theta^q)^{s_2}} (x_1 - \alpha_1 \omega^{s_1+s_2})$. Thus, with $f = (t - \theta^q)^{-s_2}$, $g = -\alpha_1 \Omega^{s_1+s_2} (t - \theta^q)^{-s_2}$ and $h = \omega^{-1} \tau(\omega)^{-s_2}$. If v_1 is such that $\exp_C(v_1) = -\alpha_1 \Omega^{s_1-1}$, then

$$x_1 = h \exp_C \left(\frac{v_1}{\theta - t} \right) = \omega^{-1} \Omega^{s_2} \exp_C \left(\frac{v_1}{\theta - t} \right).$$

In particular, we can loosely write

$$x_1 = -\Omega^{s_2} \omega^{-1} T(\alpha_1 \Omega^{s_1-1}),$$

where

$$T(x) = \exp_C \left(\frac{v}{\theta - t} \right),$$

with v such that $\exp_C(v) = x$.

Computation of x_2 . We have $\tau^{-1}(x_2) = (t - \theta)^{s_2} \tau^{-1}(\alpha_2) \Omega^{s_2} + x_2$ which is equivalent to $\tau(x_2) = x_2 - \alpha_2 \Omega^{s_2}$. Hence, if $v_2 \in \mathbb{T}$ is such that $\exp_C(v_2) = -\alpha_2 \Omega^{s_2-1}$, we can choose (remember that the notation T is loose)

$$x_2 = \omega^{-1} \exp_C \left(\frac{v_2}{\theta - t} \right) = -\omega^{-1} T(\alpha_2 \Omega^{s_2-1}).$$

Computation of x_3 . We must solve $\tau^{-1}(x_3) = (t - \theta)^{s_2} \tau^{-1}(\alpha_2) x_1 + x_3$, which is equivalent to $\tau(x_3) = x_3 - \alpha_2 x_1 + \alpha_1 \alpha_2 \Omega^{s_1+s_2}$. First of all, the equation $\tau(x) = x + \alpha_1 \alpha_2 \Omega^{s_1+s_2}$ has the solution $x = \omega^{-1} \exp_C \left(\frac{v}{\theta - t} \right)$ for v such that $\exp_C(v) = \alpha_1 \alpha_2 \Omega^{s_1+s_2-1}$, that is

$$x = \omega^{-1} T(\alpha_1 \alpha_2 \Omega^{s_1+s_2-1}).$$

We still need to compute a solution of the equation $\tau(x) = x - \alpha_2 x_1$. Here we see that

$$x_3 = \omega^{-1} T(\alpha_1 \alpha_2 \Omega^{s_1+s_2-1}) - \omega^{-1} T(\alpha_2 \Omega^{s_2-1} \omega^{-1} T(\alpha_1 \Omega^{s_1-1})).$$

In short, setting $F = \omega^{-1} T \Omega$, we have

$$x_1 = -\Omega^{s_2} F(\alpha_1 \Omega^{s_1}), \quad x_2 = -F(\alpha_2 \Omega^{s_2}), \quad x_3 = F(\alpha_1 \alpha_2 \Omega^{s_1+s_2}) - F(\alpha_2 \Omega^{s_2} F(\alpha_1 \Omega^{s_1})).$$

Remark 32. More generally, for higher depth multiple polylogarithms, one sees from the proof of Proposition 31 that the entries of the corresponding rigid analytic trivializations can be obtained by nesting the operator $F = \omega^{-1} T \Omega^{-1}$ several times with multiplication by elements α_i and by appropriate powers of Ω , and then taking linear combinations. So, we would like to find a conjectural interpretation of the algebraic dependence relations of nested expressions:

$$F(\alpha_n \Omega^{s_n} F(\alpha_{n-1} \Omega^{s_{n-1}} (F \cdots F(\alpha_1 \Omega^{s_1}))))).$$

Before Schanuel's conjecture, there was a conjectural statement by Gelfond (see appendix of Waldschmidt's paper [26]) which looked as a very complicated statement involving iteration of exponentials and logarithms. Thinking about this lets us appreciate the simplicity and the strength of Schanuel's Conjecture. It seems that, in what concerns, for example, a control of the transcendence degree of the field generated by the entries of a rigid analytic trivialization of a multiple polylogarithm motive, we are somewhat back to Gelfond's starting point.

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5.3. Appendix. Some ring structures related to base- p digits. We begin with a commutative, unitary ring R . For a symbol Y , we introduce, in the R -module $R[Y_0, Y_1, \dots]$ of polynomials in infinitely many indeterminates Y_0, Y_1, \dots with coefficients in R , the monomials

$$\langle Y \rangle^i := \prod_{j=0}^{\infty} Y_j^{i_j}, \quad i \in \mathbb{N}, \quad i = i_0 + i_1 p + \dots + i_r p^r,$$

where the last expression is the expansion of i in base p , so that $0 \leq i_j \leq p-1$ for all j (note that the product is finite for any i). For $i, j \in \mathbb{N}$, we set $k := i + j$ and:

$$\langle Y \rangle^i \cdot \langle Y \rangle^j := \langle Y \rangle^k.$$

the R -module

$$R\langle Y \rangle := \left\{ \sum_{i=0}^N c_i \langle Y \rangle^i : N \in \mathbb{N}, c_i \in R \right\}$$

is now equipped with the structure of an R -algebra, by using the product of monomials defined above. Explicitly, we have, for two polynomials $f = \sum_i f_i \langle Y \rangle^i$ and $g = \sum_j g_j \langle Y \rangle^j$,

$$f \cdot g = \sum_k \langle Y \rangle^k \sum_{i+j=k} f_i g_j$$

(note that $R\langle Y \rangle$ is a submodule but not a subring of $R[Y_0, Y_1, \dots]$). This means that $R\langle Y \rangle$ is isomorphic to the polynomial algebra $R[Z]$ in one indeterminate Z with coefficients in R . In particular, if R is a field, $R\langle Y \rangle$ is an R -algebra of dimension one.

Let R be an \mathbb{F}_p -algebra, together with an \mathbb{F}_p -linear injective endomorphism $R \xrightarrow{\mu} R$. We define a μ -difference structure over $R\langle Y \rangle$ by setting

$$\mu \left(\sum_i f_i \langle Y \rangle^i \right) := \sum_i \mu(f_i) \langle Y \rangle^{pi}.$$

It is easy to see that, in this way, the isomorphism $R\langle Y \rangle \cong R[Z]$ becomes a μ -difference algebra isomorphism.

We assume, from now on, that R is a commutative field extension of \mathbb{F}_q . Let \mathcal{P} be the ideal of $R[X]_\mu$ generated by the polynomials $X^{p^k} - \mu^k(X)$, for all $k \geq 0$. Then \mathcal{P} is a μ -invariant ideal.

Lemma 33. *The ideal \mathcal{P} is a prime ideal, and the quotient $R[X]_\mu/\mathcal{P}$ is isomorphic, as an R -algebra which also is a μ -difference algebra, to the R -algebra $R\langle Y \rangle$.*

Proof. All we need to show, is that there is an R -algebra isomorphism $R[Z] \xrightarrow{\phi} R[X]_\mu/\mathcal{P}$. Indeed, it follows in this way that $R[X]_\mu/\mathcal{P}$ is integral.

We construct ϕ in the following way. Let $i \in \mathbb{N}$ be with base- p expansion $i = i_0 + i_1p + \cdots + i_r p^r$, with $0 \leq i_j < p$ for all j . We define the map ϕ to be R -linear, with

$$\phi(Z^i) = X^{i_0} \mu(X)^{i_1} \cdots \mu^r(X)^{i_r} \pmod{\mathcal{P}_X}.$$

We need to show that ϕ is multiplicative. Let $i, j, k \in \mathbb{N}$ be such that $i + j = k$. We further expand i, j, k in base p ; let r be an integer such that $i = \sum_{n=0}^r i_n p^n, j = \sum_{n=0}^r j_n p^n, k = \sum_{n=0}^r k_n p^n$, with $0 \leq i_n, j_n, k_n < p$. We define the sequence $(b_n)_{n \geq 0}$ of integers to be that characterizing the carry over of the base- p expansion of the sum $i + j = k$, namely, $b_n \in \mathbb{N}$ is defined by $b_{-1} := 0$ and, inductively for $n \geq 0$, by:

$$i_n + j_n + b_{n-1} = k_n + p b_n, \quad n \geq 0.$$

We have:

$$\begin{aligned} \phi(Z^i)\phi(Z^j) &= X^{i_0+j_0} \mu(X)^{i_1+j_1} \cdots \mu^r(X)^{i_r+j_r} \\ &\equiv X^{k_0} \mu(X)^{i_1+j_1+b_0} \mu^2(X)^{i_2+j_2} \cdots \mu^r(X)^{i_r+j_r} \pmod{(X^p - \mu(X))} \\ &\equiv X^{k_0} \mu(X)^{k_1} \mu^2(X)^{i_2+j_2+b_1} \cdots \mu^r(X)^{i_r+j_r} \pmod{(X^p - \mu(X), \mu(X)^p - \mu^2(X))} \\ &\equiv X^{k_0} \mu(X)^{k_1} \mu^2(X)^{i_2+j_2+b_1} \cdots \mu^r(X)^{i_r+j_r} \pmod{(X^p - \mu(X), X^{p^2} - \mu^2(X))} \\ &\quad \vdots \\ &\equiv X^{k_0} \mu(X)^{k_1} \cdots \mu^r(X)^{k_r} \pmod{\mathcal{P}_X} \\ &= \phi(Z^k) \end{aligned}$$

so that ϕ is multiplicative, and it is obviously surjective and injective (this is also easy to verify on the structure of the ring $R\langle Y \rangle$). The verification of the remaining properties is left to the reader. \square

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