Solvability of Mixed Problems With Integral Condition for Singular Parabolic Equations

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Abstract

In this paper we proved the existance and uniqness of strong generalized solution of mixed problems wih integral condition for singular parabolic equaions depending on a theorem proved in [1] in which a priori estimaion of the solution for such problems was derived.

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1 Introduction

Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [3], Byszewski et al [4] and [5], Gasymov [7], Ionkin [8]-[9], Lazhar [11], and Said-Nadia [12]. The results and the method used here are a further elaboration of those in [2]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [6] and [13].

In [1] the author considered the following mixed problem in the rectangle $Q = (0, l) \times (0, T)$

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{x^m}\frac{\partial}{\partial x}(x^m \frac{\partial u}{\partial x}) = f(x,t),\ m > 0, \tag{1}$$

$$lu = u(x,0)\varphi(x),\ \ |u(0,t)| < \infty,\ \ \iint_{\alpha}^{l} x^m u(x,t)dx = 0,\ \alpha > 0. \tag{2}$$

and he proved the following theorem

Theorem 1: *For any function* $u \in E$ *such that* $x^{\frac{m}{2}}\frac{\partial u}{\partial t} \in L_2(Q)$ *and* $x^{-\frac{m}{2}}\frac{\partial}{\partial x}x^m\frac{\partial u}{\partial x} \in L_2(Q)$*, the following inequality holds*

$$\|u\|_E^2 \leq c\,\|\mathcal{F}\|_F^2\,, \tag{3}$$

where $c = 2\left(l + \exp(\frac{T}{2\alpha^2})\right)$.

2 The Main Result

we consider the operator $L = (\mathcal{L}, l)$ with the following domain

$$D(L) = \left\{ u \in E : x^{\frac{m}{2}}\frac{\partial u}{\partial t} \in L_2(Q), x^{\frac{m}{2}}\frac{\partial}{\partial x}(x^m\frac{\partial u}{\partial x}) \in L_2(Q) \right\},$$

acting from E into F by the rule $Lu = (\mathcal{L}u, u(x,0))$.

In a standard way its proved [10] that the operator is closable which we denote by $\overline{L}$ with the domain $D(\overline{L})$.

Definition: The solution of the equation $\overline{L}u = \mathcal{F}$ is called strong generalized solution of the problem (1)-(2). In other words, the function u is called strong generalized solution of the problem (1)-(2) if there exist a sequence of functions $u_n \in D(L)$, such that the $\|u_n - u\|_E \to 0$ and $\|Lu_n - \mathcal{F}\|_F \to 0$ at $n \to \infty$.

Theorem 2: Strong generalized solution of the problem (1)-(2) exist and unique for any $\mathcal{F} = (f, \varphi) \in F$.

Proof. For the sequence $u_n \in D(L)$ the following inequality holds

$$\|u_n\|_E^2 \leq 2\left(l + \exp\left(\frac{T}{2\alpha^2}\right)\right)\|Lu_n\|_F^2, \tag{4}$$

which implies from theorem 1. Passing in (4) to the limit at $n \to \infty$, we get the following inequality

$$\|u_n\|_E^2 \leq 2\left(l + \exp\left(\frac{T}{2\alpha^2}\right)\right)\left\|\overline{L}u_n\right\|_F^2, u \in D(\overline{L}). \tag{5}$$

From (5) implies that strong generalized solution of problem (1)-(2) is unique, the range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and $R(\overline{L}) = \overline{R(L)}$. Therefore for the proof of existence of strong generalized solution of (1)-(2) we need to prove that the range $R(L)$ of the operator L is dense in F.

Since the range of the operator L is dense in a space with the norm $(\iint\limits_0^l x^n\left(\varphi'\right)^2 dx + \frac{m}{l-\alpha}\iint\limits_\alpha^l x^{m-1}\varphi^2 dx)^{\frac{1}{2}}$, its sufficient to show that the equality

$$\iint\limits_Q x^m \mathcal{L}uv dxdt = 0, \forall u \in D_0(L) = \{u \in D(L) : u(x,0) = 0\}, x^{\frac{m}{2}} v \in L_2(Q), \tag{6}$$

imply the equality $v = 0$.

We set in (6) $x^m v = Mh$, where $Mh = x^m h$ at $0 \leq x \leq \alpha$ and

$$Mh = \frac{l-x}{l-\alpha}x^m h(x,t) + \frac{1}{l-\alpha}\iint\limits_\alpha^x \xi^m h(\xi,t)d\xi, \tag{7}$$

at $\alpha \leq x \leq l$. This holds if the function $h(x,t) = v(x,t)$ at $0 \leq x \leq \alpha$ and

$$h(x,t) = v(x,t) - \frac{1}{(l-\alpha)x^m}\iint\limits_\alpha^l \xi^m v(\xi,t)d\xi,$$

at $\alpha \leq x \leq l$. It is not hard to see that the function h satisfy the third condition from (2), that is

$$\iint\limits_\alpha^l x^m h(x,t)dx = 0. \tag{8}$$

So for any function $u \in D_0(L)$ and given function h we get the equality

$$\iint\limits_Q \frac{\partial u}{\partial t}Mhdxdt = \iint\limits_Q \frac{1}{x^m}\frac{\partial}{\partial x}(x^m\frac{\partial u}{\partial x})Mhdxdt. \tag{9}$$

We set in (9) $u = \iint_0^t w(x,r)dr$ where w is any function such that $x^{\frac{m}{2}} w \in L_2(Q)$,

$$\frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m \frac{\partial w}{\partial x}) \in L_2(Q), \iint\limits_{\alpha}^{l} x^m w(x,t)dx = 0, \left|\frac{\partial}{\partial x} w(0,t)\right| < \infty.$$

Then we get the equality

$$\iint\limits_{Q} wMhdxdt = \iint\limits_{Q} \frac{1}{x^m}\frac{\partial}{\partial x}(x^m \frac{\partial w}{\partial x})Mgdxdt, \tag{10}$$

where $g(x,t) = \iint_t^T h(x,\tau)d\tau$. The left side of (10) show that the map

$$L_2(Q) \ni x^{\frac{m}{2}} w \to \iint\limits_{Q} \frac{1}{x^m}\frac{\partial}{\partial x}(x^m \frac{\partial w}{\partial x})Mgdxdt \in R$$

is linear continuos functional. Consequently

$$x^{\frac{m}{2}}\frac{\partial}{\partial x}\frac{Mg}{x^m} \in L_2(Q), \frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m \frac{\partial}{\partial x}(\frac{Mg}{x^m})) \in L_2(Q),$$

and by virtue of (5)

$$x^{\frac{m}{2}}\frac{\partial}{\partial x} g \in L_2(Q), \frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m \frac{\partial}{\partial x} g) \in L_2(Q), \left|\frac{\partial g(0,t)}{\partial x}\right| < \infty. \tag{11}$$

Integrating by parts the right hand side of (10) and taking into account (7) and (8), we get

$$\iint\limits_{Q} wMhdxdt = -\iint\limits_{Q} x^m \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\frac{Mg}{x^m}dxdt. \tag{12}$$

On the basis of (11) we set in (12)

$$w(x,t) = \iint\limits_{0}^{t} e^{c(\tau - T)} g(x,\tau)d\tau. \tag{13}$$

Then

$$\iint\limits_{Q} e^{c(t-T)} gMgdxdt = -\iint\limits_{Q} e^{c(\tau - t)} x^m \frac{\partial w}{\partial x}\frac{\partial^2}{\partial x \partial t}(\frac{Mw}{x^m})dxdt. \tag{14}$$

By analogy of formula (12) in [1] we get

$$\begin{aligned} \iint\limits_{Q} e^{c(t-T)} gMgdxdt &= \int\limits_{Q} e^{c(t-T)}\psi(x)\left|g\right|^2 dxdt \\ &+ \int\limits_{0}^{T}\int\limits_{\alpha}^{l} \frac{me^{c(t-T)}}{2(l-\alpha)x^{m+1}}\left|\int\limits_{\alpha}^{x} \xi^m g(\xi,t)d\xi\right|^2 dxdt. \end{aligned} \tag{15}$$

Further

$$\begin{aligned}
\int\limits_Q e^{c(T-t)}x^m\frac{\partial w}{\partial x}\frac{\partial^2}{\partial t\partial x}\frac{Mw}{x^m}dxdt &= \int\limits_Q e^{c(T-t)}x^m\psi(x)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}dxdt \\
&- \int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}m}{(l-\alpha)x}\frac{\partial w}{\partial x}\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt \\
&= \frac{1}{2}\int\limits_0^l x^m\psi(x)\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right|^2 dx \\
&+ \frac{c}{2}\int\limits_Q e^{c(T-t)}x^m\psi(x)\left|w(x,t)\right|^2 dxdt \\
&- \int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}m}{(l-\alpha)x^2}w\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt \\
&+ \frac{1}{2}\int\limits_\alpha^l \frac{x^{m-1}m}{(l-\alpha)}\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right| dx \\
&+ \frac{cm}{2}\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2 dxdt, \qquad (16)
\end{aligned}$$

where $\psi(x) = \begin{cases} 1, & 0 \le x \le \alpha, \\ \frac{l-x}{l-\alpha}, & \alpha \le x \le l. \end{cases}$

By analogy of (18) in [1] we get

$$\begin{aligned}
\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}w}{(l-\alpha)x^2}\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt &\le \frac{1}{2}\int\limits_0^T\int\limits_0^l \frac{e^{c(T-t)}}{(l-\alpha)x^{m+1}}\left|\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi\right|^2 dxdt \\
&+ \frac{1}{2\alpha^2}\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2 dxdt. \qquad (17)
\end{aligned}$$

From (14) and by virtue of (13),(15)-(17) implies the following inequality

$$\int\limits_Q e^{c(t-T)}\psi(x)\left|g\right|^2 dxdt + \frac{1}{2}\int\limits_0^l x^m\psi(x)\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right|^2 dx$$

$$+\frac{c}{2}\int\limits_Q e^{c(T-t)}x^m\psi(x)\left|w(x,t)\right|^2 dxdt + \frac{m}{2}\int\limits_\alpha^l \frac{x^{m-1}}{(l-\alpha)}\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right| dx$$

$$+\frac{m}{2}(c-\frac{1}{\alpha^2})\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2(x,t)dxdt \leq 0. \tag{18}$$

We set in (13) $c \geq \frac{1}{\alpha^2}$. Then from (18) implies that $g \equiv 0$. Since $x^m v = M\frac{\partial g}{\partial t}$ then $v \equiv 0$, and theorem 2 is proved.

[1] R. Almomani; A Priori Estimation of the Solution for Mixed Problems with Integral Condition for Singular Parabolic Equations, Mathematical Analysis, Vol. 2014, Article ID 735875, 5 pages, http://dx.doi.org/10.1155/2014/735875.

[2] N. E. Benouar and N. I. Yurchuk; Mixed problem with an integral condition for parabolique equation with the Bessel operator, Differentsial'nye Uravneniya, 27 (1991), p. 2094 - 2098.

[3] A. Bouziani; On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces. IJMMS., 30:7 (2002), p. 435 - 447.

[4] L. Byszewski; Existence and uniqueness of solutions of nonlocal problem for hyperbolic equation, J. Appl. Math. Stochastic. Anal.,3 (1990), p. 163-168.

[5] L. Byszewski; Differential and functional-differential problems together with nonlocal conditions, Cracow University of technology Monograph, 184, Cracow 1995.

[6] Moussa Zakari Djibibe, Kokou Tcharie, Nikolay Iossifovich Yurchuk, Continuous Dependence of Solutions to Mixed Boundary Value Problems for Parabolic Equation, Electronic Journal of Differential Equations, Vol. 2008(2008), No. 17, pp. 1–10.

[7] E. A. Gasymov; Mixed problems on the conjugation of parabolic system of different order with nonlocal boundary conditions, Differentsial'nye Uravnenya, 26, No 8, (1991), p. 1003 - 1012.

[8] N. I. Ionkin; Solution of boundary value problems in heat conduction theory with nonlocal boundary conditions, Differents. Uravn., 13:2 (1977), p. 294 - 304.

[9] N. I. Ionkin and E. I. Moiceev; Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions, Differents. Uravn., 13 (1977), p. 294 - 304.

[10] Krzysztof Maurin, Methods of Hilbert Spaces, Warszawa, Polish Scientific Publishers, 1967.

[11] B. Lazhar; Parabolic equations with nonlocal conditions, Applied Mathematical Sciences, Vol. 1,(2007), No. 21, p. 1041 - 1048.

[12] M. Said and L. Nadia; On a nonlocal singular mixed evolution problem, 53 (2001), p. 79 - 89.

[13] N. I. Yurchuk; Mixed problem with an integral condition for certain parabolic equations, Differents. Uravn., 22:12, (1986), p. 2117 - 2126.