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## Solvability of Mixed Problems With Integral Condition for Singular Parabolic Equations

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### RÉSUMÉ.

**ABSTRACT.** In this paper we proved the existence and uniqueness of strong generalized solution of mixed problems with integral condition for singular parabolic equations depending on a theorem proved in [1] in which a priori estimation of the solution for such problems was derived.

### MOTS-CLÉS :

**KEYWORDS :** Priori estimate; mixed problem; Singular parabolic equations; integral boundary conditions.

**2000 Mathematics Subject Classification.** 35K20, 35K25, 35K30.



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## 1. Introduction

Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [3], Byszewski et al [4] and [5], Gasymov [7], Ionkin [8]-[9], Lazhar [11], and Said-Nadia [12]. The results and the method used here are a further elaboration of those in [2]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [6] and [13].

In [1] the author considered the following mixed problem in the rectangle  $Q = (0, l) \times (0, T)$

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{x^m} \frac{\partial}{\partial x} \left( x^m \frac{\partial u}{\partial x} \right) = f(x, t), \quad m > 0, \quad (1)$$

$$lu = u(x, 0)\varphi(x), \quad |u(0, t)| < \infty, \quad \int_{\alpha}^l x^m u(x, t) dx = 0, \quad \alpha > 0. \quad (2)$$

and he proved the following theorem

**Theorem 1 :** For any function  $u \in E$  such that  $x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q)$  and  $x^{-\frac{m}{2}} \frac{\partial}{\partial x} x^m \frac{\partial u}{\partial x} \in L_2(Q)$ , the following inequality holds

$$\|u\|_E^2 \leq c \|\mathcal{F}\|_F^2, \quad (3)$$

where  $c = 2 \left( l + \exp\left(\frac{T}{2\alpha^2}\right) \right)$ .

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## 2. The Main Result

we consider the operator  $L = (\mathcal{L}, l)$  with the following domain

$$D(L) = \left\{ u \in E : x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q), x^{\frac{m}{2}} \frac{\partial}{\partial x} \left( x^m \frac{\partial u}{\partial x} \right) \in L_2(Q) \right\},$$

acting from  $E$  into  $F$  by the rule  $Lu = (\mathcal{L}u, u(x, 0))$ .

In a standard way its proved [10] that the operator is closable which we denote by  $\bar{L}$  with the domain  $D(\bar{L})$ .

Definition : The solution of the equation  $\bar{L}u = \mathcal{F}$  is called strong generalized solution of the problem (1)-(2). In other words, the function  $u$  is called strong generalized solution of the problem (1)-(2) if there exist a sequence of functions  $u_n \in D(L)$ , such that the  $\|u_n - u\|_E \rightarrow 0$  and  $\|Lu_n - \mathcal{F}\|_F \rightarrow 0$  at  $n \rightarrow \infty$ .

**Theorem 2 :** Strong generalized solution of the problem (1)-(2) exist and unique for any  $\mathcal{F} = (f, \varphi) \in F$ .

**Proof.** For the sequence  $u_n \in D(L)$  the following inequality holds

$$\|u_n\|_E^2 \leq 2 \left( l + \exp\left(\frac{T}{2\alpha^2}\right) \right) \|Lu_n\|_F^2, \quad (4)$$

which implies from theorem 1. Passing in (4) to the limit at  $n \rightarrow \infty$ , we get the following inequality

$$\|u_n\|_E^2 \leq 2 \left( l + \exp \left( \frac{T}{2\alpha^2} \right) \right) \|\bar{L}u_n\|_F^2, u \in D(\bar{L}). \quad (5)$$

From (5) implies that strong generalized solution of problem (1)-(2) is unique, the range  $R(\bar{L})$  of the operator  $\bar{L}$  is closed in  $F$  and  $R(\bar{L}) = \overline{R(L)}$ . Therefore for the proof of existence of strong generalized solution of (1)-(2) we need to prove that the range  $R(L)$  of the operator  $L$  is dense in  $F$ .

Since the range of the operator  $L$  is dense in a space with the norm  $(\int_0^l x^n (\varphi')^2 dx + \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} \varphi^2 dx)^{\frac{1}{2}}$ , its sufficient to show that the equality

$$\iint_Q x^m \mathcal{L}u v dx dt = 0, \forall u \in D_0(L) = \{u \in D(L) : u(x, 0) = 0\}, x^{\frac{m}{2}} v \in L_2(Q), \quad (6)$$

imply the equality  $v = 0$ .

We set in (6)  $x^m v = Mh$ , where  $Mh = x^m h$  at  $0 \leq x \leq \alpha$  and

$$Mh = \frac{l-x}{l-\alpha} x^m h(x, t) + \frac{1}{l-\alpha} \iint_\alpha^x \xi^m h(\xi, t) d\xi, \quad (7)$$

at  $\alpha \leq x \leq l$ . This holds if the function  $h(x, t) = v(x, t)$  at  $0 \leq x \leq \alpha$  and

$$h(x, t) = v(x, t) - \frac{1}{(l-\alpha)x^m} \iint_\alpha^l \xi^m v(\xi, t) d\xi,$$

at  $\alpha \leq x \leq l$ . It is not hard to see that the function  $h$  satisfy the third condition from (2), that is

$$\iint_\alpha^l x^m h(x, t) dx = 0. \quad (8)$$

So for any function  $u \in D_0(L)$  and given function  $h$  we get the equality

$$\iint_Q \frac{\partial u}{\partial t} Mh dx dt = \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} (x^m \frac{\partial u}{\partial x}) Mh dx dt. \quad (9)$$

We set in (9)  $u = \int_0^t w(x, r) dr$  where  $w$  is any function such that  $x^{\frac{m}{2}} w \in L_2(Q)$ ,

$$\frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} (x^m \frac{\partial w}{\partial x}) \in L_2(Q), \iint_\alpha^l x^m w(x, t) dx = 0, \left| \frac{\partial}{\partial x} w(0, t) \right| < \infty.$$

Then we get the equality

$$\iint_Q w Mh dx dt = \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} (x^m \frac{\partial w}{\partial x}) Mh dx dt, \quad (10)$$

where  $g(x, t) = \int_t^T h(x, \tau) d\tau$ . The left side of (10) show that the map

$$L_2(Q) \ni x^{\frac{m}{2}} w \rightarrow \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} (x^m \frac{\partial w}{\partial x}) M g dx dt \in R$$

is linear continuous functional. Consequently

$$x^{\frac{m}{2}} \frac{\partial}{\partial x} \frac{Mg}{x^m} \in L_2(Q), \frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} (x^m \frac{\partial}{\partial x} (\frac{Mg}{x^m})) \in L_2(Q),$$

and by virtue of (5)

$$x^{\frac{m}{2}} \frac{\partial}{\partial x} g \in L_2(Q), \frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} (x^m \frac{\partial}{\partial x} g) \in L_2(Q), \left| \frac{\partial g(0, t)}{\partial x} \right| < \infty. \quad (11)$$

Integrating by parts the right hand side of (10) and taking into account (7) and (8), we get

$$\iint_Q w M h dx dt = - \iint_Q x^m \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \frac{Mg}{x^m} dx dt. \quad (12)$$

On the basis of (11) we set in (12)

$$w(x, t) = \int_0^t e^{c(\tau-T)} g(x, \tau) d\tau. \quad (13)$$

Then

$$\iint_Q e^{c(t-T)} g M g dx dt = - \iint_Q e^{c(\tau-t)} x^m \frac{\partial w}{\partial x} \frac{\partial^2}{\partial x \partial t} (\frac{Mw}{x^m}) dx dt. \quad (14)$$

By analogy of formula (12) in [1] we get

$$\begin{aligned} \iint_Q e^{c(t-T)} g M g dx dt &= \int_Q e^{c(t-T)} \psi(x) |g|^2 dx dt \\ &+ \int_0^T \int_\alpha^l \frac{m e^{c(t-T)}}{2(l-\alpha)x^{m+1}} \left| \int_\alpha^x \xi^m g(\xi, t) d\xi \right|^2 dx dt. \end{aligned} \quad (15)$$

Further

$$\begin{aligned}
\int_Q e^{c(T-t)} x^m \frac{\partial w}{\partial x} \frac{\partial^2}{\partial t \partial x} \frac{Mw}{x^m} dx dt &= \int_Q e^{c(T-t)} x^m \psi(x) \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial t \partial x} dx dt \\
&- \int_0^T \int_\alpha^l \frac{e^{c(T-t)} m}{(l-\alpha)x} \frac{\partial w}{\partial x} \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt \\
&= \frac{1}{2} \int_0^l x^m \psi(x) \left| \int_0^T e^{c(t-T)} g(x, t) dt \right|^2 dx \\
&+ \frac{c}{2} \int_Q e^{c(T-t)} x^m \psi(x) |w(x, t)|^2 dx dt \\
&- \int_0^T \int_\alpha^l \frac{e^{c(T-t)} m}{(l-\alpha)x^2} w \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt \\
&+ \frac{1}{2} \int_\alpha^l \frac{x^{m-1} m}{(l-\alpha)} \left| \int_0^T e^{c(t-T)} g(x, t) dt \right|^2 dx \\
&+ \frac{cm}{2} \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2 dx dt, \quad (16)
\end{aligned}$$

where  $\psi(x) = \begin{cases} 1, & 0 \leq x \leq \alpha, \\ \frac{l-x}{l-\alpha}, & \alpha \leq x \leq l. \end{cases}$

By analogy of (18) in [1] we get

$$\begin{aligned}
\int_0^T \int_\alpha^l \frac{e^{c(T-t)} w}{(l-\alpha)x^2} \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt &\leq \frac{1}{2} \int_0^T \int_0^l \frac{e^{c(T-t)}}{(l-\alpha)x^{m+1}} \left| \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi \right|^2 dx dt \\
&+ \frac{1}{2\alpha^2} \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2 dx dt. \quad (17)
\end{aligned}$$

From (14) and by virtue of (13),(15)-(17) implies the following inequality

$$\begin{aligned} & \int_Q e^{c(t-T)} \psi(x) |g|^2 dxdt + \frac{1}{2} \int_0^l x^m \psi(x) \left| \int_0^T e^{c(t-T)} g(x,t) dt \right|^2 dx \\ & + \frac{c}{2} \int_Q e^{c(T-t)} x^m \psi(x) |w(x,t)|^2 dxdt + \frac{m}{2} \int_\alpha^l \frac{x^{m-1}}{(l-\alpha)} \left| \int_0^T e^{c(t-T)} g(x,t) dt \right|^2 dx \\ & + \frac{m}{2} \left( c - \frac{1}{\alpha^2} \right) \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2(x,t) dxdt \leq 0. \end{aligned} \quad (18)$$

We set in (13)  $c \geq \frac{1}{\alpha^2}$ . Then from (18) implies that  $g \equiv 0$ . Since  $x^m v = M \frac{\partial g}{\partial t}$  then  $v \equiv 0$ , and theorem 2 is proved.

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### 3. Bibliographie

- [1] . Almomani ; A Priori Estimation of the Solution for Mixed Problems with Integral Condition for Singular Parabolic Equations, *Mathematical Analysis*, Vol. 2014, Article ID 735875, 5 pages, <http://dx.doi.org/10.1155/2014/735875>.
- [2] . E. Benouar and N. I. Yurchuk ; Mixed problem with an integral condition for parabolique equation with the Bessel operator, *Differentsial'nye Uravneniya*, 27 (1991), p. 2094 - 2098.
- [3] . Bouziani ; On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces. *IJMMS.*, 30 :7 (2002), p. 435 - 447.
- [4] . Byszewski ; Existence and uniqueness of solutions of nonlocal problem for hyperbolic equation, *J. Appl. Math. Stochastic. Anal.*,3 (1990), p. 163-168.
- [5] . Byszewski ; Differential and functional-differential problems together with nonlocal conditions, *Cracow University of technology Monograph*, 184, Cracow 1995.
- [6] oussa Zakari Djibibe, Kokou Tcharie, Nikolay Iossifovich Yurchuk, Continuous Dependence of Solutions to Mixed Boundary Value Problems for Parabolic Equation, *Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 17, pp. 1-10.
- [7] . A. Gasmov ; Mixed problems on the conjugation of parabolic system of different order with nonlocal boundary conditions, *Differentsial'nye Uravneniya*, 26, No 8, (1991), p. 1003 - 1012.
- [8] . I. Ionkin ; Solution of boundary value problems in heat conduction theory with nonlocal boundary conditions, *Differents. Uravn.*, 13 :2 (1977), p. 294 - 304.
- [9] . I. Ionkin and E. I. Moiceev ; Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions, *Differents. Uravn.*, 13 (1977), p. 294 - 304.
- [10] rzysztof Maurin, *Methods of Hilbert Spaces*, Warszawa, Polish Scientific Publishers, 1967.
- [11] . Lazhar ; Parabolic equations with nonlocal conditions, *Applied Mathematical Sciences*, Vol. 1,(2007), No. 21, p. 1041 - 1048.
- [12] . Said and L. Nadia ; On a nonlocal singular mixed evolution problem, 53 (2001), p. 79 - 89.
- [13] . I. Yurchuk ; Mixed problem with an integral condition for certain parabolic equations, *Differents. Uravn.*, 22 :12, (1986), p. 2117 - 2126.