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Support properties of the intertwining and the mean value operators in Dunkl's analysis

Léonard GALLARDO* and Chaabane REJEB†

Abstract

In this paper we show that the Dunkl intertwining operator has a compact support which is invariant by the associated Coxeter-Weyl group. This property enables us to determine explicitly the support of the volume mean value operator, a fundamental tool for the study of harmonic functions relative to the Dunkl-Laplacian operator.

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1 Introduction and statement of the results

Let R be a (finite) root system in \mathbb{R}^d with associated Coxeter-Weyl group W (see [7] or [9] for details on root systems) and for $\xi \in \mathbb{R}^d$, let D_ξ be the Dunkl operator defined by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^d),$$

where R_+ is a subsystem of positive roots, σ_α is the reflection directed by the root $\alpha \in R_+$, k is a nonnegative multiplicity function (defined on R) and $\partial_\xi f$ is the usual ξ -directional derivative of f .

These operators, introduced by C. F. Dunkl in the nineties (see [1]), are related to partial derivatives by means of an intertwining operator V_k (see [3] or [4]) as follows

$$\forall \xi \in \mathbb{R}^d, \quad D_\xi V_k = V_k \partial_\xi. \tag{1.1}$$

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We know that V_k is a topological isomorphism from the space $\mathcal{C}^\infty(\mathbb{R}^d)$ (carrying its usual Fréchet topology) onto itself satisfying (1.1) and $V_k(1) = 1$ (see [15]) and V_k commutes with the W -action (see [14]) i.e.

$$\forall f \in \mathcal{C}^\infty(\mathbb{R}^d), \forall g \in W, g^{-1}.V_k(g.f) = V_k(f), \quad (1.2)$$

where $g.f(x) = f(g^{-1}x)$.

A fundamental fact due to M. Rösler (see [11] or [14]) is that for every $x \in \mathbb{R}^d$, there exists a unique compactly supported probability measure μ_x^k on \mathbb{R}^d with

$$\text{supp } \mu_x^k \subset C(x) := \text{co}\{gx, g \in W\} \quad (1.3)$$

(the convex hull of the orbit of x under the group W) such that

$$\forall f \in \mathcal{C}^\infty(\mathbb{R}^d), V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x^k(y). \quad (1.4)$$

Note that the property (1.3) has been proved in [8].

Throughout this paper, the notation $k > 0$ means that $k(\alpha) > 0$ for all $\alpha \in R$.

Concerning the measure μ_x^k (which we call Rösler's measure at point x), the first result of our paper is the following

Theorem A. For every $x \in \mathbb{R}^d$, we have

- 1) $x \in \text{supp } \mu_x^k$,
- 2) If $k > 0$, the support of μ_x^k is a W -invariant set,
- 3) If $k > 0$ then $W.x$ (the W -orbit of x) is contained in $\text{supp } \mu_x^k$.

A question strongly related to the support of Rösler's measures concerns the Dunkl-mean value operator introduced by the authors in [6] in the study of harmonic functions for the Dunkl-Laplacian operator $\Delta_k = \sum_{i=1}^d D_i^2$ where $D_i = D_{e_i}$ with $(e_i)_{1 \leq i \leq d}$ an orthonormal basis of \mathbb{R}^d . Precisely for $x \in \mathbb{R}^d$ and $r > 0$, the mean value of a continuous function f at (x, r) is defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} f(y) h_k(r, x, y) \omega_k(y) dy,$$

where $y \mapsto h_k(r, x, y)$ is a compactly supported measurable function (see Section 2) given by

$$h_k(r, x, y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0, r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle}) d\mu_y^k(z), \quad (1.5)$$

m_k is the measure $dm_k(x) := \omega_k(x) dx$ and ω_k is the weight function

$$\omega_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \quad (1.6)$$

In particular we have shown that a $\mathcal{C}^2(\mathbb{R}^d)$ -function u is Δ_k -harmonic in \mathbb{R}^d if and only if for all $(x, r) \in \mathbb{R}^d \times \mathbb{R}_+$, $u(x) = M_B^r(u)(x)$. For a further thorough study of Δ_k -harmonicity on a general W -invariant open set, it would be crucial to get information on the support of the mean value operators. We already know that the measures

$$d\eta_{x,r}^k = \frac{1}{m_k(B(0,r))} h_k(r, x, y) \omega_k(y) dy \quad (x \in \mathbb{R}^d, r > 0), \quad (1.7)$$

are probability measures with compact support equal to $\text{supp } h_k(r, x, \cdot)$ and satisfying the following inclusion ([6]):

$$\text{supp } h_k(r, x, \cdot) \subset B^W(x, r) := \cup_{g \in W} B(gx, r), \quad (1.8)$$

where $B(x, r)$ denotes the usual closed ball of radius r centered at x .

In fact, the second main result of this paper, intimately related to Theorem A, is a precise description of the support of $h_k(r, x, \cdot)$. It states that

Theorem B: Let $x \in \mathbb{R}^d$ and $r > 0$.

- 1) We have $B(x, r) \subset \text{supp } h_k(r, x, \cdot)$.
- 2) If $k > 0$, then we have

$$\text{supp } h_k(r, x, \cdot) = B^W(x, r) := \cup_{g \in W} B(gx, r).$$

We will call $B^W(x, r)$ the closed Dunkl ball centered at x and with radius $r > 0$ associated to the Coxeter-Weyl group W .

2 The harmonic kernel and the mean value operator

In this section we recall some results of [6].

Let $(r, x, y) \mapsto h_k(r, x, y)$ be the harmonic kernel defined by (1.5). We note that in the classical case (i.e. $k = 0$), we have $\mu_y^k = \delta_y$ and $h_0(r, x, y) = \mathbf{1}_{[0,r]}(\|x - y\|) = \mathbf{1}_{B(x,r)}(y)$. The harmonic kernel satisfies the following properties (see [6]):

1. For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r, x, y) \leq 1$.
2. For all fixed $x, y \in \mathbb{R}^d$, the function $r \mapsto h_k(r, x, y)$ is right-continuous and non decreasing on $]0, +\infty[$.
3. Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon > 0$,

$$0 \leq \varphi_\varepsilon \leq 1, \quad \varphi_\varepsilon = 1 \text{ on } B(0, r) \quad \text{and} \quad \forall y \in \mathbb{R}^d, \quad \lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y),$$

we have

$$\forall y \in \mathbb{R}^d, \quad h_k(r, x, y) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \tilde{\varphi}_\varepsilon(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle}) d\mu_y^k(z),$$

where $\tilde{\varphi}_\varepsilon$ is the profile function of φ_ε i.e. $\varphi_\varepsilon(x) = \tilde{\varphi}_\varepsilon(\|x\|)$.

4. For all $r > 0$, $x, y \in \mathbb{R}^d$ and $g \in W$, we have

$$h_k(r, x, y) = h_k(r, y, x) \quad \text{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y). \quad (2.1)$$

5. For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r, x, \cdot)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r, x, y) \omega_k(y) dy = m_k(B(0, r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \quad (2.2)$$

where d_k is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi) d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1} \Gamma(d/2+\gamma)}.$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of \mathbb{R}^d and c_k is the Macdonald-Mehta constant (see [10], [5]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx.$$

6. Let $r > 0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r, x, \cdot)$ is upper semi-continuous on \mathbb{R}^d .

7. The harmonic kernel satisfies the following geometric inequality: if $\|a - b\| \leq 2r$ with $r > 0$, then

$$\forall \xi \in \mathbb{R}^d, \quad h_k(r, a, \xi) \leq h_k(4r, b, \xi)$$

(see [6], Lemma 4.1). Note that in the classical case (i.e. $k = 0$), this inequality says that if $\|a - b\| \leq 2r$, then $B(a, r) \subset B(b, 4r)$.

8. Let $x \in \mathbb{R}^d$. Then the family of probability measures $d\eta_{x,r}^k(y)$ defined by (1.7) is an approximation of the Dirac measure δ_x as $r \rightarrow 0$. That is

$$\forall \alpha > 0, \quad \lim_{r \rightarrow 0} \int_{\|x-y\| > \alpha} d\eta_{x,r}^k(y) = 0$$

and if f is a locally bounded measurable function on a W -invariant open neighborhood of x and if f is continuous at x , then (see [6], Proposition 3.2):

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(y) d\eta_{x,r}^k(y) = \lim_{r \rightarrow 0} M_B^r(f)(x) = f(x). \quad (2.3)$$

3 Proof of the results

For convenience we group together the first points of Theorem A and Theorem B in the following Proposition.

Proposition 3.1 *Let $x \in \mathbb{R}^d$. Then*

i) *for every $r > 0$, $x \in \text{supp } h_k(r, x, \cdot)$,*

ii) $x \in \text{supp } \mu_x^k$,

iii) for every $r > 0$, $B(x, r) \subset \text{supp } h_k(r, x, \cdot)$.

Proof: i) Suppose that there exists $r > 0$ such that $x \notin \text{supp } h_k(r, x, \cdot)$. Then we can find $\varepsilon > 0$ such that $h_k(r, x, y) = 0$, for all $y \in B(x, \varepsilon)$. Let f be a nonnegative continuous functions on \mathbb{R}^d such that $\text{supp } f \subset B(x, \varepsilon)$ and $f = 1$ on $B(x, \varepsilon/2)$.

Since $t \mapsto h_k(t, x, y)$ is increasing on $]0, +\infty[$, we deduce that

$$\forall t \in]0, r], \quad 0 \leq M_B^t(f)(x) \leq \frac{1}{m_k[B(0, t)]} \int_{\mathbb{R}^d} f(y) h_k(r, x, y) \omega_k(y) dy = 0.$$

Hence, we obtain $M_B^t(f)(x) = 0$, for all $t \in]0, r]$. Letting $t \rightarrow 0$ and using the relation (2.3), we get a contradiction.

ii) Let $x \in \mathbb{R}^d$ be fixed. At first, we claim that

$$\forall r > 0, \quad \forall y \in \mathbb{R}^d, \quad h_k(r, x, y) \leq \mu_x^k[B(y, r)]. \quad (3.1)$$

Indeed, from the inclusion $\text{supp } \mu_x^k \subset B(0, \|x\|)$, we see that

$$\forall y \in \mathbb{R}^d, \quad \forall z \in \text{supp } \mu_x^k, \quad \|y - z\|^2 \leq \|y\|^2 + \|x\|^2 - 2\langle y, z \rangle.$$

This implies for any $y \in \mathbb{R}^d$ and $r > 0$ that

$$\forall z \in \text{supp } \mu_x^k, \quad \mathbf{1}_{[0, r]}(\sqrt{\|y\|^2 + \|x\|^2 - 2\langle y, z \rangle}) \leq \mathbf{1}_{[0, r]}(\|y - z\|) = \mathbf{1}_{B(y, r)}(z).$$

If we integrate the two terms of the previous inequality with respect to the measure μ_x^k , we obtain $h_k(r, y, x) \leq \mu_x^k(B(y, r))$ and then (3.1) follows from (2.1).

Now, if $x \notin \text{supp } \mu_x^k$, there exists $\varepsilon > 0$ such that $\mu_x^k(B(x, \varepsilon)) = 0$. Thus, we have $\mu_x^k(B(y, \varepsilon/2)) = 0$ whenever $y \in B(x, \varepsilon/2)$. Using (3.1), we deduce that $h_k(\varepsilon/2, x, \cdot) = 0$ on $B(x, \varepsilon/2)$, a contradiction with the result of i).

iii) Let $y \in \mathbb{R}^d$ such that $\|x - y\| < r$. As $\lim_{z \rightarrow y} (\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle) = \|x - y\|^2$, there exists $\eta > 0$ such that

$$\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle} \leq r \quad \text{for every } z \in B(y, \eta).$$

Therefore, by using the fact that $y \in \text{supp } \mu_y^k$ we obtain $h_k(r, x, y) \geq \mu_y^k[B(y, \eta)] > 0$. \square

Remark 3.1 For $\alpha \in R$, let

$$H_\alpha := \{x \in \mathbb{R}^d, \langle x, \alpha \rangle = 0\}$$

be the hyperplane directed by α . Note that in [12] (Corollary 3.6) and under the condition $x \notin \cup_{\alpha \in R} H_\alpha$, Rösler has proved that $x \in \text{supp } \mu_x^k$ by using the asymptotic behavior of the Dunkl kernel $E_k(x, y)$ which is defined by

$$E_k(x, y) := V_k(e^{\langle \cdot, y \rangle})(x) = \int_{\mathbb{R}^d} e^{\langle z, y \rangle} d\mu_x^k(z).$$

We turn now to the second statement of Theorem A that we recall below:

Theorem 3.1 *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then the set $\text{supp } \mu_x^k$ is W -invariant.*

Proof: In order to simplify the formulas, we will assume here that the root system R is normalized i.e. $\|\alpha\|^2 = 2$ for all $\alpha \in R$. In particular, for reflections we have $\sigma_\alpha x = x - \langle \alpha, x \rangle \alpha$.

We will prove that if $y \in \text{supp } \mu_x^k$, then $\sigma_\alpha y \in \text{supp } \mu_x^k$ for every $\alpha \in R$. Let then $y \in \text{supp } \mu_x^k$ and suppose that there is a root $\alpha \in R$ such that $\sigma_\alpha y \notin \text{supp } \mu_x^k$. Write $y' := \sigma_\alpha y$ to simplify notations. There is a ball $B(y', \epsilon)$ ($\epsilon > 0$) such that for all $f \in C^\infty(\mathbb{R}^d)$ with compact support included in $B(y', \epsilon)$, we have

$$\int_{\mathbb{R}^d} f(z) \mu_x(dz) = V_k f(x) = 0.$$

Let us denote by $C_{y', \epsilon}^\infty$ (resp. $C_{y', \epsilon}$) the set of all functions $f \in C^\infty(\mathbb{R}^d)$ (resp. $f \in C(\mathbb{R}^d)$) with compact support in $B(y', \epsilon)$. For all $\xi \in \mathbb{R}^d$ and all $f \in C_{y', \epsilon}^\infty$, we also have $\partial_\xi f \in C_{y', \epsilon}^\infty$. By the intertwining relation (1.1) we obtain

$$\forall \xi \in \mathbb{R}^d, \quad \forall f \in C_{y', \epsilon}^\infty, \quad D_\xi V_k f(x) = 0.$$

Suppose $f \in C_{y', \epsilon}^\infty$ and $f \geq 0$ and let $g := V_k f$. We have $g \geq 0$ on \mathbb{R}^d (because V_k preserves positivity) and

$$\forall \xi \in \mathbb{R}^d, \quad D_\xi g(x) = \partial_\xi g(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha \rangle} = 0. \quad (3.2)$$

But as $g(x) = 0$, x is a minimum of g so $\partial_\xi g(x) = 0$ and relation (3.2) implies

$$\forall \xi \in \mathbb{R}^d, \quad \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha \rangle} = 0. \quad (3.3)$$

Now, consider the set

$$R_x := \{\alpha \in R_+; x \in H_\alpha\}.$$

There are two possible locations for x :

• **First case:** Suppose that $R_x = \emptyset$ i.e. $x \notin \cup_{\alpha \in R} H_\alpha$ (i.e. for all root $\alpha \in R$, $\langle x, \alpha \rangle \neq 0$). Applying (3.3) with $\xi = x$ and using the fact that $g(x) = 0$, we get

$$\sum_{\alpha \in R_+} k(\alpha) g(\sigma_\alpha x) = 0.$$

As $g \geq 0$ and $k > 0$, we obtain that $g(\sigma_\alpha x) = V_k f(\sigma_\alpha x) = 0$ for all $\alpha \in R_+$ and all $f \in C_{y', \epsilon}^\infty$ and $f \geq 0$. By uniform approximation, we deduce that for all $f \in C_{y', \epsilon}$ and $f \geq 0$, we also have $V_k f(\sigma_\alpha x) = 0$. Finally for every $f \in C_{y', \epsilon}$, by decomposing $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ and using the linearity and W -equivariance of V_k (relation (1.2)), we obtain that

$$\forall f \in C_{y', \epsilon}, \quad \forall \alpha \in R_+, \quad V_k f(\sigma_\alpha x) = V_k(\sigma_\alpha \cdot f)(x) = 0,$$

where $\sigma_\alpha.f$ is the function $z \mapsto f(\sigma_\alpha z)$. As it is easy to see that $\sigma_\alpha.C_{y',\epsilon} = C_{\sigma_\alpha y',\epsilon}$, we deduce that

$$\forall \alpha \in R_+, \quad \forall f \in C_{\sigma_\alpha y',\epsilon}, \quad V_k f(x) = 0.$$

But this implies in particular that $V_k f(x) = 0$ for all $f \in C_{y,\epsilon}$ in contradiction with the hypothesis $y \in \text{supp } \mu_x^k$. The result of the theorem follows in the first case.

• **Second case:** Suppose that $R_x \neq \emptyset$. For every $\beta \in R_x$, clearly we have $x = \sigma_\beta x$. Therefore, since $g(x) = 0$, we get $g(\sigma_\beta x) = 0$, for all $\beta \in R_x$. But, as x is a minimum of g , we have

$$\forall \beta \in R_x, \quad \frac{g(x) - g(\sigma_\beta x)}{\langle x, \beta \rangle} = \int_0^1 \partial_\beta g(x - t \langle x, \beta \rangle \beta) dt = \partial_\beta g(x) = 0.$$

Hence, the relation (3.3) with $\xi = x$ implies

$$\sum_{\alpha \in R_+ \setminus R_x} k(\alpha) g(\sigma_\alpha x) = 0.$$

Consequently, we obtain $g(\sigma_\alpha x) = 0$ for all $\alpha \in R$. The end of the proof of the first case applies and gives also the result in this case. This completes the proof of the theorem. \square

From the W -invariance property of the support of μ_x^k and the fact that $x \in \text{supp } \mu_x^k$, we obtain immediately the last assertion of Theorem A:

Corollary 3.1 *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then, for all $g \in W$, $gx \in \text{supp } \mu_x^k$.*

Now, we can turn to the proof of the second statement of Theorem B.

Corollary 3.2 *Let $x \in \mathbb{R}^d$ and $r > 0$. If $k > 0$, then*

$$\text{supp } h_k(r, x, \cdot) = B^W(x, r) := \cup_{g \in W} B(gx, r). \quad (3.4)$$

Proof: Let $g \in W$ and $y \in \mathbb{R}^d$ such that $\|gx - y\| < r$. We will proceed as in the proof of Proposition 3.1, iii). We have

$$\lim_{z \rightarrow g^{-1}y} \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle} = \|x - g^{-1}y\|.$$

Hence, there exists $\eta > 0$ such that for all $z \in B(g^{-1}y, \eta)$, $\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle} \leq r$ and thus $h_k(r, x, y) \geq \mu_y^k[B(g^{-1}y, \eta)]$.

But, from the fact that $g^{-1}y \in \text{supp } \mu_y^k$ we deduce that $y \in \text{supp } h_k(r, x, \cdot)$.

This completes the proof. \square

Remark 3.2 *When $k \geq 0$, we will say that a root $\alpha \in R$ is active if $k(\alpha) > 0$. Let us denote by $R_A = \{\alpha \in R; k(\alpha) > 0\}$ the set of active roots and F the vector subspace of \mathbb{R}^d generated by $\{\alpha, \alpha \in R_A\}$. Then we can generalize the results of Theorems A and B in the following form*

a) The set R_A is a root system. Indeed, using the fact that k is W -invariant, we can see that for every $\alpha, \beta \in R_A$, $k(\sigma_\alpha \beta) = k(\beta) > 0$. Thus

$$\forall \alpha \in R_A, \quad R_A \cap \mathbb{R}\alpha = \{\pm\alpha\} \quad \text{and} \quad \sigma_\alpha(R_A) = R_A.$$

b) Let W_A be the Coxeter-Weyl group associated to the root system R_A . Then the restriction k_A of k to R_A is clearly invariant under the W_A -action. In other words, it is a multiplicity function.

c) For any $\xi \in \mathbb{R}^d$, we will use the notation $\xi = \xi' + \xi'' \in F + F^\perp = \mathbb{R}^d$ (where F^\perp is the orthogonal complement of F in \mathbb{R}^d).

• Let $x \in \mathbb{R}^d$. Rösler's measure μ_x^k is of the form (see [13])

$$\mu_x^k = \mu_{x'}^{k_A} \otimes \delta_{x''}, \quad (3.5)$$

where $\mu_{x'}^{k_A}$ is Rösler's measure associated to (R_A, k_A) and $\delta_{x''}$ is the Dirac measure at x'' . We have

$$\text{supp } \mu_x^k = x'' + \text{supp } \mu_{x'}^{k_A}.$$

From (1.3), the support of $\mu_{x'}^{k_A}$ is contained in the convex hull of $W_A \cdot x'$ (the W_A -orbit of x'). Furthermore, by Theorem A, it is invariant under the action of the group W_A and contains the whole orbite $W_A \cdot x'$.

• Let $x \in \mathbb{R}^d$ and $r > 0$. According to (1.5) and (3.5) the harmonic kernel is given by

$$h_k(r, x, y) = \int_{\mathbb{R}^d} \mathbf{1}_{[0, r]}(\sqrt{\|x'' - y''\|^2 + \|x'\|^2 + \|y'\|^2 - 2\langle x', z' \rangle}) d\mu_{y'}^{k_A}(z'), \quad y \in \mathbb{R}^d.$$

The support of $h_k(r, x, \cdot)$ takes the following form

$$\text{supp } h_k(r, x, \cdot) = x'' + B^{W_A}(x', r) = x'' + \cup_{g \in W_A} B(gx', r) = \cup_{g \in W_A} B(gx, r).$$

Example 3.1 Let (e_1, e_2) be the canonical basis of \mathbb{R}^2 . Then, the set $R := \{\pm e_1, \pm e_2\}$ is a root system in \mathbb{R}^2 , its Coxeter-Weyl group is \mathbb{Z}_2^2 and the multiplicity function can be identified to a pair $k = (k_1, k_2)$, with $k_i = k(e_i) \geq 0$, $i = 1, 2$. Take $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1, x_2 > 0$. In this case, according to [16], Rösler's measure is given by $\mu_x^k = \mu_{x_1}^{k_1} \otimes \mu_{x_2}^{k_2}$, where $\mu_{x_i}^{k_i} = \delta_{x_i}$ if $k_i = 0$ and

$$\langle \mu_{x_i}^{k_i}, f \rangle = \frac{\Gamma(k_i + 1/2)}{\sqrt{\pi} \Gamma(k_i)} \int_{-1}^1 f(tx_i) (1-t)^{k_i-1} (1+t)^{k_i} dt.$$

if $k_i > 0$ (see [2]).

• If $k = (0, 0)$, $\mu_x^k = \delta_x$ and $h_k(r, x, y) = \mathbf{1}_{B(x, r)}(y)$.

• If $k = (k_1, 0)$ with $k_1 > 0$, then $\text{supp } \mu_x^k$ is the line segment between x and $\sigma_{e_1} x = (-x_1, x_2)$ and

$$\text{supp } h_k(r, x, \cdot) = B(x, r) \cup B(\sigma_{e_1} x, r).$$

• If $k_1, k_2 > 0$, the support of μ_x^k is the convex hull of $\mathbb{Z}_2^2 \cdot x$ and the closed Dunkl ball is given by

$$B^{\mathbb{Z}_2^2}(x, r) = \text{supp } h_k(r, x, \cdot) = B((x_1, x_2), r) \cup B((-x_1, x_2), r) \cup B((x_1, -x_2), r) \cup B((-x_1, -x_2), r).$$

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