



HAL
open science

Homogenization of a discrete model for a bifurcation and application to traffic flow

N Forcadel, Wilfredo Salazar

► **To cite this version:**

N Forcadel, Wilfredo Salazar. Homogenization of a discrete model for a bifurcation and application to traffic flow. *Journal de Mathématiques Pures et Appliquées*, 2020, 136, pp.356-414. hal-01332787v2

HAL Id: hal-01332787

<https://hal.science/hal-01332787v2>

Submitted on 5 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Homogenization of a discrete model for a bifurcation and application to traffic flow

N. Forcadel¹, W. Salazar¹

September 5, 2018

Abstract

The goal of this paper is to derive rigorously a macroscopic traffic flow model, for a simple bifurcation, from a microscopic model. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider the case of a very simple bifurcation in which one road separates into two and one vehicle over two goes to the right and the other goes to the left. At the bifurcation, we then have to add a phase of transition because the vehicle in front will change. Moreover, we assume that the velocity on each roads can be different. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation on each road and a junction condition (in the sense of [22]) located at the bifurcation. From this case of a simple bifurcation, we then extend to more general scenarios. For instance, the case of a different distribution of the vehicles at the bifurcation or even to consider more than two outgoing roads. For these extensions we only present the results and explain how to adapt the proofs from the case of a simple bifurcation.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords: specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

1 Introduction

Traffic flow can be characterized at different scales. The first one is the microscopic scale in which we describe the dynamics of each vehicles individually. Another one is the macroscopic scale that describes the collective dynamics of the vehicles with macroscopic quantities such as the density of vehicles and the average speed. The link between microscopic and macroscopic models has been extensively studied specially in the case of a single road. However, there are not many results concerning the homogenization of microscopic traffic flow models in networks.

In the case of a single road, we refer to [5, 13, 20, 28] where the authors rescaled the empirical measure in order to obtain a Lighthill-Whitham-Richards (LWR) model (see [30, 33]) at the macroscopic scale. Other works have been done by rescaling the primitive of the empirical measure in order to obtain a Hamilton-Jacobi equation (which is the primitive of a LWR model) at the macroscopic scale. Among those works we refer the readers to the papers [14, 15, 16, 17], the last one being directly applied to traffic flow.

Moreover, the recent works [1, 19], and the ideas in the lectures of Lions at the "College de France" [31], concerning specified homogenization enabled us to consider a traffic flow model with a local perturbation and to deduce a macroscopic model with a junction condition (see [18]). We would also like to refer to the paper of Colombo and Goatin [4], where the authors present an

¹Normandie Univ, INSA de Rouen, LMI (EA 3226 - FR CNRS 3335), 76000 Rouen, France, 685 Avenue de l'Université, 76801 St Etienne du Rouvray cedex. France

homogenization result from a LWR model with a discontinuous flow in space to a LWR model with a flow limiting condition at a single point.

Concerning homogenization results on networks, we would like to mention the recent work of Cristiani and Sahu [10] where the authors present a first order microscopic model on a network and show the link to a multi-path model (see [7, 8]). In fact they consider for each possible path a different population of vehicles. Their homogenization result is set in a very general network, however, they assume the convergence of the empirical measure (of each population) and they prove that the limit satisfies a multi-path model.

The present work focuses on obtaining a macroscopic model from a microscopic model for traffic flow in the case of a simple bifurcation. The schematic representation of the microscopic model is given in Figure 1.

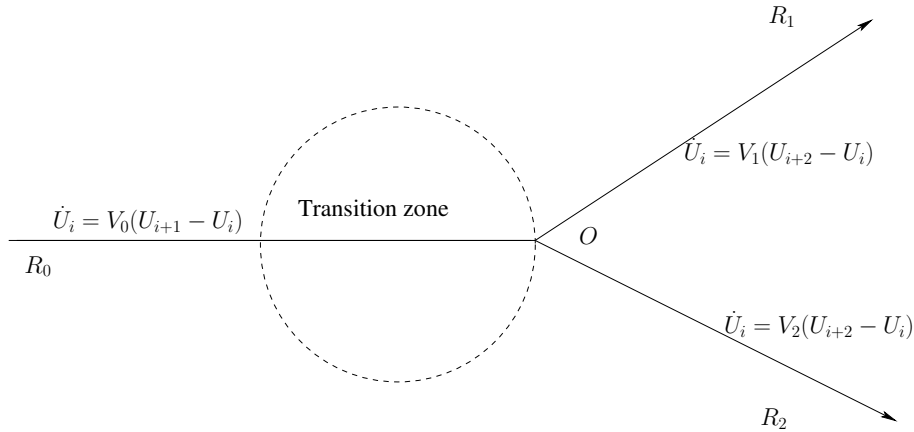


Figure 1: Simplified schematic representation of the microscopic model.

More precisely, we consider one incoming road R_0 which separates into two outgoing roads R_1 and R_2 . We denote by $U_i(t)$ the position of the i th vehicle, and we assume that the vehicles with an odd index go to R_1 while the ones with an even index go to R_2 . Finally, we assume that on each road R_i the velocity of each vehicle is given by a function V_i . In order to obtain our result we will rescale the microscopic model which describes the dynamics of each vehicle, in order to get a macroscopic model that describes the dynamics of the density of vehicles. At the macroscopic scale we will obtain an Hamilton-Jacobi equation on each branch and a junction condition at the origin (see Figure 2, where u_x^0 is related to the density of vehicles and the effective Hamiltonians \bar{H}_i are defined in the next section). At the intersection, the junction condition can be seen as a flux limiting condition (see [22]).

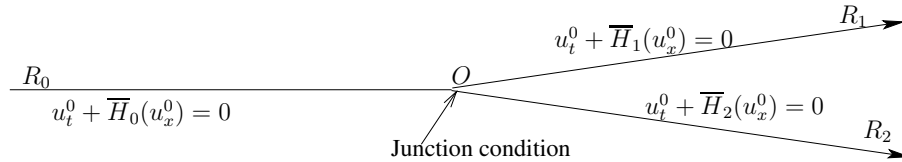


Figure 2: Schematic representation of the macroscopic (homogenized) model.

Finally, we give some extensions from the case of a simple bifurcation. We present the results in the case of more than two outgoing roads, and in the case we have a more general (but still periodic) distribution of the vehicles on each road. For the extensions, we do not give the details of the proofs, we only give some tips on how to adapt the proofs already presented in this paper.

In this paper we will use the recent developments on Hamilton-Jacobi equations on networks, particularly the paper of Imbert and Monneau [22] which gives a suitable definition of viscosity solutions at the junction (see also the work of Lions and Souganidis [32]).

1.1 General first order microscopic model for a junction

In this paper we are interested in a first order microscopic model for a simple bifurcation located at the origin, where we consider that the vehicles with odd indexes go to the left and vehicles with even indexes go to the right. We denote by U_i the position of the i th vehicle and \dot{U}_i its velocity. For $i = 0, 1, 2$ we call V_i an optimal velocity function. Before the bifurcation (i.e for $U_i(t) \ll 0$) we assume that the velocity of each vehicle is given by

$$\dot{U}_i(t) = V_0(U_{i+1}(t) - U_i(t)),$$

while after the bifurcation (i.e for $U_i(t) \gg 0$) we assume that

$$\dot{U}_i(t) = \begin{cases} V_1(U_{i+2}(t) - U_i(t)) & \text{if } i \text{ is odd} \\ V_2(U_{i+2}(t) - U_i(t)) & \text{if } i \text{ is even.} \end{cases}$$

In order to pass from the velocity V_0 to V_i , $i = 1$ or 2 , we introduce a transition function $\bar{\phi}$ and we consider the following system for all $t > 0$:

$$\dot{U}_i(t) = \begin{cases} \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_1(U_{i+2}(t) - U_i(t))\right) & \text{if } i \text{ is odd} \\ \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+2}(t) - U_i(t))\right) & \text{if } i \text{ is even.} \end{cases} \quad (1.1)$$

The function $\bar{\phi}$ allows a transition near the junction, and is defined by

$$\bar{\phi}(x, a, b) = \begin{cases} a & \text{if } x < -h_0 - h_{max} - 1, \\ (h_0 + h_{max} - |x|)(\min(a, b) - a) + \min(a, b) & \text{if } -1 \leq x + h_0 + h_{max} < 0, \\ \min(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \frac{|x|}{h_0} \min(a, b) + \left(1 - \frac{|x|}{h_0}\right) b & \text{if } -h_0 \leq x \leq 0, \\ b & \text{if } x > 0, \end{cases} \quad (1.2)$$

with $h_{max} > h_0 > 0$. Figure 3 represents model (1.1).

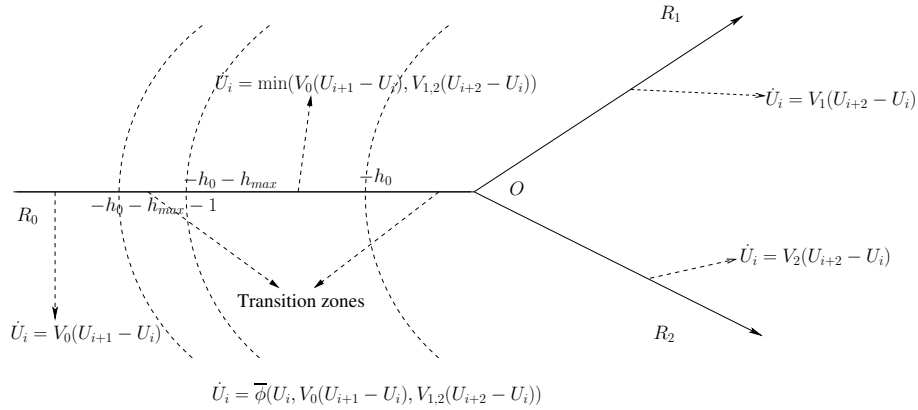


Figure 3: Schematic representation of the microscopic model.

In model (1.1), the vehicles with an odd index go to the road R_1 and the others go to the road R_2 . Since we work with a first order model, on R_0 the velocity of each vehicle depends on the distance $U_{i+1} - U_i$, but on R_1 and R_2 it depends on $U_{i+2} - U_i$.

We will call the transition zone, the interval $[-h_0 - h_{max} - 1, 0]$, where the vehicles will change from one model to the other. Concerning the optimal velocity functions V_i , for $i = 0, 1, 2$, and $\bar{\phi}$, we do the following assumptions.

Assumption (A)

- (A1) $V_i : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V_i is non-decreasing on \mathbb{R} .
- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V_i(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V_i(h) = V_i(h_{max}) =: V_{max}^i$.
- (A5) There exists a unique p_0 (resp. p_1 and p_2) such that the function $p \mapsto pV_0(-1/p)$ (resp. $p \mapsto pV_j(-2/p)$, for $j=1,2$) is decreasing in $[-1/h_0, p_0]$ (resp. $[-2/h_0, p_i]$) and increasing in $[p_0, 0]$ (resp. $[p_i, 0]$).
- (A6) The function $\bar{\phi} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to each variable. For all $(x, a, b) \in \mathbb{R}$, the functions $\bar{\phi}(x, \cdot, b)$ and $\bar{\phi}(x, a, \cdot)$ are non-decreasing. We denote by $\|\bar{\phi}'\|_\infty$, the smallest constant such that for all $(x, a, b), (x', a', b') \in \mathbb{R}^3$, we have

$$|\bar{\phi}(x, a, b) - \bar{\phi}(x', a', b')| \leq \|\bar{\phi}'\|_\infty (|x - x'| + |a - a'| + |b - b'|).$$

Remark 1.1 (An extra perturbation). *In order to make the microscopic model more realistic, it is possible to modify it and add a local perturbation around the junction point (by multiplying the velocity with a certain function like in [18]), that would decrease the speed of the vehicles near the origin. However, in the rest of the paper the function $\bar{\phi}$ is already treated like a local perturbation; adding an extra one would just complicate the notations without adding a mathematical interest (see [18] for how to treat a local perturbation in a microscopic model).*

Remark 1.2 (The transition zone). *The optimal velocity functions V_i describe the dynamics of the vehicles on each branch. The role of the transition zone is to do a continuous transition from one dynamic to the next one. Notice that given the form of the transition (1.2), if initially the vehicles have enough space between them, there will always be at least a distance h_0 between two vehicles. Meaning that in model (1.1) there is always a safety distance h_0 that avoids any collisions.*

2 Main results

Like in [15, 18], we inject the system of ODE into a system of PDE. To do this, we separate the vehicles into two groups, those going into R_1 and those going into R_2 .

2.1 Injecting the system of ODEs into a system of PDEs

We introduce two functions, the first one is the rescaled "cumulative distribution function" of vehicles with an odd index

$$\rho_1^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=1, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=1, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right) \quad (2.1)$$

and the second one is the rescaled "cumulative distribution function" of vehicles with an even index

$$\rho_2^\varepsilon(t, x) = -2\varepsilon \cdot \left(\sum_{i[2]=0, i \geq 0} H(x - \varepsilon U_i(t/\varepsilon)) + \sum_{i[2]=0, i < 0} (-1 + H(x - \varepsilon U_i(t/\varepsilon))) \right) - \varepsilon, \quad (2.2)$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (2.3)$$

Here and in the rest of this paper, $i[2]$ denotes the rest of the euclidean division of i by 2 (either 0 or 1). Under assumption (A), the function $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ is a (discontinuous viscosity) solution (see Theorem 7.1) of the following non-local equation, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}, \left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_1^\varepsilon\left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{\xi^\varepsilon}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_2^\varepsilon\left[\frac{\xi^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\xi_x^\varepsilon| = 0, \end{cases} \quad (2.4)$$

where N_0^ε and M_i^ε for $i = 1, 2$, are non-local operators defined by

$$N_0^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_0(z) F(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}, \quad (2.5)$$

and

$$M_i^\varepsilon[U](x) = \int_{\mathbb{R}} J_i(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}, \quad (2.6)$$

with $J_i = V_i'$ for $i = 0, 1, 2$ and

$$F(z) = \begin{cases} 0 & \text{if } z \geq 1, \\ 1/2 & \text{if } -1 \leq z < 1, \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and} \quad E(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ 1/2 & \text{if } -2 \leq z < 0, \\ 3/2 & \text{if } z < -2. \end{cases} \quad (2.7)$$

Finally, the function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$, is defined by $\phi(x, a, b) = -\bar{\phi}(x, -a, -b)$ for all $(x, a, b) \in \mathbb{R}^3$. In particular it has the following form

$$\phi(x, a, b) = \begin{cases} a & \text{if } x < -h_0 - h_{max} - 1, \\ (h_0 + h_{max} - |x|)(\max(a, b) - a) + \max(a, b) & \text{if } -1 \leq x + h_0 + h_{max} < 0, \\ \max(a, b) & \text{if } -h_0 - h_{max} \leq x < -h_0, \\ \frac{|x|}{h_0} \max(a, b) + \left(1 - \frac{|x|}{h_0}\right) b & \text{if } -h_0 \leq x \leq 0, \\ b & \text{if } x > 0. \end{cases} \quad (2.8)$$

Remark 2.1 (Choice of the "cumulative distribution functions"). *Contrary to [18], it is impossible to work with the integral of the empirical measure of the positions of all the vehicles. Indeed, near the bifurcation, the vehicles stop being in order (some go to R_1 and others to R_2), which makes it impossible to recover the distance $U_{i+1} - U_i$. To overcome this difficulty, we consider a modified version of the integral of the empirical measure of the two types of vehicles (the ones going on R_1 and the ones going to R_2). This modification simply allows us to obtain the index of the vehicles more easily: for $\varepsilon = 1$, if $i[2] = 1$ then $\rho_1^\varepsilon(t, U_i(t)) = -(i+1)$ and if $i[2] = 0$ then $\rho_2^\varepsilon(t, U_i(t)) = -(i+1)$.*

In particular, this implies (see Section 7) that if for instance $i[2] = 1$ and $\varepsilon = 1$, we have

$$\begin{cases} N_0^\varepsilon(\rho_1^\varepsilon(t, \cdot), [\rho_2^\varepsilon])(U_i(t)) = -V_0(U_{i+1}(t) - U_i(t)) \\ M_1^\varepsilon[\rho_1^\varepsilon(t, \cdot)](U_i(t)) = -V_1(U_{i+1}(t) - U_i(t)). \end{cases}$$

We obtain a similar result if $i[2] = 0$. This results helps us inject the system of ODEs into a system of PDEs.

The new transition function ϕ comes from the fact that with the non-local operators we recover the opposite of the velocity of the vehicles.

Remark 2.2. *Given the definition of the cumulative distribution function ρ_1^ε (resp. ρ_2^ε) we expect that $(\rho_1)_x$ (resp. $(\rho_2)_x$) (the gradient of the limit of ρ_1^ε (resp. ρ_2^ε) as ε goes to 0) is going to be the density of vehicles on R_0 and twice the density on R_1 (resp. R_2).*

We complete (2.4) with the following initial conditions

$$\begin{cases} u^\varepsilon(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \\ \xi^\varepsilon(0, x) = \xi_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (2.9)$$

and we make the following assumptions:

(A0) (Initial condition). For all $x \leq 0$,

$$u_0(x) = \xi_0(x).$$

Moreover, we assume for all $x \in \mathbb{R}$

$$-2k_0 = -\frac{2}{h_0} \leq (u_0)_x \leq 0 \quad \text{and} \quad -2k_0 \leq (\xi_0)_x \leq 0.$$

2.2 Convergence result

2.2.1 The effective Hamiltonians

Here, we introduce three Hamiltonians, \bar{H}_0 , \bar{H}_1 , and \bar{H}_2 that we will use in the rest of the paper. They are the effective Hamiltonians on each of the branches R_0 , R_1 , and R_2 . We define $k_0 = 1/h_0$ and $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_0(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_0 \left(\frac{-1}{p} \right) \cdot |p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.10)$$

We also define, for $i = 1, 2$, $\bar{H}_i : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_i(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_i \left(\frac{-2}{p} \right) \cdot |p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.11)$$

For $i = 0, 1, 2$, let us notice that such \bar{H}_i is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_i(p) = +\infty \right)$ and because of (A5), there exists a unique point $p_i \in [-2k_0, 0]$ such that

$$\begin{cases} \bar{H}_i \text{ is non-increasing on } (-\infty, p_i), \\ \bar{H}_i \text{ is increasing on } (p_i, +\infty), \end{cases} \quad (2.12)$$

We denote by

$$H_0 = \max_{i \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p). \quad (2.13)$$

Figure 4 gives a schematic representation of \bar{H}_0 , \bar{H}_1 , and \bar{H}_2 .

2.2.2 The macroscopic model

The main objective of this article is to obtain an homogenisation result when the number of vehicles per unit length goes to infinity, that is to say what is the behavior of $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ as ε goes to 0. First we can notice that the radius of the transition zone will go to 0, therefore at the macroscopic scale we will obtain a junction condition in 0 in the sense of Imbert and Monneau [22]. We consider one incoming road R_0 (isometric to $(-\infty, 0]$) and two outgoing roads R_1 and R_2 (isometric to $[0, +\infty)$) and all the branches are glued at the origin. Moreover, we define

$$\bar{R} = R_0 \cup R_1 \cup R_2 \quad \text{and} \quad R_i \cap R_j = \{0\} \text{ for } i \neq j.$$

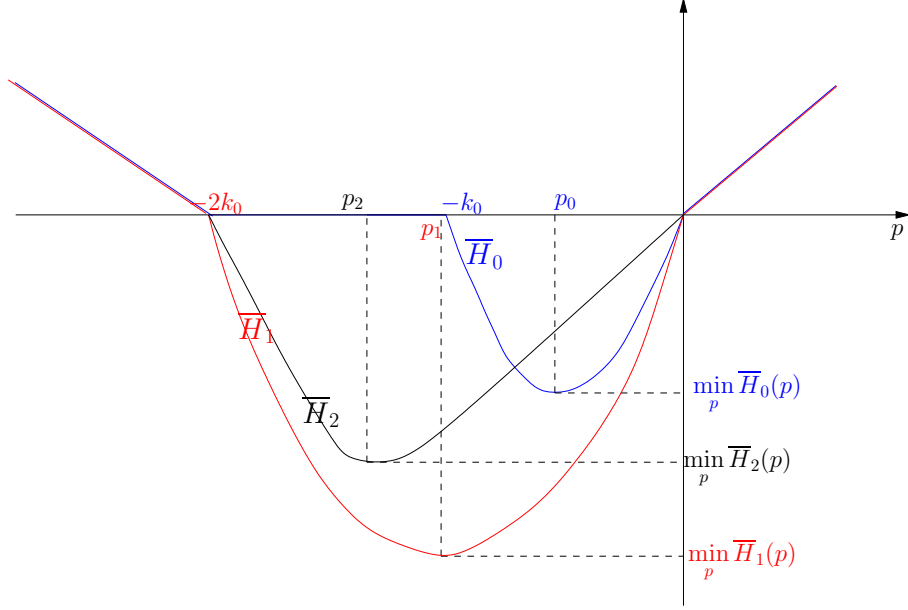


Figure 4: Schematic representation of the effective Hamiltonians.

In order to give a more physical interpretation, we consider that the branches R_0, R_1 and R_2 are generated respectively by the vectors \vec{e}_0, \vec{e}_1 and \vec{e}_2 , such that

$$R_0 = (-\infty, 0] \cdot \vec{e}_0 \quad \text{and} \quad R_{1,2} = [0, +\infty) \cdot \vec{e}_{1,2}.$$

For a smooth function $u : [0, T] \times \bar{R} \rightarrow \mathbb{R}$, we denote by $\partial_i u(t, x)$ the spatial derivative of u at $x \in R_i$, for $i = 0, 1, 2$ and we define

$$u_x(t, x) := \begin{cases} \partial_i u(t, x) & \text{if } x \in R_i^* := R_i \setminus \{0\}, \\ (\partial_0 u(t, 0), \partial_1 u(t, 0), \partial_2 u(t, 0)) & \text{if } x = 0. \end{cases} \quad (2.14)$$

Finally, we introduce for all $x, y \in \bar{R}$ the distance $d(x, y)$ on \bar{R} ,

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases} \quad (2.15)$$

The main result of this article is to prove that the function \tilde{u}^ε defined by

$$\tilde{u}^\varepsilon(t, x) = \begin{cases} u^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ u^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_1^*, \\ \xi^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_2^*, \end{cases} \quad (2.16)$$

converges locally uniformly on $(0, +\infty) \times \bar{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_1(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_1^*, \\ u_t^0 + \bar{H}_2(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_2^*, \\ u_t^0 + F_{\bar{A}}(\partial_0 u^0(t, 0), \partial_1 u^0(t, 0), \partial_2 u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_0(d(0, x)) & \text{for } x \in R_0 \cup R_1, \\ \xi_0(d(0, x)) & \text{for } x \in R_2^*, \end{cases} \end{cases} \quad (2.17)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_0, p_1, p_2) = \max(\bar{A}, \bar{H}_0^+(p_0), \bar{H}_1^-(p_1), \bar{H}_2^-(p_2)), \quad (2.18)$$

and for $i = 0, 1, 2$, we define

$$\overline{H}_i^-(p) = \begin{cases} \overline{H}_i(p) & \text{if } p \leq p_i, \\ \overline{H}_i(p_i) & \text{if } p \geq p_i, \end{cases} \quad \text{and} \quad \overline{H}_i^+(p) = \begin{cases} \overline{H}_i(p_i) & \text{if } p \leq p_i, \\ \overline{H}_i(p) & \text{if } p \geq p_i. \end{cases} \quad (2.19)$$

Remark 2.3. In (2.16), we can replace u^ε by ξ^ε for $x \in R_0$ and we obtain the same homogenisation result. The coefficient 2 that appears in \overline{H}_i for $i = 1, 2$ and not in \overline{H}_0 comes from the fact that we look one vehicle over two on R_i and therefore the density of the vehicles is divided by 2.

Remark 2.4. As explained before, $(\rho_i)_x$ is related to the density of vehicles (see Remark 2.2). In this way, (2.17) is a macroscopic model for traffic flow and the constant \overline{A} can be interpreted as a flux limiter at the junction. We refer to [22, 24] for this kind of model. In fact, in each road the effective equation is equivalent to a LWR model.

2.2.3 The homogenization result

Theorem 2.5 (Junction condition by homogenization). *Assume (A0) and (A). For $\varepsilon > 0$, let $(u^\varepsilon, \xi^\varepsilon)$ be the solution of (2.4). Then there exists a unique constant $\overline{A} \in [H_0, 0]$ such that the function \tilde{u}^ε defined by (2.16) converges locally uniformly to the unique viscosity solution u^0 of (2.17).*

The previous result will allow us to get the following homogenization result for the vehicles.

Theorem 2.6. *Assume (A) and that at initial time we have, for all $i \in \mathbb{Z}$, if the vehicles i and $i + 1$ are both in R_0 ,*

$$U_i(0) \leq U_{i+1}(0) - h_0, \quad (2.20)$$

and if not

$$U_i(0) \leq U_{i+2}(0) - h_0. \quad (2.21)$$

We also assume that there exists a constant $R > 0$ such that for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$

$$U_{i+1}(0) - U_i(0) = \begin{cases} h_1 & \text{if } i[2] = 1 \\ h_2 & \text{if } i[2] = 0 \end{cases} \quad (2.22)$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h, \quad (2.23)$$

with $h, h_1, h_2 \geq h_0$. We define two functions u_0 and ξ_0 (satisfying (A0)) by

$$u_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_1 & \text{if } x > 0 \end{cases} \quad \text{and} \quad \xi_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_2 & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

Then there exists a unique constant $\overline{A} \in [H_0, 0]$ such that the function

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_1^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_1^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_1^*, \\ \rho_2^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_2^*, \end{cases} \quad (2.24)$$

converges locally uniformly to the unique solution u^0 of (2.17).

Remark 2.7. Conditions (2.22) and (2.23) mean that the initial condition is well-prepared.

The goal of the following theorem is to show that the effective Hamiltonians in (2.17) are only evaluated for values in $[-2k_0, 0]$. However, it is convenient to work with the extended Hamiltonians presented in (2.10)-(2.11) because it is necessary to have coercive Hamiltonians, in order to apply the results developed by Imbert and Monneau in [22].

Theorem 2.8. Assume (A0)-(A). Let u^0 be the unique solution of (2.17), then we have for all $(t, x) \in [0, +\infty) \times \bar{R}$,

$$-2k_0 \leq u_x^0 \leq 0,$$

with k_0 defined in (A0).

To construct the effective flux limiter \bar{A} , we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution (v, ζ) of the following Hamilton-Jacobi equation

$$\begin{cases} \phi(x, N_0(v, [\zeta])(x), M_1[v](x)) \cdot |v_x| = \lambda, & \text{for } x \in \mathbb{R}, \\ \phi(x, N_0(\zeta, [v])(x), M_2[\zeta](x)) \cdot |\zeta_x| = \lambda, & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.25)$$

Theorem 2.9 (Effective flux limiter). Assume (A). We define the following set of functions,

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ two Lipschitz continuous functions } m_1 \text{ and } m_2 \text{ s.t. } m_1(0) = m_2(0) = 0 \\ \text{and a constant } C > 0 \text{ s.t. } \|v - m_1\|_\infty, \|\zeta - m_2\|_\infty \leq C\}.$$

Then \bar{A} is given by

$$\bar{A} = \inf\{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (2.25)}\}. \quad (2.26)$$

2.2.4 Link with macroscopic models

Notice that the homogenization of (2.4), does not directly give the dynamics of the density of vehicles. In fact, in R_0 , u^0 is the primitive of the density of vehicles, but in R_1 and R_2 it is twice the primitive of the density of vehicles. Therefore, the integral of the density of vehicles in \bar{R} is given by

$$\bar{\rho}^0(t, x) = \begin{cases} u^0(t, x) & \text{for } x \in R_0, \\ u^0(t, x)/2 & \text{for } x \in R_1^* \cup R_2^*. \end{cases} \quad (2.27)$$

However, we cannot explicit the dynamics of $\bar{\rho}^0$ because of its definition at the origin.

2.3 Notations and organization of the paper

We recall the definition of the non-local operators that we use in this paper,

$$N_0(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_0(z) F(\Xi(x+z) - U(x)) dz - \frac{3}{2} V_{max}^0, \quad (2.28)$$

and for $i = 1, 2$,

$$M_i[U](x) = \int_{-\infty}^{+\infty} J_i(z) E(U(x+z) - U(x)) dz - \frac{3}{2} V_{max}^i. \quad (2.29)$$

To each operator N and M , we associate the operators \tilde{N} and \tilde{M} , which are defined in the same way except that the functions F and E are replaced by the functions \tilde{F} and \tilde{E} defined by

$$\tilde{F}(z) = \begin{cases} 0 & \text{if } z > 1, \\ 1/2 & \text{if } -1 < z \leq 1, \\ 3/2 & \text{if } z \leq -1, \end{cases} \quad \text{and } \tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0, \\ 1/2 & \text{if } -2 < z \leq 0, \\ 3/2 & \text{if } z \leq -2. \end{cases} \quad (2.30)$$

Remark 2.10. Using the fact that E , F and V are bounded, there exists a constant M_0 such that for every U , Ξ and every $x \in \mathbb{R}$, we have for $i = 1, 2$,

$$\begin{cases} -M_0 \leq -\frac{3}{2} V_{max}^0 \leq N_0(U, [\Xi])(x) \leq 0, \\ -M_0 \leq -\frac{3}{2} V_{max}^i \leq M_i[U](x) \leq 0, \end{cases}$$

with

$$M_0 = \max_{i \in \{0,1,2\}} \frac{3}{2} V_{max}^i.$$

Remark 2.11 (Choice of the transition function). *In some computation it is useful to specify the explicit dependence of the function $\phi(x, a, b)$ on $\max(a, b)$. That is why we use the following notation,*

$$\tilde{\phi}(x, a, \max(a, b), b) = \phi(x, a, b) \quad \text{for all } (x, a, b) \in \mathbb{R}^3. \quad (2.31)$$

Finally, for a given point $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}$, we define for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{P}_{\bar{r}, \bar{R}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \bar{R}, x_0 + \bar{R}) \quad (2.32)$$

and

$$B(x_0, \bar{r}) = \{x \in \mathbb{R}, \text{ s.t. } |x - x_0| \leq \bar{r}\}.$$

Also for a given point $(t_0, y_0) \in (0, +\infty) \times \bar{R}$, we define for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \bar{R}}(t_0, y_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times \mathcal{B}_{\bar{R}}(y_0),$$

with

$$\mathcal{B}_{\bar{R}}(y_0) = \{y \in \bar{R} \text{ s.t. } d(y, y_0) < \bar{R}\}.$$

We denote by $C > 0$ a generic constant that may vary from one line to the next.

Organization of the paper. In Section 3, we introduce the definition of viscosity solutions for the considered problems and give stability, existence, and uniqueness results. Section 4 contains the results concerning the effective Hamiltonians and the correctors for the junction (Theorem 4.3) that are necessary for the proof of convergence (Theorem 2.5) which proof is located in Section 5. In Section 6, we present the proof of Theorem 4.3. In Section 7, we show the link between the ODEs and the system of PDEs. Finally, in Section 8, from the case of a simple bifurcation, we extend to more general scenarios.

3 Viscosity solutions

In this section we give the definition of viscosity solution of the equations we treat in this paper. We refer the reader to the user's guide of Crandall, Ishii, Lions [9] and the book of Barles [6] for an introduction to viscosity solutions. We also refer the reader to [25, 26, 29] for results concerning viscosity solutions for weakly coupled systems and to [22] for viscosity solutions on networks.

3.1 Definitions

In order to give a more general definition for all the non-local equations we will consider in this paper, we give the definition of a viscosity solution of the following equation, with $p \in \mathbb{R}$,

$$\begin{cases} u_t + G_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], u_x) = 0 \\ \xi_t + G_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \xi_x) = 0 \end{cases} \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}, \quad (3.1)$$

with, for $i = 1, 2$,

$$G_p^i(x, U, [\Xi], [U], q) = (1 - \psi^+(x)) \bar{H}_i(q) + (1 - \psi^-(x)) \bar{H}_0(q) \\ \psi^+(x) \psi^-(x) \phi(x, N_p^0(U, [\Xi])(x), M_p^i[U](x)) |q + p|,$$

and $\psi^\pm : \mathbb{R} \rightarrow [0, 1]$ two Lipschitz continuous functions. To G_p^i we associate \tilde{G}_p^i defined in the same way but with the non-local operators \tilde{N}_p^0 and \tilde{M}_p^i . For $p \in [-2k_0, 0]$, the non-local operators N_p^0 and M_p^i are defined by

$$N_p^0(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_0(z)F(\Xi(x+z) - U(x) + p \cdot z)dz - \frac{3}{2}V_{max}^0, \quad (3.2)$$

and for $i = 1, 2$

$$M_p^i[U](x) = \int_{-\infty}^{+\infty} J_i(z)E(U(x+z) - U(x) + p \cdot z)dz - \frac{3}{2}V_{max}^i. \quad (3.3)$$

We also consider the following initial condition

$$\begin{cases} u(0, x) = u_0(x) \\ \xi(0, x) = \xi_0(x), \end{cases} \quad (3.4)$$

that satisfies (A0).

We recall the definition of the upper and lower semi-continuous envelopes, u^* and u_* , of a locally bounded function u , for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y). \quad (3.5)$$

We can now give the definition of viscosity solutions for (3.1).

Definition 3.1 (Definition of viscosity solutions for (3.1)). *Let $T > 0$, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$, $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (A0). Let $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, $\xi : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ upper-semi continuous (resp. lower-semi continuous) locally bounded functions. We set $\Omega = (0, T) \times \mathbb{R}$. A function (u, ξ) is a sub-solution (resp. a super-solution) of (3.1) if for all $(t, x) \in \Omega$ and for any test function $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a local maximum (resp. local minimum) at the point $(t, x) \in \Omega$, we have*

$$\varphi_t(t, x) + G_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], \varphi_x) \leq 0$$

$$(resp. \varphi_t(t, x) + \tilde{G}_p^1(x, u(t, x), [\xi(t, \cdot)], [u(t, \cdot)], \varphi_x) \geq 0),$$

and if for all $(t, x) \in \Omega$ and for any test function φ such that $\xi - \varphi$ attains a local maximum (resp. local minimum) at (t, x) , we have

$$\varphi_t(t, x) + G_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \varphi_x) \leq 0$$

$$(resp. \varphi_t(t, x) + \tilde{G}_p^2(x, \xi(t, x), [u(t, \cdot)], [\xi(t, \cdot)], \varphi_x) \geq 0).$$

-A function (u, ξ) is a sub-solution (resp. super-solution) of (3.1)-(3.4) if (u, ξ) is a sub-solution (resp. super-solution) of (3.1) on Ω and satisfies

$$u(0, x) \leq u_0(x) \quad (resp. \geq) \quad \text{and} \quad \xi(0, x) \leq \xi_0(x) \quad (resp. \geq). \quad (3.6)$$

-A function (u, ξ) is a viscosity solution of (3.1) (resp. of (3.1)-(3.4)) if (u^*, ξ^*) is a sub-solution and (u_*, ξ_*) is a super-solution of (3.1) (resp. of (3.1)-(3.4)).

Remark 3.2. *We use this definition in order to have a stability result for the non-local operators (see [12, 34] for similar definitions). We refer to [15, Proposition 4.2] for the corresponding stability result.*

Definition 3.3 (Class of test functions for (2.17)). We denote $R^\infty := (0, +\infty) \times \bar{R}$ and $R_i^\infty := (0, +\infty) \times R_i^*$ for $i = 0, 1, 2$. We define a class of test functions on R^∞ by

$$C^2(R^\infty) = \{\varphi \in C(R^\infty), \text{ the restriction of } \varphi \text{ to } R_i^\infty, i = 0, 1, 2 \text{ is } C^2\}.$$

Definition 3.4 (Definition of viscosity solutions for (2.17)). Let \bar{H}_0 and \bar{H}_i , for $i = 1, 2$ be given respectively by (2.10) and (2.11). Let $\bar{A} \in \mathbb{R}$. An upper semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \bar{R} \rightarrow \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.17) if $u(0, x) \leq \bar{u}_0(x)$ (resp. $u(0, x) \geq \bar{u}_0(x)$) and for all $(t, x) \in R^\infty$ and for all $\varphi \in C^2(R^\infty)$ such that

$$u \leq \varphi \text{ (resp. } u \geq \varphi) \text{ in a neighbourhood of } (t, x) \in R^\infty \text{ and } u(t, x) = \varphi(t, x),$$

we have

$$\begin{cases} \varphi_t(t, x) + \bar{H}_0(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_0^*, \\ \varphi_t(t, x) + \bar{H}_1(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_1^*, \\ \varphi_t(t, x) + \bar{H}_2(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x \in R_2^*, \\ \varphi_t(t, x) + F_{\bar{A}}(\varphi_x(t, x)) \leq 0 \text{ (resp. } \geq 0) & \text{if } x = 0. \end{cases}$$

We say that u is a viscosity solution of (2.17) if u^* and u_* are respectively a sub-solution and a super-solution of (2.17). We refer to this solution as \bar{A} -flux limited solution.

Thanks to the work of Imbert and Monneau [22], we have the following result which gives an equivalent definition of viscosity solutions for (2.17). We use this equivalent definition in the proof of Theorem 2.5 in Section 5.

Theorem 3.5 (Equivalent definition for sub/super-solutions). Let \bar{H}_0 and \bar{H}_i for $i = 1, 2$ be given by (2.10)-(2.11) and consider $A \in [H_0, +\infty)$ with $H_0 = \max_{i \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_i(p)$. Given arbitrary solutions $p_0 \in \mathbb{R}$ and $p_i^A \in \mathbb{R}$ for $i = 1, 2$ of

$$\bar{H}_0(p_0^A) = \bar{H}^-(p_0^A) = A \quad \text{and} \quad \bar{H}_i^+(p_i^A) = \bar{H}_i(p_i^A) = A, \quad (3.7)$$

let us fix any time independent test function $\phi^0(x)$ satisfying, for $i = 0, 1, 2$,

$$\partial_i \phi^0(0) = p_i^A.$$

Given a function $u : (0, T) \times \bar{R} \rightarrow \mathbb{R}$, the following properties hold true.

i) If u is an upper semi-continuous sub-solution of (2.17) with $A = H_0$, for $x \neq 0$, satisfying

$$u(t, 0) = \limsup_{(s, y) \rightarrow (t, 0), y \in R_i^*} u(s, y), \quad (3.8)$$

then u is a H_0 -flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0, T)$, if u is an upper semi-continuous sub-solution of (2.17) for $x \neq 0$, satisfying (3.8), and if for any test function φ touching u from above at $(t_0, 0)$ with

$$\varphi(t, x) = \psi(t) + \phi^0(x), \quad (3.9)$$

for some $\psi \in C^2(0, +\infty)$, we have

$$\varphi_t + F_A(\varphi_x) \leq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited sub-solution at $(t_0, 0)$.

iii) Given $t_0 \in (0, T)$, if u is a lower semi-continuous super-solution of (2.17) for $x \neq 0$ and if for any test function φ satisfying (3.9) touching u from above at $(t_0, 0)$ we have

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at } (t_0, 0),$$

then u is a A -flux limited super-solution at $(t_0, 0)$.

3.2 Results for viscosity solutions of (2.4)

Lemma 3.6 (Existence of barriers for (3.1)). *Assume (A0) and (A). There exists a constant $K_1 > 0$ such that the functions (u^+, ξ^+) and (u^-, ξ^-) defined by*

$$(u^+(t, x), \xi^+(t, x)) = (K_1 t + u_0(x), K_1 t + \xi_0(x)) \quad \text{and} \quad (u^-(t, x), \xi^-(t, x)) = (u_0(x), \xi_0(x))$$

are respectively a super and sub-solution of (3.1).

Proof. We define $K_1 = M_0 \cdot (|p| + 2k_0) + 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right)$, where M_0 is defined in Remark 2.10. Let us prove that (u^+, ξ^+) is a super-solution of (3.1). Using assumption (A0) and the form of the non-local operators and of \overline{H}_i , $i = 0, 1, 2$ we have

$$\begin{aligned} \tilde{G}_p^1(x, u^+(t, x), [\xi^+(t, \cdot)], [u^+(t, \cdot)], (u_0)_x) &\geq -M_0 \cdot |p + (u_0)_x| \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right) \\ &\geq -M_0(|p| + 2k_0) \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{q \in [-2k_0, 0]} |\overline{H}_i(q)| \right) \\ &= -K_1, \end{aligned}$$

where we have used Remark (2.10). Similarly, we have

$$\begin{aligned} \tilde{G}_p^2(x, \xi^+(t, x), [u^+(t, \cdot)], [\xi^+(t, \cdot)], (\xi_0)_x) &\geq -M_0(|p| + 2k_0) \\ &\quad - 2 \max_{i=0,1,2} \left(\max_{x \in [-2k_0, 0]} |\overline{H}_i(x)| \right) \\ &= -K_1. \end{aligned} \tag{3.10}$$

The proof for the sub-solution is similar using that the non-local operators and $\overline{H}_i((u_0)_x)$ for $i = 0, 1, 2$ are non-positive. This ends the proof of Lemma 3.6. \square

Proposition 3.7 (Existence and uniqueness for (3.1)). *Assume (A). Let (u, ξ) (resp. (v, ζ)) be a sub-solution (resp. a super-solution) of (3.1)-(3.4). We also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$\begin{aligned} u(t, x) &\leq u_0(x) + K(t + 1), & \xi(t, x) &\leq \xi_0(x) + K(1 + t) \\ -v(t, x) &\leq -u_0(x) + K(1 + t), & \zeta(t, x) &\leq -\xi_0(x) + K(1 + t). \end{aligned} \tag{3.11}$$

If

$$u(0, x) \leq v(0, x) \quad \text{and} \quad \xi(0, x) \leq \zeta(0, x) \quad \text{for all } x \in \mathbb{R},$$

then

$$u(t, x) \leq v(t, x) \quad \text{and} \quad \xi(t, x) \leq \zeta(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}.$$

In particular, the previous result combined with the Lemma 3.6 imply that there exists a unique solution (u, ξ) of (3.1)-(3.4). Moreover, the functions u and ξ are continuous and there exists a constant $K_1 > 0$ such that for all $(t, x) \in [0, +\infty) \times \mathbb{R}$

$$0 \leq u(t, x) - u_0(x) \leq K_1 t \quad \text{and} \quad 0 \leq \xi(t, x) - \xi_0(x) \leq K_1 t.$$

Proof. The first part of this result (comparison principle) is classical and uses the monotonicity properties of G (see for example [11, 15] for a similar result).

To prove the rest of Proposition 3.7, we apply Perron's method (see [23, Proof of Theorem 6], [3, 21] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle. \square

We now give a comparison principle on bounded sets, we use the notations from (2.32).

Proposition 3.8 (Comparison principle on bounded sets for (3.1)). *Assume (A). Let (u, ξ) be a sub-solution of (3.1) and let (v, ζ) be a super-solution of (3.1) on the open set $\mathcal{P}_{\bar{r}, \bar{R}} \subset (0, T) \times \mathbb{R}$. Also assume that*

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{R}},$$

then

$$u \leq v \quad \text{and} \quad \xi \leq \zeta \quad \text{on } \mathcal{P}_{\bar{r}, \bar{R}}.$$

We now give a result on the control of the oscillations for the solution of (2.4) (with $\varepsilon = 1$). This result will be used in particular to prove Theorem 2.8.

Theorem 3.9 (Control of the oscillations). *Let $T > 0$. Assume (A0)-(A) and let (u, ξ) be a solution of (2.4)-(2.9), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - 2 \leq u(t, x) - u(t, y) &\leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - 2 \leq \xi(t, x) - \xi(t, y) &\leq 0, \end{aligned} \quad (3.12)$$

with $K_0 := 2k_0$.

Proof. Using the barriers constructed in Lemma 3.6 (with $p = 0$, $\psi^+ \equiv 1$ and $\psi^- \equiv 1$) we deduce that (u, ξ) satisfies for all $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$0 \leq u(t, x) - u_0(x) \leq M_0 K_0 t \quad \text{and} \quad 0 \leq \xi(t, x) - \xi_0(x) \leq M_0 K_0 t. \quad (3.13)$$

In the rest of the proof we use the following notation

$$\tilde{\Omega} = \{(t, x, y) \in [0, T) \times \mathbb{R}^2 \text{ s.t. } x \geq y\}.$$

Step 1: proof of the bound of the time derivative. For all $h \geq 0$, we have

$$u(0, x) \leq u(h, x) \leq M_0 K_0 h + u(0, x) \quad \text{and} \quad \xi(0, x) \leq \xi(h, x) \leq M_0 K_0 h + \xi(0, x).$$

Using the fact that (2.4) is invariant by addition of constant to the solution and by translation in time, we deduce by the comparison principle that, for all $(t, x) \in [0, +\infty) \times \mathbb{R}$, we have

$$u(t, x) \leq u(t + h, x) \leq M_0 K_0 h + u(t, x) \quad \text{and} \quad \xi(t, x) \leq \xi(t + h, x) \leq M_0 K_0 h + \xi(t, x).$$

We deduce the result by choosing $C_1 = M_0 K_0$.

Step 2: proof of the upper inequality for the control of the space oscillations. We introduce

$$M = \sup_{(t, x, y) \in \tilde{\Omega}} \max(u(t, x) - u(t, y), \xi(t, x) - \xi(t, y)). \quad (3.14)$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 2.1: the test functions. For $\eta, \alpha > 0$, small parameters, we define

$$\begin{cases} \varphi_1(t, x, y) = u(t, x) - u(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2, \\ \varphi_2(t, x, y) = \xi(t, x) - \xi(t, y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2. \end{cases}$$

Using (3.13), we have that

$$\lim_{|x|,|y| \rightarrow \infty} \varphi_i(t, x, y) = -\infty.$$

Since φ_1, φ_2 are upper semi-continuous, the function $\psi = \max(\varphi_1, \varphi_2)$ reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \tilde{\Omega}$. Classically, we have for η and α small enough,

$$\begin{cases} 0 < \frac{M}{2} \leq \psi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases}$$

Step 2.2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Then we have,

$$\frac{\eta}{T} \leq u_0(x) - u_0(y) \leq 0 \quad \text{or} \quad \frac{\eta}{T} \leq \xi_0(x) - \xi_0(y) \leq 0,$$

where we have used (A0), and we get a contradiction. The fact that $\bar{x} > \bar{y}$ comes directly from the fact that $\psi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 2.3: utilisation of the equation. By doing a duplication of the time variable and passing to the limit we get, if $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_1(\bar{t}, \bar{x}, \bar{y})$,

$$\frac{\eta}{(T - \bar{t})^2} \leq -\phi(\bar{x}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{x}), M_1[u(\bar{t}, \cdot)](x)) \cdot |2\alpha\bar{x}| \leq 2M_0|\alpha\bar{x}|$$

or, if $\psi(\bar{t}, \bar{x}, \bar{y}) = \varphi_2(\bar{t}, \bar{x}, \bar{y})$,

$$\frac{\eta}{(T - \bar{t})^2} \leq -\phi(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](x)) \cdot |2\alpha\bar{x}| \leq 2M_0|\alpha\bar{x}|,$$

where we have used the fact that the non-local operators are negative. This is a contradiction for α small enough.

Step 3: proof of the lower inequality for the control of the space oscillations. In order to do this part of the proof, we use the following lemma, which proof is postponed.

Lemma 3.10. *Assume (A0)-(A). Let $T > 0$, then the solution (u, ξ) of (2.4)-(2.9) (with $\varepsilon = 1$) satisfies*

$$|\xi(t, x) - u(t, x)| \leq 1 \quad \text{for } (t, x) \in [0, T] \times (-\infty, -h_0]. \quad (3.15)$$

We now introduce,

$$M = \sup_{(t, x, y) \in \tilde{\Omega}} \max(u(t, y) - u(t, x) - K_0(x - y) - 2)$$

and we want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 3.1: the test functions. For η and α small parameters, we introduce

$$\varphi(t, x, y) = u(t, y) - u(t, x) - K_0(x - y) - 2 - \frac{\eta}{T - t} - \alpha x^2$$

Using (3.13), we deduce that the function φ reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in \tilde{\Omega}$ and that for η and α small enough, we have

$$\begin{cases} 0 < \frac{M}{2} \leq \psi(\bar{t}, \bar{x}, \bar{y}), \\ \alpha|\bar{x}| \rightarrow 0 \text{ as } \alpha \rightarrow 0. \end{cases} \quad (3.16)$$

Moreover, as in Step 2.2 we get that $\bar{t} > 0$ and $\bar{x} > \bar{y}$.

Step 3.2: getting a contradiction. By doubling the time variable and passing to the limit, we get

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq \phi(\bar{x}, \tilde{N}_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{x}), \tilde{M}_1[u(\bar{t}, \cdot)](\bar{x})) \cdot | -K_0 - 2\alpha\bar{x} | \\ &\quad - \phi(\bar{y}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})) \cdot |K_0| \\ &\leq -\phi(\bar{y}, N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})) \cdot K_0, \end{aligned} \quad (3.17)$$

where we have used the fact that the non-local operators are non-positive.

Let us now prove that

$$M_1[u(\bar{t}, \cdot)](\bar{y}) = \int_{h_0}^{h_{max}} J_1(z) E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max}^1 = 0. \quad (3.18)$$

It is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) < -2.$$

Let $z \in (h_0, h_{max}]$, if $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, we observe that

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq -K_0 z < -2.$$

If $\bar{y} + z \geq \bar{x}$, using the fact that u is non-increasing in space and that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we obtain

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -K_0 z - 2 < -2.$$

We recall that (Remark 2.11) $\phi(x, a, b) = \tilde{\phi}(x, a, \max(a, b), b)$. In particular, given that the non-local operators are non-positive (Remark 2.10), we have

$$\max(N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), M_1[u(\bar{t}, \cdot)](\bar{y})) = 0. \quad (3.19)$$

Therefore, given the definition of ϕ in (2.8), the right-hand side term in (3.17) is equal to zero unless $\bar{y} \leq -h_0 - h_{max}$. Let us now prove that if $\bar{y} \leq -h_0 - h_{max}$, we have $N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}) = 0$, which will directly give us a contradiction from (3.17).

We claim that

$$N_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}) = \int_{h_0}^{h_{max}} J_0(z) F(\xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max}^0 = 0. \quad (3.20)$$

In fact it is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$\xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) < -1.$$

Let $z \in (h_0, h_{max}]$, if $\bar{y} + z < \bar{x}$, we obtain

$$\begin{aligned} \xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) &\leq u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{x}) + 1 \\ &\leq 1 - K_0 z \\ &< -1, \end{aligned}$$

where we have used Lemma 3.10 for the first line (since $\bar{y} + z \leq \bar{y} + h_{max} \leq -h_0$), the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$ for the second line and the fact that $K_0 z > 2$ for the third line.

If $\bar{y} + z \geq \bar{x}$, we then have

$$\begin{aligned} \xi(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) &\leq u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) + 1 \\ &\leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) + 1 \\ &\leq 1 - K_0(\bar{x} - \bar{y}) - 2 < -1, \end{aligned}$$

where we have used Lemma 3.10 for the first line, the fact that u is non-increasing in space for the second line, and the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ for the third line. Injecting (3.18), (3.19), and (3.20) in (3.17), we obtain $\eta/T^2 \leq 0$, which is a contradiction.

The proof of the lower bound on the control of the space oscillations of ξ is done similarly and we skip it. This ends the proof of Theorem 3.9. \square

Proof of Lemma 3.10. We only prove that $\xi(t, x) - u(t, x) \leq 1$ for $x \leq -h_0$ and $t \in [0, T)$, the proof for the second part of the result is similar so we skip it. We will prove this result under a different form using the technique introduced by Ishii and Lions in [27] for the proof of local gradient estimates. Let $z \in (-\infty, -h_0)$ and $\delta > 0$ a small parameter be such that $z \leq -h_0 - \delta$. We introduce

$$\Delta = [0, T) \times B(z, \delta).$$

We will prove that for a constant $L := (4K_1T + 8k_0\delta)/\delta^2$, we have

$$\xi(t, x) - u(t, x) \leq 1 + L(x - z)^2 \quad \text{for all } (t, x) \in \Delta. \quad (3.21)$$

In fact, if we take $x = z$ in (3.21), we get

$$\xi(t, z) - u(t, z) \leq 1 \quad \forall (t, z) \in [0, T) \times (-\infty, -h_0).$$

Using the continuity of the solution (u, ξ) , we can pass to the limit in the previous inequality as z goes to $-h_0$ and we obtain the complete result. In order to prove (3.21), we introduce

$$M = \sup_{(t, x) \in \Delta} (\xi(t, x) - u(t, x) - 1 - L(x - z)^2) \quad (3.22)$$

and we want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For η and ε small parameters, we define

$$\varphi(t, x, y) = \xi(t, x) - u(t, y) - 1 - \frac{L}{2}(x - z)^2 - \frac{L}{2}(y - z)^2 - \frac{(x - y)^2}{2\varepsilon} - \frac{\eta}{T - t}.$$

As in the previous proof, we have that φ reaches a maximum at a finite point that we denote by

$$(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times B(z, \delta) \times B(z, \delta).$$

By classical arguments, we have that

$$\begin{cases} 0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{cases}$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction assume that $\bar{t} = 0$, then we have

$$\frac{\eta}{T} < \xi_0(x) - u_0(x) - 1 + 2k_0|x - y| \leq o_\varepsilon(1),$$

where we have used assumption (A0) and we get a contradiction for ε small enough.

Step 3: $|\bar{x} - z| < \delta$ and $|\bar{y} - z| < \delta$. By contradiction assume that either $|\bar{x} - z| = \delta$ or $|\bar{y} - z| = \delta$, in which case, we have

$$\begin{aligned} 0 < \varphi(\bar{t}, \bar{x}, \bar{y}) &= \xi(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \\ &\leq 2K_1T + \xi_0(\bar{x}) - \xi_0(\bar{y}) + \xi_0(\bar{y}) - u_0(\bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \\ &\leq 2K_1T + 2k_0|\bar{x} - \bar{y}| - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} - \frac{L}{2}\delta^2 \\ &\leq 2K_1T + 4k_0\delta - \frac{L}{2}\delta^2 \leq 0, \end{aligned} \quad (3.23)$$

where we have used the barriers from Lemma 3.6 for the second line, assumption (A0) for the third line, and for the fourth line the fact that $L = (4K_1T + 8k_0\delta)/\delta^2$. This gives us a contradiction.

Step 3: utilisation of the equation. By doubling the time variable and passing to the limit, we obtain that there exists two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2},$$

and

$$\begin{aligned} a + \phi(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x})) \cdot |p_\varepsilon + L(\bar{x} - z)| &\leq 0, \\ b + \phi(\bar{y}, \tilde{N}_0(u(\bar{t}, \cdot), [\xi(\bar{t}, \cdot)])(\bar{y}), \tilde{M}_1(\bar{t}, \cdot)(\bar{y})) \cdot |p_\varepsilon - L(\bar{y} - z)| &\geq 0, \end{aligned}$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$. Combining these inequalities we obtain

$$\frac{\eta}{T^2} \leq -\phi(\bar{x}, N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x})) \cdot |p_\varepsilon + L(\bar{x} - z)|. \quad (3.24)$$

Notice that because $\bar{x} < z + \delta \leq -h_0$, and because of the form of ϕ (Remark 2.11), the right-hand side term in (3.24) is only a combination of the non-local term $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x})$ and of $\max(N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}), M_2[\xi(\bar{t}, \cdot)](\bar{x}))$.

Let us prove that $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}) = 0$, which, given that the non-local operators are non-positive, will imply that the entire right-hand side term in (3.24) is equal to zero.

As a matter of fact, using that $|\bar{x} - \bar{y}| \rightarrow 0$ as ε goes to 0, for all $h \in [h_0, h_{max}]$, we can assume that $\bar{x} + h > \bar{y}$. Using the fact that u is decreasing in space we deduce that

$$u(\bar{t}, \bar{x} + h) - \xi(\bar{t}, \bar{x}) \leq u(\bar{t}, \bar{y}) - \xi(\bar{t}, \bar{x}) < -1, \quad (3.25)$$

where we have used the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$. This implies that $N_0(\xi(\bar{t}, \cdot), [u(\bar{t}, \cdot)])(\bar{x}) = 0$. This gives us a contradiction and ends the proof of Lemma 3.10. \square

4 Effective Hamiltonians and correctors for the junction

This section is devoted to the construction of correctors at the junction and far from the junction.

4.1 Correctors far from the junction

The following propositions explicit the construction of the effective Hamiltonians and of the correctors far from the junction.

Proposition 4.1 (Homogenization on R_0). *Assume (A). Then for $p \in [-2k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution (v, ζ) of*

$$\begin{cases} N_p^0(v, [\zeta])(x) \cdot |v_x + p| = \lambda, \\ N_p^0(\zeta, [v])(x) \cdot |\zeta_x + p| = \lambda, \\ v \text{ and } \zeta \text{ are } \mathbb{Z}\text{-periodic.} \end{cases} \quad (4.1)$$

Moreover, for $p \in [-2k_0, 0]$, we have $\lambda = \bar{H}_0(p) = -V_0 \left(\frac{-1}{p} \right) |p|$.

Proposition 4.2 (Homogenization on R_1 and R_2). *Assume (A). Then for $i = 1, 2$ and for $p \in [-2k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of*

$$\begin{cases} M_p^i[v](x) \cdot |v_x + p| = \lambda, \\ v \text{ is } \mathbb{Z}\text{-periodic,} \end{cases} \quad (4.2)$$

with M_p^i defined in (3.3). Moreover, for $p \in [-2k_0, 0]$ we have $\lambda = \bar{H}_i(p) = -V_i \left(\frac{-2}{p} \right) |p|$.

To prove the previous propositions, it is only necessary to notice that $v = \zeta \equiv 0$ are obvious solutions to each problem.

4.2 Correctors at the junction

Like in [18, 19] in order to prove the convergence result we need to construct correctors for the junction. We now present the existence result of the appropriate correctors. The proof of this result is postponed to Section 6. Given $\bar{A} \in \mathbb{R}$, $\bar{A} \geq H_0$, we introduce the real numbers $\bar{p}_+^1, \bar{p}_+^2 \in \mathbb{R}$ such that

$$\bar{H}_1(\bar{p}_+^1) = \bar{H}_1^+(\bar{p}_+^1) = \bar{A} \quad \text{and} \quad \bar{H}_2(\bar{p}_+^2) = \bar{H}_2^+(\bar{p}_+^2) = \bar{A} \quad (4.3)$$

thanks to the form of \bar{H}_1 and of \bar{H}_2 , there is only one couple of real numbers satisfying (4.3). We also introduce two real number \hat{p}_-^0 and \bar{p}_-^0 defined by

$$\begin{cases} \hat{p}_-^0 = \min\{p, p \in E_0\} \\ \bar{p}_-^0 = \max\{p, p \in E_0\} \end{cases} \quad \text{with} \quad E_0 = \{p \in [-2k_0, 0], \bar{H}_0^-(p) = \bar{H}_0(p) = \bar{A}\}. \quad (4.4)$$

Note that if $\bar{A} \neq 0$, then $\hat{p}_-^0 = \bar{p}_-^0$.

Theorem 4.3 (Existence of global corrector for the junction). *Assume (A).*

i) (General properties) *There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution (w, χ) of (2.25) with $\lambda = \bar{A}$ and such that there exists a constant $C > 0$ and two globally Lipschitz continuous functions m_1 and m_2 with $m_1(0) = m_2(0) = 0$ such that, for all $x \in \mathbb{R}$,*

$$|w(x) - m_1(x)| \leq C \quad \text{and} \quad |\chi(x) - m_2(x)| \leq C, \quad (4.5)$$

and for all $x \leq h_0$,

$$|w(x) - \chi(x)| \leq C. \quad (4.6)$$

ii) (Bound from below at infinity) *There exist two constants $\gamma_0 > 0$ and $C > 0$ such that for every $\gamma \in (0, \gamma_0)$, we have for $x > 0$ and $h \geq 0$,*

$$w(x+h) - w(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad \chi(x+h) - \chi(x) \geq (\bar{p}_+^2 - \gamma)h - C \quad (4.7)$$

and for $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$

$$w(x-h) - w(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad \chi(x-h) - \chi(x) \geq (-\bar{p}_-^0 - \gamma)h - C. \quad (4.8)$$

The first inequality in (4.7) is valid only if $\bar{A} > \min \bar{H}_1$ and the second inequality is valid only if $\bar{A} > \min \bar{H}_2$. Similarly, the inequalities in (4.8) are valid only if $\bar{A} > \min \bar{H}_0$.

iii) (Rescaling) *For $\varepsilon > 0$, we set*

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon \chi\left(\frac{x}{\varepsilon}\right),$$

then (up to a sub-sequence $\varepsilon_n \rightarrow 0$) we have that w^ε and χ^ε converge locally uniformly respectively towards W and X which satisfy respectively

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_1(W_x) = \bar{A} & \text{for all } x > 0, \\ \bar{H}_0(W_x) = \bar{A} & \text{for all } x < 0. \end{cases} \quad (4.9)$$

and

$$\begin{cases} |X(x) - X(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_2(X_x) = \bar{A} & \text{for all } x > 0, \\ \bar{H}_0(X_x) = \bar{A} & \text{for all } x < 0. \end{cases} \quad (4.10)$$

In particular, we have (with $W(0) = 0 = X(0)$),

$$\bar{p}_+^1 x 1_{\{x>0\}} + \bar{p}_-^0 x 1_{\{x<0\}} \leq W(x) \leq \bar{p}_+^1 x 1_{\{x>0\}} + \hat{p}_-^0 x 1_{\{x<0\}} \quad (4.11)$$

and

$$\bar{p}_+^2 x 1_{\{x>0\}} + \bar{p}_-^0 x 1_{\{x<0\}} \leq X(x) \leq \bar{p}_+^2 x 1_{\{x>0\}} + \hat{p}_-^0 x 1_{\{x<0\}}. \quad (4.12)$$

5 Proof of convergence

This section contains the proof of Theorem 2.5, which relies on the existence of correctors provided by Propositions 4.1, 4.2 and Theorem 4.3.

We will use the following lemmas for the proof of Theorem 2.5, the first one is a direct consequence of Perron's method and Lemma 3.6 and the second one is a direct consequence of Theorem 3.9.

Lemma 5.1 (Barriers uniform in ε). *Assume (A0) and (A). Then there exists a constant $C > 0$, such that for all $\varepsilon > 0$, the solution $(u^\varepsilon, \xi^\varepsilon)$ of (2.4)-(2.9) satisfies for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct \quad \text{and} \quad |\xi^\varepsilon(t, x) - \xi_0(x)| \leq Ct.$$

Lemma 5.2 (Control of the space oscillations). *Assume (A0) and (A). Then the solution $(u^\varepsilon, \xi^\varepsilon)$ of (2.4)-(2.9) satisfies for all $t > 0$, $x, y \in \mathbb{R}$, $x \geq y$,*

$$\begin{cases} -2k_0(x - y) - 2\varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0, \\ -2k_0(x - y) - 2\varepsilon \leq \xi^\varepsilon(t, x) - \xi^\varepsilon(t, y) \leq 0. \end{cases} \quad (5.1)$$

Before doing the proof of Theorem 2.5, let us show how it will allow us to prove Theorem 2.8.

Proof of Theorem 2.8. We want to prove that for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_1$, $x \geq y$ (or for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_2$, $x \geq y$)

$$-2k_0 \cdot d(x, y) \leq u^0(t, x) - u^0(t, y) \leq 0.$$

Using Lemma 5.2, we have that \tilde{u}^ε , defined by (2.16), satisfies for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_1$, $x \geq y$ (or for all $t \in [0, +\infty)$, $x, y \in R_0 \cup R_2$, $x \geq y$),

$$-2k_0 \cdot d(x, y) - 2\varepsilon \leq \tilde{u}^\varepsilon(t, x) - \tilde{u}^\varepsilon(t, y) \leq 0.$$

Now using Theorem 2.5, and passing to the limit as $\varepsilon \rightarrow 0$, we obtain the result. □

Proof of Theorem 2.5. Let us introduce for $(t, x) \in (0, T) \times \mathbb{R}$,

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon(t, x) \quad \text{and} \quad \bar{\xi}(t, x) = \limsup_{\varepsilon \rightarrow 0}^* \xi^\varepsilon(t, x), \quad (5.2)$$

$$\underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon(t, x) \quad \text{and} \quad \underline{\xi}(t, x) = \liminf_{\varepsilon \rightarrow 0}^* \xi^\varepsilon(t, x). \quad (5.3)$$

Thanks to Lemma 5.1 these functions are well defined. We also introduce

$$\bar{v}(t, y) = \begin{cases} \max(\bar{u}(t, -d(0, y)), \bar{\xi}(t, -d(0, y))) & \text{for } (t, y) \in (0, T) \times R_0, \\ \bar{u}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_1^*, \\ \bar{\xi}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_2^*, \end{cases}$$

and

$$\underline{v}(t, y) = \begin{cases} \min(\underline{u}(t, -d(0, y)), \underline{\xi}(t, -d(0, y))) & \text{for } (t, y) \in (0, T) \times R_0, \\ \underline{u}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_1^*, \\ \underline{\xi}(t, d(0, y)) & \text{for } (t, y) \in (0, T) \times R_2^*. \end{cases}$$

We want to prove that \bar{v} and \underline{v} are respectively a sub-solution and a super-solution of (2.17). In this case, the comparison principle will imply that $\bar{v} \leq \underline{v}$, but by construction we have $\underline{v} \leq \bar{v}$, hence we will get $\underline{v} = \bar{v} = u^0$, the unique solution of (2.17).

Let us prove that \bar{v} is a sub-solution of (2.17) (the proof for v is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi \in C^2((0, +\infty) \times \bar{R})$ (in the sense of Definition 3.3), and a point $(\bar{t}, \bar{y}) \in (0, +\infty) \times \bar{R}$ such that

$$\begin{cases} \bar{v}(\bar{t}, \bar{y}) = \varphi(\bar{t}, \bar{y}) \\ \bar{v} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}) & \text{with } \bar{r} > 0 \\ \bar{v} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}) & \text{with } \eta > 0 \\ \varphi_t + \bar{H}(\bar{y}, \varphi_y(\bar{t}, \bar{y})) = \theta & \text{with } \theta > 0, \end{cases} \quad (5.4)$$

where

$$\bar{H}(\bar{y}, \varphi_y(\bar{t}, \bar{y})) = \begin{cases} \bar{H}_0(\partial_0 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_0^*, \\ \bar{H}_1(\partial_1 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_1^*, \\ \bar{H}_2(\partial_2 \varphi(\bar{t}, \bar{y})) & \text{if } \bar{y} \in R_2^*, \\ F_{\bar{A}}(\partial_0 \varphi(\bar{t}, 0), \partial_1 \varphi(\bar{t}, 0), \partial_2 \varphi(\bar{t}, 0)) & \text{if } \bar{y} = 0. \end{cases}$$

We denote by $\bar{x} = \text{sign}(\bar{y})d(0, \bar{y})$, with $\text{sign} : \bar{R} \rightarrow \mathbb{R}$,

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y \in R_1^* \cup R_2^*, \\ -1 & \text{if } y \in R_0^*, \\ 0 & \text{if } y = 0. \end{cases}$$

Given Lemma 5.2 and (5.2)-(5.3), we can assume (up to changing φ at infinity) that for ε small enough, we have

$$\begin{aligned} u^\varepsilon(\tau, \text{sign}(y)d(0, y)) &\leq \varphi(\tau, y) - \eta & \text{for } (\tau, y) \in (\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}))^c \cap (R_0 \cup R_1), \\ \xi^\varepsilon(\tau, \text{sign}(y)d(0, y)) &\leq \varphi(\tau, y) - \eta & \text{for } (\tau, y) \in (\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}))^c \cap (R_0 \cup R_2). \end{aligned} \quad (5.5)$$

Using the previous lemmas we get that the function \bar{v} satisfies for all $t > 0$, $x, y \in R_0^*$ (or $x, y \in R_1^*$ or $x, y \in R_2^*$), such that $\text{sign}(x)d(0, x) \geq \text{sign}(y)d(0, y)$,

$$-2k_0 \cdot d(x, y) \leq \bar{v}(t, x) - \bar{v}(t, y) \leq 0. \quad (5.6)$$

First case: $\bar{y} \neq 0$. We only consider $\bar{y} \in R_0^*$, since the other cases ($\bar{y} \in R_1^*$ and $\bar{y} \in R_2^*$) can be treated in the same way. We define $p = \partial_0 \varphi(\bar{t}, \bar{y})$ which, according to (5.6) satisfies

$$-2k_0 \leq p \leq 0. \quad (5.7)$$

We choose, $\bar{r} > 0$ small enough so that $\bar{x} + \bar{r} < 0$. Let us prove that the test function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(t, x) = \varphi(t, x\bar{e}_0) \quad \text{for } x < 0,$$

(notice that $\tilde{\varphi}_x(\bar{t}, \bar{x}) = \partial_0 \varphi(\bar{t}, \bar{y})$ and $\tilde{\varphi}_t(\bar{t}, \bar{x}) = \varphi_t(\bar{t}, \bar{y})$) satisfies in the viscosity sense, the following inequality for $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$,

$$\tilde{\varphi}_t + \phi \left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x), \tilde{M}_1^\varepsilon \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] (x) \right) \cdot |\tilde{\varphi}_x| \geq \frac{\theta}{2}. \quad (5.8)$$

Let us notice that for ε small enough we have for all $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, for $k = 1, 2$,

$$\phi \left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x), \tilde{M}_k^\varepsilon \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] (x) \right) = \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x). \quad (5.9)$$

For all $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have for \bar{r} small enough

$$\begin{aligned} \tilde{\varphi}_t(t, x) + \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |\tilde{\varphi}_x(t, x)| &= \tilde{\varphi}_t(\bar{t}, \bar{x}) + o_{\bar{r}}(1) \\ &+ \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |p| \\ &= \theta - \bar{H}_0(p) + o_{\bar{r}}(1) \\ &+ \tilde{N}_0^\varepsilon \left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (x) \cdot |p| \\ &=: \Delta, \end{aligned} \quad (5.10)$$

where we have used (5.4) and the definition of p . We recall that for $-2k_0 \leq p \leq 0$, we have

$$\bar{H}_0(p) = N_p^0(0, [0])(0) \cdot |p| = \tilde{N}_p^0(0, [0])(0) \cdot |p|.$$

Moreover, for all $z \in [h_0, h_{max}]$, and for ε and \bar{r} small enough we have that

$$\begin{aligned} \frac{\tilde{\varphi}(t, x + \varepsilon z) - \tilde{\varphi}(t, x)}{\varepsilon} &= z\tilde{\varphi}_x(t, x) + O(\varepsilon) \\ &\leq pz + o_{\bar{r}}(1) + c\varepsilon, \end{aligned}$$

with $c > 0$ a fixed constant. Now using the fact that \tilde{F} is decreasing we have

$$\tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) \leq \tilde{F}\left(\frac{\tilde{\varphi}(t, x + \varepsilon z) - \tilde{\varphi}(t, x)}{\varepsilon}\right).$$

Using this result and replacing the non-local operators in (5.10) by their definition we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}} + |p| \int_{h_0}^{h_{max}} J_0(z) \tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) dz \\ &\quad - |p| \int_{h_0}^{h_{max}} J_0(z) \tilde{F}(pz) dz. \end{aligned} \tag{5.11}$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-2k_0 \leq p < 0$,

$$\begin{aligned} \int_{\mathbb{R}} J_0(z) \tilde{F}(pz + c\varepsilon + o_{\bar{r}}(1)) dz &= -V_0 \left(\frac{-1 - c\varepsilon - o_{\bar{r}}(1)}{p} \right) - \frac{1}{2} V_0 \left(-\frac{-1 + c\varepsilon + o_{\bar{r}}(1)}{p} \right) + \frac{3}{2} V_{max}^0, \\ \int_{\mathbb{R}} J_0(z) \tilde{F}(pz) dz &= -V_0 \left(\frac{-1}{p} \right) + \frac{3}{2} V_{max}^0. \end{aligned} \tag{5.12}$$

Injecting (5.12) in (5.11) and choosing ε and \bar{r} small enough, we obtain

$$\begin{aligned} \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \cdot \left[-V_0 \left(\frac{-1 - c\varepsilon + o_{\bar{r}}(1)}{p} \right) + V_0 \left(\frac{-1}{p} \right) \right] \\ &\geq \theta + o_{\bar{r}}(1) - \|V_0'\|_{\infty} \cdot (c\varepsilon + o_{\bar{r}}(1)) \\ &\geq \frac{\theta}{2}, \end{aligned} \tag{5.13}$$

where we have used assumption (A1) for the second line.

Getting the contradiction. By definition, for ε small enough and using (5.5), we have

$$u^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{and} \quad \xi^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Using the comparison principle on bounded sets for (2.4), we get

$$u^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{and} \quad \xi^\varepsilon \leq \tilde{\varphi} - \eta \quad \text{on } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

Passing to the limit as $\varepsilon \rightarrow 0$, this implies $\bar{v} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$ and this contradicts the fact that $\bar{v}(\bar{t}, \bar{y}) = \varphi(\bar{t}, \bar{y})$.

Remark 5.3. In the case $\bar{y} \in R_1^*$ or $\bar{y} \in R_2^*$, (2.4) is completely decoupled for ε small enough in $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$ and therefore we can consider each line of (2.4) as independent.

Second case: $\bar{y} = 0$. Thanks to (5.6) (which implies (3.8)), we may use Theorem 3.5 and assume that the test function has the following form

$$\varphi(t, y) = g(t) + \bar{p}_-^0 y 1_{\{y \in R_0^*\}} + \bar{p}_+^1 y 1_{\{y \in R_1^*\}} + \bar{p}_+^2 y 1_{\{y \in R_2^*\}} \quad \text{on } \mathcal{Q}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \quad (5.14)$$

where g is a C^1 function defined in $(0, +\infty)$. The last line in condition (5.4) becomes

$$g'(t) + F_{\bar{A}}(\bar{p}_-^0, \bar{p}_+^1, \bar{p}_+^2) = g'(t) + \bar{A} = \theta \quad \text{at } (\bar{t}, 0). \quad (5.15)$$

Let us consider (w, χ) the solution of (2.25) provided by Theorem 4.3, and let us denote

$$\varphi^\varepsilon(t, x) = \begin{cases} g(t) + \varepsilon w\left(\frac{x}{\varepsilon}\right) & \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, x \vec{e}_0 1_{\{x \leq 0\}} + x \vec{e}_1 1_{\{x \geq 0\}}) & \text{outside } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \end{cases} \quad (5.16)$$

and

$$\psi^\varepsilon(t, x) = \begin{cases} g(t) + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) & \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, x \vec{e}_0 1_{\{x \leq 0\}} + x \vec{e}_2 1_{\{x \geq 0\}}) & \text{outside } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (5.17)$$

We claim that these functions satisfy in the viscosity sense, for \bar{r} and ε small enough, on $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$,

$$\begin{cases} \varphi_t^\varepsilon(t, x) + \phi\left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\varphi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right), \tilde{M}_1^\varepsilon\left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot)\right](x)\right) \cdot |\varphi_x^\varepsilon| \geq \frac{\theta}{2}, \\ \psi_t^\varepsilon(t, x) + \phi\left(\frac{x}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\varphi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right), \tilde{M}_2^\varepsilon\left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot)\right](x)\right) \cdot |\psi_x^\varepsilon| \geq \frac{\theta}{2}, \end{cases}$$

We only prove that φ^ε is a super-solution (the case for ψ^ε is similar and we skip it), let h be a test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$w\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),$$

and

$$w(y) \geq \frac{1}{\varepsilon} (h(t_1, \varepsilon y) - g(t_1)),$$

for y in a neighbourhood of $\frac{x_1}{\varepsilon}$. Since w does not depend on time, we have

$$h_t(t_1, x_1) = g'(t_1).$$

Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0(w, [\chi])\left(\frac{x_1}{\varepsilon}\right), \tilde{M}_1[w]\left(\frac{x_1}{\varepsilon}\right)\right) \cdot |h_x(t_1, x_1)| \geq \bar{A}.$$

This implies that (using (5.15) and taking \bar{r} small enough)

$$h_t(t_1, x_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0(w, [\chi])\left(\frac{x_1}{\varepsilon}\right), \tilde{M}_1[w]\left(\frac{x_1}{\varepsilon}\right)\right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon h_{max} \leq \bar{r}$, we deduce from the previous inequality that for all $(t, x) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x})$, we have

$$h_t(t_1, x_1) + \phi\left(\frac{x_1}{\varepsilon}, \tilde{N}_0^\varepsilon\left(\frac{\varphi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x_1), \tilde{M}_1^\varepsilon\left[\frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot)\right](x_1)\right) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}. \quad (5.18)$$

Getting the contradiction. Using (5.5) and (5.14), we have for ε small enough

$$\begin{cases} u^\varepsilon + \eta \leq g(t) + \bar{p}_-^0 x 1_{\{x < 0\}} + \bar{p}_+^1 x 1_{\{x > 0\}} \\ \xi^\varepsilon + \eta \leq g(t) + \bar{p}_-^0 x 1_{\{x < 0\}} + \bar{p}_+^2 x 1_{\{x > 0\}}, \end{cases} \quad \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0),$$

Using the fact that $w^\varepsilon \rightarrow W$ and $\zeta^\varepsilon \rightarrow X$, and using (4.11), we have for ε small enough

$$\begin{cases} u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \\ \xi^\varepsilon + \frac{\eta}{2} \leq \psi^\varepsilon, \end{cases} \quad \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Combining this result with (5.16)-(5.17), we get

$$\begin{cases} u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \\ \xi^\varepsilon + \frac{\eta}{2} \leq \psi^\varepsilon, \end{cases} \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

By the comparison principle on bounded subsets the previous inequality holds in $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$\bar{v} + \frac{\eta}{2} \leq \varphi \quad \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Evaluating this inequality in $(\bar{t}, 0)$, we obtain a contradiction.

Remark 5.4. *In order to prove that \underline{v} is a super-solution, the test function in the case $\bar{y} = 0$, needs to be*

$$\varphi(t, y) = g(t) + \hat{p}_-^0 y 1_{\{y \in R_0^*\}} + \bar{p}_+^1 y 1_{\{y \in R_1^*\}} + \bar{p}_+^2 y 1_{\{y \in R_2^*\}} \quad \text{on } \mathcal{Q}_{\bar{r}, 2\bar{r}}(\bar{t}, 0). \quad (5.19)$$

This ends the proof of Theorem 2.5. □

6 Truncated cell problem

This section contains the proof of Theorem 4.3. We proceed as in [18, 19] and we will construct correctors on a truncated domain and then pass to the limit as the size of the domain goes to infinity. For $l \in (h_0 + h_{max} + 1, +\infty)$, $h_0 \ll l$ and $h_0 + h_{max} + 1 \leq R \ll l$, we want to find $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of

$$\begin{cases} \begin{cases} G_R^1(x, w^{l,R}(x), [\chi^{l,R}], [w^{l,R}], w_x^{l,R}) = \lambda_{l,R} \\ G_R^2(x, \chi^{l,R}(x), [w^{l,R}], [\chi^{l,R}], \chi_x^{l,R}) = \lambda_{l,R} \end{cases} & \text{if } x \in (-l, l) \\ \begin{cases} \bar{H}_1^+(w_x^{l,R}) = \lambda_{l,R} \\ \bar{H}_2^+(\chi_x^{l,R}) = \lambda_{l,R} \end{cases} & \text{if } x = l \\ \begin{cases} \bar{H}_0^-(w_x^{l,R}) = \lambda_{l,R} \\ \bar{H}_0^-(\chi_x^{l,R}) = \lambda_{l,R} \end{cases} & \text{if } x = -l \end{cases} \quad (6.1)$$

with, for $j = 1, 2$,

$$G_R^j(x, w(x), [\chi], [w], q) = \psi_R^-(x) \psi_R^+(x) \phi(x, N_0(w, [\chi])(x), M_j[w](x)) |q| + (1 - \psi_R^+(x)) \bar{H}_j(q) + (1 - \psi_R^-(x)) \bar{H}_0(q), \quad (6.2)$$

and $\psi_R^\pm \in C^\infty$, $\psi_R^\pm : \mathbb{R} \rightarrow [0, 1]$, with $\psi_R^-(x) = \psi_R^+(-x)$ and

$$\psi_R^\pm \equiv \begin{cases} 1 & \text{on } (-\infty, R] \\ 0 & \text{on } [R + 1, +\infty), \end{cases} \quad \text{and } \psi_R(x) < 1 \quad \forall x \notin (-\infty, R]. \quad (6.3)$$

To G_R^j we associate \tilde{G}_R^j which is defined in the same way by replacing the non-local operators N_0 and M_j respectively by \tilde{N}_0 and \tilde{M}_j .

6.1 Comparison principle for a truncated problem

Proposition 6.1 (Comparison principle on a truncated domain). *Let us consider the following problem for $0 < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with $l_2 \gg R$.*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \tilde{G}_R^1(x, u(x), [\xi], [u], u_x) \geq \lambda \\ \tilde{G}_R^2(x, \xi(x), [u], [\xi], \xi_x) \geq \lambda \end{array} \right. & \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \overline{H}_1^+(u_x) \geq \lambda \\ \overline{H}_2^+(\xi_x) \geq \lambda \end{array} \right. & \text{if } x = l_2 \end{array} \right. \quad (6.4)$$

and for $\varepsilon_0 > 0$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} G_R^1(x, v(x), [\zeta], [v], v_x) \leq \lambda - \varepsilon_0 \\ G_R^2(x, \zeta(x), [v], [\zeta], \zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. & \text{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \overline{H}_1^+(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}_2^+(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. & \text{if } x = l_2 \end{array} \right. \quad (6.5)$$

Then if $v(l_1) \leq u(l_1)$ and $\zeta(l_1) \leq \xi(l_1)$, we have $v \leq u$ and $\zeta \leq \xi$ in $[l_1, l_2]$.

Proof. Like in [18], the only new difficulty to prove this proposition is the comparison at l_2 . But since near l_2 , the system decouples itself, we can proceed as in [19, Proposition 4.1]. \square

Remark 6.2. *We have a similar result if we exchange the boundary conditions, that is to say for $l_1 < l_2 < -h_0 - h_{max} - 1$, and if for all $x \in [l_2, l_2 + h_{max}]$ $v(x) \leq u(x)$ and $\zeta(x) \leq \xi(x)$ and the following conditions are imposed at $x = l_1$,*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \overline{H}_0^-(u_x) \geq \lambda \\ \overline{H}_0^-(\xi_x) \geq \lambda \end{array} \right. & \text{if } x = l_1 \\ \left\{ \begin{array}{l} \overline{H}_0^-(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}_0^-(\zeta_x) \leq \lambda - \varepsilon_0 \end{array} \right. & \text{if } x = l_1. \end{array} \right. \quad (6.6)$$

6.2 Existence of correctors on a truncated domain

Proposition 6.3 (Existence of correctors on a truncated domain). *There exists a constant $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of (6.1). Moreover, there exists a constant C (depending only on k_0) and two Lipschitz continuous functions $m_1^{l,R}$ and $m_2^{l,R}$, such that*

$$\left\{ \begin{array}{l} H_0 = \max_{i \in \{0,1,2\}} \min_{p \in \mathbb{R}} \overline{H}_i(p) \leq \lambda_{l,R} \leq 0, \\ |w^{l,R}(x) - m_1^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |\chi^{l,R}(x) - m_2^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |m_i^{l,R}(x) - m_i^{l,R}(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \quad \text{for } i = 1, 2 \\ |w^{l,R}(x) - \chi^{l,R}(x)| \leq C & \text{for all } x \in [-l, h_0]. \end{array} \right. \quad (6.7)$$

Proof. We consider the approximated truncated cell problem,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \delta v^\delta + G_R^1(x, v^\delta(x), [\zeta^\delta], [v^\delta], v_x^\delta) = 0 \\ \delta \zeta^\delta + G_R^2(x, \zeta^\delta(x), [v^\delta], [\zeta^\delta], \zeta_x^\delta) = 0 \end{array} \right. & \text{if } x \in (-l, l) \\ \left\{ \begin{array}{l} \delta v^\delta + \overline{H}_1^+(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}_2^+(\zeta_x^\delta) = 0 \end{array} \right. & \text{if } x = l \\ \left\{ \begin{array}{l} \delta v^\delta + \overline{H}_0^-(v_x^\delta) = 0 \\ \delta \zeta^\delta + \overline{H}_0^-(\zeta_x^\delta) = 0 \end{array} \right. & \text{if } x = -l \end{array} \right. \quad (6.8)$$

Step 1: existence of a solution. Using that $(0, 0)$ and $(C_0/\delta, C_0/\delta)$ are respectively sub and super-solution of (6.8), with $C_0 = |\min_{p \in \mathbb{R}} \overline{H}_0(p)|$ and that we have a comparison principle, we deduce that there exists a continuous viscosity solution (v^δ, ζ^δ) of (6.8) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta} \quad \text{and} \quad 0 \leq \zeta^\delta \leq \frac{C_0}{\delta}. \quad (6.9)$$

Step 2: control of the oscillations of v^δ and ζ^δ .

Lemma 6.4. *The functions v^δ and ζ^δ satisfy for all $x, y \in [-l, l]$, $x \geq y$,*

$$-K_0(x - y) - 2 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{and} \quad -K_0(x - y) - 2 \leq \zeta^\delta(x) - \zeta^\delta(y) \leq 0, \quad (6.10)$$

with $K_0 = 2k_0$.

Proof of Lemma 6.4. In the rest of the proof we will use the following notation

$$\hat{\Omega} = \{(x, y) \in [-l, l]^2 \text{ s.t. } x \geq y\}.$$

Proof of the upper inequality. We want to prove that

$$M = \sup_{(x, y) \in \hat{\Omega}} \max(v^\delta(x) - v^\delta(y), \zeta^\delta(x) - \zeta^\delta(y)) \leq 0. \quad (6.11)$$

We argue by contradiction and assume that $M > 0$. Since Ω is compact and v^δ and ζ^δ are continuous, we can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \hat{\Omega}$. Given that $M > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (6.8).

Let us assume for instance that $M = v^\delta(\bar{x}) - v^\delta(\bar{y})$ (the other case is similar so we skip it).

We distinguish 3 cases.

-If $(\bar{x}, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], 0) &\leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], 0) &\geq 0 \end{aligned}$$

combining these inequalities with the fact that $G(x, U, [\Xi], [U], 0) = 0$, we obtain

$$\delta M \leq 0.$$

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain similarly

$$\delta M \leq 0, \quad (6.12)$$

using the fact that $\overline{H}_1^+(0) = 0$.

-If $\bar{x} \in (-l, l]$ and $\bar{y} = -l$, we obtain

$$\delta M \leq \overline{H}_0^-(0) \leq 0.$$

For every value of \bar{x}, \bar{y} we obtain a contradiction, therefore $M \leq 0$.

Proof of the lower inequality. In order to proof this inequalities, we will use the following lemma which proof is postponed.

Lemma 6.5. *For all $x \in [-l, -h_0]$, we have*

$$|\zeta^\delta(x) - v^\delta(x)| \leq 1. \quad (6.13)$$

We want to prove that

$$M = \sup_{(x,y) \in \hat{\Omega}} \max (v^\delta(y) - v^\delta(x) - K_0(x-y) - 2, \zeta^\delta(y) - \zeta^\delta(x) - K_0(x-y) - 2) \leq 0.$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \hat{\Omega}$. Since $M > 0$, we deduce that $\bar{x} > \bar{y}$. Therefore, we can use the viscosity inequalities from (6.8).

Let us for instance assume that $M = v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2$ (the other case is similar and we skip it).

We distinguish 4 cases.

-If $\bar{x}, \bar{y} \in (-l, l)$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) &\geq 0, \end{aligned}$$

combining these inequalities and using the definition of M , we obtain

$$\delta M \leq \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) - G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0). \quad (6.14)$$

We recall that the non-local operators are non-positive (see Remark 2.10) and that $\bar{H}_i(-K_0) = 0$ for $i = 0, 1, 2$. This implies that

$$\delta M \leq -G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) = \psi_R^-(\bar{y})\psi_R^+(\bar{y})\phi(\bar{y}, N_0(v^\delta, [\zeta^\delta])(\bar{y}), M_1[v^\delta](\bar{y})) \cdot K_0. \quad (6.15)$$

To treat (6.15), we will compute $M_1[v^\delta](\bar{y})$ and $N_0(v^\delta, [\zeta^\delta])(\bar{y})$.

Let us prove that $M_1[v^\delta](\bar{y}) = 0$. In fact, it is sufficient to prove that for all $z \in (h_0, h_{max}]$, we have

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) < -2.$$

Let $z \in (h_0, h_{max}]$, if $\bar{y} + z \leq \bar{x}$, then using the fact that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y} - z) - 2 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2,$$

we deduce that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq -K_0 z < -2.$$

If $\bar{y} + z > \bar{x}$, using the fact that v^δ is non-increasing and that $M > 0$, we obtain

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq v^\delta(\bar{x}) - v^\delta(\bar{y}) \leq -K_0(\bar{x} - \bar{y}) - 2 < -2.$$

Using that $N_0(v^\delta, [\zeta^\delta])(\bar{y}) \leq 0$, and that $M_1[v^\delta](\bar{y}) = 0$, we get

$$\max (N_0(v^\delta, [\zeta^\delta])(\bar{y}), M_1[v^\delta](\bar{y})) = 0.$$

We can therefore assume that $-R - 1 \leq \bar{y} \leq -h_0 - h_{max}$ (if not we get a contradiction in (6.15)).

Let us now prove that for all $-R - 1 \leq \bar{y} \leq -h_0 - h_{max}$, $N_0(v^\delta, [\zeta^\delta])(\bar{y}) = 0$. In fact, it suffices to prove that for all $z \in (h_0, h_{max}]$, we have

$$\zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) < -1.$$

Let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using the fact that ζ^δ is non-increasing and Lemma 6.5 (since $\bar{y} + z \leq \bar{y} + h_{max} \leq -h_0$), we obtain

$$\begin{aligned} \zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) &\leq v^\delta(\bar{y} + z) - v^\delta(\bar{y}) + 1 \\ &\leq v^\delta(\bar{x}) - v^\delta(\bar{y}) + 1 \\ &\leq -K_0(\bar{x} - \bar{y}) - 1 < -1, \end{aligned}$$

where we used the fact that $M > 0$ for the third line. If $\bar{y} + z < \bar{x}$, using Lemma 6.5 and that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y} - z) - 2 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - K_0(\bar{x} - \bar{y}) - 2,$$

we obtain

$$\begin{aligned} \zeta^\delta(\bar{y} + z) - v^\delta(\bar{y}) &\leq v^\delta(\bar{y} + z) - v^\delta(\bar{y}) + 1 \\ &\leq 1 - K_0 z < -1. \end{aligned}$$

Injecting the previous results into (6.15), we get $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + \overline{H}_0^-(-K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) &\geq \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], [v^\delta], -K_0) \geq 0. \end{aligned}$$

Using the fact that $\overline{H}_0^-(-K_0) = 0$ we obtain $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + G_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}_1^+(-K_0) &\geq 0, \end{aligned}$$

using the result of the first case, and the fact that $\overline{H}_1^+(-K_0) < 0$, we directly obtain $\delta M \leq 0$, which is a contradiction.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\begin{aligned} \delta v^\delta(\bar{y}) + \overline{H}_0^-(-K_0) &\leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}_1^+(-K_0) &\geq 0, \end{aligned}$$

just like before, we obtain $\delta M \leq 0$, which is a contradiction.

For every value of $\bar{x}, \bar{y} \in [-l, l]$ we obtain a contradiction, therefore we have $M \leq 0$. This ends the proof of Lemma 6.4. \square

Step 3: construction of the Lipschitz continuous function.

Lemma 6.6. *There exists two Lipschitz continuous functions m_1^δ and m_2^δ , such that there exists a constant $C > 0$ (independent of l and R) such that*

$$\begin{cases} |v^\delta(x) - m_1^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |\zeta^\delta(x) - m_2^\delta(x)| \leq C & \text{for all } x \in [-l, l], \\ |m_i^\delta(x) - m_i^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \text{ for } i = 1, 2. \end{cases} \quad (6.16)$$

Moreover, for all $x \in [-l, h_0]$, we have

$$|v^\delta(x) - \zeta^\delta(x)| \leq C. \quad (6.17)$$

Proof. We only do the construction of m_1^δ , since the construction of m_2^δ is similar and we skip it. We define m_1^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m_1^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m_1^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m_1^δ and v^δ are non-increasing, and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \leq K_0 h_0 + 2 = 4$, we deduce that for all $x \in [ih_0, (i+1)h_0]$,

$$-4 \leq v^\delta((i+1)h_0) - m_1^\delta(ih_0) \leq v^\delta(x) - m_1^\delta(x) \leq v^\delta(ih_0) - m_1^\delta((i+1)h_0) \leq 4, \quad (6.18)$$

and for all $x, y \in [-l, l]$,

$$|m_1^\delta(x) - m_1^\delta(y)| \leq 2K_0|x - y|. \quad (6.19)$$

We build m_2^δ the same way but using ζ^δ instead of v^δ .

Moreover, let $x \in [-l, h_0]$, first if $x \leq -h_0$, using Lemma 6.5, we have that

$$|v^\delta(x) - \zeta^\delta(x)| \leq 1.$$

If $x \in [-h_0, h_0]$, we have

$$\begin{aligned} |v^\delta(x) - \zeta^\delta(x)| &\leq |v^\delta(x) - m_1(x)| + |\zeta^\delta(x) - m_2(x)| + |m_1(x) - m_2(x)| \\ &\leq 8 + |m_1(x) - m_1(-h_0)| + |m_2(-h_0) - m_2(x)| + |m_1(-h_0) - m_2(-h_0)| \\ &\leq 8 + 1 + 4K_0|x + h_0| \\ &\leq 9 + 8K_0h_0 = 25, \end{aligned}$$

where we have used for the third inequality the fact that

$$|m_1(-h_0) - m_2(-h_0)| = |v^\delta(-h_0) - \zeta^\delta(-h_0)| \leq 1$$

and for the last inequality, the fact that $x \in [-h_0, h_0]$.

Choosing $C = \max(2K_0, 25)$ we obtain (6.16) and (6.17). This ends the proof of Lemma 6.6. \square

Step 4: passing to the limit as δ goes to 0. Using (6.9), Lemma 6.6 and (6.16), we deduce that there exists a subsequence $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \delta_n v^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ \delta_n \zeta^{\delta_n}(0) &\rightarrow -\lambda_{l,R} && \text{as } n \rightarrow +\infty, \\ m_i^{\delta_n} - m_i^{\delta_n}(0) &\rightarrow m_i^{l,R} && \text{as } n \rightarrow +\infty \quad \text{for } i = 1, 2. \end{aligned}$$

The last convergence being locally uniform. Let us consider,

$$\bar{w}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \rightarrow 0} (v^{\delta_n} - v^{\delta_n}(0))$$

and

$$\bar{\chi}^{l,R} = \limsup_{\delta_n \rightarrow 0}^* (\zeta^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{\chi}^{l,R} = \liminf_{\delta_n \rightarrow 0} (\zeta^{\delta_n} - v^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}$, $\bar{w}^{l,R}$, $\underline{w}^{l,R}$, $\bar{\chi}^{l,R}$, $\underline{\chi}^{l,R}$ and $m_i^{l,R}$, satisfy for $i = 1, 2$,

$$\begin{aligned} \min_{p \in \mathbb{R}} \bar{H}_0(p) &\leq \lambda_{l,R} \leq 0, \\ |\bar{w}^{l,R} - m_1^{l,R}| &\leq C, \quad |\underline{w}^{l,R} - m_1^{l,R}| \leq C, \\ |\bar{\chi}^{l,R} - m_2^{l,R}| &\leq C, \quad |\underline{\chi}^{l,R} - m_2^{l,R}| \leq C, \\ &|\partial_x m_i^{l,R}| \leq C, \end{aligned} \quad (6.20)$$

and for $x \leq h_0$,

$$|\bar{\chi}^{l,R} - \underline{w}^{l,R}| \leq C, \quad |\underline{\chi}^{l,R} - \bar{w}^{l,R}| \leq C. \quad (6.21)$$

By stability of viscosity solutions, we have that $(\bar{w}^{l,R} - 2C, \bar{\chi}^{l,R} - 2C)$ and $(\underline{w}^{l,R}, \underline{\chi}^{l,R})$ are respectively a sub-solution and a super-solution of (6.1), and

$$\bar{w}^{l,R} - 2C \leq \underline{w}^{l,R} \quad \text{and} \quad \bar{\chi}^{l,R} - 2C \leq \underline{\chi}^{l,R}.$$

By Perron's method, we can construct a solution $(w^{l,R}, \chi^{l,R})$ of (6.1) and thanks to (6.20)-(6.21), $m^{l,R}$, $w^{l,R}$, $\chi^{l,R}$ and $\lambda_{l,R}$ satisfy (6.7) in which the first inequality is replaced by

$$\min_{p \in \mathbb{R}} \overline{H}_0(p) \leq \lambda_{l,R} \leq 0.$$

The first inequality of (6.7) comes from the fact that near $\{-l\}$ and $\{l\}$, (6.1) contains only local operators \overline{H}_i for $i = 0, 1, 2$. Indeed, using [2, Lemme B.1], we can touch from above, near $\{-l\}$ or $\{l\}$, the functions $(w^{l,R}, \chi^{l,R})$ with a regular test function φ . This implies that $\overline{H}_i(\varphi_x) \leq \lambda_{l,R}$ for $i = 0, 1$ or 2 which gives

$$\max_{i \in \{0,1,2\}} \min_{p \in \mathbb{R}} \overline{H}_i(p) \leq \lambda_{l,R}. \quad (6.22)$$

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 6.3. \square

Proof of Lemma 6.5. This proof uses similar arguments to ones used in the proof of Theorem 3.9. We only prove that for all $x \in [-l, -h_0]$ we have $\zeta^\delta(x) - v^\delta(x) \leq 1$, the proof for the second part of the result is similar and we skip it. First, let us prove the inequality on the interval $[-R, -h_0]$.

Step 1: $x \in [-R, -h_0]$. Let $z \in (-R, -h_0)$ and $\alpha > 0$ be some small parameter such that $z \in [-R + \alpha, -h_0 - \alpha]$. We denote by

$$\Delta = \{x \in [-l, l] \text{ s.t. } |z - x| \leq \alpha\}.$$

We want to prove that for $L := 4C_0/(\delta\alpha^2)$, we have

$$\zeta^\delta(x) - v^\delta(x) \leq 1 + L(x - z)^2 \quad \text{for all } x \in \Delta. \quad (6.23)$$

In fact, if we take $x = z$ in the previous inequality, we obtain

$$\zeta^\delta(x) - v^\delta(x) \leq 1 \quad \text{for all } z \in (-R, -h_0)$$

Using the continuity of v^δ and ζ^δ , we deduce the result in $[-R, -h_0]$.

In order to prove (6.23), we introduce

$$M = \sup_{x \in \Delta} (\zeta^\delta(x) - v^\delta(x) - 1 - L(x - z)^2). \quad (6.24)$$

We argue by contradiction and assume that $M > 0$.

Step 1.1: the test function. We define for ε a small parameter, the function

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(y) - 1 - \frac{L}{2}(x - z)^2 - \frac{L}{2}(y - z)^2 - \frac{(x - y)^2}{2\varepsilon}.$$

The function φ reaches a maximum at a finite point (\bar{x}, \bar{y}) in the domain

$$\{(x, y) \in [-l, l]^2 \text{ s.t. } |z - x| \leq \alpha \text{ and } |z - y| \leq \alpha\}. \quad (6.25)$$

By classical arguments, we deduce that

$$\begin{cases} 0 < M \leq M_\varepsilon \\ |\bar{x} - \bar{y}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \\ \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{cases} \quad (6.26)$$

Step 1.2: $|\bar{x} - z| < \alpha$ and $|\bar{y} - z| < \alpha$. By contradiction, assume that $|\bar{x} - z| = \alpha$ or $|\bar{y} - z| = \alpha$, in which case, we have

$$\begin{aligned} 0 &< \varphi(\bar{x}, \bar{y}) = \zeta^\delta(\bar{x}) - v^\delta(\bar{y}) - 1 - \frac{L}{2}(\bar{x} - z)^2 - \frac{L}{2}(\bar{y} - z)^2 - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \\ &\leq 2\frac{C_0}{\delta} - \frac{L}{2}\alpha^2 = 0, \end{aligned} \quad (6.27)$$

where we have used (6.9) and the fact that $L = 4C_0/(\delta\alpha^2)$ for the second line, which gives us a contradiction.

Step 1.3: utilisation of the equation. Using the viscosity inequalities we obtain

$$\begin{aligned} \delta\zeta^\delta(\bar{x}) + G_R^2(\bar{x}, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], L(\bar{x} - z) + p_\varepsilon) &\leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -L(\bar{y} - z) + p_\varepsilon) &\geq 0, \end{aligned}$$

with $p_\varepsilon = (\bar{x} - \bar{y})/\varepsilon$. Combining these inequalities and using the definition of M , we obtain

$$\begin{aligned} \delta M &\leq \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], [v^\delta], -L(\bar{y} - z) + p_\varepsilon) - G_R^2(\bar{x}, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], L(\bar{x} - z) + p_\varepsilon) \\ &\leq \phi(\bar{y}, \tilde{N}_0(v^\delta, [\zeta^\delta])(\bar{y}), \tilde{M}_1^\delta[v^\delta](\bar{y})) \cdot | -L(\bar{y} - z) + p_\varepsilon | \\ &\quad - \phi(\bar{x}, N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon| \\ &\leq -\phi(\bar{x}, N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon|, \end{aligned} \quad (6.28)$$

where we have used the fact that the non-local operators are non-positive (Remark 2.10) and that $\bar{x}, \bar{y} \in [-R, -h_0]$ which implies that $\psi_R^\pm(\bar{y}) = \psi_R^\pm(\bar{x}) = 1$.

Let us prove that $N_0(\zeta^\delta, [v^\delta])(\bar{x}) = 0$. It is sufficient to prove that for all $h \in [h_0, h_{max}]$, we have

$$v^\delta(\bar{x} + h) - \zeta^\delta(\bar{x}) < -1.$$

In fact, using (6.26), for ε small enough, we can assume that for all $h \in [h_0, h_{max}]$, we have $\bar{x} + h > \bar{y}$. Using the fact that v^δ is non-increasing and that $M > 0$, we have

$$v^\delta(\bar{x} + h) - \zeta^\delta(\bar{x}) \leq v^\delta(\bar{y}) - \zeta^\delta(\bar{x}) < -1.$$

Using that the non-local operators are non-positive, we have that

$$\max(N_0(\zeta^\delta, [v^\delta])(\bar{x}), M_2[\zeta^\delta](\bar{x})) = 0.$$

Given the form of the transition function (see Remark 2.11), (6.28) becomes

$$\delta M \leq -\tilde{\phi}(\bar{x}, 0, 0, M_2[\zeta^\delta](\bar{x})) \cdot |L(\bar{x} - z) + p_\varepsilon| = 0. \quad (6.29)$$

This gives us a contradiction and therefore $M \leq 0$.

Step 2: $x \in [-l, -R]$. We introduce

$$M = \sup_{x \in [-l, -R]} (\zeta^\delta(x) - v^\delta(x) - 1). \quad (6.30)$$

We want to prove that $M \leq 0$ and we argue by contradiction and by assuming that $M > 0$. Given that ζ^δ and v^δ are continuous, M is reached for a point $\bar{x} \in [-l, -R]$. First, let us notice that thanks to Step 1, M cannot be reached for $\bar{x} = -R$. We distinguish 2 cases.

Case 1: $\bar{x} \in (-l, -R)$. We introduce

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(x) - 1 - \frac{(x - y)^2}{2\varepsilon} - \frac{1}{2}((x - \bar{x})^2 + (y - \bar{x})^2),$$

which reaches a maximum at a point $(x_\varepsilon, y_\varepsilon) \in [-l, -R]^2$. We denote by $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$. By classical arguments, we can prove that

$$\lim_{\varepsilon \rightarrow 0} M_\varepsilon = M, \quad \lim_{\varepsilon \rightarrow 0} \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} = 0, \quad \text{and} \quad (x_\varepsilon, y_\varepsilon) \rightarrow (\bar{x}, \bar{x}) \text{ as } \varepsilon \text{ goes to } 0. \quad (6.31)$$

Therefore, for ε small enough, we can assume that $x_\varepsilon, y_\varepsilon \in (-l, -R)$. Using the viscosity inequalities, we obtain

$$\begin{aligned} \delta\zeta^\delta(x_\varepsilon) + G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], [\zeta^\delta], (x_\varepsilon - \bar{x}) + p_\varepsilon) &\leq 0 \\ \delta v^\delta(y_\varepsilon) + \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], [v^\delta], -(y_\varepsilon - \bar{x}) + p_\varepsilon) &\geq 0, \end{aligned}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of M , we obtain

$$\begin{aligned} \delta M &\leq \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], [v^\delta], -(y_\varepsilon - \bar{x}) + p_\varepsilon) \\ &\quad - G_R^2(x_\varepsilon, \zeta^\delta(\bar{x}), [v^\delta], [\zeta^\delta], (x_\varepsilon - \bar{x}) + p_\varepsilon) \\ &\leq (1 - \psi_R^-(y_\varepsilon))\bar{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\bar{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon) \\ &\quad + \psi_R^-(y_\varepsilon)\phi(y_\varepsilon, \tilde{N}_0(v^\delta, [\zeta^\delta])(y_\varepsilon), \tilde{M}_1^\delta[v^\delta](y_\varepsilon)) \cdot |-(y_\varepsilon - \bar{x}) + p_\varepsilon| \\ &\quad - \psi_R^-(x_\varepsilon)\phi(x_\varepsilon, N_0(\zeta^\delta, [v^\delta])(x_\varepsilon), M_2[\zeta^\delta](x_\varepsilon)) \cdot |(x_\varepsilon - \bar{x}) + p_\varepsilon| \\ &\leq (1 - \psi_R^-(y_\varepsilon))\bar{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\bar{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon) \\ &\quad - \psi_R^-(x_\varepsilon)N_0(\zeta^\delta, [v^\delta])(x_\varepsilon) \cdot |(x_\varepsilon - \bar{x}) + p_\varepsilon|, \end{aligned} \quad (6.32)$$

where we have used that $\phi(x_\varepsilon, a, b) = a$ for $x_\varepsilon \leq -h_0 - h_{max} - 1$.

Using that $N_0(\zeta^\delta, [v^\delta])(x_\varepsilon) = 0$ (see Step 1), we get

$$\delta M \leq (1 - \psi_R^-(y_\varepsilon))\bar{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - (1 - \psi_R^-(x_\varepsilon))\bar{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon). \quad (6.33)$$

This implies that

$$\begin{aligned} \delta M &\leq (1 - \psi_R^-(y_\varepsilon))(\bar{H}_0(-(y_\varepsilon - \bar{x}) + p_\varepsilon) - \bar{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon)) \\ &\quad + \|(\psi_R^-)'\|_\infty |x_\varepsilon - y_\varepsilon| |\bar{H}_0((x_\varepsilon - \bar{x}) + p_\varepsilon)| \\ &\leq \|\bar{H}_0'\|_\infty (|x_\varepsilon - \bar{x}| + |y_\varepsilon - \bar{x}|) + \|(\psi_R^-)'\|_\infty |x_\varepsilon - y_\varepsilon| \cdot V_{max}^0 |x_\varepsilon - \bar{x}| + p_\varepsilon| \\ &\leq o_\varepsilon(1) \end{aligned} \quad (6.34)$$

where we have used the regularity of ψ_R^- for the first inequality, the form of \bar{H}_0 (the fact that $|\bar{H}_0(p)| \leq V_{max}^0 |p|$, see (2.10)), and (6.31) for the last inequality.

Injecting (6.34) into (6.33), we obtain that

$$\delta M \leq o_\varepsilon(1),$$

which gives a contradiction for ε small enough.

Case 2: $\bar{x} = -l$. We introduce

$$\varphi(x, y) = \zeta^\delta(x) - v^\delta(x) - 1 - \varepsilon G^\gamma \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) - \frac{1}{2}((x - \bar{x})^2 + (y - \bar{x})^2),$$

where G^γ is the vertex test function introduced by Imbert and Monneau in [22, Theorem 3.2]. We fix $\gamma = \delta M/2$. We refer to [22] for a detailed description on the vertex test function and on how to build it, but for the readers convenience we recall the properties we use in this proof.

1. (Regularity)

$$G^\gamma \in C([-l, l]^2) \quad \begin{cases} G^\gamma(x, \cdot) \in C^1([-l, l]) & \text{for all } x \in [-l, l] \\ G^\gamma(\cdot, y) \in C^1([-l, l]) & \text{for all } y \in [-l, l]. \end{cases} \quad (6.35)$$

2. (Bound from below) $G^\gamma \geq 0 = G(0, 0)$.

3. (Super-linearity) There exists $g : [0, +\infty) \rightarrow \mathbb{R}$ non-decreasing and such that for all $(x, y) \in [-l, l]^2$

$$g(|x - y|) \leq G^\gamma(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty.$$

4. (Compatibility condition on the gradient)

$$\bar{H}_0(y, -G_y^\gamma(x, y)) - \bar{H}_0(x, G_x^\gamma(x, y)) \leq \gamma, \quad (6.36)$$

with, for all $x \in [-l, l]$ and $p \in \mathbb{R}$,

$$\bar{H}_0(x, p) = \begin{cases} \bar{H}_0^-(p) & \text{if } x = -l \\ \bar{H}_0(p) & \text{if } x \in (-l, l] \end{cases} \quad (6.37)$$

Like in the previous case, the test function φ reaches a maximum at a point $(x_\varepsilon, y_\varepsilon)$ and (6.31) remains valid with $\bar{x} = -l$. Proceeding like before, we obtain (6.32) but for ε small enough the only terms that remain are the local ones. Therefore, we have

$$\begin{aligned} \delta M &\leq \bar{H}_0\left(y_\varepsilon, -(y_\varepsilon - \bar{x}) - G_y^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) - \bar{H}_0\left(x_\varepsilon, x_\varepsilon - \bar{x} + G_x^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) \\ &\leq \bar{H}_0\left(y_\varepsilon, -G_y^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) - \bar{H}_0\left(x_\varepsilon, G_x^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right)\right) + \|\bar{H}'_0\|_\infty (|x_\varepsilon + l| + |y_\varepsilon + l|) \\ &\leq \gamma + o_\varepsilon(1). \end{aligned} \quad (6.38)$$

This implies that $\delta M/2 \leq o_\varepsilon(1)$ and we obtain a contradiction for ε small enough. \square

Proposition 6.7 (First definition of the flux limiter). *The following limits exist (up to some sub-sequence)*

$$\begin{cases} \bar{A}_R = \lim_{l \rightarrow +\infty} \lambda_{l,R}, \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R. \end{cases} \quad (6.39)$$

Moreover, we have

$$H_0 \leq \bar{A}, \bar{A}_R \leq 0.$$

Proof. This proof is a direct consequence of (6.7). \square

Proposition 6.8 (Control of the slopes on a truncated domain). *Assume that l and R are large enough. Let $(w^{l,R}, \chi^{l,R})$ be the solution of (6.1) given by Proposition 6.3.*

If we assume up to a sub-sequence that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R} > \min \bar{H}_1$. Then there exist $\gamma_0 > 0$ and a constant $C > 0$ (independent of l and R) such that for all $\gamma \in (0, \gamma_0)$ and for all $x \geq 0$ and $h \geq 0$

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad (6.40)$$

Similarly, if we assume that $\bar{A} > \min \bar{H}_2$ then we have for all $x \geq 0$ and $h \geq 0$

$$\chi^{l,R}(x+h) - \chi^{l,R}(x) \geq (\bar{p}_+^2 - \gamma)h - C. \quad (6.41)$$

Finally, if we assume that $\bar{A} > \min \bar{H}_0$, we have for all $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad \text{and} \quad \chi^{l,R}(x-h) - \chi^{l,R}(x) \geq (-\bar{p}_-^0 - \gamma)h - C. \quad (6.42)$$

Proof. We do the proof of Proposition 6.8 in two steps.

Step 1: proof of (6.40)-(6.41). We do the proof only for $w^{l,R}$, since the truncated cell problem (6.1) decouples itself for $x > 0$. The proof for $\chi^{l,R}$ is similar and we skip it. For $\mu > 0$, small enough, we denote by p_μ^1 the real numbers defined by

$$\bar{H}_1(p_\mu^1) = \bar{H}_1^+(p_\mu^1) = \lambda_{l,R} - \mu. \quad (6.43)$$

Using that

$$\min \bar{H}_1 < \lambda_{l,R} \leq 0,$$

we deduce that p_μ^1 exists for μ small enough and $p_\mu^1 \in [-2k_0, 0)$.

Let us now consider $w^+ = p_\mu^1 x$ that satisfies

$$\bar{H}_1(w_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}. \quad (6.44)$$

We also have

$$\begin{aligned} M_1[w^+](x) &= \int_{\mathbb{R}} J_1(z) E(p_\mu^1(x+z) - p_\mu^1 x) dz - \frac{3}{2} V_{max}^1 \\ &= \int_0^{\frac{-2}{p_\mu^1}} \frac{1}{2} J_1(z) dz + \int_{\frac{-2}{p_\mu^1}}^{+\infty} \frac{3}{2} J_1(z) dz - \frac{3}{2} V_{max}^1 \\ &= -V_1 \left(\frac{-2}{p_\mu^1} \right). \end{aligned}$$

For all $x \in (0, l)$, we have

$$M_1[w^+](x) \cdot |w_x^+| = -V_1 \left(\frac{-2}{p_\mu^1} \right) \cdot |p_\mu^1| = \bar{H}_1(p_\mu^1) = \lambda_{l,R} - \mu. \quad (6.45)$$

Using (6.45), we can see that the restriction of w^+ to $(0, l]$ satisfies

$$\begin{cases} G_R^1(x, w^+, [\chi^+], [w^+], w_x^+) = \lambda_{l,R} - \mu & \text{if } x \in (0, l) \\ \bar{H}_1^+(w_x^+) = \lambda_{l,R} - \mu & \text{if } x = l, \end{cases}$$

notice that since $x > 0$ there is no actual dependence on χ^+ on the previous equation.

Let us introduce, for some $x_0 \in (0, l]$,

$$g = w^{l,R} - w^{l,R}(x_0) \quad \text{and} \quad u = w^+ - w^+(x_0) \quad (6.46)$$

Then we have

$$g(x_0) = 0 \geq u(x_0). \quad (6.47)$$

Using that g is a solution of (6.4) (in $(0, l]$ the solutions are invariant by addition of constants) and u is a solution of (6.5) (with $\varepsilon_0 = \mu$), joint to the comparison principle (Proposition 6.1), we get that

$$w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^1(x - x_0)$$

This implies that for all $h \geq 0$, and for all $x \in (0, l)$,

$$w^{l,R}(x+h) - w^{l,R}(x) \geq p_\mu^1 h$$

Finally, if we choose $\gamma_0 < |p_1 - \bar{p}_+^1|$, then we have

$$\bar{H}_1(\bar{p}_+^1 - \gamma) = \bar{H}_1^+(\bar{p}_+^1 - \gamma).$$

Choosing $\mu > 0$ such that

$$p_\mu^1 = \bar{p}_+^1 - \gamma.$$

we obtain (6.40).

Step 2: proof of (6.42). The arguments are similar to the previous ones but for the readers convenience, we detail where the two proofs differ. For $\mu > 0$, small enough, we denote by p_μ^0 a solution of

$$\bar{H}_0(p_\mu^0) = \bar{H}_0^-(p_\mu^0) = \lambda_{l,R} - \mu$$

Moreover, since $\min \bar{H}_0 < \lambda_{l,R}$, for μ small enough, we have $p_\mu^0 \in [-2k_0, 0)$. We introduce $w^- = p_\mu^0 x = \chi^-$. Like before, we can see that the restriction of w^- to $[-l, -h_0 - h_{max} - 1)$ satisfies

$$\begin{cases} G_R^1(x, w^-, [\chi^-], [w^-], w_x^-) = \lambda_{l,R} - \mu & \text{if } x \in (-l, -h_0 - h_{max} - 1) \\ \bar{H}_0^-(w_x^-) = \lambda_{l,R} - \mu & \text{if } x = -l. \end{cases}$$

Similarly, the restriction of χ^- to $[-l, -h_{max} - h_0 - 1)$ satisfies

$$\begin{cases} G_R^2(x, \chi^-, [w^-], [\chi^-], \chi_x^-) = \lambda_{l,R} - \mu & \text{if } x \in (-l, -h_0 - h_{max} - 1) \\ \bar{H}_0^-(\chi_x^-) = \lambda_{l,R} - \mu & \text{if } x = -l. \end{cases}$$

Let us introduce, for some $x_0 \in [-l, -h_0 - h_{max} - 1)$,

$$\begin{cases} g = w^{l,R} - w^{l,R}(x_0) \\ h = \chi^{l,R} - w^{l,R}(x_0), \end{cases} \quad \text{and} \quad \begin{cases} u = w^- - w^-(x_0) - \tilde{C} \\ v = \chi^- - \chi^-(x_0) - \tilde{C}, \end{cases} \quad (6.48)$$

with $\tilde{C} = Ch_{max} + 3C$ where $C > 0$ is the constant provided by Proposition 6.3. Then, using Proposition 6.3, we have for all $x \in [x_0, x_0 + h_{max}]$

$$\begin{aligned} g(x) = w^{l,R}(x) - w^{l,R}(x_0) &\geq -C|x - x_0| - 2C \geq -Ch_{max} - 2C \geq -\tilde{C} \\ &\geq -\tilde{C} + p_\mu^0(x - x_0) = u(x) \end{aligned}$$

and

$$\begin{aligned} h(x) = \chi^{l,R}(x) - w^{l,R}(x_0) &= \chi^{l,R}(x) - \chi^{l,R}(x_0) + \chi^{l,R}(x_0) - w^{l,R}(x_0) \\ &\geq -Ch_{max} - 2C - C \geq -\tilde{C} \geq -\tilde{C} + p_\mu^0(x - x_0) = v(x). \end{aligned}$$

Using the comparison principle (Proposition 6.1) with the boundary conditions (6.6) (see Remark 6.2), we get that

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - C \\ \chi^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - C. \end{cases}$$

Using Proposition 6.3, we have that $|\chi^{l,R}(x_0) - w^{l,R}(x_0)| \leq C$. Therefore, up to changing the constant $\tilde{C} > 0$, we have

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^0(x - x_0) - \tilde{C} \\ \chi^{l,R}(x) - \chi^{l,R}(x_0) \geq p_\mu^0(x - x_0) - \tilde{C}. \end{cases}$$

This implies that for all $h \geq 0$, and for all $x \in (-l, -h_0 - h_{max} - 1)$,

$$\begin{cases} w^{l,R}(x-h) - w^{l,R}(x) \geq -p_\mu^0 h - \tilde{C} \\ \chi^{l,R}(x-h) - \chi^{l,R}(x) \geq -p_\mu^0 h - \tilde{C}. \end{cases}$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_-^0|$, then we have

$$\bar{H}_0(\bar{p}_-^0 + \gamma) = \bar{H}_0^-(\bar{p}_-^0 + \gamma).$$

Choosing $\mu > 0$ such that

$$p_\mu^0 = \bar{p}_+^0 + \gamma.$$

we obtain (6.42). □

Proof of Theorem 4.3. The proof is performed in two steps.

Step 1: proof of i) and ii). We want to pass to the limit as $l \rightarrow +\infty$ and then as $R \rightarrow +\infty$ for the solution of (6.1) given by Proposition 6.3. Using (6.3), there exists $l_n \rightarrow +\infty$, such that for $i = 1, 2$, we have

$$m_i^{l_n, R} - m_i^{l_n, R}(0) \rightarrow m_i^R \quad \text{as } n \rightarrow +\infty,$$

the convergence being locally uniform. We also define

$$\begin{aligned} \bar{w}^R(x) &= \limsup_{n \rightarrow +\infty}^* (w^{l_n, R} - w^{l_n, R}(0)), & \bar{\chi}^R(x) &= \limsup_{n \rightarrow +\infty}^* (\chi^{l_n, R} - w^{l_n, R}(0)), \\ \underline{w}^R(x) &= \liminf_{n \rightarrow +\infty} (w^{l_n, R} - w^{l_n, R}(0)), & \underline{\chi}^R(x) &= \liminf_{n \rightarrow +\infty} (\chi^{l_n, R} - w^{l_n, R}(0)). \end{aligned} \quad \text{and}$$

Thanks to (6.3), we know that these limits are finite and satisfy

$$m_1^R - C \leq \underline{w}^R \leq \bar{w}^R \leq m_1^R + C. \quad \text{and} \quad m_2^R - C \leq \underline{\chi}^R \leq \bar{\chi}^R \leq m_2^R + C.$$

By stability of viscosity solutions $(\bar{w}^R - 2C, \bar{\chi}^R - 2C)$ and $(\underline{w}^R, \underline{\chi}^R)$ are respectively a sub-solution and a super-solution of

$$\begin{cases} G_R^1(x, w^R(x), [\chi^R], [w_x^R], w_x^R) = \bar{A}_R \\ G_R^2(x, \chi^R(x), [w^R], [\chi^R], \chi_x^R) = \bar{A}_R \end{cases} \quad (6.49)$$

Therefore, using Perron's method, we can construct a solution (w^R, χ^R) of (6.49), with m^R, \bar{A}_R, w^R and χ^R satisfying

$$\begin{cases} |m_i^R(x) - m_i^R(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \text{ for } i = 1, 2, \\ |w^R(x) - m_1^R(x)| \leq C, |\chi^R(x) - m_2^R(x)| \leq C & \text{for all } x \in \mathbb{R}, \\ |w^R(x) - \chi^R(x)| \leq C & \text{for all } x \leq h_0, \\ H_0 \leq \bar{A}_R \leq 0. \end{cases} \quad (6.50)$$

Using Proposition 6.8, if $\bar{A} > \bar{H}_1$ (resp. $\bar{A} > \min \bar{H}_2$), we know that there exists γ_0 and a constant $C > 0$, such that for all $\gamma \in (0, \gamma_0)$, for all $x \geq 0$, and $h \geq 0$,

$$w^R(x+h) - w^R(x) \geq (\bar{p}_+^1 - \gamma)h - C \quad \left(\text{resp. } \chi^R(x+h) - \chi^R(x) \geq (\bar{p}_+^1 - \gamma)h - C \right). \quad (6.51)$$

Similarly, if we assume $\bar{A} > \min \bar{H}_0$, then we have similarly, for all $x \leq -h_0 - h_{max} - 1$ and $h \geq 0$,

$$w^R(x-h) - w^R(x) \geq (-\bar{p}_-^0 - \gamma)h - C \quad \text{and} \quad \chi^R(x-h) - \chi^R(x) \geq (-\bar{p}_-^0 - \gamma)h - C. \quad (6.52)$$

We now pass to the limit as $R \rightarrow +\infty$. We consider (up to a sub-sequence), for $i = 1, 2$,

$$\left\{ \begin{array}{l} \bar{w}(x) = \limsup_{R \rightarrow +\infty}^* (w^R - w^R(0)), \quad \underline{w}(x) = \liminf_{R \rightarrow +\infty_*} (w^R - w^R(0)), \\ \bar{\chi}(x) = \limsup_{R \rightarrow +\infty}^* (\chi^R - w^R(0)), \quad \underline{\chi}(x) = \liminf_{R \rightarrow +\infty_*} (\chi^R - w^R(0)), \\ \bar{A} = \lim_{R \rightarrow +\infty} \bar{A}_R, \quad m_i = \lim_{R \rightarrow +\infty} (m_i^R - m_i^R(0)). \end{array} \right.$$

The last convergence being locally uniform. Thanks to (6.50), we know that the previous limits are finite and that

$$m_1 - C \leq \underline{w} \leq \bar{w} \leq m_1 + C. \quad \text{and} \quad m_2 - C \leq \underline{\chi} \leq \bar{\chi} \leq m_2 + C. \quad (6.53)$$

By stability of viscosity solutions ($\bar{w} - 2C, \bar{\chi} - 2C$) and ($\underline{w}, \underline{\chi}$) are respectively sub-solution and super-solution of (2.25) with $\lambda = \bar{A}$. Using Perron's method we can construct a solution (w, χ) of (2.25) with $\lambda = \bar{A}$ that satisfies (4.5), (4.6), (4.7) and (4.8).

Step 2: proof of iii). Let us now consider the rescaled functions $w^\varepsilon = \varepsilon w(x/\varepsilon)$ and $\chi^\varepsilon(x) = \varepsilon \chi(x/\varepsilon)$. Using (4.5), we have that

$$w^\varepsilon(x) = \varepsilon m_1 \left(\frac{x}{\varepsilon} \right) + O(\varepsilon) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon m_2 \left(\frac{x}{\varepsilon} \right) + O(\varepsilon). \quad (6.54)$$

Therefore, there exists a subsequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, such that

$$w^{\varepsilon_n} \rightarrow W \quad \text{and} \quad \chi^{\varepsilon_n} \rightarrow X \quad \text{locally uniformly as } n \rightarrow +\infty, \quad (6.55)$$

with $W(0) = X(0) = 0$. Arguing as in the proof of convergence away from the junction point, we have that (W, X) satisfies

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \bar{H}_0(W_x) = \bar{A} \\ \bar{H}_0(X_x) = \bar{A} \end{array} \right. \quad \text{for } x < 0, \\ \left\{ \begin{array}{l} \bar{H}_1(W_x) = \bar{A} \\ \bar{H}_2(X_x) = \bar{A} \end{array} \right. \quad \text{for } x > 0. \end{array} \right.$$

This proves (4.9). Let us now prove (4.11).

For $x < 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\bar{A} > \min \bar{H}_0$,

$$W_x \leq \bar{p}_-^0 + \gamma \quad \text{and} \quad X_x \leq \bar{p}_-^0 + \gamma,$$

where we have used (4.8). Therefore, we have

$$\hat{p}_-^0 \leq W_x, X_x \leq \bar{p}_-^0, \quad (6.56)$$

this inequality remains valid if $\bar{A} = \min \bar{H}_0$ (in which case, given the form of \bar{H}_0 , we have $W_x = X_x = \hat{p}_-^0 = \bar{p}_-^0 = p_0$).

For $x > 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\bar{A} > \min \bar{H}_1$ (resp. $\bar{A} > \min \bar{H}_2$),

$$W_x \geq \bar{p}_+^1 - \gamma \quad (\text{resp. } X_x \geq \bar{p}_+^2 - \gamma),$$

where we have used (4.7). Therefore given the form of \bar{H}_1 (resp. of \bar{H}_2), we have that $W_x = \bar{p}_+^1$ (resp. $X_x = \bar{p}_+^2$) for $x > 0$, this result is still valid if $\bar{A} = \min \bar{H}_1$ (resp. $\bar{A} = \bar{H}_2$).

Combining these results, we obtain (4.11). □

Proof of Theorem 2.9. Up to a sub-sequence, we assume that $\bar{A} = \lim_{R \rightarrow +\infty} \lim_{l \rightarrow +\infty} \lambda_{l,R}$. We want to prove that $\bar{A} = \inf E$, with

$$E = \{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of (2.25)}\},$$

and

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ two Lipschitz continuous functions } m_1 \text{ and } m_2 \text{ s.t. } m_1(0) = m_2(0) = 0 \\ \text{and a constant } C > 0 \text{ s.t. } \|v - m_1\|_\infty, \|\zeta - m_2\|_\infty \leq C\}.$$

We argue by contradiction and assume that there exists $\lambda \in E$ such that $\lambda < \bar{A}$, associated with $(v^\lambda, \zeta^\lambda)$ solution of (2.25). Arguing as in the proof of Theorem 4.3, we deduce that the functions

$$v_\lambda^\varepsilon(x) = \varepsilon v^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \zeta_\lambda^\varepsilon(x) = \varepsilon \zeta^\lambda\left(\frac{x}{\varepsilon}\right) \quad (6.57)$$

have as limits, as ε goes to 0, W^λ and X^λ (with $W^\lambda(0) = X^\lambda(0) = 0$) which satisfies

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \bar{H}_0(W_x^\lambda) = \lambda \\ \bar{H}_0(X_x^\lambda) = \lambda \end{array} \right. \quad \text{for } x < 0, \\ \left\{ \begin{array}{l} \bar{H}_1(W_x^\lambda) = \lambda \\ \bar{H}_2(X_x^\lambda) = \lambda \end{array} \right. \quad \text{for } x > 0. \end{array} \right.$$

This means that for all $x > 0$, we have

$$W_x^\lambda \leq p_+^{\lambda,1} < \bar{p}_+^1 \quad \text{with} \quad \bar{H}_1(p_+^{\lambda,1}) = \bar{H}_1^+(p_+^{\lambda,1}) = \lambda, \quad (6.58)$$

and

$$X_x^\lambda \leq p_+^{\lambda,2} < \bar{p}_+^2 \quad \text{with} \quad \bar{H}_2(p_+^{\lambda,2}) = \bar{H}_2^+(p_+^{\lambda,2}) = \lambda. \quad (6.59)$$

Similarly, for all $x < 0$, we have

$$W_x^\lambda, X_x^\lambda \geq p_-^\lambda > \bar{p}_-^0 \quad \text{with} \quad p_-^\lambda = \min\{p \in [-2k_0, 0] : \bar{H}_0(p) = \bar{H}_0^-(p) = \lambda\}. \quad (6.60)$$

These inequalities imply that for all $\gamma > 0$, there exists a constant \tilde{C}_γ such that

$$v^\lambda(x) \leq \begin{cases} (p_+^{\lambda,1} + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \quad \text{and} \quad \zeta^\lambda(x) \leq \begin{cases} (p_+^{\lambda,2} + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \quad (6.61)$$

In fact if $(v^\lambda, \zeta^\lambda)$ does not satisfy (6.61), we cannot have (6.58), (6.59), and (6.60). Using Theorem 4.3, we have for γ small enough and for $\tilde{R} > 0$ big enough,

$$v^\lambda \leq v \quad \text{and} \quad \zeta^\lambda < \zeta \quad \text{for } |x| \geq \tilde{R}.$$

This implies that there exists a constant $C_{\tilde{R}}$ such that for all $x \in \mathbb{R}$, we have

$$v^\lambda(x) < v(x) + C_{\tilde{R}} \quad \text{and} \quad \zeta^\lambda(x) < \zeta(x) + C_{\tilde{R}}.$$

Let us now introduce two functions (u, ξ) and (u^λ, ξ^λ) , defined by

$$\begin{cases} u(t, x) = v(x) + C_{\tilde{R}} - \bar{A}t, \\ \xi(t, x) = \zeta(x) + C_{\tilde{R}} - \bar{A}t, \end{cases} \quad \text{and} \quad \begin{cases} u^\lambda(t, x) = v^\lambda(x) - \lambda t, \\ \xi^\lambda(t, x) = \zeta^\lambda(x) - \lambda t. \end{cases}$$

Both functions are solutions of (2.4) (with $\varepsilon = 1$) and

$$u^\lambda(0, x) \leq u(0, x) \quad \text{and} \quad \xi^\lambda(0, x) \leq \xi(0, x).$$

Using the comparison principle (Proposition 3.7), we obtain

$$v^\lambda(x) - \lambda t \leq v(x) - \bar{A}t + C_{\tilde{R}}.$$

Dividing by t and passing to the limit as t goes to infinity, we get $\bar{A} \leq \lambda$, which is a contradiction. \square

7 Link between the system of ODEs and the system of PDEs

Theorem 7.1. *For $\varepsilon = 1$, the cumulative distribution function (ρ_1, ρ_2) defined by (2.1)-(2.2), is a discontinuous viscosity solution of*

$$\begin{cases} (\rho_1)_t + \phi(x, N_0(\rho_1, [\rho_2(t, \cdot)])(x), M_1[\rho_1(t, \cdot)](x)) \cdot |\partial_x \rho_1| = 0 & \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}, \\ (\rho_2)_t + \phi(x, N_0(\rho_2, [\rho_1(t, \cdot)])(x), M_2[\rho_2(t, \cdot)](x)) \cdot |\partial_x \rho_2| = 0 & \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \end{cases} \quad (7.1)$$

Conversely, if (u, ξ) is a bounded and continuous viscosity solution of (7.1) satisfying for some $T > 0$, and for all $t \in (0, T)$,

$$u(t, x) \text{ and } \xi(t, x) \text{ are decreasing in } x,$$

then the points $U_i(t)$, defined by $u(t, U_i(t)) = -(i+1)$ for $i \in \mathbb{Z}$ if i odd, and defined by $\xi(t, U_i(t)) = -(i+1)$ for $i \in \mathbb{Z}$ if i even, satisfy the system (1.1) on $(0, T)$.

Before the proof of Theorem 7.1, let us do the proof of Theorem 2.6.

Proof of Theorem 2.6. We recall that in Theorem 2.6, we have

$$u_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_1 & \text{if } x > 0 \end{cases} \quad \text{and} \quad \xi_0(x) = \begin{cases} -x/h & \text{if } x \leq 0 \\ -2x/h_2 & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

First, we would like to prove that for all $\varepsilon > 0$, we have

$$|\rho_1^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{for all } x \in \mathbb{R} \quad (7.2)$$

and

$$|\rho_1^\varepsilon(0, x) - \xi_0(x)| \leq g(\varepsilon) \quad \text{for all } x \in \mathbb{R}, \quad (7.3)$$

with $f(\varepsilon), g(\varepsilon) \rightarrow 0$ as ε goes to 0. Let us begin by proving (7.2). To do this, we consider a piece-wise affine function v such that

$$\rho_1^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z} \text{ such that } i[2] = 1. \quad (7.4)$$

First, given that for all $i \in \mathbb{Z}$, $U_{i+1}(0) - U_i(0) \geq h_0$, we notice that v is $2k_0$ -Lipschitz continuous and by definition of ρ_1^1 , we have

$$|\rho_1^1(0, x) - v(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}. \quad (7.5)$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z}, \text{ s.t. } i[2] = 1, U_i(0) \leq -R\}.$$

Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \leq -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \leq U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho_1^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z}, \text{ s.t. } i[2] = 1, U_i(0) \geq R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $i[2] = 1$ and $U_i(0) \geq R$ we have $U_{i+2}(0) - U_i(0) = h_1$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{2x}{h_1} + \frac{2U_{i_1}(0)}{h_1} + \rho_1^1(0, U_{i_1}(0)) = -\frac{2x}{h_1} + \frac{2U_{i_1}(0)}{h_1} - (i_1 + 1).$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho_1^\varepsilon(0, x) = \varepsilon \rho_1^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho_1^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho_1^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x) \right| \\ &\leq 2\varepsilon + \varepsilon \max \left(\left| \frac{2U_{i_1}(0)}{h_1} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right). \end{aligned} \quad (7.6)$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho_1^\varepsilon(0, x) - u_0(x)| &\leq \left| \rho_1^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right) \right| \\ &\leq 2\varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \end{aligned} \quad (7.7)$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (7.6) and (7.7) and choosing

$$f(\varepsilon) = 2\varepsilon + \varepsilon \max \left(\max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \left| \frac{2U_{i_1}(0)}{h_1} - i_1 - 1 \right|, \left| \frac{U_{i_0}(0)}{h} - i_0 - 1 \right| \right)$$

we deduce (7.2). Similarly, we can construct $g(\varepsilon)$ such that (7.3) is satisfied. Notice also that thanks to (7.2) and (7.3), we have

$$|(\rho_1^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon \quad \text{and} \quad |(\rho_2^\varepsilon)^*(0, x) - \xi_0(x)| \leq g(\varepsilon) + \varepsilon. \quad (7.8)$$

Therefore, we have

$$\begin{cases} u_0(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_1^\varepsilon(0, x) \leq (\rho_1^\varepsilon)^*(0, x) \leq u_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \\ \xi_0(x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_2^\varepsilon(0, x) \leq (\rho_2^\varepsilon)^*(0, x) \leq \xi_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Using the fact that $(\rho_1^\varepsilon, \rho_2^\varepsilon)$ is a viscosity solution of (2.4) and the comparison principle (Proposition 3.7) we deduce that (with $(u^\varepsilon, \xi^\varepsilon)$ a continuous solution of (2.4) associated to the initial condition (u_0, ξ_0))

$$\begin{cases} u^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_1^\varepsilon(t, x) \leq (\rho_1^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \\ \xi^\varepsilon(t, x) - \max(f(\varepsilon), g(\varepsilon)) \leq \rho_2^\varepsilon(t, x) \leq (\rho_2^\varepsilon)^*(t, x) \leq \xi^\varepsilon(t, x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Passing to the limit as $\varepsilon \rightarrow 0$ and using Theorem 2.5 we get that

$$\tilde{\rho}^\varepsilon(t, x) = \begin{cases} \rho_1^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ \rho_1^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_1^*, \\ \rho_2^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_2^*, \end{cases} \quad (7.9)$$

converges locally uniformly to u^0 (unique solution of (2.17)), which ends the proof of Theorem 2.6. \square

Proof of Theorem 7.1. Theorem 7.1 is a consequence of the following lemma.

Lemma 7.2 (Link between the velocities). *Assume (A). Let $((U_i)_i)$ be the solution of (1.1) with*

$$\begin{cases} U_i(0) \leq U_{i+1}(0) - h_0 & \text{if } U_i(0), U_{i+1}(0) \in R_0 \\ U_i(0) \leq U_{i+2}(0) - h_0 & \text{otherwise.} \end{cases} \quad (7.10)$$

Then we have for all $i \in \mathbb{Z}$, such that $i[2] = 1$,

$$\dot{U}_i(t) = -\phi(U_i(t), N_0(u(t, \cdot), [\xi(t, \cdot)])(U_i(t)), M_1[u(t, \cdot)](U_i(t))), \quad (7.11)$$

and for all $i \in \mathbb{Z}$ such that $i[2] = 0$,

$$\dot{U}_i(t) = -\phi(U_i(t), N_0(\xi(t, \cdot), [u(t, \cdot)])(U_i(t)), M_2[\xi(t, \cdot)](U_i(t))), \quad (7.12)$$

where E and F are defined in (2.7), $J_i = V_i'$ for $i = 0, 1, 2$, and u and ξ are continuous functions, decreasing in x , such that

$$\begin{cases} u(t, x) = \rho_1(t, x) = (\rho_1)_*(t, x) & \text{for } x = U_i(t), \text{ for all } i \in \mathbb{Z}, \text{ such that } i[2] = 1, \\ \xi(t, x) = \rho_2(t, x) = (\rho_2)_*(t, x) & \text{for } x = U_i(t), \text{ for all } i \in \mathbb{Z}, \text{ such that } i[2] = 0. \end{cases} \quad (7.13)$$

with (ρ_1, ρ_2) defined in (2.1)-(2.2) (with $\varepsilon = 1$).

Proof. We drop the time dependence to simplify the presentation. We only do the proof in the case $i \in \mathbb{Z}$, $i[2] = 0$, the other case being identical.

In the case $U_i \leq 0$, all the vehicles remain in order, meaning that $\xi(U_i) = -(i+1)$ and $-(i+2) = u(U_{i+1}) < u(U_i) < u(U_{i-1}) = -i$, using this and (7.13) we have for all $z \in [0, +\infty)$,

$$\begin{cases} -1 < u(U_i + z) - \xi(U_i) < 1 & \text{if } z \in [0, U_{i+1} - U_i), \\ u(U_i + z) - \xi(U_i) \leq -1 & \text{if } z \in [U_{i+1} - U_i, +\infty). \end{cases}$$

Given that u and ξ are continuous, this implies that

$$N_0(\xi, [u])(U_i) = \int_0^{U_{i+1}-U_i} \frac{1}{2} J_0(z) dz + \int_{U_{i+1}-U_i}^{+\infty} \frac{3}{2} J_0(z) dz - \frac{3}{2} V_{max}^0 = -V_0(U_{i+1} - U_i). \quad (7.14)$$

Now using (7.13), and the fact that $\xi(U_i) = -(i+1)$, we obtain

$$\begin{cases} -2 < \xi(U_i + z) - \xi(U_i) \leq 0 & \text{if } z \in [0, U_{i+2} - U_i), \\ \xi(U_i + z) - \xi(U_i) \leq -2 & \text{if } z \in [U_{i+2} - U_i, +\infty). \end{cases}$$

Again, given that ξ is continuous, this implies that

$$M_2[\xi](U_i) = \int_0^{U_{i+2}-U_i} \frac{1}{2} J_2(z) dz + \int_{U_{i+2}-U_i}^{+\infty} \frac{3}{2} J_2(z) dz - \frac{3}{2} V_{max}^2 = -V_2(U_{i+2} - U_i). \quad (7.15)$$

Combining (7.14) and (7.15), we get (7.12). □

Thanks to (7.13), we have for $x = U_i(t)$, for all $i \in \mathbb{Z}$, $i[2] = 1$,

$$\begin{aligned} \tilde{N}_0((\rho_1)_*(t, \cdot), [(\rho_2)_*(t, \cdot)])(x) &= \tilde{N}_0(u(t, \cdot), [\xi(t, \cdot)])(x) = N_0(u(t, \cdot), [\xi(t, \cdot)])(x), \\ \tilde{M}_1[(\rho_1)_*(t, \cdot)](x) &= M_1[u(t, \cdot)](x) = M_1[u(t, \cdot)](x). \end{aligned}$$

Similarly, we have for all $x = U_i(t)$, for all $i \in \mathbb{Z}$, $i[2] = 0$,

$$\begin{aligned} \tilde{N}_0((\rho_2)_*(t, \cdot), [(\rho_1)_*(t, \cdot)])(x) &= \tilde{N}_0(\xi(t, \cdot), [u(t, \cdot)])(x) = N_0(\xi(t, \cdot), [u(t, \cdot)])(x), \\ \tilde{M}_2[(\rho_2)_*(t, \cdot)](x) &= \tilde{M}_2[\xi(t, \cdot)](x) = M_2[\xi(t, \cdot)](x) \end{aligned}$$

and using Lemma 7.2, and Definition 3.1, we can see that $((\rho_1)_*, (\rho_2)_*)$ is a discontinuous viscosity super-solution of (7.1). We obtain a similar result for $((\rho_1)^*, (\rho_2)^*)$ therefore, (ρ_1, ρ_2) is a discontinuous viscosity solution of (7.1).

We prove the converse. Using from [15, Proposition 11], we can conclude that $((\rho_1)_* = [u], (\rho_2)_* = [\xi])$ (resp. $(\rho_1^* = [u], \rho_2^* = [\xi])$) is a viscosity super-solution (resp. sub-solution) of

$$\begin{cases} (\rho_1)_t - \tilde{c}_1(t, x)(\rho_1)_x = 0 & \text{with } \tilde{c}_1(t, x) = \phi(x, N_0(u(t, \cdot), [\xi(t, \cdot)])(x), M_1[u(t, \cdot)](x)), \\ (\rho_2)_t - \tilde{c}_2(t, x)(\rho_2)_x = 0 & \text{with } \tilde{c}_2(t, x) = \phi(x, N_0(\xi(t, \cdot), [u(t, \cdot)])(x), M_2[\xi(t, \cdot)](x)). \end{cases} \quad (7.16)$$

Using the fact that u and ξ are decreasing, we define for all $i \in \mathbb{Z}$, $i[2] = 1$,

$$U_i(t) = \inf \{x, u(t, x) \leq -(i+1)\} = (u(t, \cdot))^{-1}(-i-1), \quad (7.17)$$

and for all $i \in \mathbb{Z}$, $i[2] = 0$,

$$U_i(t) = \inf \{x, \xi(t, x) \leq -(i+1)\} = (\xi(t, \cdot))^{-1}(-i-1), \quad (7.18)$$

and we consider the functions $t \mapsto U_i(t)$. They are continuous because u and ξ are decreasing in x and continuous in (t, x) .

We now prove that the functions U_i are viscosity solutions of (1.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -(i+1) + \varphi(t) - x$. Let us for instance consider $i \in \mathbb{Z}$, such that $i[2] = 0$ then we have

$$\hat{\varphi}(t_0, U_i(t_0)) = (\rho_2)_*(t_0, U_i(t_0))$$

and

$$\hat{\varphi}(t, x) \leq (\rho_2)_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+2}(t).$$

This implies that

$$\begin{aligned} \varphi_i(t_0) + \tilde{c}_2(t_0, U_i(t_0)) &\geq 0 \\ \Leftrightarrow \varphi_i(t_0) &\geq -\tilde{c}_2(t_0, U_i(t_0)) = -\phi(U_i(t_0), -V_0(U_{i+1}(t_0) - U_i(t_0)), -V_2(U_{i+2}(t_0) - U_i(t_0))) \\ \Leftrightarrow \varphi_i(t_0) &\geq \bar{\phi}(U_i(t_0), V_0(U_{i+1}(t_0) - U_i(t_0)), V_2(U_{i+2}(t_0) - U_i(t_0))). \end{aligned}$$

We obtain a similar result in the case $i[2] = 1$. This proves that U_i are viscosity super-solutions of (1.1). The proof for sub-solutions is similar and we skip it. Moreover, $\tilde{c}_i(\cdot, U_i(\cdot))$, for $i = 1, 2$, is continuous, we deduce that $U_i \in C^1$, and is therefore a classical solution of (1.1). \square

8 Extensions

In this section, we will introduce some extensions of model (1.1) for which an homogenization result is possible by using the same arguments as the ones used in the case presented in this paper. However, since the models we introduce in this section are more complex, many technical difficulties appear. Particularly, we will no longer be working with a system of two equations but with a more general system. We will not go into details of the proofs for each of the models, however we give some guidelines for any reader that would like to do the proofs in detail.

8.1 One incoming road, n outgoing roads

8.1.1 General model

Let us begin by considering a model where we have one incoming road R_0 and $n \in \mathbb{N} \setminus \{0\}$ outgoing roads that we denote by R_k , for $k = 1, \dots, n$. We consider a simple periodic setting and assume that the vehicle $i \in \mathbb{Z}$ such that $i[n] = k - 1$ goes into R_k , where $i[n]$ denotes the rest of the euclidean division of i by n (therefore $i[n] \in \{0, \dots, n-1\}$). We consider the following model, if $i[n] = k - 1$, for all $t > 0$,

$$\dot{U}_i(t) = \bar{\phi}(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_k(U_{i+n}(t) - U_i(t))), \quad (8.1)$$

where U_i denotes the position of the i *th* vehicle and \dot{U}_i its velocity. For $i = 0, \dots, n$, V_i is an optimal velocity function. The function $\bar{\phi}$ is the same as before and is defined in (1.2).

In Figure 5, we have an schematic representation of model (8.1).

We assume that the optimal velocities satisfy assumption (A). However, we need to change assumption (A5), to take into account the fact that we have n possible exits.

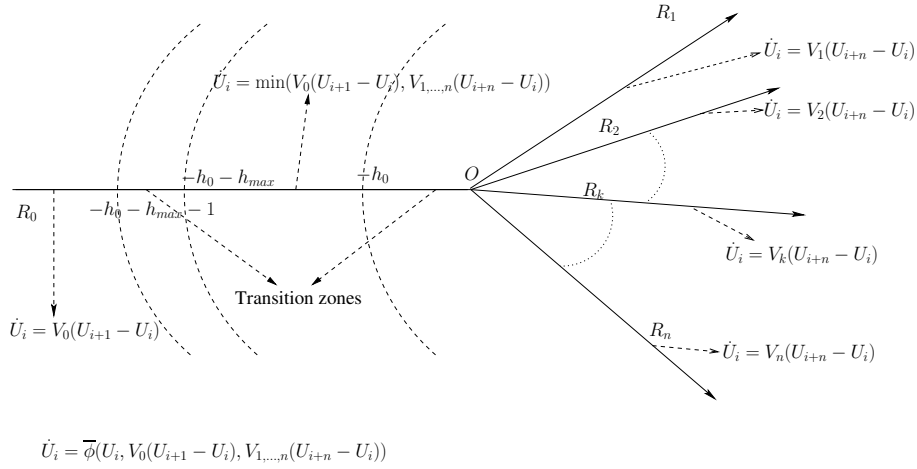


Figure 5: Schematic representation of the microscopic model.

- (A5') There exists a unique p_0 (resp. p_1, \dots, p_k) such that the function $p \mapsto pV_0(-1/p)$ (resp. $p \mapsto pV_j(-2/p)$, for $j=1, \dots, k$) is decreasing in $[-1/h_0, p_0]$ (resp. $[-2/h_0, p_i]$) and increasing in $[p_0, 0]$ (resp. $[p_i, 0]$).

To simplify we call (A') assumption (A) with (A5') instead of (A5).

8.1.2 Injecting the system of ODEs into a system of PDEs

The technique remains the same as before and we inject the system of ODE (8.1) into a system of PDEs by considering n "cumulative distribution functions". For $j = 0, \dots, n-1$,

$$\rho_j^\varepsilon(t, x) = -n\varepsilon \cdot \left(\sum_{i[n]=j, i \geq 0} H \left(x - \varepsilon U_i \left(\frac{t}{\varepsilon} \right) \right) + \sum_{i[n]=j, i < 0} \left(-1 + H \left(x - \varepsilon U_i \left(\frac{t}{\varepsilon} \right) \right) \right) \right) + \varepsilon(n-1-j), \quad (8.2)$$

with H the heaviside function defined in (2.3).

Remark 8.1. We choose this type of "cumulative distribution function", to simplify the computations, because if $i[n] = j$, then we have $\rho_j(t, U_i(t)) = -(i+1)$.

Under assumption (A') we can prove that $(\rho_j)_{j=0, \dots, n-1}$ is a (discontinuous viscosity) solution of the following non-local system of PDEs, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} \partial_t u_j^\varepsilon + \phi \left(\frac{x}{\varepsilon}, N_0^\varepsilon \left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon} \right] \right) (x), M_{j+1}^\varepsilon \left[\frac{u_j^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \right) \cdot |\partial_x u_j^\varepsilon| = 0 \\ \text{for } j = 0, \dots, n-2 \\ \partial_t u_{n-1}^\varepsilon + \phi \left(\frac{x}{\varepsilon}, N_0^\varepsilon \left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_0^\varepsilon(t, \cdot)}{\varepsilon} \right] \right) (x), M_n^\varepsilon \left[\frac{u_{n-1}^\varepsilon(t, \cdot)}{\varepsilon} \right] (x) \right) \cdot |\partial_x u_{n-1}^\varepsilon| = 0, \end{cases} \quad (8.3)$$

where N_0^ε and M_j^ε for $j = 1, \dots, n$ are non-local operators defined by

$$N_0^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_0(z) F(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^0, \quad (8.4)$$

and

$$M_j^\varepsilon[U](x) = \int_{\mathbb{R}} J_j(z) E(U(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^j, \quad (8.5)$$

with $J_j = V_j'$ for $j = 0, 1, \dots, n$,

$$F(z) = \begin{cases} 0 & \text{if } z \geq n-1, \\ 1/2 & \text{if } -1 \leq z < n-1, \\ 3/2 & \text{if } z < -1, \end{cases} \quad \text{and} \quad E(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ 1/2 & \text{if } -n \leq z < 0, \\ 3/2 & \text{if } z < -n. \end{cases} \quad (8.6)$$

Finally, the function ϕ is the same as the one in (2.8).

We will consider the following initial conditions,

$$u_j^\varepsilon(0, x) = u_{j,0}(x) \quad \text{for } x \in \mathbb{R} \text{ and for } j = 0, \dots, n-1, \quad (8.7)$$

and we make the following assumptions.

(A0') (Initial condition). For all $x \leq 0$, and all $j = 1, \dots, n-1$,

$$u_{0,0}(x) = u_{j,0}(x).$$

Moreover, we assume that for all $x \in \mathbb{R}$

$$-nk_0 = -\frac{n}{h_0} \leq (u_{j,0})_x \leq 0 \quad \text{for all } j = 0, \dots, n-1.$$

8.1.3 The effective Hamiltonians

Like in the case of a simple bifurcation, we will have an effective Hamiltonian on each road. We introduce \bar{H}_k , for $k = 0, \dots, n$ which are the effective Hamiltonian on each of the roads R_k for $k = 0, \dots, n$. We define $k_0 = 1/h_0$ and $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_0(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0, \\ -V_0 \left(\frac{-1}{p} \right) \cdot |p| & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.8)$$

We also define, for $k = 1, \dots, n$, $\bar{H}_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\bar{H}_k(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0, \\ -V_k \left(\frac{-n}{p} \right) \cdot |p| & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.9)$$

For $k = 0, \dots, n$, let us notice that \bar{H}_k is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_k(p) = +\infty \right)$ and because of (A5'), there exists a unique point $p_k \in [-nk_0, 0]$ such that

$$\begin{cases} \bar{H}_k \text{ is non-increasing on } (-\infty, p_k), \\ \bar{H}_k \text{ is increasing on } (p_k, +\infty), \end{cases} \quad (8.10)$$

We denote by

$$H_0 = \max_{k \in \{0, \dots, n\}} \min_{p \in \mathbb{R}} \bar{H}_k(p). \quad (8.11)$$

8.1.4 Convergence result

Let $(u_j^\varepsilon)_j$ be the solution of (8.3)-(8.7). It is possible to prove that the function defined by

$$\tilde{u}^\varepsilon(t, x) = \begin{cases} u_0^\varepsilon(t, -d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_0, \\ u_{k-1}^\varepsilon(t, d(0, x)) & \text{for } (t, x) \in (0, +\infty) \times R_k^*, \quad k \in \{1, \dots, n\}, \end{cases} \quad (8.12)$$

converges locally uniformly on compact subsets of $(0, +\infty) \times \bar{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_k(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_k^*, \\ u_t^0 + F_{\bar{A}}^-(\partial_0 u^0(t, 0), \dots, \partial_n u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_{0,0}(-d(0, x)) & \text{for } x \in R_0, \\ u_{k-1,0}(d(0, x)) & \text{for } x \in R_k \end{cases} \end{cases} \quad (8.13)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}^-$ is defined by

$$F_{\bar{A}}^-(p_0, \dots, p_n) = \max(\bar{A}, \bar{H}_0^+(p_0), \max_{k=1, \dots, n} \bar{H}_k^-(p_k)), \quad (8.14)$$

where for $k = 0, \dots, n$,

$$\bar{H}_k^-(p) = \begin{cases} \bar{H}_k(p) & \text{if } p \leq p_k \\ \bar{H}_k(p_k) & \text{if } p \geq p_k \end{cases} \quad \text{and} \quad \bar{H}_k^+(p) = \begin{cases} \bar{H}_k(p_k) & \text{if } p \leq p_k \\ \bar{H}_k(p) & \text{if } p \geq p_k. \end{cases} \quad (8.15)$$

Theorem 8.2 (Junction condition by homogenization). *Assume $(A0')$ and (A') . For $\varepsilon > 0$, let $(u_j^\varepsilon)_j$ be the solution of (8.3)-(8.7). Then there exists $\bar{A} \in [H_0, 0]$ such that the function defined by (8.12) converges locally uniformly to the unique viscosity solution u^0 of (8.13).*

Theorem 8.3. *Assume (A') and that at initial time we have, for all $i \in \mathbb{Z}$, if the vehicle $i+1$ is in R_0 ,*

$$U_i(0) \leq U_{i+1}(0) - h_0, \quad (8.16)$$

and if not

$$U_i(0) \leq U_{i+n}(0) - h_0. \quad (8.17)$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$

$$U_{i+n}(0) - U_i(0) = h_j \quad \text{if } i[n] = j - 1$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h \geq h_0$ and for $j \in \{1, \dots, n\}$, $h_j \geq h_0$. We define the functions $(u_{j,0})_j$ (satisfying $(A0')$) by

$$u_{j-1,0}(x) = \begin{cases} \frac{-x}{h} & \text{if } x \leq 0 \\ \frac{-nx}{h_j} & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R} \text{ for } j = 1, \dots, n.$$

Then there exists a constant $\bar{A} \in [H_0, 0]$ such that the function

$$\bar{\rho}^\varepsilon(t, y) = \begin{cases} \rho_0^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_{k-1}^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_k^*, \quad k \in \{1, \dots, n\}, \end{cases} \quad (8.18)$$

converges locally uniformly to u^0 the unique solution of (8.13).

Theorem 8.4. *Assume $(A0')$ - (A') . Let u^0 be the unique solution of (8.13), then we have for all $(t, x) \in [0, +\infty) \times \bar{R}$,*

$$-nk_0 \leq u_x^0 \leq 0,$$

with k_0 defined in $(A0')$.

Remark 8.5. First, notice that, at the macroscopic scale, we obtain a similar result like the one from the case of a simple bifurcation (and of course if $n = 2$ we find the same result).

In the introduction we mentioned that the main difficulty to obtain an homogenization result was to build the correctors at the junction since we are in a non-periodic setting. However, notice that in the proof of Proposition 6.3 (in the core of the proof of Theorem 4.3 which gives the existence of correctors at the junction) the key element is that we are able to control the oscillations in space of the solutions in the truncated domain for the approximated cell problem (Lemma 6.4). Notice also that the arguments used in that lemma are actually similar to the ones used to prove Theorem 3.9. That is why we will give the equivalent theorem in the case of n outgoing roads and then give some guidelines on how to prove it.

Theorem 8.6 (Control of the oscillations). *Let $T > 0$. Assume $(A0')$ - (A') and let $(u_j)_j$ be a solution of (8.3)-(8.7), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - n \leq u(t, x) - u(t, y) &\leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - n \leq \xi(t, x) - \xi(t, y) &\leq 0, \end{aligned} \quad (8.19)$$

with $K_0 := nk_0$.

Sketch of the proof of Theorem 8.6. We only give the ideas necessary to prove Theorem 8.6, and the order in which the proof must be done.

1. Prove the control of the oscillations in time using the barriers and the fact that the solutions are invariant by additions of constants and by translations in time.
2. Like in Theorem 3.9, prove that the functions u_j are non-increasing in space.
3. Now we need a comparison between the functions $(u_j)_j$ solution of (8.3) (with $\varepsilon = 1$) for all $x \leq -h_0$. We want to prove that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$,

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \leq n - 1 \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x) \leq n - 1. \quad (8.20)$$

To prove (8.20), we need to proceed in the following order.

- (a) Using a localisation argument, like in the proof of Lemma 3.10, we prove that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$,

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x).$$

- (b) Using the previous result, we deduce that for all $x \leq -h_0$, all $t \in [0, T]$ and for $j = 0, \dots, n - 2$

$$u_{j+1}(x) - u_j(x) \leq n - 1 \quad \text{and} \quad u_0(x) - u_{n-1}(x) \leq n - 1.$$

4. Using the previous results, proceeding like in the proof of Theorem 3.9 prove the lower bounds on the control of the space oscillations.

□

8.2 A more general distribution of vehicles

Let us consider the case we have one incoming road R_0 and two outgoing roads R_1 and R_2 like in the case treated in detail in the previous sections. However, instead of considering one vehicle going to the left and one to the right, we consider a more general distribution of the vehicles (but still a periodic distribution). More precisely, let $n \geq 2$ and assume that the vehicle $i \in \mathbb{Z}$ such that $i[n] = 0$ goes into R_1 and the rest of the vehicles go into R_2 (one vehicle goes left and $n - 1$ go right). We then consider the following model for all $i \in \mathbb{Z}$ and all $t > 0$,

-if $i[n] = 0$,

$$\dot{U}_i(t) = \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_1(U_{i+n}(t) - U_i(t))\right), \quad (8.21)$$

-if $i[n] = k \in \{1, \dots, n-2\}$,

$$\dot{U}_i(t) = \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+1}(t) - U_i(t))\right), \quad (8.22)$$

-if $i[n] = n-1$,

$$\dot{U}_i(t) = \bar{\phi}\left(U_i(t), V_0(U_{i+1}(t) - U_i(t)), V_2(U_{i+2}(t) - U_i(t))\right), \quad (8.23)$$

where U_i denotes the position of the i -th vehicle and \dot{U}_i its velocity. For $i = 0, 1, 2$, V_i is an optimal velocity function. The function $\bar{\phi}$ is the same as before (see (1.2)).

In Figure 6, we have a schematic representation of model (8.21)-(8.22)-(8.23).

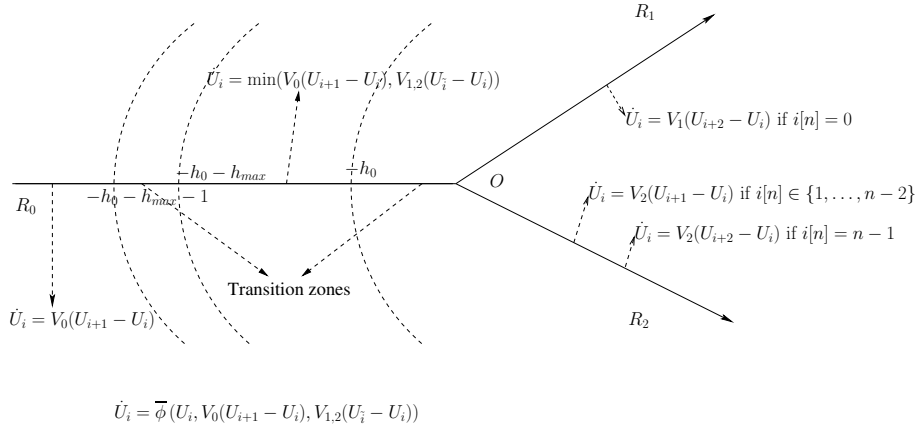


Figure 6: Schematic representation of the microscopic model (8.21)-(8.22)-(8.23). The index \tilde{i} denotes the index of the vehicle that will be in front of the vehicle i after the bifurcation.

We assume that the optimal velocities satisfy assumption (A), however, we need to change assumption (A5), to take into account the fact that we have a different distribution of vehicles.

- (A5'') There exists a unique p_0 (resp. p_1 and p_2) such that the function $p \mapsto pV_0(-1/p)$ (resp. $p \mapsto pV_1(-n/p)$ and $p \mapsto pV_2(-n/(p(n-1)))$) is decreasing in $[-1/h_0, p_0]$ (resp. $[-n/h_0, p_1]$ and $[-nk_0/(n-1), p_2]$) and increasing in $[p_0, 0]$ (resp. $[p_i, 0]$).

To simplify we call (A'') assumption (A) with (A5'') instead of (A5).

As before, for $j = 0, \dots, n-1$, we define the following "cumulative distribution functions"

$$\rho_j^\varepsilon(t, x) = -n\varepsilon \cdot \left(\sum_{i[n]=j, i \geq 0} H\left(x - \varepsilon U_i\left(\frac{t}{\varepsilon}\right)\right) + \sum_{i[n]=j, i < 0} \left(-1 + H\left(x - \varepsilon U_i\left(\frac{t}{\varepsilon}\right)\right)\right) \right) + \varepsilon(n-1-j). \quad (8.24)$$

Under assumption (A'') we can prove that $(\rho_j)_{j=0, \dots, n-1}$ is a (discontinuous viscosity) solution

of the following non-local system of PDEs, for $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} \partial_t u_0^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_0^\varepsilon}{\varepsilon}, \left[\frac{u_1^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), M_1^\varepsilon\left[\frac{u_0^\varepsilon(t, \cdot)}{\varepsilon}\right](x)\right) \cdot |\partial_x u_0^\varepsilon| = 0 \\ \partial_t u_j^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), N_2^\varepsilon\left(\frac{u_j^\varepsilon}{\varepsilon}, \left[\frac{u_{j+1}^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x)\right) \cdot |\partial_x u_j^\varepsilon| = 0 \\ \text{for } j = 1, \dots, n-2 \\ \partial_t u_{n-1}^\varepsilon + \phi\left(\frac{x}{\varepsilon}, N_0^\varepsilon\left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_1^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x), L_2^\varepsilon\left(\frac{u_{n-1}^\varepsilon}{\varepsilon}, \left[\frac{u_1^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x)\right) \cdot |\partial_x u_{n-1}^\varepsilon| = 0, \end{cases} \quad (8.25)$$

where N_0^ε and M_1^ε are defined respectively in (8.4) and (8.5). The non-local operator N_2^ε is defined like N_0^ε but with J_2 and V_{max}^2 instead of J_0 and V_{max}^0 (in order to recover the velocity V_2 on R_2). Finally, the non-local operator L_2^ε is defined by

$$L_2^\varepsilon(U, [\Xi])(x) = \int_{\mathbb{R}} J_2(z) G(\Xi(x + \varepsilon z) - U(x)) dz - \frac{3}{2} V_{max}^2, \quad (8.26)$$

with $J_2 = V_2'$ and

$$G(z) = \begin{cases} 0 & \text{if } z \geq n-2, \\ 1/2 & \text{if } -2 \leq z < n-2, \\ 3/2 & \text{if } z < -2. \end{cases} \quad (8.27)$$

The function ϕ is the same as the one in (2.8).

8.2.1 The effective Hamiltonians

Like in the previous scenarios, we have an effective Hamiltonian on each road. The effective Hamiltonians $\bar{H}_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\bar{H}_1 : \mathbb{R} \rightarrow \mathbb{R}$ are defined respectively in (8.8) and (8.9) while \bar{H}_2 is defined by (with $k_0 = 1/h_0$)

$$\bar{H}_2(p) = \begin{cases} -p - nk_0 & \text{for } p < -nk_0 \\ -V_2\left(\frac{-n}{p(n-1)}\right) & \text{for } -nk_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (8.28)$$

For $k = 0, 1, 2$, let us notice that such \bar{H}_k is continuous, coercive $\left(\lim_{|p| \rightarrow +\infty} \bar{H}_k(p) = +\infty\right)$ and because of (A5''), there exists a unique point $p_k \in [-nk_0, 0]$ such that

$$\begin{cases} \bar{H}_k \text{ is non-increasing on } (-\infty, p_k), \\ \bar{H}_k \text{ is increasing on } (p_k, +\infty), \end{cases} \quad (8.29)$$

We denote by

$$H_0 = \max_{k \in \{0, 1, 2\}} \min_{p \in \mathbb{R}} \bar{H}_k(p).$$

8.2.2 Convergence result

Theorem 8.7. *Assume (A'') and that at initial time between two consecutive vehicles there is atleast a distance of h_0 between them. We also assume that there exists a constant $R > 0$ such that for all $i \in \mathbb{Z}$, if $U_i(0) \geq R$*

$$U_{i+n}(0) - U_i(0) = h_1 \quad \text{if } i[n] = 0,$$

$$\begin{cases} U_{i+1}(0) - U_i(0) = h_2 & \text{if } i[n] \neq 0, n-1 \\ U_{i+2}(0) - U_i(0) = h_2 & \text{if } i[n] = n-1, \end{cases}$$

and if $U_i(0) \leq -R$

$$U_{i+1}(0) - U_i(0) = h,$$

with $h, h_1, h_2 \geq h_0$. We define the functions $u_{0,0}$ and $u_{1,0}$ (satisfying (A0')) by

$$u_{k-1,0}(x) = \begin{cases} \frac{-x}{h} & \text{if } x \leq 0 \\ \frac{-nx}{h_k} & \text{if } x > 0 \end{cases} \quad \text{for all } x \in \mathbb{R} \text{ and } k = 1 \text{ or } 2.$$

Then there exists a constant $\bar{A} \in [H_0, 0]$ such that, for any $k \in \{0, \dots, n-1\}$ and any $j \in \{1, \dots, n-1\}$, the function

$$\tilde{\rho}^\varepsilon(t, y) = \begin{cases} \rho_k^\varepsilon(t, -d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_0, \\ \rho_0^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_1^*, \\ \rho_j^\varepsilon(t, d(0, y)) & \text{for } (t, y) \in (0, +\infty) \times R_2^*, \end{cases} \quad (8.30)$$

converges locally uniformly to u^0 the unique solution of

$$\begin{cases} u_t^0 + \bar{H}_0(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_0^*, \\ u_t^0 + \bar{H}_1(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_1^*, \\ u_t^0 + \bar{H}_2(u_x^0) = 0 & \text{for } (t, x) \in [0, +\infty) \times R_2^*, \\ u_t^0 + F_{\bar{A}}^-(\partial_0 u^0(t, 0), \partial_1 u^0(t, 0), \partial_2 u^0(t, 0)) = 0 & \text{for } (t, x) \in [0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) = \begin{cases} u_{0,0}(-d(0, x)) & \text{for } x \in R_0, \\ u_{0,0}(d(0, x)) & \text{for } x \in R_1 \\ u_{1,0}(d(0, x)) & \text{for } x \in R_2 \end{cases} \end{cases} \quad (8.31)$$

where \bar{A} is a constant to be determined and $F_{\bar{A}}^-$ is defined by

$$F_{\bar{A}}^-(p_0, p_1, p_2) = \max(\bar{A}, \bar{H}_0^+(p_0), \bar{H}_1^-(p_1), \bar{H}_2^-(p_2)), \quad (8.32)$$

and for $k = 0, 1, 2$, we define

$$\bar{H}_k^-(p) = \begin{cases} \bar{H}_k(p) & \text{if } p \leq p_k, \\ \bar{H}_k(p_k) & \text{if } p \geq p_k, \end{cases} \quad \text{and} \quad \bar{H}_k^+(p) = \begin{cases} \bar{H}_k(p_k) & \text{if } p \leq p_k, \\ \bar{H}_k(p) & \text{if } p \geq p_k. \end{cases}$$

Moreover, the unique solution u^0 of (8.31) satisfies

$$-nk_0 \leq u_x^0 \leq 0,$$

with $k_0 = 1/h_0$.

Remark 8.8. Let us notice that if we choose $n = 2$, we recover the same result as the one from the case of a simple bifurcation (one vehicle goes left, the other goes right).

Like in the previous extension, we will give the equivalent of Theorem 3.9 in the case of the more general distribution of vehicles. Contrary to the case of n outgoing roads, there is a slight difference, in this case, when building the correctors away from the junction in R_2 . We will explain how to correctly choose them and how to obtain the effective Hamiltonian \bar{H}_2 .

Theorem 8.9 (Control of the oscillations). *Let $T > 0$. Assume that (A'') and (A0') are valid and let $(u_j)_j$ be a solution of (8.25), with $\varepsilon = 1$. Then there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0, T]$, $t \geq s$, we have*

$$\begin{aligned} 0 \leq u(t, x) - u(s, x) &\leq C_1(t - s), & \text{and} & & -K_0(x - y) - n \leq u(t, x) - u(t, y) &\leq 0, \\ 0 \leq \xi(t, x) - \xi(s, x) &\leq C_1(t - s) & & & -K_0(x - y) - n \leq \xi(t, x) - \xi(t, y) &\leq 0, \end{aligned} \quad (8.33)$$

with $K_0 := nk_0$.

Sketch of the proof of Theorem 8.9. We only give the ideas necessary to prove Theorem 8.9, and the order in which the proof must be done.

1. Prove the control of the oscillations in time using the barriers and the fact that the solutions are invariant by additions of constants and by translations in time.
2. Like in Theorem 3.9, prove that the functions u_j are non-increasing in space.
3. Now we need a comparison between the functions $(u_j)_j$ solution of (8.25) (with $\varepsilon = 1$). We want to prove that for all $x \in \mathbb{R}$, all $t \in [0, T]$ and for $j = 1, \dots, n-2$, we have

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \leq n-1. \quad (8.34)$$

We also want to prove that for all $x \leq -h_0$ and all $t \in [0, T]$,

$$-1 \leq u_1(t, x) - u_0(t, x) \leq n-1 \quad \text{and} \quad -1 \leq u_0(t, x) - u_{n-1}(t, x) \leq n-1. \quad (8.35)$$

Finally, we want to prove that for all $x \geq -h_0$ and all $t \in [0, T]$, we have

$$-2 \leq u_1(t, x) - u_{n-1}(t, x) \leq n-2. \quad (8.36)$$

To prove (8.36)-(8.35)-(8.34), we need to proceed in the following order.

(a) First, prove that

$$-1 \leq u_{j+1}(t, x) - u_j(t, x) \quad \text{for all } x \in \mathbb{R}, \text{ all } t \in [0, T] \text{ and for } j = 1, \dots, n-2. \quad (8.37)$$

To do this, we argue classically by contradiction and assume that the supremum of $-1 - u_{j+1}(t, x) + u_j(t, x)$ over $[0, T] \times \mathbb{R}$ is strictly positive.

(b) Using a localisation term, like in the proof of Lemma 3.10, prove that

$$-1 \leq u_1(t, x) - u_0(t, x) \quad \text{for all } x \leq -h_0 \text{ and all } t \in [0, T].$$

(c) Similarly, prove that

$$-1 \leq u_0(t, x) - u_{n-1}(t, x) \quad \text{for all } x \leq -h_0 \text{ and all } t \in [0, T].$$

(d) Using the previous results, deduce that for all $x \leq -h_0$, all $t \in [0, T]$, and for $j = 0, \dots, n-2$, we have

$$u_{j+1}(t, x) - u_j(t, x) \leq n-1 \quad \text{and} \quad u_0(t, x) - u_{n-1}(t, x) \leq n-1.$$

(e) Using a localisation term, prove that for all $x \geq -h_0$ and all $t \in [0, T]$, we have

$$-2 \leq u_1(t, x) - u_{n-1}(t, x).$$

(f) Using the previous result and (8.37), deduce that for all $x \geq -h_0$, all $t \in [0, T]$ and for $j = 1, \dots, n-2$, we have

$$u_{j+1}(t, x) - u_j(t, x) \leq n-1 \quad \text{and} \quad u_1(t, x) - u_{n-1}(t, x) \leq n-2.$$

4. Using the previous results, proceeding like in the proof of Theorem 3.9 prove the lower bounds on the control of the space oscillations.

□

Proposition 8.10 (Homogenization on R_2). *Assume (A''). Then for $p \in [-nk_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution $(v_j)_{j=1, \dots, n-1}$ of*

$$\begin{cases} N_p^2(v_j, [v_{j+1}])(x) \cdot |\partial_x v_j + p| = \lambda & \text{for } j=1, \dots, n-2, \\ L_p^2(v_{n-1}, [v_1])(x) \cdot |\partial_x v_{n-1} + p| = \lambda, \\ v_j \text{ for } j = 1, \dots, n-1 \text{ is } \mathbb{Z}\text{-periodic,} \end{cases}$$

with

$$N_p^2(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_2(z) F(\Xi(x+z) - U(x) + pz) dz - \frac{3}{2} V_{max}^2$$

and

$$L_p^2(U, [\Xi])(x) = \int_{-\infty}^{+\infty} J_2(z) G(\Xi(x+z) - U(x) + pz) dz - \frac{3}{2} V_{max}^2.$$

Moreover, for $p \in [-nk_0, 0]$ we have $\lambda = \bar{H}_2(p) = -V_2 \left(\frac{-n}{p(n-1)} \right) |p|$.

Proof. Contrary to the construction of the correctors on R_0 and R_1 , we cannot consider the correctors equal to zero. However, we can see that choosing for $j = 1, \dots, n-1$,

$$v_j \equiv \frac{j-1}{n-1},$$

we obtain for $j = 1, \dots, n-2$

$$L_p^2(v_{n-1}, [v_1])(x) = N_p^2(v_j, [v_{j+1}])(x) = -V_2 \left(\frac{-n}{p(n-1)} \right). \quad (8.38)$$

This ends the proof of Proposition (8.10). □

Remark 8.11. *From the previous extensions, we can imagine even more complex scenarios. For instance having a bifurcation with n outgoing roads, with a very general (but still periodic) distribution of vehicles. The technique remains the same for that type of problem but one has to consider a great number of "cumulative distribution functions".*

ACKNOWLEDGMENTS

This project was co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council via the M2NUM project and by ANR HJNet (ANR-12-BS01-0008-01) and ANR MFG (ANR-16-CE40-0015-01)

References

- [1] Y. ACHDOU AND N. TCHOU, *Hamilton-jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction*, Communications in Partial Differential Equations, 40 (2015), pp. 652–693.
- [2] M. AL HAJ, N. FORCADEL, AND R. MONNEAU, *Existence and uniqueness of traveling waves for fully overdamped frenkel–kontorova models*, Archive for Rational Mechanics and Analysis, 210 (2013), pp. 45–99.
- [3] O. ALVAREZ AND A. TOURIN, *Viscosity solutions of nonlinear integro-differential equations*, in Annales de l'Institut Henri Poincaré. Analyse non linéaire, vol. 13, Elsevier, 1996, pp. 293–317.

- [4] B. ANDREIANOV, K. H. KARLSEN, AND N. H. RISEBRO, *A theory of l 1-dissipative solvers for scalar conservation laws with discontinuous flux*, Archive for rational mechanics and analysis, 201 (2011), pp. 27–86.
- [5] A. AW, A. KLAR, M. RASCLE, AND T. MATERNE, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 259–278.
- [6] G. BARLES, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer Verlag, 1994.
- [7] G. BRETTI, M. BRIANI, AND E. CRISTIANI, *An easy-to-use algorithm for simulating traffic flow on networks: numerical experiments*, Discrete Contin. Dyn. Syst. Ser. S, 7 (2014), pp. 379–394.
- [8] M. BRIANI AND E. CRISTIANI, *An easy-to-use algorithm for simulating traffic flow on networks: theoretical study*, Netw. Heterog. Media, 9 (2014), pp. 519–552.
- [9] M. G. CRANDALL, H. ISHII, AND P.-L. LIONS, *User’s guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), pp. 1–67.
- [10] E. CRISTIANI AND S. SAHU, *On the micro-to-macro limit for first-order traffic flow models on networks*, Netw. Heterog. Media, 11 (2016), pp. 395–413.
- [11] F. DA LIO, N. FORCADEL, AND R. MONNEAU, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics*, J. Eur. Math. Soc. (JEMS), 10 (2008), pp. 1061–1104.
- [12] F. DA LIO, C. I. KIM, AND D. SLEPČEV, *Nonlocal front propagation problems in bounded domains with neumann-type boundary conditions and applications*, Asymptotic Analysis, 37 (2004), pp. 257–292.
- [13] M. DI FRANCESCO AND M. D. ROSINI, *Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit*, Arch. Ration. Mech. Anal., 217 (2015), pp. 831–871.
- [14] N. FORCADEL, C. IMBERT, AND R. MONNEAU, *Homogenization of fully overdamped frenkel-kontorova models*, Journal of Differential Equations, 246 (2009), pp. 1057–1097.
- [15] ———, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete Contin. Dyn. Syst., 23 (2009), pp. 785–826.
- [16] ———, *Homogenization of accelerated frenkel-kontorova models with n types of particles*, Transactions of the American Mathematical Society, 364 (2012), pp. 6187–6227.
- [17] N. FORCADEL AND W. SALAZAR, *Homogenization of second order discrete model and application to traffic flow*, Differential Integral Equations, 28 (2015), pp. 1039–1068.
- [18] N. FORCADEL, W. SALAZAR, AND M. ZAYDAN, *Specified homogenization of a discrete traffic model leading to an effective junction condition*, Commun. Pure Appl. Anal., 17 (2018), pp. 2173–2206.
- [19] G. GALISE, C. IMBERT, AND R. MONNEAU, *A junction condition by specified homogenization and application to traffic lights*, Anal. PDE, 8 (2015), pp. 1891–1929.
- [20] D. HELBING, *From microscopic to macroscopic traffic models*, in A perspective look at non-linear media, Springer, 1998, pp. 122–139.
- [21] C. IMBERT, *A non-local regularization of first order hamilton-jacobi equations*, Journal of Differential Equations, 211 (2005), pp. 218–246.

- [22] C. IMBERT AND R. MONNEAU, *Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks*, Ann. Sci. Éc. Norm. Supér. (4), 50 (2017), pp. 357–448.
- [23] C. IMBERT, R. MONNEAU, AND E. ROUY, *Homogenization of first order equations with (u/ε) -periodic hamiltonians part ii: Application to dislocations dynamics*, Communications in Partial Differential Equations, 33 (2008), pp. 479–516.
- [24] C. IMBERT, R. MONNEAU, AND H. ZIDANI, *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, ESAIM Control Optim. Calc. Var., 19 (2013), pp. 129–166.
- [25] H. ISHII, *Perron’s method for monotone systems of second-order elliptic partial differential equations*, Differential and Integral Equations, 5 (1992), pp. 1–24.
- [26] H. ISHII AND S. KOIKE, *Viscosity solutions for monotone systems of second-order elliptic pdes*, Communications in partial differential equations, 16 (1991), pp. 1095–1128.
- [27] H. ISHII AND P.-L. LIONS, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, Journal of Differential equations, 83 (1990), pp. 26–78.
- [28] H. LEE, H.-W. LEE, AND D. KIM, *Macroscopic traffic models from microscopic car-following models*, Physical Review E, 64 (2001), p. 056126.
- [29] S. M. LENHART, *Viscosity solutions for weakly coupled systems of first-order partial differential equations*, Journal of mathematical analysis and applications, 131 (1988), pp. 180–193.
- [30] M. J. LIGHTHILL AND G. B. WHITHAM, *On kinematic waves. ii. a theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 229 (1955), pp. 317–345.
- [31] P. L. LIONS, *Lectures at collège de france*, 2013-2014.
- [32] P.-L. LIONS AND P. SOUGANIDIS, *Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), pp. 807–816.
- [33] P. I. RICHARDS, *Shock waves on the highway*, Operations research, 4 (1956), pp. 42–51.
- [34] D. SLEPČEV, *Approximation schemes for propagation of fronts with nonlocal velocities and neumann boundary conditions*, Nonlinear Analysis: Theory, Methods & Applications, 52 (2003), pp. 79–115.