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Solvability of Mixed Problems With Integral Condition for Singular Parabolic Equations

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Abstract

In this paper we proved the existence and uniqueness of strong generalized solution of mixed problems with integral condition for singular parabolic equations depending on a theorem proved in [1] in which a priori estimation of the solution for such problems was derived.

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Keywords: Priori estimate; mixed problem; Singular parabolic equations; integral boundary conditions.

1 Introduction

Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [3], Byszewski et al [4] and [5], Gasymov [7], Ionkin [8]-[9], Lazhar [11], and Said-Nadia [12]. The results and the method used here are a further elaboration of those in [2]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [6] and [13].

In [1] the author considered the following mixed problem in the rectangle $Q = (0, l) \times (0, T)$

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{x^m} \frac{\partial}{\partial x} (x^m \frac{\partial u}{\partial x}) = f(x, t), \quad m > 0, \quad (1)$$

$$lu = u(x, 0)\varphi(x), \quad |u(0, t)| < \infty, \quad \int_{\alpha}^l x^m u(x, t) dx = 0, \quad \alpha > 0. \quad (2)$$

and he proved the following theorem

Theorem 1: *For any function $u \in E$ such that $x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q)$ and $x^{-\frac{m}{2}} \frac{\partial}{\partial x} x^m \frac{\partial u}{\partial x} \in L_2(Q)$, the following inequality holds*

$$\|u\|_E^2 \leq c \|\mathcal{F}\|_F^2, \quad (3)$$

where $c = 2(l + \exp(\frac{T}{2\alpha^2}))$.

2 The Main Result

we consider the operator $L = (\mathcal{L}, l)$ with the following domain

$$D(L) = \left\{ u \in E : x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q), x^{\frac{m}{2}} \frac{\partial}{\partial x} (x^m \frac{\partial u}{\partial x}) \in L_2(Q) \right\},$$

acting from E into F by the rule $Lu = (\mathcal{L}u, u(x, 0))$.

In a standard way its proved [10] that the operator is closable which we denote by \bar{L} with the domain $D(\bar{L})$.

Definition: The solution of the equation $\bar{L}u = \mathcal{F}$ is called strong generalized solution of the problem (1)-(2). In other words, the function u is called strong generalized solution of the problem (1)-(2) if there exist a sequence of functions $u_n \in D(L)$, such that the $\|u_n - u\|_E \rightarrow 0$ and $\|Lu_n - \mathcal{F}\|_F \rightarrow 0$ at $n \rightarrow \infty$.

Theorem 2: Strong generalized solution of the problem (1)-(2) exist and unique for any $\mathcal{F} = (f, \varphi) \in F$.

Proof. For the sequence $u_n \in D(L)$ the following inequality holds

$$\|u_n\|_E^2 \leq 2 \left(l + \exp \left(\frac{T}{2\alpha^2} \right) \right) \|Lu_n\|_F^2, \quad (4)$$

which implies from theorem 1. Passing in (4) to the limit at $n \rightarrow \infty$, we get the following inequality

$$\|u_n\|_E^2 \leq 2 \left(l + \exp \left(\frac{T}{2\alpha^2} \right) \right) \|\bar{L}u_n\|_F^2, u \in D(\bar{L}). \quad (5)$$

From (5) implies that strong generalized solution of problem (1)-(2) is unique, the range $R(\bar{L})$ of the operator \bar{L} is closed in F and $R(\bar{L}) = \overline{R(L)}$. Therefore for the proof of existence of strong generalized solution of (1)-(2) we need to prove that the range $R(L)$ of the operator L is dense in F .

Since the range of the operator L is dense in a space with the norm $(\int_0^l x^n (\varphi')^2 dx + \frac{m}{l-\alpha} \int_0^l x^{m-1} \varphi^2 dx)^{\frac{1}{2}}$, its sufficient to show that the equality

$$\iint_Q x^m \mathcal{L}uv dx dt = 0, \forall u \in D_0(L) = \{u \in D(L) : u(x, 0) = 0\}, x^{\frac{m}{2}} v \in L_2(Q), \quad (6)$$

imply the equality $v = 0$.

We set in (6) $x^m v = Mh$, where $Mh = x^m h$ at $0 \leq x \leq \alpha$ and

$$Mh = \frac{l-x}{l-\alpha} x^m h(x, t) + \frac{1}{l-\alpha} \iint_{\alpha}^x \xi^m h(\xi, t) d\xi, \quad (7)$$

at $\alpha \leq x \leq l$. This holds if the function $h(x, t) = v(x, t)$ at $0 \leq x \leq \alpha$ and

$$h(x, t) = v(x, t) - \frac{1}{(l-\alpha)x^m} \iint_{\alpha}^l \xi^m v(\xi, t) d\xi,$$

at $\alpha \leq x \leq l$. It is not hard to see that the function h satisfy the third condition from (2), that is

$$\iint_{\alpha}^l x^m h(x, t) dx = 0. \quad (8)$$

So for any function $u \in D_0(L)$ and given function h we get the equality

$$\iint_Q \frac{\partial u}{\partial t} Mh dx dt = \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} (x^m \frac{\partial u}{\partial x}) Mh dx dt. \quad (9)$$

We set in (9) $u = \int_0^t w(x, r) dr$ where w is any function such that $x^{\frac{m}{2}} w \in L_2(Q)$,

$$\frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} \left(x^m \frac{\partial w}{\partial x} \right) \in L_2(Q), \int_{\alpha}^l \int x^m w(x, t) dx = 0, \left| \frac{\partial}{\partial x} w(0, t) \right| < \infty.$$

Then we get the equality

$$\iint_Q w M h dx dt = \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial w}{\partial x} \right) M g dx dt, \quad (10)$$

where $g(x, t) = \int_t^T h(x, \tau) d\tau$. The left side of (10) show that the map

$$L_2(Q) \ni x^{\frac{m}{2}} w \rightarrow \iint_Q \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial w}{\partial x} \right) M g dx dt \in R$$

is linear continuos functional. Consequently

$$x^{\frac{m}{2}} \frac{\partial}{\partial x} \frac{Mg}{x^m} \in L_2(Q), \frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} \left(x^m \frac{\partial}{\partial x} \left(\frac{Mg}{x^m} \right) \right) \in L_2(Q),$$

and by virtue of (5)

$$x^{\frac{m}{2}} \frac{\partial}{\partial x} g \in L_2(Q), \frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} \left(x^m \frac{\partial}{\partial x} g \right) \in L_2(Q), \left| \frac{\partial g(0, t)}{\partial x} \right| < \infty. \quad (11)$$

Integrating by parts the right hand side of (10) and taking into account (7) and (8), we get

$$\iint_Q w M h dx dt = - \iint_Q x^m \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \frac{Mg}{x^m} dx dt. \quad (12)$$

On the basis of (11) we set in (12)

$$w(x, t) = \int_0^t \int e^{c(\tau-T)} g(x, \tau) d\tau. \quad (13)$$

Then

$$\iint_Q e^{c(t-T)} g M g dx dt = - \iint_Q e^{c(\tau-t)} x^m \frac{\partial w}{\partial x} \frac{\partial^2}{\partial x \partial t} \left(\frac{Mw}{x^m} \right) dx dt. \quad (14)$$

By analogy of formula (12) in [1] we get

$$\begin{aligned} \iint_Q e^{c(t-T)} g M g dx dt &= \int_Q e^{c(t-T)} \psi(x) |g|^2 dx dt \\ &+ \int_0^T \int_{\alpha}^l \frac{m e^{c(t-T)}}{2(l-\alpha)x^{m+1}} \left| \int_{\alpha}^x \xi^m g(\xi, t) d\xi \right|^2 dx dt. \end{aligned} \quad (15)$$

Further

$$\begin{aligned}
\int_Q e^{c(T-t)} x^m \frac{\partial w}{\partial x} \frac{\partial^2}{\partial t \partial x} \frac{Mw}{x^m} dx dt &= \int_Q e^{c(T-t)} x^m \psi(x) \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial t \partial x} dx dt \\
&- \int_0^T \int_\alpha^l \frac{e^{c(T-t)} m}{(l-\alpha)x} \frac{\partial w}{\partial x} \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt \\
&= \frac{1}{2} \int_0^l x^m \psi(x) \left| \int_0^T e^{c(t-T)} g(x, t) dt \right|^2 dx \\
&+ \frac{c}{2} \int_Q e^{c(T-t)} x^m \psi(x) |w(x, t)|^2 dx dt \\
&- \int_0^T \int_\alpha^l \frac{e^{c(T-t)} m}{(l-\alpha)x^2} w \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt \\
&+ \frac{1}{2} \int_\alpha^l \frac{x^{m-1} m}{(l-\alpha)} \left| \int_0^T e^{c(t-T)} g(x, t) dt \right|^2 dx \\
&+ \frac{cm}{2} \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2 dx dt, \quad (16)
\end{aligned}$$

where $\psi(x) = \begin{cases} 1, & 0 \leq x \leq \alpha, \\ \frac{l-x}{l-\alpha}, & \alpha \leq x \leq l. \end{cases}$

By analogy of (18) in [1] we get

$$\begin{aligned}
\int_0^T \int_\alpha^l \frac{e^{c(T-t)} w}{(l-\alpha)x^2} \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi dx dt &\leq \frac{1}{2} \int_0^T \int_0^l \frac{e^{c(T-t)}}{(l-\alpha)x^{m+1}} \left| \int_\alpha^x \xi^m \frac{\partial w}{\partial t} d\xi \right|^2 dx dt \\
&+ \frac{1}{2\alpha^2} \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2 dx dt. \quad (17)
\end{aligned}$$

From (14) and by virtue of (13),(15)-(17) implies the following inequality

$$\begin{aligned}
& \int_Q e^{c(t-T)} \psi(x) |g|^2 dxdt + \frac{1}{2} \int_0^l x^m \psi(x) \left| \int_0^T e^{c(t-T)} g(x,t) dt \right|^2 dx \\
& + \frac{c}{2} \int_Q e^{c(T-t)} x^m \psi(x) |w(x,t)|^2 dxdt + \frac{m}{2} \int_\alpha^l \frac{x^{m-1}}{(l-\alpha)} \left| \int_0^T e^{c(t-T)} g(x,t) dt \right|^2 dx \\
& + \frac{m}{2} \left(c - \frac{1}{\alpha^2} \right) \int_0^T \int_\alpha^l \frac{e^{c(T-t)} x^{m-1}}{(l-\alpha)} w^2(x,t) dxdt \leq 0. \tag{18}
\end{aligned}$$

We set in (13) $c \geq \frac{1}{\alpha^2}$. Then from (18) implies that $g \equiv 0$. Since $x^m v = M \frac{\partial g}{\partial t}$ then $v \equiv 0$, and theorem 2 is proved.

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