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Symmetric differentials on complex hyperbolic manifolds with cusps

Benoît Cadorel

Abstract

Let (X, D) be a logarithmic pair, and let h be a smooth metric on $T_{X \setminus D}$. We give sufficient conditions on the curvature of h for $\Omega_X(\log D)$ and Ω_X to be big. As an application, we give a metric proof of the bigness of $\Omega_X(\log D)$ on any toroidal compactification of a bounded symmetric domain. Then, we use this singular metric approach to study the bigness and the nefness of Ω_X in the more specific case of the ball. We obtain effective ramification orders for a cover $X' \rightarrow X$, étale outside the boundary, to have all its subvarieties with big cotangent bundle. We also prove that $\Omega_{X'}$ is nef if the ramification is high enough. Moreover, the ramification orders we obtain do not depend on the dimension of the quotient of the ball we consider.

1 Introduction

For any compact quotient X of a bounded symmetric domain, we know from the work of Brunebarbe, Klingler and Totaro [BKT13], that the cotangent bundle Ω_X must be a big vector bundle. The method they use to prove this result consists mainly in computing the curvature of the Bergman metric to show that the bundle must be nef, and then that its higher Segre class must be positive. In the case where X is merely a compactification of a quotient of a bounded symmetric domain, with boundary D , the general philosophy of logarithmic pairs says that $\Omega_X(\log D)$ should have positivity properties similar to the compact case. In this spirit, Brunebarbe proves the following in [Bru16] :

Theorem 1 ([Bru16]). *Let (X, D) be a toroidal compactification of a quotient of a bounded symmetric domain. Then $\Omega_X(\log D)$ is big.*

Brunebarbe's proof relies on a close study of some well suited variations of Hodge structure. One purpose of this paper is to give another approach to this result, generalizing the one of [BKT13]. Indeed, a theorem of Boucksom [Bou02] indicates that we can estimate the volume of a given pseudo-effective line bundle, by the maximal power of the curvature of a suitable singular metric, integrated outside its singularities. Applying these ideas, we can prove the following result.

Theorem 2. *Let (X, D) be a logarithmic pair. Assume that $T_X|_{X \setminus D}$ admits a smooth Kähler metric h satisfying the following hypothesis :*

1. *h has negative holomorphic sectional curvature on $X \setminus D$, bounded by a constant $-A$;*
2. *h has non-positive bisectional curvature;*

Then $\Omega_X(\log D)$ is a big vector bundle. In addition, if

3. h , seen as a metric on T_X , is locally bounded;

then Ω_X is big.

Remark that this result, coupled with a theorem of Campana and Păun [CP15], implies that a logarithmic pair (X, D) with a Kähler metric satisfying the first two hypothesis of Theorem 2, must have $K_X + D$ big. This can be seen as a weak logarithmic version of a recent theorem of Wu and Yau [WY16], stating that a projective manifold admitting a Kähler metric with negative holomorphic sectional curvature must have an ample canonical bundle.

In the case of a quotient of a bounded symmetric domain, the Bergman metric on the open part of our compactification satisfies all the properties we need to apply Theorem 2. Thus, it seems that the simple use of singular metrics is well suited to study the positivity properties of the toroidal compactifications of bounded symmetric domains. In particular, we will see that for toroidal compactification of a quotient of the *ball*, we can obtain effective results for the general notions of positivity of the cotangent bundle.

If a quotient $X = \mathbb{B}^n / \Gamma$ is compact, it is well known that the Bergman metric on \mathbb{B}^n will induce negativity properties on T_X . In particular, the bundles K_X, Ω_X will be ample, X will be Kobayashi hyperbolic, and so on. If the group Γ is not co-compact, it is legitimate to ask to what extent these properties are preserved under the toroidal compactification. More precisely, given such a toroidal compactification $\overline{X} = \overline{\mathbb{B}^n} / \Gamma$, we would like to study the general notions of positivity for the classical bundles supported by \overline{X} .

In the simple case of curves, we know that $K_{\overline{X}}$ has *a priori* no reason to be even nef (i.e. to have non-negative degree) : it suffices to consider $\overline{X} = \mathbb{P}^1$, and $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is a quotient of the unit disk. In the case of surfaces, Hirzebruch considers in [Hir84] the blowing-up of a product of two elliptic curves at a point. By using logarithmic Yau-Miyaoka's inequality, he shows that such a manifold is a toroidal compactification of a quotient of \mathbb{B}^2 . This provides an example of a toroidal compactification of a quotient of the ball for which $K_{\overline{X}}$ is neither big nor nef. However, this particular feature of $K_{\overline{X}}$ is specific to the dimensions 1 and 2 : Di Cerbo and Di Cerbo prove in [CC15] that $K_{\overline{X}}$ must always be nef for $n \geq 3$. Using their work, Bakker and Tsimerman show in turn that $K_{\overline{X}}$ is big for $n \geq 3$, and even ample if $n \geq 6$.

In this paper, we propose to study the various notions of positivity for the cotangent bundle $\Omega_{\overline{X}}$, on a given toroidal compactification \overline{X} of a quotient of the ball. First of all, the results of [Bou02] will permit us to estimate the intersection numbers of the logarithmic tautological bundle with curves $C \subset \mathbb{P}(T_{\overline{X}}(-\log D))$. The nefness of the logarithmic cotangent bundle of \overline{X} will follow naturally.

Theorem 3. *For any toroidal compactification of a quotient of the ball (\overline{X}, D) , the logarithmic cotangent bundle $\Omega_{\overline{X}}(\log D)$ is nef.*

Using singular metrics related to the Bergman metric on \mathbb{B}^n permits us to compare the curvature of Ω_X and K_X on the open part $X \subset \overline{X}$. The results of [BT15] provide us with effective estimate on the positivity of $K_{\overline{X}}$, and of its linear combinations with D , that we can transpose to $\Omega_{\overline{X}}$. Since the cotangent bundle behaves well under restriction to subvarieties, we can prove the following statement.

Theorem 4. *Let X' be a quotient of \mathbb{B}^n , and let $X \rightarrow X'$ be an étale cover ramifying at order at least l on any boundary component. Assume that*

- (i) $l \geq 9$ if $n = 3$,
- (ii) $l \geq 11$ if $n = 4$,
- (iii) $l \geq 13$ if $n \geq 5$.

Then, for any subvariety V of \overline{X} , not included in D , any resolution of V has big cotangent bundle. In particular, by [CP15], any such subvariety is of general type.

Now that we know that the logarithmic cotangent bundle is nef on a compactification of a quotient of \mathbb{B}^n , we want to prove similar claims on the standard cotangent bundle. One natural way to complete our study, is to resolve the birational transformation $\mathbb{P}(T_{\overline{X}}(-\log D)) \rightarrow \mathbb{P}(T_{\overline{X}})$, and to use this resolution to relate the two tautological bundles on these projectivized spaces. This will be our work in Section 4. Studying this resolution will provide us with useful identities of intersection numbers, which will give us a bound on the ramification needed for $\Omega_{\overline{X}}$ to be nef.

Theorem 5. *Let $X \xrightarrow{\sigma} X'$ be a finite covering of a quotient of the ball, ramifying to an order larger than 7 on the boundary $D' \subset \overline{X}'$. Then $\Omega_{\overline{X}}$ is nef.*

Finally, Theorem 5 will permit us to use the classical criterion for bigness of nef divisors, to obtain the following refined version of Theorem 4, dealing with *embedded* subvarieties of \overline{X} .

Theorem 6. *Let $X \xrightarrow{\sigma} X'$ be a finite étale covering, ramifying to an order larger than 7 on the boundary $D' \subset \overline{X}'$. Then all immersed submanifolds $\overline{W} \rightarrow \overline{X}$ not included in D have big cotangent bundle. In particular, they are of general type by [CP15].*

We see that the theorems 4 and 5 can be related to a result of [BT15] about the Green-Griffiths conjecture on the pairs (\overline{X}, D) . In their article, Bakker and Tsimerman actually use a theorem of Nadel [Nad89] to prove that if $\dim X = 3$ (resp. $\dim X = 4, 5$, resp. $\dim X \geq 6$), \overline{X} will verify the Green-Griffiths conjecture when the ramification order l satisfies $l \geq 2$ (resp. $l \geq 3$, resp. $l \geq 4$). In particular, it implies that, with the same ramification orders, all curves not included in the boundary are hyperbolic. The bounds of [BT15] are consequently smaller than ours in the case of curves, but our method has the advantage of working for submanifolds of any dimension.

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2 Compactifications of quotients of the ball

2.1 Construction of the toroidal compactification

We recall some results on the structure of the toroidal compactification of a quotient of the complex unit ball. Let $\Gamma \subset \mathrm{PU}(n, 1)$ be a group of automorphisms of the ball. As explained in [Mok12] and [DCDC15], if we assume that all parabolic isometries of Γ are unipotent, it is possible to compactify the quotient \mathbb{B}^n/Γ using a construction similar to the one of [AMRT10], which we can find in full detail in [Mok12]. If Γ is supposed to be a neat arithmetic subgroup of $\mathrm{Aut}(\mathbb{B}^n)$, this assumption will always be verified.

From now on, we will assume that Γ is a group of automorphisms of \mathbb{B}^n with unipotent parabolic isometries. Let $X = \mathbb{B}^n/\Gamma$. The toroidal compactification of X consists in adding to it a finite number of abelian varieties at its cusps, to obtain a smooth manifold \overline{X} . Let us describe the structure of \overline{X} in the neighborhood of such a cusp.

For any $N > 0$, let

$$S^{(N)} = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}; l(z', z_n) > N\}, \quad (1)$$

with $l(z', z_n) = \text{Im} z_n - \|z'\|^2$. The open set $S^{(0)}$ is a Siegel domain representation of \mathbb{B}^n with respect to a given base point $b \in \partial\mathbb{B}^n$, and the family $(S^{(N)})_N$ represents the family of horoballs of \mathbb{B}^n at the point b .

There exists a finite number of conjugacy classes of maximal parabolic subgroups $\Gamma_i \subset \Gamma$, each one of them corresponding to a cusp C_i of X . Let $\Gamma_b \subset \Gamma$ be such a group, fixing some $b \in \partial\mathbb{B}^n$. Then, for a certain $N > 0$, Γ_b fixes the horoball $S^{(N)}$, where the Siegel representation (1) is taken so that $0 \in \mathbb{C}^{n-1} \times \mathbb{C}$ corresponds to b .

The stabilizer of b in Γ acts on $S^{(N)}$ as the semi-direct product of two group actions, that we will now describe. The first one of these is an action of \mathbb{Z} , defined by

$$k \cdot (z', z_n) = (z', z_n + k\tau),$$

where $\tau \in \mathbb{R}_+^*$ is some parameter depending on b . Let $G^{(N)} = S^{(N)}/\mathbb{Z}$, with its natural analytic structure.

We have $G^{(N)} \cong \{(w', w_n) \in \mathbb{C}^{n-1} \times \mathbb{C}^*; \|(w', w_n)\|_\mu < e^{-\frac{2\pi}{\tau}N}\}$, where $\|(w', w_n)\|_\mu = |w_n|e^{\frac{2\pi}{\tau}\|w'\|^2}$. The projection is realized by the following holomorphic application :

$$\begin{aligned} S^{(N)} & \xrightarrow{\Psi} G^{(N)} \\ (w', w_n) & \mapsto \left(z', e^{\frac{2i\pi z_n}{\tau}}\right). \end{aligned}$$

Let $\widehat{G^{(N)}} = \{(w', w_n) \in \mathbb{C}^{n-1} \times \mathbb{C}; \|(w', w_n)\|_\mu < e^{-\frac{2\pi}{\tau}N}\}$. Thus, if we note $D_0 = \{w_n = 0\} \subset \widehat{G^{(N)}}$, we see easily that the differential of Ψ send surjectively $T_{S^{(N)}}$ onto $T_{G^{(N)}}(-\log D_0)$.

The second group action comes from a lattice $\Lambda_b \subset \mathbb{C}^{n-1}$, and can be written

$$a \cdot (z', z_n) = (z' + a, z_n + i\|a\|^2 + 2i\bar{a} \cdot z'),$$

for $a \in \Lambda_b$, $(z', z_n) \in S^{(N)}$. The stabilizer of b in Γ acts on $S^{(N)}$ as the semi-direct product of these two previous actions. Thus, the action of Λ_b goes to the quotient $S^{(N)}/\mathbb{Z} \cong G^{(N)}$, and we can write its action on $G^{(N)}$ as

$$a \cdot (w', w_n) = \left(w' + a, e^{-\frac{2\pi}{\tau}\|a\|^2} e^{-4\pi\frac{\bar{a} \cdot z'}{\tau}} w_n\right). \quad (2)$$

The action of Λ_b on $G^{(N)}$ extends naturally as an action on $\widehat{G^{(N)}}$. We can thus define the open manifold $\Omega_b^{(N)}$ to be the quotient $\widehat{G^{(N)}}/\Lambda_b$.

The subspace $D_0 \subset \widehat{G^{(N)}}$ goes to the quotient by Λ_b , to give an abelian variety $D_b = D_0/\Lambda_b \hookrightarrow \Omega_b^{(N)}$. Moreover, the embedding of the horoball $S^{(N)} \hookrightarrow \mathbb{B}^n$ induces an embedding of the quotient

$$\Omega^{(N)} \setminus D_b = G^{(N)}/\Lambda_b \hookrightarrow X.$$

The *toroidal compactification* of X is defined to be the gluing of the manifolds $\Omega_{b_i}^{(N)}$ on X along the open subsets $\Omega_{b_i}^{(N)} \setminus T_{b_i}$, where the $b_i \in \partial\mathbb{B}^n$ span a family of representatives of the cusps. Let us denote by \overline{X} this compactification. We see that, as sets, we have

$$\overline{X} = X \sqcup \bigsqcup_i D_{b_i}.$$

Let us denote by $D = \bigsqcup_i D_{b_i}$ the compactifying divisor of \overline{X} . This divisor is a disjoint union of abelian varieties.

Terminology.

1. In the rest of this paper, a *quotient of the ball* will always mean a quotient of \mathbb{B}^n by a subgroup of $\mathrm{PU}(n, 1)$ with unipotent parabolic isometries.
2. Unless otherwise specified (e.g. in Section 3), a *toroidal compactification* will always be a toroidal compactification of a quotient of the ball, as defined in this section.

2.2 Local coordinates. Bergman metric

Let D_b be a component of D , and let $w_0 \in D_b$ be any point of this component. In some neighborhood U of x_0 , we can consider local coordinates (w', w_n) , coming from the global coordinates on $\overline{G_b^{(N)}}$. We will describe explicitly the action of Λ_b on the logarithmic tangent bundle to U in these coordinates.

First, we study the action of this group on $T_{G^{(N)}}(-\log D_0)$. By (2), it can be expressed as

$$\begin{cases} a \cdot \frac{\partial}{\partial w'_i} \Big|_x &= \sum_j \frac{\partial w'_j}{\partial w'_i} \frac{\partial}{\partial w'_j} \Big|_{a \cdot x} + \frac{\partial w'_j}{\partial w'_i} \frac{\partial}{\partial w'_j} \Big|_{a \cdot x} = \frac{\partial}{\partial w'_i} - \frac{4\pi\bar{a}_i}{\tau} w'_n \left(\frac{\partial}{\partial w'_n} \right)_{a \cdot x} \\ a \cdot \left(w'_n \frac{\partial}{\partial w'_n} \right)_x &= \left(w'_n \frac{\partial}{\partial w'_n} \right)_{a \cdot x} \end{cases},$$

where (w'_i) is the family of coordinates at the point $a \cdot x$. After taking the quotient by Λ_b , we see that

$$(e_j)_{1 \leq j \leq n} = \left(\left(\frac{\partial}{\partial w_j} - \frac{4\pi}{\tau} w_j \left(w_n \frac{\partial}{\partial w_n} \right) \right)_{1 \leq j \leq n-1}, w_n \frac{\partial}{\partial w_n} \right)$$

is well defined on the whole $\Omega_b^{(N)}$, and realizes a *smooth* frame for $T_{\overline{X}}(-\log D)$ on $\Omega_b^{(N)}$ for some $N > 0$ large enough.

Recall that on the ball \mathbb{B}^n , with standard coordinates (z_j) , the Bergman metric is given by, up to a normalization choice

$$h_{\mathrm{Berg}} = \frac{(1 - \|z\|^2) \sum_j dz_j \otimes d\bar{z}_j + \left(\sum_j \bar{z}_j dz_j \right) \otimes \left(\sum_j z_j d\bar{z}_j \right)}{(1 - \|z\|^2)^2}. \quad (3)$$

With this particular choice of normalization, the metric has constant holomorphic sectional curvature equal to -2 , and we also have $\mathrm{Ric}(h_{\mathrm{Berg}}) = -(n+1)\omega_{\mathrm{Berg}}$, where ω_{Berg} is the Kähler form associated with the Bergman metric.

The smooth frame $(e_j)_j$ permits to express the Bergman metric on $\Omega^{(N)} \setminus D_b$. Indeed, as we can see from [Mok12], we have the following proposition :

Proposition 2.2.1. *The Bergman metric on \mathbb{B}^n induces a singular hermitian metric on $T_{\overline{X}}(-\log D)$, whose expression in the frame $(e_j)_j$ admits the diagonal form*

$$(H_{ij}) = (h_{Berg}(e_i, e_j)) = \text{diag}(l(w)^{-1}, \dots, l(w)^{-2}),$$

with, for any $w = (w', w_n) \in \Omega_b^{(N)} \setminus D_b$, $l(w) = \frac{\tau}{4\pi} \log \|w\|_\mu^2$.

Remark. Even though the metric $\|\cdot\|_\mu$ is a priori defined only on $S^{(N)}$, it is invariant under the actions of \mathbb{Z} and Λ_b , so it is legitimate to express the norm $\|w\|_\mu$ for any $w \in \Omega_b^{(N)} \setminus D_b$.

Later on, we will need to compute the intersection numbers of $K_{\overline{X}} + D$ in terms of the Bergman metric on $X \subset \overline{X}$. The following proposition, which comes from Mumford's work [Mum77], will be useful for this purpose.

Proposition 2.2.2. *Let (\overline{X}, D) be a toroidal compactification, and let $\overline{V} \xrightarrow{f} \overline{X}$ be a generically injective holomorphic map, from a complex manifold of dimension p , such that $f(\overline{V}) \not\subset D$. Let $V = f^{-1}(D)$. Then we have*

$$(K_{\overline{X}} + D)^p \cdot [f(\overline{V})] = \int_V \left(\frac{i}{2\pi} f^* \Theta(\det h_{Berg}^*) \right)^p = \frac{(n+1)^p}{\pi^p} \int_V f^* \omega_{Berg}^p.$$

The first equality actually comes from the fact that h_{Berg}^* is a *good* metric on $\Omega_{\overline{X}}(\log D)$ in the sense of [Mum77]. The second equality is just an application of the fact that h_{Berg} has constant sectional curvature.

3 Bigness of the cotangent bundles

In this section, we use singular metrics to study the bigness of the standard and logarithmic cotangent bundle of a log-pair (X, D) . We will see that general assumptions on the negativity of the curvature of $X \setminus D$, are already sufficient to prove that $\Omega_X(\log D)$ is big.

Terminology. We call a *log-pair* the data of a pair (X, D) , where X is a smooth complex projective manifold, and $D \subset X$ a divisor with simple normal crossings. If D is smooth, we say that the log-pair (X, D) has *smooth boundary*.

3.1 Singular metrics on the tangent bundles

The following result relates the bigness of the standard and logarithmic cotangent bundles of a given log-pair (X, D) , to the negativity of the curvature of a given metric Kähler metric on the open part $X \setminus D$. This result is as a generalization of a theorem of [BKT13] : we will use a criterion for bigness of [Bou02], coupled with the well known Ahlfors-Schwarz lemma, to extend the field of application of their proof. This will give a proof of the following theorem, which is a slightly more general version of Theorem 2.

Theorem 7. *Let (X, D) be a logarithmic pair. Assume that $T_X|_{X \setminus D}$ admits a smooth metric h (not necessarily Kähler) satisfying the following hypothesis :*

1. h has negative holomorphic sectional curvature H on $X \setminus D$, bounded by a constant $-A$;

2. h has non-positive bisectional curvature B ;
3. h has negative bisectional curvature at some point of $\mathbb{P}(T_X|_{X \setminus D})$ i.e. there exist $x_0 \in X \setminus D$, $v_0 \in T_{x_0}X \setminus \{0\}$ such that

$$\forall w \in T_{x_0}X \setminus \{0\}, B(v_0, w) < 0.$$

Then $\Omega_X(\log D)$ is a big vector bundle. In addition, if

4. h , seen as a metric on T_X , is locally bounded;

then Ω_X is big.

Remark. By [BKT13], if the metric h is supposed to be Kähler, the first two hypothesis of Theorem 7 actually imply the third one. Thus, Theorem 2 is a consequence of Theorem 7.

Before proving Theorem 7, let us begin by recalling some well known growth properties of metrics with negative holomorphic sectional curvature, derived from the Ahlfors-Schwarz lemma.

Proposition 3.1.1 (Ahlfors-Schwarz lemma). *Let \mathbb{H} be a model of the Poincaré half-plane, with its canonical metric ω_P . Let h be another smooth metric on $T_{\mathbb{H}}$, with negative sectional curvature bounded by a constant $-A$. Then, there exists a constant $C > 0$, depending only on A , such that*

$$h \leq C\omega_P.$$

In particular, if Δ is the unit disk of \mathbb{C} , and if h is a metric on T_{Δ} with bounded negative curvature as above, there exists $C > 0$ such that

$$h(z) \leq \frac{C}{(1 - |z|^2)^2}. \quad (4)$$

Similarly, if Δ^ is the punctured unit disk, any such metric on T_{Δ^*} is bounded as*

$$h(z) \leq \frac{C}{|z|^2 |\log |z||^2}.$$

Now, let Δ^n be the unit polydisk in \mathbb{C}^n , with the coordinates (z_1, \dots, z_n) , and let $U = (\Delta^*)^m \times \Delta^{n-m}$ be the complementary of $D = \{z_1 \dots z_m = 0\}$. We introduce the Poincaré metric $h^{(p)}$ on U , defined by its Kähler form

$$\omega^{(p)} = \sum_{k=1}^m \frac{i}{2} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2 |\log |z_k||^2} + \sum_{k=m+1}^n \frac{i}{2} dz_k \wedge d\bar{z}_k.$$

Proposition 3.1.2. *Let h be a smooth metric on T_U , with holomorphic sectional curvature bounded from above by a negative constant $-A$. Then h has Poincaré growth, i.e. for any $x \in D$, there exists a constant C (depending only on A) such that for any vector fields ξ and η on U , we have*

$$|h(\xi, \eta)|^2 \leq C\omega^{(p)}(\xi, \xi)\omega^{(p)}(\eta, \eta). \quad (5)$$

in the neighborhood of x .

Proof. Applying Cauchy-Schwarz, we see that it suffices to prove that for any vector field η , we locally have $\|\eta\|_h \leq C\|\eta\|_{(p)}$. Moreover, we can clearly suppose η constant.

Let $\eta = \sum_j a_j \frac{\partial}{\partial z_j}$ be such a constant vector field. Then

$$\|\eta\|_h^2 \leq n^2 \sum_j \left\| a_j \frac{\partial}{\partial z_j} \right\|_h^2.$$

Thus, it suffices to prove the result for $\eta = \frac{\partial}{\partial z_j}$ for any $j \in [1, n]$. Let $x_0 \in U$, and let ω be a neighborhood of x_0 on which $\|x\|_\infty$ is bounded by a constant B , for any $x \in \omega$.

If $j \in [1, m]$, we apply the Ahlfors-Schwarz lemma to the punctured disk passing through x and directed by $\frac{\partial}{\partial z_j}$ to get that

$$\left\| \frac{\partial}{\partial z_j} \right\|_h(x) \leq C \frac{1}{|z_j|^2 |\log |z_j||^2},$$

on ω , for some C depending only on A .

Similarly, if $j \in [m+1, n]$ we see from (4) that $\left\| \frac{\partial}{\partial z_j} \right\|_h$ must be bounded from above by

$$\left\| \frac{\partial}{\partial z_j} \right\|_h(x) \leq C \frac{1}{(1-|z|^2)^2} \leq \frac{C}{(1-B^2)^2},$$

with C depending only on A . This proves the result. \square

Corollary 3.1.1. *Let Δ^n and $D \subset \Delta^n$ be as above, and let h be a smooth metric on $T_{\Delta^n \setminus D}$, which we suppose to have negative sectional curvature bounded by $-A$. Then for any vector field ξ of $T_X(-\log D)$, $\|\xi\|_h$ is bounded in the neighborhood of any point of D .*

Proof. It suffices to apply (5) on the vectors of the canonical frame $\left(\left(z_j \frac{\partial}{\partial z_j} \right)_{1 \leq j \leq m}, \left(\frac{\partial}{\partial z_j} \right)_{m \leq j \leq n} \right)$, and to remark that $\omega^{(p)}$ is bounded on these vectors. \square

We now prove that under the first three assumptions of Theorem 7, $\Omega_X(\log D)$ is big.

Let $Y = \mathbb{P}(T_X(-\log D)) \xrightarrow{p} X$ and let $\mathcal{O}(1)$ be the tautological bundle of this projectivized space.

Lemma 3.1.1. *The line bundle $\mathcal{O}(1)$ is pseudo-effective on Y .*

Proof. Let \widehat{h} be the metric induced by h on the tautological bundle $\mathcal{O}(-1) \rightarrow Y$. Remark that \widehat{h} is not defined on $p^{-1}(D)$. Denote by \widehat{h}^* the dual of this metric; we see that \widehat{h}^* is determined locally by the norm of a non-vanishing section of $\mathcal{O}(1)$. More specifically, if $(x, [v]) \in p^{-1}(D)$, choose a section σ of $T_X(-\log D)$, non vanishing around x , such that $\sigma(x) = v$. Then σ induces a local section $\widehat{\sigma}$ of $\mathcal{O}(-1)$ around $(x, [v])$, whose dual section we will denote by $\widehat{\sigma}^*$. Locally, the norm of $\widehat{\sigma}^*$ is given by

$$\|\sigma^*\|_{\widehat{h}^*} = \frac{1}{\|\sigma\|_h},$$

where $\widehat{\sigma}^*$ is the section of $\mathcal{O}(1)$ dual to $\widehat{\sigma}$. Then, on $p^{-1}(X \setminus D)$, the curvature of $(\mathcal{O}(1), \widehat{h}^*)$ is determined near $(x, [v])$ by

$$\frac{i}{2} \Theta(\widehat{h}^*) \stackrel{loc}{=} \frac{i}{2} \bar{\partial} \partial \log \|\widehat{\sigma}^*\|_{\widehat{h}^*} = \frac{i}{2} \bar{\partial} \partial \log \|\sigma\|_h.$$

We can develop this expression, to obtain

$$\frac{i}{2}\Theta(\widehat{h}^*) \cdot (\xi, \xi) \stackrel{loc}{=} -\frac{i}{2} \frac{\langle \sigma, \Theta(h) \cdot (p_*\xi, p_*\xi)\sigma \rangle_h}{\|\sigma\|_h^2} + \omega_h^{FS}(\xi^{vert}, \xi^{vert}). \quad (6)$$

The first term appearing in the right hand side of this equation is equal to $B(\sigma, p_*\xi)\|p_*\xi\|_h$, where B is the bisectional curvature of h . It is non-negative by our hypothesis. The second term, associated with the Fubini-Study metric on the fibers, is also non-negative. This implies that $i\bar{\partial}\partial \log \|\widehat{\sigma}^*\|_{\widehat{h}^*}^2 \geq 0$, i.e. that $-\log \|\widehat{\sigma}^*\|_{\widehat{h}^*}^2$ is plurisubharmonic on $Y \setminus p^{-1}(D)$. Moreover, $\|\widehat{\sigma}^*\|_{\widehat{h}^*}^2 = \frac{1}{\|\sigma\|_h^2}$ is locally bounded from below by Corollary 3.1.1, so $-\log \|\sigma^*\|_{\widehat{h}^*}^2$ is bounded from above. By the usual properties of bounded plurisubharmonic functions, we see that this last function extends uniquely on $p^{-1}(D)$ to a plurisubharmonic function, defined locally on Y .

Consequently, we can write $\widehat{h}^* \stackrel{loc}{=} e^{-\Psi}$, with Ψ plurisubharmonic. This implies in particular that \widehat{h}^* is a *singular* metric on $\mathcal{O}(1)$, with positive curvature in the sense of currents. By [Dem92], this implies in turn that $\mathcal{O}(1)$ is a pseudo-effective line bundle. \square

To conclude, we will use the following theorem of [Bou02] :

Theorem 8. (Boucksom [Bou02]) *Let L be a pseudo-effective line bundle on a compact Kähler manifold M of dimension n . Then, for any closed positive current $T \in c_1(L)$, if we denote by T_{ac} the absolutely continuous part of T , the powers T_{ac}^k have bounded mass on M .*

Moreover, the volume of L is equal to

$$\text{vol}(L) = \max_T \int_M T_{ac}^n,$$

where T ranges among the positive closed $(1, 1)$ -currents representing $c_1(L)$.

Proof of Theorem 7. Let $T = \frac{i}{2}\Theta_c(\widehat{h}^*)$, where by Θ_c we mean the curvature in the sense of currents. Since $p^{-1}(D)$ has zero Lebesgue measure, for any k , T_{ac}^k is the current of integration against $\Theta(\widehat{h}^*)^k$ on $Y \setminus p^{-1}(D)$. In particular,

$$\int_Y T_{ac}^{2n-1} = \int_{Y \setminus p^{-1}(D)} \left(\frac{i}{2}\Theta(\widehat{h}^*) \right)^{2n-1}.$$

Remark that Theorem 8 implies that this last integral converges. By (6), we have

$$\frac{i}{2}\Theta(\widehat{h}^*)_{(x, [v])}(\xi, \xi) = -\|p_*\xi\|^2 B(v, p_*\xi) + \omega_h^{FS}(\xi^{vert}, \xi^{vert})$$

and since h has non-positive bisectional curvature, the $(2n-1, 2n-1)$ -form $\left(\frac{i}{2}\Theta(\widehat{h}^*)\right)^{2n-1}$ is non-negative on $Y \setminus p^{-1}(D)$. Moreover, by our third hypothesis, this form is positive at the point $(x_0, [v_0]) \in Y \setminus p^{-1}(D)$.

This means, because of Theorem 8, that

$$\text{vol}(\mathcal{O}(1)) \geq \int_{Y \setminus p^{-1}(D)} \left(\frac{i}{2}\Theta(\widehat{h}^*) \right)^{2n-1} > 0.$$

Thus, $\mathcal{O}(1)$ has positive volume, hence is big on Y . This proves the first assertion of Theorem 2.

Now, assume that h , seen as a metric on T_X , is locally bounded near D . As before, it follows from our second hypothesis that h induces a metric \widehat{h}_0^* on the tautological bundle $\mathcal{O}(1) \rightarrow \mathbb{P}(T_X)$, with positive curvature above $X \setminus D$. If $p_0 : \mathbb{P}(T_X) \rightarrow X$ is the canonical projection, we see that \widehat{h}_0^* can locally be written

$$\widehat{h}_0^* \stackrel{loc}{=} e^{-\Psi_0},$$

with Ψ_0 plurisubharmonic on $p^{-1}(X \setminus D)$. Because of our fourth hypothesis, we see that Ψ_0 must be bounded from above near any point of $p^{-1}(D)$, and thus must extend into a plurisubharmonic function near any such point. This implies that the tautological bundle $\mathcal{O}(1)$ is pseudo-effective, and, applying Theorem 8, we obtain as before that this line bundle has positive volume. This ends the proof. \square

We can now give a proof of Theorem 1. If Ω is a bounded symmetric domain, its Bergman metric h_Ω is a Kähler metric satisfying the first two hypothesis of Theorem 2. Therefore, for any quotient X of Ω by a subgroup $\Gamma \subset \text{Aut}(\Omega)$, the metric h_X induced on X by h_Ω satisfies those same hypothesis. If $\bar{X} = X \sqcup D$ is any smooth compactification of X , with D a divisor with simple normal crossings, Theorem 2 implies that $\Omega_{\bar{X}}(\log D)$ is big. This proves Theorem 1.

We finish this section by a result which will be central in our study of the nefness of the cotangent bundles of a toroidal compactification.

Proposition 3.1.3. *Let (X, D) be a pair satisfying the hypothesis 1 and 2 of Theorem 2. Let $Y = \mathbb{P}(T_X(-\log D))$ with its canonical projection p onto X . Let $f : V \rightarrow Y$ a generically finite morphism from a smooth complex manifold onto a subvariety $f(V) \subset Y$, not included in $p^{-1}(D)$. Then $f^*\widehat{h}^*$ induces a singular metric on $\mathcal{O}(1)$, and*

$$\text{vol}(f^*\mathcal{O}(1)) \geq \int_{f^{-1}(Y \setminus p^{-1}(D)) \cap V_S} \left[\frac{i}{2\pi} f^*\Theta(\widehat{h}^*) \right]^{\dim V},$$

where V_S is the locus of points where f is immersive.

Proof. We saw in the proof of Theorem 7 that we can locally write $\widehat{h}^* \stackrel{loc}{=} e^{-\Psi}$, with Ψ plurisubharmonic and nowhere equal to $-\infty$ on $Y \setminus p^{-1}(D)$. Consequently, we can write

$$f^*\widehat{h}^* \stackrel{loc}{=} e^{-\Psi \circ f},$$

with $\Psi \circ f$ plurisubharmonic, and nowhere equal to $-\infty$ outside $f^{-1}(p^{-1}(D))$. Since $f(V)$ is not included in $p^{-1}(D)$, this implies that $\Psi \circ f \in \text{Psh} \cap L_{loc}^1$, hence that $f^*\widehat{h}^*$ induces a singular metric on $f^*\mathcal{O}(1)$, with positive curvature. Therefore, the line bundle $f^*\mathcal{O}(1)$ is pseudo-effective, and we can estimate its volume using Theorem 8. Since $V_S \cup f^{-1}(p^{-1}(D))$ has zero Lebesgue measure, the absolutely continuous part of $\Theta_c(f^*\widehat{h}^*)$ is equal to $f^*\Theta(\widehat{h}^*)$ almost everywhere, which gives the result. \square

3.2 Bigness of the standard cotangent bundle of a compactification of a quotient of the ball

In this section, we prove Theorem 4. We start by specifying some results that we can find in [BT15]. Let us resume the notations and conventions introduced in Section 2.

Proposition 3.2.1. *Let X' be a quotient of \mathbb{B}^n , and let $X \rightarrow X'$ be an étale cover, ramifying at order l on the boundary. Then, for any $\beta > 0$ such that*

1. $\beta \leq l$ if $n \in [3, 5]$;
2. $\beta \leq \frac{n+1}{2\pi}l$ if $n \geq 6$,

the divisor $K_{\overline{X}} + (1 - \beta)D$ is nef and big.

Proof. Denote by π the projection $\overline{X} \rightarrow \overline{X}'$. Then if $n \geq 6$, we have :

$$\begin{aligned} \pi^* \left(K_{\overline{X}'} + \left(1 - \frac{n+1}{2\pi} D' \right) \right) &= K_{\overline{X}} + D - \frac{n+1}{2\pi} \pi^* D' \\ &\leq K_{\overline{X}} + D - \frac{n+1}{2\pi} l D \\ &\leq K_{\overline{X}} + (1 - \beta) D, \end{aligned}$$

where by " \leq " we mean that the two divisors differ by some linear combination with non-negative coefficients of components of D .

The divisor of the left hand side is big by [BT15], thus so is the right hand side divisor. Moreover, because of our definition of the relation \leq , it is easy to see that the intersection number of $K_{\overline{X}} + (1 - \beta)D$ with any curve not included in D is non-negative. Besides, since $\mathcal{O}_{\overline{X}}(K_{\overline{X}} + D)|_D \cong \mathcal{O}_D$ and $D|_D$ being negative (see [Mok12]), we see that for any curve $C \subset D$, we must have $[K_{\overline{X}} + (1 - \beta)D] \cdot C \geq 0$. This proves that $K_{\overline{X}} + (1 - \beta)D$ is nef.

In the case $n \in [3, 5]$, we reason similarly, writing

$$\pi^* K_{\overline{X}'} \leq K_{\overline{X}} + (1 - \beta)D,$$

and using the fact that $K_{\overline{X}'}$ is nef and big. □

Lemma 3.2.1. *With the same hypothesis as in Proposition 3.2.1, assume that β is a rational number satisfying $\beta < l$ if $n \in [3, 5]$, and $\beta < \frac{n+1}{2\pi}l$ if $n \geq 6$. Then, for any $m \in \mathbb{N}^*$ large enough, the divisor $m[K_{\overline{X}} + (1 - \beta)D]$ is base-point free.*

Proof. This is a simple application of the base-point free theorem (see [KM98]). The divisor $K_{\overline{X}} + (1 - \beta)D$ is big, so for m large enough, $m[K_{\overline{X}} + (1 - \beta)D] - K_{\overline{X}}$ is big.

In addition, we have

$$\begin{aligned} m[K_{\overline{X}} + (1 - \beta)D] - K_{\overline{X}} &= (m - 1) \left[K_{\overline{X}} + \left(\frac{m(1 - \beta)}{m - 1} \right) D \right] \\ &= (m - 1) \left[K_{\overline{X}} + \left(1 - \beta + \frac{(1 - \beta)}{m - 1} \right) D \right], \end{aligned}$$

so if m is large enough, this last divisor will fall in the case of Proposition 3.2.1, and consequently, be nef.

The divisor $K_{\overline{X}} + (1 - \beta)D$ satisfies the hypothesis of the base-point free theorem, so the result follows. □

From now on, we will assume that X and X' are as in Theorem 4. Then, $l > 4\pi$ if $n \geq 5$ and $l > 2(n+1)$ if $n \in \{3, 4\}$, so it is possible to find a rational number β such that

$$\beta \in \left] 2(n+1), \max\left(l, \frac{n+1}{2\pi}l\right) \right[.$$

In that case, because of Lemma 3.2.1, we can write $\beta = \frac{p}{q}$, with p, q large enough so that $L = q(K_{\overline{X}} + D) - pD$ is base-point free.

Consider a subvariety V of \overline{X} , not included in D . Because of the base-point freeness of L , there exists a section $s \in H^0(\overline{X}, p(K_{\overline{X}} + D) - qD)$, which does not vanish identically on V .

Since $\frac{p}{q} > 2(n+1)$, we have $\frac{2}{p} < \frac{1}{(n+1)q}$. Choose a real number $\alpha \in \left] \frac{2}{p}, \frac{1}{(n+1)q} \right[$. Let g be the metric induced by h_{Berg} on the line bundle $\mathcal{O}(q(K_{\overline{X}} + D))$, and let

$$\phi = \|s\|_g^\alpha.$$

We can see in [Mok12, Proposition 1] that near the boundary, the metric g is bounded in the canonical frame $dz_1 \wedge dz_2 \wedge \dots \wedge \frac{dz_n}{z_n}$ as

$$\left\| dz_1 \wedge \dots \wedge \frac{dz_n}{z_n} \right\|_g^2 \leq C |\log |z_n||^{n+1}, \quad (7)$$

where $z_n = 0$ is a local equation for D .

Consider the singular metric \tilde{h} defined on $T_{\overline{X}}$ by $\tilde{h} = \phi h_{Berg}$, and let h_V be its restriction to T_V (at the points where it is defined).

Lemma 3.2.2. *On $X \setminus s^{-1}(0)$, \tilde{h} has negative holomorphic sectional curvature, bounded by a constant $-A$, and negative bisectional curvature.*

Proof. Locally on $X \setminus s^{-1}(0)$, we can write

$$\frac{i}{2} \Theta(\tilde{h}) \stackrel{loc}{=} \frac{i}{2} \bar{\partial} \partial \log \phi \otimes I_n + \frac{i}{2} \Theta(h), \quad (8)$$

so, $s|_{X \setminus s^{-1}(0)}$ being a non-vanishing section of the line bundle $\mathcal{O}(q(K_{\overline{X}} + D))$, we have

$$\begin{aligned} \frac{i}{2} \bar{\partial} \partial \log \phi &= \frac{i}{2} \alpha \bar{\partial} \partial \log \|s\|_g \\ &= \frac{i}{2} \alpha q \Theta_{K_{\overline{X}+D}} \\ &= -q\alpha \operatorname{Ric}(h_{Berg}) \\ &= q\alpha(n+1) \omega_{Berg} \end{aligned}$$

To study the negativity of (8), we can reason locally, in the neighborhood of a point of X corresponding to $0 \in \mathbb{B}^n$, where ω_{Berg} admits the expression (3). Then, we can write $\Theta(h_{Berg})$ matricially as

$$\Theta(h_{Berg})_0 = -\omega_{Berg} I_n + {}^t \overline{T} \wedge T,$$

with $T = (dz_1 \dots dz_n)$. Since $q\alpha(n+1) < 1$, an easy calculation gives the result. \square

Let $V_1 \xrightarrow{f_1} V$ be any resolution of V . If we let $Z = V_{\text{sing}} \cup D \cup s^{-1}(0)$, it is possible to find a resolution $\tilde{V} \xrightarrow{f} V$, dominating f_1 , such that the reduced divisor $f^{-1}(Z)_{\text{red}}$ has simple normal crossings. Since the sectional and bisectional holomorphic curvatures decrease on submanifolds, we see from Lemma 3.2.2 that h_V has bounded negative sectional curvature and negative bisectional curvature on $\tilde{V} \setminus f^{-1}(Z)$.

Lemma 3.2.3. *For any $x \in \tilde{V}$, for any local vector field ξ of $T_{\tilde{V}}$ defined on a neighborhood of x , $\|\xi\|_{h_V}$ is bounded in a neighborhood of x .*

Proof. If $x \notin f^{-1}(D)$, h_{Berg} , considered as a metric on $T_{\overline{X}}$, is bounded in a neighborhood of $f(x)$, so the result is clear.

If $x \in f^{-1}(D)$, h_{Berg} having Poincaré growth with respect to D , we can write for any p near x :

$$\|f_*(\xi)\|_{\text{Berg}}(f(p)) \leq \frac{C}{|z_n| |\log |z_n||^2},$$

where z_n is some local coordinate around $f(x)$, defining D . Thus,

$$\begin{aligned} \|\xi\|_{h_V} &= \phi \|f_*(\xi)\| \\ &\leq C \frac{\|s\|_g^\alpha}{|w_n|^2 |\log |w_n||^2}. \end{aligned}$$

Since s , seen as a section of $\mathcal{O}(q(K_{\overline{X}} + D))$, vanishes at order p on D , this last function is bounded by $\frac{|z_n|^{p\alpha}}{|z_n|^2 |\log |z_n||^{2-(n+1)q\alpha}}$, because of (7). Since $p\alpha > 2$, this gives the result. \square

The proof of Theorem 4 is now straightforward.

Proof of Theorem 4. Because of Lemma 3.2.2 and Lemma 3.2.3, the metric h_V satisfies all four hypothesis of Theorem 7 on \tilde{V} . Therefore, $\Omega_{\tilde{V}}$ is big. Since the morphism $\tilde{V} \rightarrow V_1$ is proper and birational, it follows that Ω_{V_1} is big, which ends the proof. \square

4 Birational transformation between logarithmic and standard projectivized tangent bundles

In this section, we introduce a construction that will reveal useful in Section 5, when we study the nefness of the cotangent bundle of a toroidal compactification.

The plan of our work in the next sections is straightforward : we will first show that the logarithmic cotangent bundle of a toroidal compactification is nef, using Proposition 3.1.3, and then use this result to study the standard cotangent bundle. To do this, we will resolve the birational map $\mathbb{P}(T_{\overline{X}}(-\log D)) \rightarrow \mathbb{P}(T_{\overline{X}})$ into a sequence of two birational morphisms :

$$\mathbb{P}(T_{\overline{X}}(-\log D)) \xleftarrow{\pi} \tilde{Y} \xrightarrow{\pi_0} \mathbb{P}(T_{\overline{X}}). \quad (9)$$

With this construction, it will not be hard to express the pullbacks of the two tautological line bundles onto \tilde{Y} , in term of each other. Therefore, we will be able later on to deduce a condition

for $\Omega_{\overline{X}}$ to be nef (i.e. for the line bundle $\mathcal{O}(1)_0 \rightarrow \mathbb{P}(T_{\overline{X}}(-\log D))$ to be nef), knowing that $\Omega_{\overline{X}}(\log D)$ is nef (i.e. that the tautological bundle $\mathcal{O}(1)_{\log} \rightarrow \mathbb{P}(T_{\overline{X}}(-\log D))$ is nef).

In the rest of the section, we describe the resolution (9) : in fact, it holds for more general log-pairs than the toroidal compactifications. We show actually that for any log-pair (X, D) with *smooth boundary*, there is a canonical way to resolve the map $\mathbb{P}(T_X(-\log D)) \rightarrow \mathbb{P}(T_X)$, by blowing up a single smooth analytic subset in each of these two manifolds.

Terminology and conventions. For our purposes, we will have to switch regularly between the algebraic and geometric conventions for projectivized bundles. Let us briefly sum up these conventions.

Let X denote a general scheme, and let \mathcal{E} be a graded \mathcal{O}_X -algebra. Define the X -scheme $\text{Proj}_X(\mathcal{E})$ in the usual way (see for example [Har77]). Then, if E is a vector bundle on X , $\mathbb{P}_X(E)$ will denote the projectivized bundle of *lines* of E , and $\mathbb{P}_X^*(E)$ will denote its projectivized bundle of *hyperplanes*. We have the following natural isomorphisms :

$$\mathbb{P}_X(E) \cong \mathbb{P}_X^*(E^*) \cong \text{Proj}_X(\text{Sym } E^*).$$

4.1 Resolution of the rational maps

For the rest of the section, (X, D) will be a log-pair with smooth boundary. We will denote by $Y = \mathbb{P}(T_X(-\log D))$ the projectivized bundle of the logarithmic tangent bundle, with its associated tautological bundle $\mathcal{O}_Y(1)$. In the same way, let $Y_0 = \mathbb{P}(T_X)$, and let $\mathcal{O}_{Y_0}(1)$ be its tautological bundle. We will denote by $p : Y \rightarrow X$ and $p_0 : Y_0 \rightarrow X$ the canonical projections.

As announced before, we will prove that the natural birational map $Y \dashrightarrow Y_0$ can be resolved by blowing-up only one subvariety on Y and on Y_0 . Let us first describe these two subvarieties.

Logarithmic exact sequence. Rational map $Y \dashrightarrow Y_0$

On (X, D) , we have the usual logarithmic cotangent exact sequence :

$$0 \rightarrow \Omega_X \rightarrow \Omega_X(\log D) \xrightarrow{\text{res}} \mathcal{O}_D \rightarrow 0, \quad (10)$$

the last arrow being the Poincaré residue map.

The surjective morphism $\Omega_X(\log D) \rightarrow \mathcal{O}_D$ induces a section of the projection $p^{-1}(D) = \mathbb{P}^*(\Omega_X(\log D)|_D) \rightarrow D$, whose image we will denote by Z .

The sequence (10) permits to write the following exact sequence of graded \mathcal{O}_X -algebras :

$$\Omega_X \odot \text{Sym } \Omega_X(\log D) \rightarrow \text{Sym } \Omega_X(\log D) \xrightarrow{v} \text{Sym } \mathcal{O}_D \rightarrow 0. \quad (11)$$

The injection $Z \hookrightarrow Y$ is actually induced by the surjective morphism v , which determines a morphism of projective spaces over X :

$$D \simeq \mathbb{P}_X^*(\mathcal{O}_D) \hookrightarrow \mathbb{P}_X^*(\Omega_X(\log D)).$$

By functoriality of the tautological bundles, we see that $\mathcal{O}_Y(1)|_Z$ is isomorphic to the tautological bundle of $\mathbb{P}_X^*(\mathcal{O}_D)$, i.e. is trivial.

Therefore, the sequence (11) gives the following exact sequence of \mathcal{O}_Y -modules

$$p^*\Omega_X \longrightarrow \mathcal{O}_Y(1) \longrightarrow \mathcal{O}_Y(1) \otimes \mathcal{O}_Z \longrightarrow 0,$$

or, twisting by $\mathcal{O}_Y(-1)$:

$$p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1) \xrightarrow{\iota} \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0, \quad (12)$$

and we see in particular that \mathcal{I}_Z , the ideal sheaf of Z , is the image of the morphism ι .

The morphism of \mathcal{O}_Y -modules $p^*\Omega_X \xrightarrow{\iota \otimes Id_{\mathcal{O}(1)}} \mathcal{O}_Y(1)$ induces a map $\mathbb{P}(T_X(-\log D)) \xrightarrow{\Psi_0} \mathbb{P}(T_X)$ which is the rational map we are studying. Since this morphism is not surjective, this map is not globally defined. Actually, by (12), the image of $\iota \otimes Id_{\mathcal{O}(1)}$ is $\mathcal{I}_Z \otimes \mathcal{O}_Y(1)$. This shows that the locus of indeterminacy of Ψ_0 is Z , and that this map can be resolved by blowing up at Z , to give a well-defined morphism

$$\mathrm{Bl}_Z Y \xrightarrow{\Psi} Y_0.$$

Let $\tilde{Y} \xrightarrow{\pi} Y$ be the blowing-up of Y at Z , with its canonical projection, and let $E \subset \tilde{Y}$ be the exceptional divisor.

Remark. Let (\bar{X}, D) be a toroidal compactification. With the notations of Section 2, the embedding $\mathbb{P}(\mathcal{O}_{D_b}) \hookrightarrow \mathbb{P}(T_{\bar{X}}(-\log D))$ is given locally by

$$x \in D = \mathbb{P}(\mathcal{O}_{D_b}) \mapsto \left(x, \left[w_n \frac{\partial}{\partial w_n} \right] \right),$$

so the indeterminacy locus of $\mathbb{P}_{\bar{X}}(T_{\bar{X}}(-\log D)) \rightarrow \mathbb{P}_{\bar{X}}(T_{\bar{X}})$ is

$$Z = \bigsqcup_b \left\{ \left(x, \left[w_n \frac{\partial}{\partial w_n} \right] \right); x \in D_b \right\}.$$

The rational map $Y_0 \rightarrow Y$.

In a similar way, we can write the following exact sequence :

$$0 \longrightarrow \Omega_X(\log D) \otimes \mathcal{O}(-D) \longrightarrow \Omega_X \longrightarrow \Omega_D \longrightarrow 0, \quad (13)$$

whose last arrow is given by the restriction to T_D , and the first arrow is given in local coordinates by

$$\left(\sum_i v_i dz_i + v_n \frac{dz_n}{z_n} \right) \otimes z_n \mapsto \sum_i (z_n v_i) dz_i + v_n dz_n,$$

where (z_1, \dots, z_n) are local coordinates such that z_n is an equation for D . Exactly as before, the last arrow induces a closed immersion $\mathbb{P}_D(T_D) \cong \mathbb{P}_X^*(\Omega_D) \hookrightarrow \mathbb{P}_X(T_X)$, whose image we will denote by Z_0 . The first arrow also induces a rational map $\Phi_0 : \mathbb{P}_X(T_X) \rightarrow \mathbb{P}_X(T_X(-\log D) \otimes \mathcal{O}(D))$, with indeterminacy locus Z_0 , which can be resolved by a blowing-up at Z_0 . Let $\Phi : \mathrm{Bl}_{Z_0} Y_0 \rightarrow \mathbb{P}(T_X(-\log D) \otimes \mathcal{O}(D))$ be the induced map.

We will now prove that the morphism $\mathrm{Bl}_Z Y \rightarrow Y_0$ factors through the blowing-up $\mathrm{Bl}_{Z_0} Y_0 \rightarrow Y_0$. To see this, we use the universal property of blowing-ups (see [Har77]).

Let $\Psi^* \mathcal{I}_{Z_0} \cdot \mathcal{O}_{\tilde{Y}}$ be the sheaf of ideals of $\mathcal{O}_{\tilde{Y}}$ generated by the image of \mathcal{I}_{Z_0} under the natural map $\Psi^* \mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{\tilde{Y}}$.

Proposition 4.1.1. *The sheaf of ideals $\Psi^* \mathcal{I}_{Z_0} \cdot \mathcal{O}_{\tilde{Y}} \subset \mathcal{O}_{\tilde{Y}}$ is invertible on \tilde{Y} .*

Proof. We have the following commutative diagram :

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\Psi} & Y_0 \\ \downarrow \pi & & \downarrow p_0 \\ Y & \xrightarrow{p} & X \end{array}$$

Let $j = \iota \otimes Id_{\mathcal{O}(1)} : p^* \Omega_X \longrightarrow \mathcal{O}_Y(1)$. The morphism Ψ is determined by the surjective morphism of $\mathcal{O}_{\tilde{Y}}$ -modules

$$\pi^* p^* \Omega_X \xrightarrow{\pi^* j} \pi^* \mathcal{I}_Z \otimes \pi^* \mathcal{O}_Y(1) = \mathcal{O}_{\tilde{Y}}(-E) \otimes \pi^* \mathcal{O}_Y(1).$$

By general properties of projective morphisms, $\pi^* j$ factors through $\Psi^* \mathcal{O}_{Y_0}(1)$, into a composite morphism

$$\pi^* p^* \Omega_X \longrightarrow \Psi^* \mathcal{O}_{Y_0}(1) \xrightarrow{v} \mathcal{O}_{\tilde{Y}}(-E) \otimes \pi^* \mathcal{O}_Y(1). \quad (14)$$

Remark first that v is a surjective morphism of invertible sheaves on \tilde{Y} : it is consequently an isomorphism.

Let us determine \mathcal{J} , the image of $\Psi^* (\mathcal{I}_{Z_0} \otimes \mathcal{O}_{Y_0}(1))$ by v . Similarly to what we see for \mathcal{I}_Z , the \mathcal{O}_{Y_0} -module $\mathcal{I}_{Z_0} \otimes \mathcal{O}_{Y_0}(1)$ is the image of the morphism

$$p_0^* [\Omega_X(\log D) \otimes \mathcal{O}(-D)] \longrightarrow \mathcal{O}_{Y_0}(1),$$

so, since $p \circ \pi = p_0 \circ \Psi$, the image of $\Psi^* [\mathcal{I}_{Z_0} \otimes \mathcal{O}_{Y_0}(1)]$ under v is the image of the composition

$$\pi^* p^* [\Omega_X(\log D) \otimes \mathcal{O}(-D)] \longrightarrow \Psi^* \mathcal{O}_{Y_0}(1) \xrightarrow{\simeq} \mathcal{O}_{\tilde{Y}}(-E) \otimes \pi^* \mathcal{O}_Y(1).$$

Now, we have the following commutative diagram

$$\begin{array}{ccccc} \pi^* p^* [\Omega_X(\log D) \otimes \mathcal{O}(-D)] & \longrightarrow & \pi^* p^* \Omega & \longrightarrow & \mathcal{O}_{\tilde{Y}}(-E) \otimes \pi^* \mathcal{O}_Y(1) \\ \downarrow = & & \downarrow & & \downarrow \\ \pi^* p^* [\Omega_X(\log D) \otimes \mathcal{O}(-D)] & \longrightarrow & \pi^* p^* \Omega_X(\log D) & \longrightarrow & \pi^* \mathcal{O}_Y(1) \end{array}$$

According to what we just said, the image of the composition of the two maps at the top is \mathcal{J} . The image of the composition of the two maps at the bottom is

$$\begin{aligned} \pi^* \mathcal{O}_Y(1) \otimes \pi^* p^* \mathcal{O}(-D) &= \pi^* \mathcal{O}_Y(1) \otimes \pi^* \mathcal{O}(-p^{-1}(D)) \\ &= \pi^* \mathcal{O}_Y(1) \otimes \mathcal{O}(-\widetilde{p^{-1}(D)} - E), \end{aligned}$$

where $\widetilde{p^{-1}(D)}$ is the strict transform of $p^{-1}(D)$ under the blowing-up π . Hence, we find

$$\mathcal{J} = \pi^* \mathcal{O}_Y(1) \otimes \mathcal{O}(-E) \otimes \mathcal{O}(-\widetilde{p^{-1}(D)}).$$

Twisting by $\Psi^* \mathcal{O}_{Y_0}(-1) \simeq \mathcal{O}(E) \otimes \pi^* \mathcal{O}_Y(-1)$, we obtain

$$\Psi^* \mathcal{I}_{Z_0} \cdot \mathcal{O}_{\tilde{Y}} = \mathcal{J} \otimes \mathcal{O}_{\tilde{Y}}(E) \otimes \pi^* \mathcal{O}_Y(1) = \mathcal{O}(-\widetilde{p^{-1}(D)}).$$

Since $p^{-1}(D)$ is a divisor on Y , the subscheme $\widetilde{p^{-1}(D)}$ is also a divisor, and its ideal sheaf is invertible. This proves the result. \square

By the previous proposition, the universal property of blowing-ups (see [Har77]) implies that the morphism $\Psi : \tilde{Y} = \text{Bl}_Z(Y) \rightarrow Y_0$ factors through $\text{Bl}_{Z_0}(Y_0) \rightarrow Y_0$.

Hence, there is a morphism $\text{Bl}_Z(Y) \xrightarrow{\tilde{\Psi}} \text{Bl}_{Z_0}(Y)$, lifting and resolving the birational map Ψ_0 . In the same manner, we prove that there exists a morphism

$$\text{Bl}_{Z_0}(Y) \xrightarrow{\tilde{\Phi}} \text{Bl}_{\mathbb{P}(\mathcal{O}_D \otimes \mathcal{O}(D))} \mathbb{P}(T_X(-\log D) \otimes \mathcal{O}(D))$$

lifting and resolving the map Φ_0 .

Since $\mathbb{P}_X(T_X(-\log D) \otimes \mathcal{O}(D))$ is canonically isomorphic to Y , and since the subvarieties $\mathbb{P}(\mathcal{O}_D \otimes \mathcal{O}(-D))$ and $\mathbb{P}(\mathcal{O}_D)$ are image of each other *via* this isomorphism, we see that $\tilde{\Phi}$ and $\tilde{\Psi}$ must be biholomorphic, and inverse of each other modulo the natural isomorphism

$$\text{Bl}_{\mathbb{P}(\mathcal{O}_D(D))} \mathbb{P}(T_X(-\log D) \otimes \mathcal{O}(D)) \xrightarrow{\simeq} \text{Bl}_Z Y.$$

Therefore, we obtain the following result.

Proposition 4.1.2. *Let (X, D) be a log-pair with smooth boundary, let $Y = \mathbb{P}(T_X(-\log D)) \xrightarrow{p} X$ and $Y_0 = \mathbb{P}(T_X) \xrightarrow{p_0} X$, the logarithmic and standard projectivized tangent bundles. The projection map $p|_{p^{-1}(D)}$ admits a section, realizing a closed immersion $D \rightarrow Y$, whose image we will denote by Z . Denote by Z_0 the image of the canonical closed immersion $\mathbb{P}(T_D) \hookrightarrow \mathbb{P}(T_X)$. Then the natural birational map $Y \rightarrow Y_0$ induces an isomorphism of projective manifolds :*

$$\text{Bl}_Z Y \xrightarrow{\simeq} \text{Bl}_{Z_0} Y_0.$$

Moreover, if $\pi : \text{Bl}_Z Y \rightarrow Y$ and $\pi_0 : \text{Bl}_{Z_0} Y_0 \rightarrow Y_0$ denote the respective blowing-ups, then the strict transform of $p^{-1}(D)$ corresponds under this isomorphism to the exceptional divisor of π_0 . In the same manner, the strict transform of $p_0^{-1}(D)$ under π_0 corresponds to the exceptional divisor of π .

Proof. We proved the first part of the proposition in the previous discussion. The claim about the exceptional divisor of π_0 follows easily from our proof of the invertibility of $\Psi^* \mathcal{I}_{Z_0} \cdot \mathcal{O}_{\text{Bl}_Z Y}$, with the same notations than before. The result for the exceptional divisor of π follows in the same way. \square

Keep the same notations as before, and let E, E_0 be the exceptional divisors of the respective projections π, π_0 .

Proposition 4.1.3. *On \tilde{Y} , we have the following isomorphism of line bundles :*

$$\pi^* \mathcal{O}_Y(1) \simeq \pi_0^* \mathcal{O}_{Y_0}(1) \otimes_{\mathcal{O}_{\tilde{Y}}} \mathcal{O}(E)$$

Proof. This isomorphism is just the morphism v appearing in (14), restated with the new notation $\pi_0 = \Psi$. \square

Proposition 4.1.4. *The restriction of $\pi^* \mathcal{O}_Y(1)$ to E is trivial.*

Proof. We already saw that $\mathcal{O}_Y(1)|_Z$ is trivial. Thus, it is clear that $\pi^* \mathcal{O}_Y(1)$ is trivial when restricted to $E = \pi^{-1}(Z)$. \square

We see from this result that if $W \subset Y_0$ is a subvariety with strict transform under π_0 denoted by \widetilde{W} , we can compute the maximal intersection of $\mathcal{O}_{Y_0}(1)$ with W in terms of intersection numbers of \widetilde{W} with $\pi^*\mathcal{O}_Y(1)$ and E . Indeed, we have

$$\begin{aligned} c_1(\mathcal{O}_{Y_0})^{\dim W} \cdot W &= c_1(\pi_0^*\mathcal{O}_{Y_0})^{\dim W} \cdot \widetilde{W} \\ &= c_1(\pi^*\mathcal{O}_Y \otimes \mathcal{O}(-E))^{\dim W} \cdot \widetilde{W} \\ &= c_1(\pi^*\mathcal{O}_Y)^{\dim W} \cdot \widetilde{W} + (-1)^{\dim W} E^{\dim W} \cdot \widetilde{W} \\ &= c_1(\mathcal{O}_Y)^{\dim W} \cdot \pi(\widetilde{W}) + (-1)^{\dim W} E^{\dim W} \cdot \widetilde{W}. \end{aligned}$$

We will see in the next sections that in the case where (\overline{X}, D) is a toroidal compactification, we can estimate the first term of the right hand side of this last equation, in terms of the Bergman metric on $\overline{X} \setminus D$. As for the second member, we can prove a more general result, for any log-pair with smooth boundary. To estimate the intersection numbers with $E \subset \widetilde{Y}$, we must first determine the normal bundle $N_{E/\widetilde{Y}}$.

Proposition 4.1.5. *There is a canonical isomorphism*

$$N_{Z/Y}^* \simeq p^*(\Omega_X|_D). \quad (15)$$

Proof. As we saw earlier, there is an exact sequence of \mathcal{O}_Y -modules

$$p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1) \xrightarrow{t} \mathcal{I}_Z \longrightarrow 0,$$

which gives, after taking the tensor product with \mathcal{O}_Z :

$$(p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1)) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \longrightarrow \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \longrightarrow 0.$$

Because of Proposition 4.1.4, $\mathcal{O}_Y(-1) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \cong \mathcal{O}_Z$, so the previous surjective map becomes

$$p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \longrightarrow \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \longrightarrow 0,$$

We have $\mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \simeq N_{Z/Y}^*$ as \mathcal{O}_Z -modules, and the following isomorphism of locally free sheaves holds on Z :

$$p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Z \simeq p^*(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_D).$$

Therefore, we get the following surjective morphism of locally free sheaves of \mathcal{O}_Z -modules :

$$p^*(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_D) \twoheadrightarrow N_{Z/Y}^*.$$

Since a surjective morphism between two locally free \mathcal{O}_Z -modules of the same rank is an isomorphism, this gives the result. \square

The exceptional divisor is isomorphic, as a D -scheme, to $\mathbb{P}(N_{Z/Y}) = \mathbb{P}^*(N_{Z/Y}^*)$. We saw in Proposition 4.1.2 that the canonical isomorphism $\text{Bl}_Z Y \cong \text{Bl}_{Z_0} Y_0$ associates E with the strict transform of $p_0^{-1}(D)$ under π_0 . Since $\mathbb{P}(T_D)$ has codimension one in $p_0^{-1}(D)$, this strict transform is actually isomorphic to $p_0^{-1}(D)$. Therefore,

$$\Psi|_E : E \longrightarrow p_0^{-1}(D) \simeq \mathbb{P}(\Omega_X|_D) \quad (16)$$

is an isomorphism. It makes sense to ask whether it is induced by the isomorphism of vector bundles of (15). The next proposition shows that it is actually the case.

Proposition 4.1.6. *When restricted to E , the morphism π_0 induces an isomorphism $E \rightarrow p_0^{-1}(D) \cong \mathbb{P}(\Omega_X|_D)$, determined by the isomorphism of \mathcal{O}_E -modules (15).*

Proof. The morphism π_0 is induced by the following morphism of $\mathcal{O}_{\bar{Y}}$ -modules :

$$\pi^*(p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1)) \xrightarrow{\pi^* \iota} \mathcal{O}_{\bar{Y}} \cong \pi^*\mathcal{I}_Z.$$

If we take the tensor product with \mathcal{O}_E , since $\pi^*\mathcal{O}_Y(-1)|_E$ is trivial, we get a morphism of \mathcal{O}_E -modules

$$\pi^*p^*\Omega_X \otimes_{\mathcal{O}_{\bar{Y}}} \mathcal{O}_E \rightarrow \pi^*\mathcal{I}_Z \otimes_{\mathcal{O}_{\bar{Y}}} \mathcal{O}_E.$$

which, according to the projection formula, can be seen as a morphism of the form

$$\pi^*(p^*\Omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Z) \rightarrow \pi^*(\mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Z).$$

This is induced by the isomorphism of Proposition 4.1.5. Consequently, we see that the isomorphism $\pi_0|_E : E \rightarrow p_0^{-1}(D) = \mathbb{P}(\Omega_X|_D)$ is induced by (15), as we claimed. \square

5 Nefness of the cotangent bundles

In the rest of the text, (\bar{X}, D) will be a toroidal compactification.

With what has been introduced until now, we can use the results of [BT15] to determine a condition for $\Omega_{\bar{X}}$ to be nef. For this, we let $\bar{Y} = \mathbb{P}_{\bar{X}}(T_{\bar{X}}(-\log D))$, with its canonical projection p and tautological bundle $\mathcal{O}(1)_{\log}$, and $\bar{Y}_0 = \mathbb{P}_{\bar{X}}(T_{\bar{X}})$, with its projection p_0 and tautological bundle $\mathcal{O}(1)_0$.

We start by proving that $\Omega_{\bar{X}}(\log D)$ is always nef.

Proof of Theorem 3. Let $C \subset \bar{Y}$ be an irreducible curve. If $C \not\subset p^{-1}(D)$, it follows from Proposition 3.1.3 that $c_1\mathcal{O}(1)_{\log} \cdot C \geq 0$.

If $C \subset p^{-1}(D) = \mathbb{P}(T_{\bar{X}}(-\log D)|_D)$, the result is given by the next lemma. \square

Lemma 5.0.1. *The restriction $\Omega_{\bar{X}}(\log D)|_D$ is nef.*

Proof. This is a basic application of the properties of the logarithmic conormal sequence. Recall that we have the following exact sequence of \mathcal{O}_X -modules :

$$0 \rightarrow \Omega_X \rightarrow \Omega_X(\log D) \xrightarrow{res} \mathcal{O}_D \rightarrow 0.$$

If we take the tensor product with \mathcal{O}_D , we get

$$0 \rightarrow \mathcal{T}or_{\mathcal{O}_{\bar{X}}}^1(\mathcal{O}_D, \mathcal{O}_D) \rightarrow \Omega_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_D \rightarrow \Omega_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_D \rightarrow \mathcal{O}_D \rightarrow 0. \quad (17)$$

We have $\mathcal{T}or_{\mathcal{O}_{\bar{X}}}^1(\mathcal{O}_D, \mathcal{O}_D) = N_{D/\bar{X}}^*$, and on D , we can express the conormal exact sequence as

$$0 \rightarrow N_{D/\bar{X}}^* \rightarrow \Omega_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_D \rightarrow \Omega_D \rightarrow 0.$$

Therefore, the sequence (17) breaks to give the logarithmic conormal sequence

$$0 \rightarrow \Omega_D \rightarrow \Omega_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_D \rightarrow \mathcal{O}_D \rightarrow 0.$$

Since the boundary is made of abelian varieties, Ω_D is trivial on any component of D . Consequently, the vector bundle $\Omega_{\bar{X}}(\log D)|_D$ is an extension of trivial bundles, hence is nef. \square

Let us mention the following result, first step in our study of the nefness of $\Omega_{\overline{X}}$.

Proposition 5.0.1. *When restricted to D , the cotangent bundle $\Omega_{\overline{X}}^1$ is nef.*

Proof. As stated in [Mok12], for any component D_b of D , the neighborhoods $\Omega_b^{(N)}$ introduced in Section 2 are isomorphic to tubular neighborhoods of the zero section of the normal bundle $N_b \rightarrow D_b$. Consequently, we have

$$\Omega_{\overline{X}}|_{D_b} \simeq N_b^* \oplus \Omega_{D_b} \simeq N_b^* \oplus \mathbb{C}^{n-1},$$

since D_b is an abelian variety. Moreover, for any such component D_b , the conormal bundle N_b^* is positive ([Mok12]). Thus, $\Omega_{\overline{X}}|_{D_b}$ is sum of a trivial bundle and of an ample bundle on D_b , hence is nef. \square

We will now make use of the results we proved in Section 4 to estimate the intersection numbers of the type $c_1\mathcal{O}(1)_0 \cdot C$, where C is a curve of \overline{Y}_0 , not included in the boundary. To do this, we will pull back all our objects to the blowing-up $\text{Bl}_Z\overline{Y}$. Let \tilde{Y} denotes this blowing-up, that we endow with its natural projections π and π_0 , respectively onto \overline{Y} and \overline{Y}_0 .

Proposition 5.0.2. *Let $C \subset \overline{Y}$ be a curve such that $p_0(C) \not\subset D$. Then*

$$c_1\mathcal{O}(1)_0 \cdot C \geq \left(\frac{1}{n+1} (K_{\overline{X}} + D) - D \right) \cdot p_0(C).$$

Proof. We denote by \tilde{C} the proper transform of the curve C by the blowing-up π_0 . Then Proposition 4.1.3 gives

$$\begin{aligned} c_1\mathcal{O}(1)_0 \cdot C &= \pi_0^* (c_1\mathcal{O}(1)_0) \cdot \tilde{C} \\ &= \pi^* (c_1\mathcal{O}(1)_{\log} - E) \cdot \tilde{C}, \end{aligned}$$

Moreover, thanks to Proposition 3.1.3, we obtain

$$\begin{aligned} \pi^* (c_1\mathcal{O}(1)_{\log}) \cdot \tilde{C} &= c_1\mathcal{O}(1)_{\log} \cdot \pi(\tilde{C}) \\ &\geq \int_{Y \cap \pi(\tilde{C})} \frac{i}{2\pi} \Theta(\widehat{h}^*) \\ &= \int_{Y \cap C} \frac{i}{2\pi} \Theta(\widehat{h}^*). \end{aligned}$$

The Bergman metric being of constant sectional curvature equal to -2 with our choice of normalization, the following equality is true at any point $(x, [v]) \in Y$, for any $\xi \in T_{(x, [v])}Y$:

$$\begin{aligned} \frac{i}{2\pi} \Theta(\widehat{h}^*)_{(x, [v])} \cdot (\xi, \xi) &\geq -\frac{i}{2\pi} \frac{\Theta_{T_X}(v, v, \xi, \xi)}{\|v\|^2} \\ &\geq \frac{1}{\pi} \omega_{Berg}(\xi, \xi), \end{aligned}$$

so

$$\pi^* (c_1\mathcal{O}(1)_{\log}) \cdot \tilde{C} \geq \int_{X \cap p_0(C)} \frac{1}{\pi} \omega_{Berg}.$$

However, because of Proposition 2.2.2, we obtain

$$\int_{X \cap p_0(C)} \frac{1}{\pi} \omega_{Berg} = \frac{1}{n+1} (K_{\overline{X}} + D) \cdot C.$$

Besides, since E is an irreducible component of $(p \circ \pi)^{-1}(D)$, we have

$$E \cdot \tilde{C} \geq D \cdot (p \circ \pi)(\tilde{C}) = D \cdot p_0(C).$$

□

We can now prove our main result on the nefness of $\Omega_{\overline{X}}$.

Proof of Theorem 5 . Consider an irreducible curve $C \subset \overline{Y}_0$. First assume that $p_0(C) \subset D$. According to Proposition 5.0.1, the bundle $\Omega_{\overline{X}}|_D$ is nef. Since C can be seen as a curve of the projective space $\mathbb{P}(T_{\overline{X}}|_D)$, we see that $\int_C c_1(\mathcal{O}(1))_{\overline{Y}_0} \geq 0$.

Assume now that $C \cap Y \neq \emptyset$. Then, according to Proposition 5.0.2, we have

$$\int_C c_1(\mathcal{O}(1))_{\overline{Y}_0} \geq \frac{1}{n+1} \int_{p_0(C)} c_1(K_{\overline{X}} + (1 - (n+1))D).$$

In addition, since σ ramifies to an order larger than 7 along the boundary, we have

$$\int_{p_0(C)} c_1(K_{\overline{X}} + (1 - (n+1))D) \geq \int_{p_0(C)} c_1\left(\sigma^*(K_{\overline{X}} + D) - \frac{n+1}{7}\sigma^*D\right).$$

Therefore,

$$\begin{aligned} \int_{p_0(C)} c_1\left(K_{\overline{X}} + \left(1 - \frac{n+1}{2}\right)D\right) &\geq (\deg \sigma) \int_{\sigma(p_0(C))} c_1\left(K_{\overline{X}'} + \left(1 - \frac{n+1}{7}\right)D'\right) \\ &\geq (\deg \sigma) \int_{\sigma(p_0(C))} c_1\left(K_{\overline{X}'} + \left(1 - \frac{n+1}{2\pi}\right)D'\right), \end{aligned} \quad (18)$$

since $(D \cdot (\sigma \circ p_0(C))) \geq 0$ (the divisor D and the curve $\sigma \circ p_0(C)$ are in normal intersection). The line bundle $K_{\overline{X}} + (1 - \frac{n+1}{2\pi})D$ is nef by [BT15], so the last term of (18) is non-negative, which gives the result. □

6 Immersed submanifolds of \overline{X}

As we explained earlier in Section 3, we can prove Theorem 4 using simple estimates on singular metrics. We want to show now that if we restrict ourselves to the study of immersed submanifolds, we can use the results of the sections 4 and 5 to obtain better bounds on the ramifications needed.

Let \overline{W} be a smooth manifold of dimension p , and let $q_{\overline{W}} : \mathbb{P}(T_{\overline{W}}) \rightarrow \overline{W}$ be its projectivized tangent space. Suppose we have an immersion $f : \overline{W} \rightarrow \overline{X}$ (which we do not assume to be injective), such that $f(\overline{W}) \not\subset D$. Let $W = f^{-1}(X)$ be the open part of \overline{W} .

The immersion f induces a well defined morphism $\mathbb{P}(T_{\overline{W}}) \xrightarrow{\tilde{f}} \overline{Y}_0$. Let $\overline{W}_1 \xrightarrow{q} \mathbb{P}(T_{\overline{W}})$ be the blowing-up of $\mathbb{P}(T_{\overline{W}})$ with respect to $\tilde{f}^{-1}(Z_0)$. All these manifolds take place in the following fibre square :

$$\begin{array}{ccc}
\overline{W}_1 & \xrightarrow{\tilde{g}} & \tilde{Y} \\
\downarrow q & & \downarrow \pi_0 \\
\mathbb{P}(T_{\overline{W}}) & \xrightarrow{\tilde{f}} & \overline{Y}
\end{array}$$

Then, we have the following equalities of intersection numbers on $\mathbb{P}(T_{\overline{W}})$:

$$\begin{aligned}
[c_1 \mathcal{O}(1)_{\mathbb{P}(T_{\overline{W}})}]^{2p-1} &= [\tilde{f}^* c_1 \mathcal{O}(1)_0]^{2p-1} \\
&= [q^* \tilde{f}^* c_1 \mathcal{O}(1)_0]^{2p-1} \\
&= [\tilde{g}^* \pi_0^* c_1 \mathcal{O}(1)_0]^{2p-1} \\
&= [\tilde{g}^* (\pi^* c_1 \mathcal{O}(1)_{\log} - E)]^{2p-1}
\end{aligned}$$

where the last equality comes from Proposition 4.1.3. Since $\pi^* \mathcal{O}(1)_{\log}|_E \simeq \mathcal{O}_E$, we finally obtain

$$[c_1 \mathcal{O}(1)_{\mathbb{P}(T_{\overline{W}})}]^{2p-1} = [\tilde{g}^* \pi^* c_1 \mathcal{O}(1)_{\log}]^{2p-1} + (-E)^{2p-1} \cdot \overline{W}_1. \quad (19)$$

The line bundle $\mathcal{O}(1)_{\log}$ is nef on \overline{Y} by Theorem 3, so $\tilde{g}^* \pi^* \mathcal{O}(1)_{\log}$ is nef on \tilde{Y} , and

$$[\tilde{g}^* \pi^* c_1 \mathcal{O}(1)_{\log}]^{2p-1} = \text{vol}(\tilde{g}^* \pi^* \mathcal{O}(1)_{\log}).$$

Consequently, Proposition 3.1.3 gives the inequality

$$[\tilde{g}^* \pi^* c_1 \mathcal{O}(1)_{\log}]^{2p-1} \geq \int_{W_1} \left[\tilde{g}^* \left(\frac{i}{2\pi} \Theta(\widehat{h}^*) \right)^{2p-1} \right],$$

where we denoted by W_1 the open part $q^{-1}(\mathbb{P}(T_W)) \subset \overline{W}_1$.

Since blowing-up along $\tilde{f}^{-1}(Z_0)$ induces an isomorphism on $q_W^{-1}(f^{-1}(X)) = \tilde{f}^{-1}(Y)$, this is actually means that

$$[\tilde{g}^* \pi^* c_1 \mathcal{O}(1)_{\log}]^{2p-1} \geq \int_{\tilde{f}^{-1}(Y)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1} \quad (20)$$

Proposition 6.0.1. *The following inequality holds :*

$$\int_{\tilde{f}^{-1}(Y)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1} \geq \deg(f) \binom{2p-1}{p} \frac{(K_{\overline{X}} + D)^p \cdot f(W)}{(n+1)^p}.$$

Proof. We can project the integral we want to compute onto W :

$$I := \int_{\tilde{f}^{-1}(Y)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1} = \int_{f^{-1}(X)} (q_W)_* \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1}. \quad (21)$$

We will follow the ideas of [Div16] to estimate the integral of $\tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1}$ in terms of the pull-back of the Bergman metric onto W . Take $P \in f^{-1}(X)$, and choose a coordinate neighborhood

U of P such that the map $f|_U$ lifts to a map which takes its values in \mathbb{B}^n . We will also denote this lift by f . We can assume that the coordinate on U are centered at $P = 0$, and that $f(P) = f(0) = 0$. In addition, up to a linear change of coordinates on U and a unitary change of coordinates on \mathbb{B}^n , we can assume that around 0, f is of the form

$$(z_1, \dots, z_p) \mapsto (z_1, \dots, z_p, 0, \dots, 0) + O(|z|^2).$$

The Bergman metric in the usual coordinates (z_1, \dots, z_n) on \mathbb{B}^n admits the canonical form of (3) :

$$h_{Berg} = \frac{(1 - \|z\|^2) \sum_i dz_i \otimes \overline{dz_i} + (\sum_i dz_i) \otimes (\sum_i \overline{dz_i})}{(1 - \|z\|^2)^2}.$$

Thus, at 0, we have $h_{Berg} = I + O(|z|^2)$.

Let us use (6) to evaluate the integral of $\tilde{f}^* \Theta(\widehat{h}^*)$ along the fiber of q_W at P . We get:

$$\begin{aligned} \int_{\mathbb{P}(T_P W)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1} &= \int_{\mathbb{P}(T_P W)} \left[\frac{i}{2\pi} \Theta(\widehat{h}|_{T_U}^*) \right]^{2p-1} \\ &= \int_{\mathbb{P}(T_P W)} \left[\frac{1}{\pi} \omega_{h_P}^{FS} - \frac{i}{2\pi} \frac{\langle v, \Theta_P(h)v \rangle}{\|v\|_{h_P}^2} \right]^{2p-1} \\ &\geq \binom{2p-1}{p} \int_{\mathbb{P}(T_P W)} \left[\frac{\omega_{h_P}^{FS}}{\pi} \right]^{p-1} \left[-\frac{i}{2\pi} \frac{\langle v, \Theta_P(h)v \rangle}{\|v\|_{h_P}^2} \right]^p, \end{aligned} \quad (22)$$

where the last inequality holds since all the forms appearing in the Newton binomial are positive. Now, remark that in the coordinates we chose on U , we have

$$-\Theta_P(h) = \left(\sum_i dz_i \wedge \overline{dz_i} \right) I_n + \left(\sum_i dz_i \otimes e_i^* \right) \wedge \left(\sum_i \overline{dz_i} \otimes e_i \right).$$

Hence, when evaluated on any unitary vector $v \in T_P U$, we obtain

$$-\frac{i}{2} \langle v, \Theta_P(h)v \rangle = 2v^* \wedge \overline{v^*} + \omega_{Berg}|_{v^\perp}, \quad (23)$$

where $v^\perp \subset T_P U$ denotes the subspace of vectors orthogonal to v for h_P . Since all the forms appearing in this equation are positive, we have

$$\left[-\frac{i}{2} \langle v, \Theta_P(h)v \rangle \right]^p \geq [v^* + \overline{v^*} + \omega|_{v^\perp}]^p = \omega_{Berg}^p.$$

This gives

$$\int_{\mathbb{P}(T_P W)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^{2p-1} \geq \binom{2p-1}{p} \left[\frac{\omega_{Berg}}{\pi} \right]^p \int_{\mathbb{P}(T_P W)} \left[\frac{\omega_{h_P}^{FS}}{\pi} \right]^{p-1} = \binom{2p-1}{p} \left[\frac{\omega_{Berg}}{\pi} \right]^p$$

Finally, we obtain, integrating on $f^{-1}(X)$:

$$I \geq \binom{2p-1}{p} \int_{f^{-1}(X)} \left[\frac{f^* \omega_{Berg}}{\pi} \right]^p.$$

Proposition 2.2.2 allows us to write

$$I \geq \binom{2p-1}{p} \left[\frac{f^*(K_{\overline{X}} + D)}{n+1} \right]^p = \deg(f) \binom{2p-1}{p} \frac{(K_{\overline{X}} + D)^p \cdot f(W)}{(n+1)^p}$$

□

Remark. When $p \in \{1, n\}$, we can be more precise in this inequality. Suppose first that $p = 1$. The equation (23) can be replaced with

$$-\frac{i}{2} \langle v, \Theta_P(h)v \rangle = 2v^* \wedge \overline{v^*} = 2 f^* \omega_{Berg},$$

so, when continuing the computations, we finally find

$$\int_{\tilde{f}^{-1}(Y)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right] \geq 2 \deg(f) \frac{(K_{\overline{X}} + D) \cdot f(W)}{n+1}.$$

Suppose now that $p = n$. Then (22) becomes an equality, and in addition

$$\left[-\frac{i}{2} \langle v, \Theta_P(h)v \rangle \right]^n = [2v^* + \overline{v^*} + \omega|_{v^\perp}]^n = 2 \omega_{Berg}^p.$$

Thus, finishing the computations gives the equality

$$\int_{\tilde{f}(Y)} \tilde{f}^* \left[\frac{i}{2\pi} \Theta(\widehat{h}^*) \right]^n = 2 \binom{2n-1}{n} \deg(f) \frac{(K_{\overline{X}} + D)^n \cdot f(W)}{(n+1)^n}.$$

In particular, we obtain the following inequality of intersection numbers, using the fact that $2 \binom{2n-1}{n} = \binom{2n}{n}$:

$$c_1 \mathcal{O}(1)_{\log}^{2n-1} \geq \binom{2n}{n} \frac{(K_{\overline{X}} + D)^n}{(n+1)^n}. \quad (24)$$

Remark. Using Hirzebruch's proportionality principle, we can prove that the inequality (24) is actually an equality. See [Mum77].

We will now compute the term $(-E)^{2p-1} \cdot \overline{W}_1$ appearing in (19).

Let E_W be the proper transform of $\mathbb{P}(T_{\overline{W}}|_{f^{-1}(D)}) \hookrightarrow \mathbb{P}(f^*T_X|_D) = \tilde{f}^{-1}(p_0^{-1}(D))$ under the blowing-up q . We have then $E_W = \tilde{g}^{-1}(E)$, and

$$(-E)^{2p-1} \cdot \overline{W}_1 = -(E|_E)^{2(p-1)} \cdot (\overline{W}_1|_E) = -(g^* E|_E)^{2(p-1)} \cdot E_W.$$

According to Proposition 4.1.6, there is an isomorphism $E \simeq \mathbb{P}(T_X|_D)$, and $\mathcal{O}_E(-1)$, the tautological line bundle of this projectivized space, is isomorphic to the normal bundle $\mathcal{O}_E(E)$. Moreover, E_W is isomorphic to $\mathbb{P}(T_W|_{W_D})$ as a blowing-up along a smooth divisor, and the morphism $E_W \rightarrow E$ is given by

$$E_W = \mathbb{P}(T_{\overline{W}}|_{f^{-1}(D)}) \xrightarrow{\tilde{f}_D} \mathbb{P}(T_X|_D) \simeq E,$$

where $\tilde{f}_D = \tilde{f}|_{\tilde{f}^{-1}(p_0^{-1}(D))}$. Thus, we can use the functoriality of tautological bundles under pull-backs to write $\tilde{f}_D^* \mathcal{O}_E(1) = \mathcal{O}_{E_W}(1)$, which gives the formula

$$(g^* E|_E)^{2(p-1)} \cdot E_W = \int_{E_W} c_1(\tilde{f}_D^* \mathcal{O}_E(1))^{2(p-1)} = \int_{E_W} c_1(\mathcal{O}_{E_W}(1))^{2(p-1)}. \quad (25)$$

We will now estimate this last intersection number in terms of $c_1(N_{D/\bar{X}})$. Since D admits a tubular neighborhood in \bar{X} , we can write the following isomorphism of vector bundles over D :

$$T_{\bar{X}}|_D \simeq T_D \oplus N_{D/\bar{X}} \simeq \mathcal{O}_D^{n-1} \oplus N_{D/\bar{X}}. \quad (26)$$

We see that we can choose a metric h_D on $T_{\bar{X}}|_D$ whose curvature decomposes locally in the following way in the splitting (26) :

$$\Theta(T_{\bar{X}}|_D) = \begin{pmatrix} 0 & 0 \\ 0 & \Theta(N_{D/\bar{X}}) \end{pmatrix}.$$

Thus, according to (6), we can write

$$\frac{i}{2\pi} \Theta(\mathcal{O}_{E_W}(1))_{(x,[v])} = -\frac{i}{2\pi} \Theta(N_{D/\bar{X}})_y \frac{\langle \tau(w), \tau(w) \rangle_{h_D}}{\|w\|_{h_D}^2} + \frac{1}{\pi} \tilde{f}_D^* \omega^{FS},$$

where $(y, [w]) = \tilde{f}_D(x, [v])$, and $\tau : T_{\bar{X}}|_D \rightarrow N_{D/\bar{X}}$ denotes the canonical projection, which by our choice of metric is orthogonal.

Integrating $\left(\frac{i}{2\pi} \Theta(\mathcal{O}_{E_W}(1))_{(x,[v])}\right)^{2(p-1)}$ over a fibre $E_{W,x}$ of the projection $E_W \rightarrow f^{-1}(D)$, for some $x \in f^{-1}(D)$, we get

$$\int_{E_{W,x}} c_1(\mathcal{O}_{E_W}(1))^{2(p-1)} = \left[-\frac{i}{2\pi} \Theta(N_{D/\bar{X}})_y\right]^{p-1} \int_{E_{W,x}} \frac{\|\tau(w)\|_{h_D}^2}{\|w\|_{h_D}^2} \left(\frac{\omega^{FS}}{\pi}\right)^{p-1},$$

with $y = f(x)$. However, $N_{D/\bar{X}}$ is negative on D , so the form $-\frac{i}{2\pi} \Theta(N_{D/\bar{X}})$ is positive. Hence, we get

$$\int_{E_{W,x}} c_1(\mathcal{O}_{E_W}(1))^{2(p-1)} \leq \left[-\frac{i}{2\pi} \Theta(N_{D/\bar{X}})_y\right]^{p-1} \int_{E_{W,x}} \frac{\|w\|_{h_D}^2}{\|w\|_{h_D}^2} \left(\frac{\omega^{FS}}{\pi}\right)^{p-1} = \left[-\frac{i}{2\pi} \Theta(N_{D/\bar{X}})_y\right]^{p-1}.$$

Thus, integrating over $f^{-1}(D)$, we obtain

$$(-E)^{2p-1} \cdot \bar{W}_1 \geq - \int_{f^{-1}(D)} \left[-\frac{i}{2\pi} \Theta(N_{D/\bar{X}})\right]^{p-1} = -(-D|_{\bar{W}})^{p-1} = (-D)^p \cdot \bar{W}. \quad (27)$$

Remark. If $\dim \bar{W} = n$, we can simplify the previous computations. Suppose for example that $\bar{W} = \bar{X}$. Then $(-E)^{2p-1} \cdot \bar{W}_1$ is just the maximal intersection of $(-E)$. We have seen that $E \simeq \mathbb{P}(T_{\bar{X}}|_D)$ and that $\mathcal{O}_E(E)$ is isomorphic to the tautological line bundle $\mathcal{O}_E(-1)$. Hence,

$$(-E)^{2n-1} = -[c_1 \mathcal{O}_E(E)|_E]^{2(n-1)} = -[c_1 \mathcal{O}_E(1)]^{2(n-1)} = - \int_D s_{n-1}(T_{\bar{X}}|_D).$$

Since $T_{\overline{X}}|_D \simeq \mathcal{O}_D^{n-1} \oplus N_{D/\overline{X}}$, we find $s_{n-1}(T_{\overline{X}}|_D) = (-1)^{n-1}c_1(N_{D/\overline{X}})^{n-1}$. This means that

$$(-E)^{2n-1} = - \int_D (-1)^{n-1}c_1(N_{D/\overline{X}})^{n-1} = (-D)^n.$$

In this case, we actually find an equality in (27).

Putting everything together, we have proved the following result :

Proposition 6.0.2. *Let $W \xrightarrow{f} \overline{X}$ be an immersion of a smooth manifold \overline{W} of dimension p , not necessarily injective, such that $f(\overline{W}) \not\subset D$. Then if $\mathcal{O}_{\overline{W}}(1)$ is the tautological bundle of $\mathbb{P}(T_{\overline{W}})$, we have the following inequality :*

$$c_1\mathcal{O}_{\overline{W}}(1)^{2p-1} \geq \left[\binom{2p-1}{p} \frac{1}{(n+1)^p} (K_{\overline{X}} + D)^p + (-D)^p \right] \cdot f_*[\overline{W}], \quad (28)$$

where $f_*[\overline{W}] = \deg(f) \cdot f(\overline{W})$ denotes the image cycle of \overline{W} . If $p = 1$, we have the more precise inequality

$$\deg K_{\overline{W}} = c_1\mathcal{O}_{\overline{W}}(1) \geq \left[\frac{2}{n+1} (K_{\overline{X}} + D) - D \right] \cdot f_*[\overline{W}], \quad (29)$$

and if $p = n$, we have the equality (via [Mum77])

$$c_1\mathcal{O}_{\overline{W}}(1)^{2n-1} = \left[\binom{2n}{n} \frac{1}{(n+1)^n} (K_{\overline{X}} + D)^n + (-D)^n \right] \cdot f_*[\overline{W}]. \quad (30)$$

We are now able to prove our refined result on the type of embedded subvarieties of \overline{X} .

Proof of Theorem 6. Let $\overline{W} \xrightarrow{f} \overline{X}$ be an immersion, whose image is not included in the boundary, with $\dim \overline{W} = p$. Let $\mathcal{O}_{\overline{W}}(1)$ be the tautological bundle on the projectivized bundle $\mathbb{P}(T_{\overline{W}})$, and let $\tilde{f}: \mathbb{P}(T_{\overline{W}}) \rightarrow Y$ be the morphism induced by f . Then (28) gives, since $(K_{\overline{X}} + D)|_D \simeq \mathcal{O}_D$:

$$\begin{aligned} c_1\mathcal{O}_{\overline{W}}(1)^{2p-1} &\geq \left[\binom{2p-1}{p} \frac{1}{(n+1)^p} (K_{\overline{X}} + D)^p + (-D)^p \right] \cdot [f(\overline{W})] \\ &= C \left[(K_{\overline{X}} + D)^p + \left(-\frac{1}{C^{1/p}}D\right)^p \right] \cdot f_*[\overline{W}] \\ &\geq C \left[(K_{\overline{X}'} + D')^p + \left(-\frac{1}{lC^{1/p}}D'\right)^p \right] \cdot \sigma_* f_*[\overline{W}] \\ &= C \left[(K_{\overline{X}'} + (1 - \frac{1}{lC^{1/p}}D'))^p \right] \cdot \sigma_* f_*[\overline{W}] \end{aligned} \quad (31)$$

$$(32)$$

with $C = \binom{2p-1}{p} \frac{1}{(n+1)^p}$. According to [BT15], $K_{\overline{X}'} + (1 - \lambda)D'$ is ample for $\lambda \in]0, \frac{n+1}{2\pi}[$, so we see that the last term of (32) is positive as soon as $lC^{1/p} > \frac{2\pi}{n+1}$, i.e. if

$$l \binom{2p-1}{p}^{1/p} > 2\pi.$$

This is true if $l > \max_k \frac{2\pi}{\binom{2k-1}{k}^{1/k}} = 2\pi$. Thus, if $l \geq 7$, $\mathcal{O}_{\overline{W}}(1)$ has positive maximal intersection.

Since $l \geq 7$, $\mathcal{O}_{\overline{W}}(1) = \tilde{f}^*\mathcal{O}_{Y_0}(1)$ is nef because of Theorem 5, so we deduce from our previous calculations that it must be a big line bundle. \square

Remark. In the case where \overline{W} is a curve, the better inequality (29) permits to prove that for any étale cover $X \rightarrow X'$ ramifying at order at least 4 on any boundary component, all curves $C \subset \overline{X}$ not included in D are hyperbolic.

References

- [AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze.
- [BKT13] Yohan Brunebarbe, Bruno Klingler, and Burt Totaro, *Symmetric differentials and the fundamental group*, *Duke Math. J.* **162** (2013), no. 14, 2797–2813.
- [Bou02] Sébastien Boucksom, *On the volume of a line bundle*, *Internat. J. Math.* **13** (2002), no. 10, 1043–1063.
- [Bru16] Yohan Brunebarbe, *Symmetric differentials and variations of Hodge structures*, *J. Reine Angew. Math.* (2016).
- [BT15] Benjamin Bakker and Jacob Tsimerman, *The Kodaira dimension of complex hyperbolic manifolds with cusps*, arXiv:1503.05654v1 (2015).
- [CC15] Gabriele Di Cerbo and Luca F. Di Cerbo, *On the canonical divisor of smooth toroidal compactifications of complex hyperbolic manifolds*, arXiv:1502.06258 (2015).
- [CP15] Frédéric Campana and Mihai Păun, *Orbifold generic semi-positivity: an application to families of canonically polarized manifolds*, *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 2, 835–861.
- [DCDC15] Gabriele Di Cerbo and Luca F. Di Cerbo, *Effective results for complex hyperbolic manifolds*, *J. Lond. Math. Soc. (2)* **91** (2015), no. 1, 89–104.
- [Dem92] Jean-Pierre Demailly, *Singular Hermitian metrics on positive line bundles*, *Complex algebraic varieties (Bayreuth, 1990)*, *Lecture Notes in Math.*, vol. 1507, Springer, Berlin, 1992, pp. 87–104.
- [Div16] Simone Diverio, *Segre forms and Kobayashi–Lübke inequality*, *Mathematische Zeitschrift* (2016), 1–15.
- [Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, *Graduate Texts in Mathematics*, No. 52.
- [Hir84] Friedrich Hirzebruch, *Chern numbers of algebraic surfaces: an example*, *Math. Ann.* **266** (1984), no. 3, 351–356.
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, *Cambridge Tracts in Mathematics*, vol. 134, Cambridge University Press, Cambridge, 1998.
- [Mok12] Ngaiming Mok, *Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume*, *Perspectives in analysis, geometry, and topology*, *Progr. Math.*, vol. 296, Birkhäuser/Springer, New York, 2012, pp. 331–354.

- [Mum77] David Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272.
- [Nad89] Alan M. Nadel, *The non-existence of certain level structures on abelian varieties over complex function fields*, Annals of Mathematics **129** (1989), 161–178.
- [WY16] Damir Wu and Shing-Tung Yau, *Negative holomorphic curvature and positive canonical bundle*, Inventiones mathematicae **204** (2016), no. 2, 595–604.

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