



**HAL**  
open science

# Stability of periodic waves of 1D cubic nonlinear Schrödinger equations

Stephen Gustafson, Stefan Le Coz, Tai-Peng Tsai

► **To cite this version:**

Stephen Gustafson, Stefan Le Coz, Tai-Peng Tsai. Stability of periodic waves of 1D cubic nonlinear Schrödinger equations. *Applied Mathematics Research eXpress*, 2017, 2017 (2), pp.431–487. 10.1093/amrx/abx004 . hal-01330828v2

**HAL Id: hal-01330828**

**<https://hal.science/hal-01330828v2>**

Submitted on 12 Oct 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABILITY OF PERIODIC WAVES OF 1D CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

STEPHEN GUSTAFSON, STEFAN LE COZ, AND TAI-PENG TSAI

ABSTRACT. We study the stability of the cnoidal, dnoidal and snoidal elliptic functions as spatially-periodic standing wave solutions of the 1D cubic nonlinear Schrödinger equations. First, we give global variational characterizations of each of these periodic waves, which in particular provide alternate proofs of their orbital stability with respect to same-period perturbations, restricted to certain subspaces. Second, we prove the spectral stability of the cnoidal waves (in a certain parameter range) and snoidal waves against same-period perturbations, thus providing an alternate proof of this (known) fact, which does not rely on complete integrability. Third, we give a rigorous version of a formal asymptotic calculation of Rowlands to establish the instability of a class of real-valued periodic waves in 1D, which includes the cnoidal waves of the 1D cubic focusing nonlinear Schrödinger equation, against perturbations with period a large multiple of their fundamental period. Finally, we develop a numerical method to compute the minimizers of the energy with fixed mass and momentum constraints. Numerical experiments support and complete our analytical results.

## CONTENTS

1. Introduction	2
2. Preliminaries	5
2.1. Spaces of Periodic Functions	5
2.2. Jacobi Elliptic Functions	6
2.3. Elliptic Integrals	7
2.4. Classification of Real Periodic Waves	8
3. Variational Characterizations and Orbital Stability	9
3.1. The Minimization Problems	10
3.2. Minimization Among Periodic Functions	11
3.3. Minimization Among Half-Anti-Periodic Functions	14
3.4. Orbital Stability	17
4. Spectral Stability	18
4.1. Spectra of $L_+$ and $L_-$	20
4.2. Orthogonality Properties	23
4.3. Spectral Stability of $sn$ and $cn$	24

---

*Date:* October 12, 2016.

*2010 Mathematics Subject Classification.* 35Q55; 35B10; 35B35.

*Key words and phrases.* Nonlinear Schrödinger equations, periodic waves, stability.

The work of S. G. is partially supported by NSERC grant 251124-12.

The work of S. L. C. is partially supported by ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02 and ANR-14-CE25-0009-01.

The work of T. T. is partially supported by NSERC grant 261356-13.

5. Linear Instability	26
5.1. Theoretical Analysis	26
5.2. Numerical Spectra	33
6. Numerics	34
6.1. Gradient Flow With Discrete Normalization	34
6.2. Discretization	36
7. Numerical Solutions of Minimization Problems	37
7.1. Minimization Among Periodic Functions	38
7.2. Minimization Among Half-Anti-Periodic Functions	39
References	41

## 1. INTRODUCTION

We consider the cubic nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + b|\psi|^2\psi = 0, \quad \psi(0, x) = \psi_0(x) \quad (1.1)$$

in one space dimension, where  $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  and  $b \in \mathbb{R} \setminus \{0\}$ . Equation (1.1) has well-known applications in optics, quantum mechanics, and water waves, and serves as a model for nonlinear dispersive wave phenomena more generally [11, 31]. It is said to be *focusing* if  $b > 0$  and *defocusing* if  $b < 0$ . Note that (1.1) is invariant under

- spatial translation:  $\psi(t, x) \mapsto \psi(t, x + a)$  for  $a \in \mathbb{R}$
- phase multiplication:  $\psi(t, x) \mapsto e^{i\alpha}\psi(t, x)$  for  $\alpha \in \mathbb{R}$ .

We are particularly interested in the spatially periodic setting

$$\psi(t, \cdot) \in H_{\text{loc}}^1 \cap P_T, \quad P_T = \{f \in L_{\text{loc}}^2(\mathbb{R}) : f(x + T) = f(x) \forall x \in \mathbb{R}\}.$$

The Cauchy problem (1.1) is globally well-posed in  $H_{\text{loc}}^1 \cap P_T$  [7]. We refer to [6] for a detailed analysis of nonlinear Schrödinger equations with periodic boundary conditions. Solutions to (1.1) conserve mass  $\mathcal{M}$ , energy  $\mathcal{E}$ , and momentum  $\mathcal{P}$ :

$$\begin{aligned} \mathcal{M}(\psi) &= \frac{1}{2} \int_0^T |\psi|^2 dx, & \mathcal{P}(\psi) &= \frac{1}{2} \text{Im} \int_0^T \psi \bar{\psi}_x dx, \\ \mathcal{E}(\psi) &= \frac{1}{2} \int_0^T |\psi_x|^2 dx - \frac{b}{4} \int_0^T |\psi|^4 dx. \end{aligned}$$

By virtue of its complete integrability, (1.1) enjoys infinitely many higher (in terms of the number of derivatives involved) conservation laws [27], but we do not use them here, in order to remain in the energy space  $H_{\text{loc}}^1$ , and with the aim of avoiding techniques which rely on integrability.

The simplest non-trivial solutions of (1.1) are the *standing waves*, which have the form

$$\psi(t, x) = e^{-iat}u(x), \quad a \in \mathbb{R}$$

and so the profile function  $u(x)$  must satisfy the ordinary differential equation

$$u_{xx} + b|u|^2u + au = 0. \quad (1.2)$$

We are interested here in those standing waves  $e^{-iat}u(x)$  whose profiles  $u(x)$  are spatially periodic – which we refer to as *periodic waves*. One can refer to the book [3]

for an overview of the role and properties of periodic waves in nonlinear dispersive PDEs.

Non-constant, real-valued, periodic solutions of (1.2) are well-known to be given by the Jacobi elliptic functions: dnoidal (dn), cnoidal (cn) (for  $b > 0$ ) and snoidal (sn) (for  $b < 0$ ) – see Section 2 for details. To make the link with Schrödinger equations set on the whole real line, one can see a periodic wave as a special case of infinite train solitons [25, 26]. Another context in which periodic waves appear is when considering the nonlinear Schrödinger equation on a Dumbbell graph [28]. Our interest here is in the stability of these periodic waves against periodic perturbations whose period is a multiple of that of the periodic wave.

Some recent progress has been made on this stability question. By Grillakis-Shatah-Strauss [18, 19] type methods, orbital stability against energy ( $H_{\text{loc}}^1$ )-norm perturbations of the same period is known for dnoidal waves [2], and for snoidal waves [13] under the additional constraint that perturbations are anti-symmetric with respect to the half-period. In [13], cnoidal waves are shown to be orbitally stable with respect to half-anti-periodic perturbations, provided some condition is satisfied. This condition is verified analytically for small amplitude cnoidal waves and numerically for larger amplitude. Remark here that the results in [13] are obtained in a broader setting, as they are also considering *non-trivially complex-valued* periodic waves. Integrable systems methods introduced in [5] and developed in [15] – in particular conservation of a higher-order functional – are used to obtain the orbital stability of the snoidal waves against  $H_{\text{loc}}^2$  perturbations of period *any* multiple of that of sn.

Our goal in this paper is to further investigate the properties of periodic waves. We follow three lines of exploration. First, we give *global* variational characterization of the waves in the class of periodic or half-anti-periodic functions. As a corollary, we obtain orbital stability results for periodic waves. Second, we prove the spectral stability of cnoidal, dnoidal and snoidal waves within the class of functions whose period is the fundamental period of the wave. Third, we prove that cnoidal waves are linearly unstable if perturbations are periodic for a sufficiently large multiple of the fundamental period of the cnoidal wave.

Our first main results concern global variational characterizations of the elliptic function periodic waves as constrained-mass energy minimizers among (certain subspaces of) periodic functions, stated as a series of Propositions in Section 3. In particular, the following characterization of the cnoidal functions seems new. Roughly stated (see Proposition 3.4 for a precise statement):

**Theorem 1.1.** *Let  $b > 0$ . The unique (up to spatial translation and phase multiplication) global minimizer of the energy, with fixed mass, among half-anti-periodic functions is a (appropriately rescaled) cnoidal function.*

Due to the periodic setting, existence of a minimizer for the problems that we are considering is easily obtained. The difficulty lies within the identification of this minimizer: is it a plane wave, a (rescaled) Jacobi elliptic function, or something else? To answer this question, we first need to be able to decide whether the minimizer can be considered real-valued after a phase change. This is far from obvious in the half-anti-periodic setting of Theorem 1.1, where we use a Fourier coefficients rearrangement argument (Lemma 3.5) to obtain this information. To identify the minimizers, we use a combination of spectral and Sturm-Liouville arguments.

As a corollary of our global variational characterizations, we obtain orbital stability results for the periodic waves. In particular, Theorem 1.1 implies the orbital stability of all cnoidal waves in the space of half-anti-periodic functions. Such orbital stability results for periodic waves were already obtained in [2, 13] as consequences of *local* constrained minimization properties. Our global variational characterizations provide alternate proofs of these results – see Corollary 3.9 and Corollary 4.7. The orbital stability of cnoidal waves was proved only for small amplitude in [13], and so we extend this result to all amplitude. Remark however once more that we are in this paper considering only real-valued periodic wave profiles, as opposed to [13] in which truly complex valued periodic waves were investigated.

Our second main result proves the linear (more precisely, *spectral*) stability of the snoidal and cnoidal (with some restriction on the parameter range in the latter case) waves against same-period perturbations, but *without* the restriction of half-period antisymmetry:

**Theorem 1.2.** *Snoidal waves and cnoidal waves (for a range of parameters) with fundamental period  $T$  are spectrally stable against  $T$ -periodic perturbations.*

See Theorem 4.1 for a more precise statement. For sn, this is already a consequence of [5, 15], whereas for cn the result was obtained in [21]. The works [5, 15] and [21] both exploit the integrable structure, so our result could be considered an alternate proof which does not use integrability, but instead relies mainly on an invariant subspace decomposition and an elementary Krein-signature-type argument. See also the recent work [16] for related arguments.

The proof of Theorem 1.2 goes as follows. The linearized operator around a periodic wave can be written as  $J\mathcal{L}$ , where  $J$  is a skew symmetric matrix and  $\mathcal{L}$  is the self-adjoint linearization of the action of the wave (see Section 4 for details). The operator  $\mathcal{L}$  is made of two Lamé operators and we are able to calculate the bottom of the spectrum for these operators. To obtain Theorem 1.2, we decompose the space of periodic functions into invariant subspaces: half-periodic and half-anti-periodic, even and odd. Then we analyse the linearized spectrum in each of these subspaces. In the subspace of half-anti-periodic functions, we obtain spectral stability as a consequence of the analysis of the spectrum of  $\mathcal{L}$  (alternately, as a consequence of the variational characterizations of Section 3). For the subspace of half-periodic functions, a more involved argument is required. We give in Lemma 4.12 an abstract argument relating coercivity of the linearized action  $\mathcal{L}$  with the number of eigenvalues with negative Krein signature of  $J\mathcal{L}$  (this is in fact a simplified version of a more general argument [20]). Since we are able to find an eigenvalue with negative Krein signature for  $J\mathcal{L}$ , spectral stability for half-periodic functions follows from this abstract argument.

Our third main result makes rigorous a formal asymptotic calculation of Rowlands [30] which establishes:

**Theorem 1.3.** *Cnoidal waves are unstable against perturbations whose period is a sufficiently large multiple of its own.*

This is stated more precisely in Theorem 5.3, and is a consequence of a more general perturbation result, Proposition 5.4, which implies this instability for any real periodic wave for which a certain quantity has the right sign. In particular, the argument does not rely on any integrability (beyond the ability to calculate the quantity in question in terms of elliptic integrals).

Perturbation argument were also used by [14], [15], but our strategy here is different. Instead of relying on abstract theory to obtain the a priori existence of branches of eigenvalues, we directly construct the branch in which we are interested. This is done by first calculating the exact terms of the formal expansion for the eigenvalue and eigenvector at the two first orders, and then obtaining the rigorous existence for the rest of the expansion using a contraction mapping argument. Note that the branch that we are constructing was described in terms of Evans function in [21].

Finally, we complete our analytical results with some numerical observations. Our motivation is to complete the variational characterizations of periodic waves, which was only partial for snoidal waves. We observe:

**Observation 1.4.** *Let  $b < 0$ . For a given period, the unique (up to phase shift and translation) global minimizer of the energy with fixed mass and 0 momentum among half-anti-periodic functions is a (appropriately rescaled) snoidal function.*

We have developed a numerical method to obtain the profile  $\phi$  as minimizer on two constraints, fixed mass and fixed (zero) momentum. We use a heat flow algorithm, where at each time step the solution is renormalized to satisfy the constraints. Mass renormalization is simply obtained by scaling. Momentum renormalization is much trickier. We define an auxiliary evolution problem for the momentum that we solve explicitly, and plug back the solution we obtain to get the desired renormalized solutions. We first have tested our algorithm in the cases where our theoretical results hold and we have a good agreement between the theoretical results and the numerical experiments. Then, we have performed experiments on snoidal waves which led to Observation 1.4.

The rest of this paper is divided as follows. In Section 2, we present the spaces of periodic functions and briefly recall the main definitions and properties of Jacobi elliptic functions and integrals. In Section 3, we characterize the Jacobi elliptic functions as global constraint minimizers and give the corresponding orbital stability results. Section 4 is devoted to the proof of spectral stability for cnoidal and snoidal waves, whereas in Section 5 we prove the linear instability of cnoidal waves. Finally, we present our numerical method in Section 6 and the numerical experiments in Section 7.

**Acknowledgments.** *We are grateful to Bernard Deconinck and Dmitri Pelinovsky for useful remarks on a preliminary version of this paper.*

## 2. PRELIMINARIES

This section is devoted to reviewing the classification of real-valued periodic waves in terms of Jacobi elliptic functions.

**2.1. Spaces of Periodic Functions.** Let  $T > 0$  be a period. Denote by  $\tau_T$  the translation operator

$$(\tau_T f)(x) = f(x + T),$$

acting on  $L^2_{\text{loc}}(\mathbb{R})$ , and its eigenspaces

$$P_T(\mu) = \{f \in L^2_{\text{loc}}(\mathbb{R}) : \tau_T f = \mu f\}$$

for  $\mu \in \mathbb{C}$  with  $|\mu| = 1$ . Taking  $\mu = 1$  yields the space of  $T$ -periodic functions

$$P_T = P_T(1) = \{f \in L^2_{\text{loc}}(\mathbb{R}) : \tau_T f = f\},$$

while for  $\mu = -1$  we get the  $T$ -anti-periodic functions

$$A_T = P_T(-1) = \{f \in L^2_{\text{loc}}(\mathbb{R}) : \tau_T f = -f\}.$$

For  $2 \leq k \in \mathbb{N}$ , letting  $\mu$  run through the  $k$ th roots of unity:  $\omega^k = 1$ , and  $\omega^j \neq 1$  for  $1 \leq j < k$ , we have

$$P_{kT} = \bigoplus_{j=0}^{k-1} P_T(\omega^j),$$

where the decomposition of  $f \in P_{kT}$  is given by

$$f = \sum_{j=0}^{k-1} f_j, \quad f_j = \frac{1}{k} \sum_{m=0}^{k-1} \omega^{-mj} \tau_{mT} f.$$

Only the case  $k = 2$  is needed here:

$$P_{2T} = P_T \oplus A_T, \quad f = \frac{1}{2}(f + \tau_T f) + \frac{1}{2}(f - \tau_T f). \quad (2.1)$$

Since the reflection  $R : f(x) \mapsto f(-x)$  commutes with  $\tau_T$  on  $P_{2T}$ , we may further decompose into odd and even components in the usual way

$$f = f^+ + f^-, \quad f^\pm = \frac{1}{2}(f \pm Rf),$$

to obtain

$$P_T = P_T^+ \oplus P_T^-, \quad A_T = A_T^+ \oplus A_T^-, \quad P_T^\pm(A_T^\pm) = \{f \in P_T(A_T) \mid f(-x) = \pm f(x)\},$$

and so

$$P_{2T} = P_T \oplus A_T = P_T^+ \oplus P_T^- \oplus A_T^+ \oplus A_T^-. \quad (2.2)$$

Each of these subspaces is invariant under (1.1), since

$$\psi \in P_T^\pm(A_T^\pm) \implies |\psi|^2 \in P_T^+ \implies \psi_{xx} + b|\psi|^2\psi \in P_T^\pm(A_T^\pm).$$

When dealing with functions in  $P_T$ , we will denote norms such as  $L^q(0, T)$  by

$$\|u\|_{L^q} = \|u\|_{L^q(0, T)} = \left( \int_0^T |u|^q \right)^{\frac{1}{q}},$$

and the *complex*  $L^2$  inner product by

$$(f, g) = \int_0^T f \bar{g} dx. \quad (2.3)$$

**2.2. Jacobi Elliptic Functions.** Here we recall the definitions and main properties of the Jacobi elliptic functions. The reader might refer to treatises on elliptic functions (e.g. [24]) or to the classical handbooks [1, 17] for more details.

Given  $k \in (0, 1)$ , the *incomplete elliptic integral of the first kind* in trigonometric form is

$$x = F(\phi, k) := \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}},$$

and the *Jacobi elliptic functions* are defined through the inverse of  $F(\cdot, k)$ :

$$\text{sn}(x, k) := \sin(\phi), \quad \text{cn}(x, k) := \cos(\phi), \quad \text{dn}(x, k) := \sqrt{1 - k^2 \sin^2(\phi)}.$$

The relations

$$1 = \text{sn}^2 + \text{cn}^2 = k^2 \text{sn}^2 + \text{dn}^2 \quad (2.4)$$

follow. For extreme value  $k = 0$  we recover trigonometric functions,

$$\operatorname{sn}(x, 0) = \sin(x), \quad \operatorname{cn}(x, 0) = \cos(x), \quad \operatorname{dn}(x, 0) = 1,$$

while for extreme value  $k = 1$  we recover hyperbolic functions:

$$\operatorname{sn}(x, 1) = \tanh(x), \quad \operatorname{cn}(x, 1) = \operatorname{dn}(x, 1) = \operatorname{sech}(x).$$

The periods of the elliptic functions can be expressed in terms of the *complete elliptic integral of the first kind*

$$K(k) := F\left(\frac{\pi}{2}, k\right), \quad K(k) \rightarrow \begin{cases} \frac{\pi}{2} & k \rightarrow 0 \\ \infty & k \rightarrow 1 \end{cases}.$$

The functions  $\operatorname{sn}$  and  $\operatorname{cn}$  are  $4K$ -periodic while  $\operatorname{dn}$  is  $2K$ -periodic. More precisely,

$$\operatorname{dn} \in P_{2K}^+, \quad \operatorname{sn} \in A_{2K}^- \subset P_{4K}, \quad \operatorname{cn} \in A_{2K}^+ \subset P_{4K}.$$

The derivatives (with respect to  $x$ ) of elliptic functions can themselves be expressed in terms of elliptic functions. For fixed  $k \in (0, 1)$ , we have

$$\partial_x \operatorname{sn} = \operatorname{cn} \cdot \operatorname{dn}, \quad \partial_x \operatorname{cn} = -\operatorname{sn} \cdot \operatorname{dn}, \quad \partial_x \operatorname{dn} = -k^2 \operatorname{cn} \cdot \operatorname{sn}, \quad (2.5)$$

from which one can easily verify that  $\operatorname{sn}$ ,  $\operatorname{cn}$  and  $\operatorname{dn}$  are solutions of

$$u_{xx} + au + b|u|^2u = 0, \quad (2.6)$$

with coefficients  $a, b \in \mathbb{R}$  for  $k \in (0, 1)$  given by

$$a = 1 + k^2, \quad b = -2k^2, \quad \text{for } u = \operatorname{sn}, \quad (2.7)$$

$$a = 1 - 2k^2, \quad b = 2k^2, \quad \text{for } u = \operatorname{cn}, \quad (2.8)$$

$$a = -(2 - k^2), \quad b = 2, \quad \text{for } u = \operatorname{dn}. \quad (2.9)$$

**2.3. Elliptic Integrals.** For  $k \in (0, 1)$ , the *incomplete elliptic integral of the second kind* in trigonometric form is defined by

$$E(\phi, k) := \int_0^\phi \sqrt{1 - k^2 \sin^2(\theta)} d\theta.$$

The *complete elliptic integral of the second kind* is defined as

$$E(k) := E\left(\frac{\pi}{2}, k\right).$$

We have the relations (using  $d\theta = \operatorname{dn}(z, k)dz$  and  $x = F(\phi, k)$ )

$$\begin{aligned} E(\phi, k) &= \int_0^x \operatorname{dn}^2(z, k) dz \\ &= x - k^2 \int_0^x \operatorname{sn}^2(z, k) dz = (1 - k^2)x + k^2 \int_0^x \operatorname{cn}^2(z, k) dz, \end{aligned} \quad (2.10)$$

relating the elliptic functions to the elliptic integral of the second kind, and

$$E(k) = K(k) - k^2 \int_0^K \operatorname{sn}^2(z, k) dz = (1 - k^2)K(k) + k^2 \int_0^K \operatorname{cn}^2(z, k) dz, \quad (2.11)$$



relating the elliptic integrals of first and second kind. We can differentiate  $E$  and  $K$  with respect to  $k$  and express the derivatives in terms of  $E$  and  $K$ :

$$\begin{aligned}\partial_k E(k) &= \frac{E(k) - K(k)}{k} < 0, \\ \partial_k K(k) &= \frac{E(k) - (1 - k^2)K(k)}{k - k^3} = \frac{k^2 \int_0^K \text{cn}^2(x, k) dx}{k - k^3} > 0.\end{aligned}$$

Note in particular  $K$  is increasing,  $E$  is decreasing. Moreover,

$$K(0) = E(0) = \frac{\pi}{2}, \quad K(1-) = \infty, \quad E(1) = 1.$$

**2.4. Classification of Real Periodic Waves.** Here we make precise the fact that the elliptic functions provide the only (non-constant) real-valued, periodic solutions of (2.6). Note that there is a two-parameter family of complex-valued, bounded, solutions for every  $a, b \in \mathbb{R}$ ,  $b \neq 0$  [12, 14].

**Lemma 2.1** (focusing case). *Fix a period  $T > 0$ ,  $a \in \mathbb{R}$ ,  $b > 0$  and  $u \in P_T$  a non-constant real solution of (2.6). By invariance under translation, and negation ( $u \mapsto -u$ ), we may suppose  $u(0) = \max u > 0$ .*

(a) *If  $0 \leq \min u < u(0)$ , then  $a < 0$ ,  $|a| < bu(0)^2 < 2|a|$ , and  $u(x) = \frac{1}{\alpha} \text{dn}(\frac{x}{\beta}, k)$ ,*

(b) *If  $\min u < 0$ , then  $\max(0, -2a) < bu(0)^2$ , and  $u(x) = \frac{1}{\alpha} \text{cn}(\frac{x}{\beta}, k)$ ,*

*for some  $\alpha > 0$ ,  $\beta > 0$ , and  $0 < k < 1$ , uniquely determined by  $T$ ,  $a$ ,  $b$  and  $\max u$ . They satisfy the  $a$ -independent relations  $b\beta^2 = 2\alpha^2$  for (a) and  $b\beta^2 = 2k^2\alpha^2$  for (b).*

Note that here  $T$  may be any multiple of the fundamental period of  $u$ . An  $a$ -independent relation is useful since  $a$  will be the unknown Lagrange multiplier for our constrained minimization problems in Section 3.

*Proof.* The first integral is constant: there exists  $C_0 \in \mathbb{R}$  such that

$$u_x^2 + au^2 + \frac{b}{2}u^4 = C_0.$$

A periodic solution has to oscillate in the energy well  $W(u) = au^2 + \frac{b}{2}u^4$  with energy level  $C_0$ . If  $0 \leq \min u$ , then  $a < 0$  and  $C_0 < 0$ . If  $\min u < 0$ , then  $C_0 > 0$ . Let  $u(x) = \frac{1}{\alpha}v(\frac{x}{\beta})$  with  $\alpha = (\max u)^{-1}$ . Then  $v$  satisfies

$$\max v = v(0) = 1, \quad v'' + a\beta^2v + \frac{b\beta^2}{\alpha^2}v^3 = 0.$$

(a) If  $0 \leq \min u$ , then  $a < 0$  and  $C_0 < 0$ . Let  $0 < y_1 < y_2$  be the roots of  $ay + \frac{b}{2}y^2 = C_0 < 0$ . Then  $u(0)^2 = y_2 \in (-a/b, -2a/b)$ .

Let  $\beta = \alpha\sqrt{2/b}$ . Then  $\frac{b\beta^2}{\alpha^2} = 2$  and  $a\beta^2 \in (-2, -1)$ , and there is a unique  $k \in (0, 1)$  so that  $a\beta^2 = -2 + k^2$ . Thus

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (-2 + k^2)v + 2v^3 = 0.$$

By uniqueness of the ODE,  $v(x) = \text{dn}(x, k)$  is the only solution. Hence  $u(x) = \frac{1}{\alpha} \text{dn}(\frac{x}{\beta}, k)$ .

(b) If  $\min u < 0$ , then  $C_0 > 0$ . Let  $y_1 < 0 < y_2$  be the roots of  $ay + \frac{b}{2}y^2 = C_0 > 0$ . Then  $u(0)^2 = y_2 > \max(0, -2a/b)$  no matter  $a < 0$  or  $a \geq 0$ . We claim we can

choose unique  $\beta > 0$  and  $k \in (0, 1)$  so that

$$a\beta^2 = 1 - 2k^2, \quad \frac{b\beta^2}{\alpha^2} = 2k^2.$$

The sum gives  $(a + \frac{b}{\alpha^2})\beta^2 = 1$ , thus  $\beta = (a + \frac{b}{\alpha^2})^{-1/2}$  noting  $(a + \frac{b}{\alpha^2}) > 0$ , and

$$k^2 = \frac{\beta^2 b}{2\alpha^2} = \frac{b}{2(b + a\alpha^2)} \in (0, 1)$$

no matter  $a < 0$  or  $a \geq 0$ . Thus

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (1 - 2k^2)v + 2k^2v^3 = 0.$$

By uniqueness of the ODE,  $v(x) = \text{cn}(x, k)$  is the only solution. Hence  $u(x) = \frac{1}{\alpha} \text{cn}(\frac{x}{\beta}, k)$ .  $\square$

**Lemma 2.2** (defocusing case). *Fix a period  $T > 0$ ,  $a \in \mathbb{R}$ ,  $b < 0$  and  $u \in P_T$  a non-constant, real solution of (2.6). By invariance under translation and negation, suppose  $u(0) = \max u > 0$ . Then  $0 < |b|u(0)^2 < a$ , and  $u(x) = \frac{1}{\alpha} \text{sn}(K(k) + \frac{x}{\beta}, k)$ , for some  $\alpha > 0$ ,  $\beta > 0$ , and  $0 < k < 1$ , uniquely determined by  $T$ ,  $a$ ,  $b$  and  $\max u$ . They satisfy the  $a$ -independent relation  $b\beta^2 = -2k^2\alpha^2$ .*

*Proof.* The first integral is constant: there exists  $C_0 \in \mathbb{R}$  such that

$$u_x^2 + au^2 + \frac{b}{2}u^4 = C_0.$$

A periodic solution has to oscillate in the energy well  $W(u) = au^2 + \frac{b}{2}u^4$  with energy level  $C_0$ . Hence  $a > 0$  and  $0 < C_0 < \max W = \frac{a^2}{-2b}$ . Let  $u(x) = \frac{1}{\alpha}v(\frac{x}{\beta})$  with  $\alpha = (\max u)^{-1}$ . Then  $v$  satisfies

$$\max v = v(0) = 1, \quad v'' + a\beta^2v + \frac{b\beta^2}{\alpha^2}v^3 = 0.$$

Let  $0 < y_1 < y_2$  be the roots of  $ay + \frac{b}{2}y^2 = C_0$ . Then  $u(0)^2 = y_1 \in (0, -a/b)$ .

Let  $\beta = (\frac{2\alpha^2}{2\alpha^2a+b})^{1/2}$  and  $k = (\frac{-b}{2\alpha^2a+b})^{1/2}$ , noting  $2\alpha^2a + b > 0$ . Then  $a\beta^2 = 1 + k^2$ ,  $\frac{b\beta^2}{\alpha^2} = -2k^2$ , and  $v$  satisfies

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (1 + k^2)v - 2k^2v^3 = 0.$$

By uniqueness of the ODE,  $v(x) = \text{sn}(K(k) + x, k)$  is the only solution. Hence  $u(x) = \frac{1}{\alpha} \text{sn}(K(k) + \frac{x}{\beta}, k)$ .  $\square$

### 3. VARIATIONAL CHARACTERIZATIONS AND ORBITAL STABILITY

Our goal in this section is to characterize the Jacobi elliptic functions as *global* constrained energy minimizers. As a corollary, we recover some known results on orbital stability, which is closely related to *local* variational information.

**3.1. The Minimization Problems.** Recall the basic conserved functionals for (1.1) on  $H_{\text{loc}}^1 \cap P_T$ :

$$\begin{aligned}\mathcal{M}(u) &= \frac{1}{2} \int_0^T |u|^2 dx, & \mathcal{P}(u) &= \frac{1}{2} \mathcal{Im} \int_0^T u \bar{u}_x dx, \\ \mathcal{E}(u) &= \frac{1}{2} \int_0^T |u_x|^2 dx - \frac{b}{4} \int_0^T |u|^4 dx.\end{aligned}$$

In this section, we consider  $L^2(0, T; \mathbb{C})$  as a *real* Hilbert space with scalar product  $\text{Re} \int_0^T f \bar{g} dx$ . This way, the functionals  $\mathcal{E}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  are  $C^1$  functionals. This also ensures that the Lagrange multipliers are real. Note that we see  $L^2(0, T; \mathbb{C})$  as a real Hilbert space only in the current section and in all the other sections it will be seen as a complex Hilbert space with the scalar product defined in (2.3).

Fix parameters  $T > 0$ ,  $a, b \in \mathbb{R}$ ,  $b \neq 0$ . Since the Jacobi elliptic functions (indeed any standing wave profiles) are solutions of (2.6), they are critical points of the *action* functional  $\mathcal{S}_a$  defined by

$$\mathcal{S}_a(u) = \mathcal{E}(u) - a\mathcal{M}(u),$$

where the values of  $a$  and  $b$  are given in (2.7)-(2.9) and the fundamental periods are  $T = 2K$  for dn,  $T = 4K$  for sn, cn. Given  $m > 0$ , the basic variational problem is to minimize the energy with fixed mass:

$$\min \{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H_{\text{loc}}^1 \cap P_T \}, \quad (3.1)$$

whose Euler-Lagrange equation

$$u'' + b|u|^2 u + au = 0, \quad (3.2)$$

with  $a \in \mathbb{R}$  arising as Lagrange multiplier, is indeed of the form (2.6). Since the momentum is also conserved for (1.1), it is natural to consider the problem with a further momentum constraint:

$$\min \{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, \mathcal{P}(u) = 0, u \in H_{\text{loc}}^1 \cap P_T \}. \quad (3.3)$$

*Remark 3.1.* Note that if a minimizer  $u$  of (3.1) is such that  $P(u) = 0$ , then it is real-valued (up to multiplication by a complex number of modulus 1). Indeed, it verifies (3.2) for some  $a \in \mathbb{R}$ . It is well known (see e.g. [13]) that the momentum density  $\mathcal{Im}(u_x \bar{u})$  is therefore constant in  $x$ , and so it is identically 0 if  $P(u) = 0$ . For  $u(x) \neq 0$  we can write  $u$  as  $u = \rho e^{i\theta}$ , and express the momentum density as  $\mathcal{Im}(u_x \bar{u}) = \theta_x \rho^2$ . Thus  $\mathcal{Im}(u_x \bar{u}) = 0$  implies  $\theta_x = 0$  and thus  $\theta(x)$  is constant as long as  $u(x) \neq 0$ . If  $u(x_0) = 0$  and  $e^{\theta(x_0^-)} \neq e^{\theta(x_0^+)}$ , we must have  $u_x(x_0) = 0$ , and hence  $u \equiv 0$  by uniqueness of the ODE.

Since (1.1) preserves the subspaces in the decomposition (2.1), it is also natural to consider variational problems restricted to anti-symmetric functions,

$$\min \{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H_{\text{loc}}^1 \cap A_{T/2} \}, \quad (3.4)$$

$$\min \{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, \mathcal{P}(u) = 0, u \in H_{\text{loc}}^1 \cap A_{T/2} \}, \quad (3.5)$$

and in light of the decomposition (2.2), further restrictions to even or odd functions may also be considered.

In general, the difficulty does not lie in proving the existence of a minimizer, but rather in identifying this minimizer with an elliptic function, since we are minimizing among *complex valued* functions, and moreover restrictions to symmetry

subspaces prevent us from using classical variational methods like symmetric rearrangements.

We will first consider the minimization problems (3.1) and (3.3) for periodic functions in  $P_T$ . Then we will consider the minimization problems (3.4) and (3.5) for half-anti-periodic functions in  $A_{T/2}$ . In both parts, we will treat separately the focusing ( $b > 0$ ) and defocusing ( $b < 0$ ) nonlinearities. For each case, we will show the existence of a unique (up to phase shift and translation) minimizer, and we will identify it with either a plane wave or a Jacobi elliptic function.

### 3.2. Minimization Among Periodic Functions.

#### 3.2.1. The Focusing Case in $P_T$ .

**Proposition 3.2.** *Assume  $b > 0$ . The minimization problems (3.1) and (3.3) satisfy the following properties.*

- (i) *For all  $m > 0$ , (3.1) and (3.3) share the same minimizers. The minimal energy is finite and negative.*
- (ii) *For all  $0 < m \leq \frac{\pi^2}{bT}$  there exists a unique (up to phase shift) minimizer of (3.1). It is the constant function  $u_{\min} \equiv \sqrt{\frac{2m}{T}}$ .*
- (iii) *For all  $\frac{\pi^2}{bT} < m < \infty$  there exists a unique (up to translations and phase shift) minimizer of (3.1). It is the rescaled function  $\text{dn}_{\alpha,\beta,k} = \frac{1}{\alpha} \text{dn}\left(\frac{\cdot}{\beta}, k\right)$  where the parameters  $\alpha$ ,  $\beta$  and  $k$  are uniquely determined. Its fundamental period is  $T$ . The map from  $m \in (\frac{\pi^2}{bT}, \infty)$  to  $k \in (0, 1)$  is one-to-one, onto and increasing.*
- (iv) *In particular, given  $k \in (0, 1)$ ,  $\text{dn} = \text{dn}(\cdot, k)$ , if  $b = 2$ ,  $T = 2K(k)$ , and  $m = \mathcal{M}(\text{dn}) = E(k)$ , then the unique (up to translations and phase shift) minimizer of (3.1) is  $\text{dn}$ .*

*Proof.* Without loss of generality, we can restrict the minimization to real-valued non-negative functions. Indeed, if  $u \in H_{\text{loc}}^1 \cap P_T$ , then  $|u| \in H_{\text{loc}}^1 \cap P_T$  and we have

$$\|\partial_x |u|\|_{L^2} \leq \|\partial_x u\|_{L^2}.$$

This readily implies that (3.1) and (3.3) share the same minimizers. Let us prove that

$$-\infty < \min \{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H_{\text{loc}}^1 \cap P_T \} < 0. \quad (3.6)$$

The last inequality in (3.6) is obtained using the constant function  $\varphi_{m,0} \equiv \sqrt{\frac{2m}{T}}$  as a test function:

$$\mathcal{E}(\varphi_{m,0}) < 0, \quad \mathcal{M}(\varphi_{m,0}) = m.$$

To prove the first inequality in (3.6), we observe that by Gagliardo-Nirenberg inequality we have

$$\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2} + \|u\|_{L^2}^4.$$

Consequently, for  $u \in H_{\text{loc}}^1 \cap P_T$  such that  $\mathcal{M}(u) = m$ , we have

$$\mathcal{E}(u) \gtrsim \|u_x\|_{L^2} \left( \|u_x\|_{L^2} - m^{3/2} \right) - m^2,$$

and  $\mathcal{E}$  has to be bounded from below. The above shows (i).

Consider now a minimizing sequence  $(u_n) \subset H_{\text{loc}}^1 \cap P_T$  for (3.1). It is bounded in  $H_{\text{loc}}^1 \cap P_T$  and therefore, up to a subsequence, it converges weakly in  $H_{\text{loc}}^1 \cap P_T$  and strongly in  $L_{\text{loc}}^2 \cap P_T$  and  $L_{\text{loc}}^4 \cap P_T$  towards  $u_\infty \in H_{\text{loc}}^1 \cap P_T$ . Therefore

$\mathcal{E}(u_\infty) \leq \mathcal{E}(u_n)$  and  $\mathcal{M}(u_\infty) = m$ . This implies that  $\|\partial_x u_\infty\|_{L^2} = \lim_{n \rightarrow \infty} \|\partial_x u_n\|_{L^2}$  and therefore the convergence from  $u_n$  to  $u_\infty$  is also strong in  $H_{\text{loc}}^1 \cap P_T$ . Since  $u_\infty$  is a minimizer of (3.1), there exists a Lagrange multiplier  $a \in \mathbb{R}$  such that

$$-\mathcal{E}'(u_\infty) + a\mathcal{M}'(u_\infty) = 0,$$

that is

$$\partial_{xx} u_\infty + bu_\infty^3 + au_\infty = 0.$$

Multiplying by  $u_\infty$  and integrating, we find that

$$a = \frac{\|\partial_x u_\infty\|_{L^2}^2 - b\|u_\infty\|_{L^4}^4}{\|u_\infty\|_{L^2}^2}.$$

Note that

$$\|\partial_x u_\infty\|_{L^2}^2 - b\|u_\infty\|_{L^4}^4 = 2\mathcal{E}(u_\infty) - \frac{b}{2}\|u_\infty\|_{L^4}^4 < 0,$$

therefore

$$a < 0.$$

We already have  $u_\infty \in \mathbb{R}$ , and we may assume  $\max u = u(0)$  by translation. By Lemma 2.1 (a), either  $u_\infty$  is constant or there exist  $\alpha, \beta \in (0, \infty)$  and  $k \in (0, 1)$  such that  $\beta = \alpha\sqrt{2/b}$  and

$$u_\infty(x) = \text{dn}_{\alpha, \beta, k}(x) = \frac{1}{\alpha} \text{dn}\left(\frac{x}{\beta}, k\right).$$

We now show that the minimizer  $u_\infty$  is of the form  $\text{dn}_{\alpha, \beta, k}$  if  $m > \frac{\pi^2}{bT}$ . Indeed, assuming by contradiction that  $u_\infty$  is a constant, we necessarily have  $u_\infty \equiv \sqrt{\frac{2m}{T}}$ . The Lagrange multiplier can also be computed and we find  $a = -bu_\infty^2 = -\frac{2bm}{T}$ . Since  $u_\infty$  is supposed to be a constrained minimizer for (3.1), the operator

$$-\partial_{xx} - a - 3bu_\infty^2 = -\partial_{xx} - \frac{4bm}{T}$$

must have Morse index at most 1, i.e. at most 1 negative eigenvalue. The eigenvalues are given for  $n \in \mathbb{Z}$  by the formula

$$\left(\frac{2\pi n}{T}\right)^2 - \frac{4bm}{T}.$$

Obviously  $n = 0$  gives a negative eigenvalue. For  $n = 1$ , the eigenvalue is non-negative if and only if

$$m \leq \frac{\pi^2}{bT},$$

which gives the contradiction. Hence when  $m > \frac{\pi^2}{bT}$  the minimizer  $u_\infty$  must be of the form  $\text{dn}_{\alpha, \beta, k}$ .

There is a positive integer  $n$  so that the fundamental period of  $u_\infty = \text{dn}_{\alpha, \beta, k}$  is  $2K(k)\beta = Tn^{-1}$ . As already mentioned, since  $u_\infty$  is a minimizer for (3.1), the operator

$$-\partial_{xx} - a - 3bu_\infty^2$$

can have at most one negative eigenvalue. The function  $\partial_x u_\infty$  is in its kernel and has  $2n$  zeros. By Sturm-Liouville theory (see e.g. [10, 29]) we have at least  $2n - 1$  eigenvalues below 0. Hence  $n = 1$  and  $2K(k)\beta = T$ .

Using  $2\alpha^2 = b\beta^2$  (see Lemma 2.1), the mass verifies,

$$m = \frac{1}{2} \int_0^T |\operatorname{dn}_{\alpha,\beta,k}(x)|^2 dx = \frac{\beta}{\alpha^2} \frac{1}{2} \int_0^{2K(k)} |\operatorname{dn}(y,k)|^2 dy = \frac{2}{b\beta} E(k)$$

where  $E(k)$  is given in Section 2.3. Using  $2K(k)\beta = T$ ,

$$m = \frac{4}{bT} E(k)K(k). \quad (3.7)$$

Note

$$\frac{\partial}{\partial k} EK(k) = \frac{E(k)^2 - (1-k^2)K(k)^2}{(1-k^2)k} > 0,$$

where the positivity of the numerator is because it vanishes at  $k = 0$  and

$$\frac{\partial}{\partial k} (E^2 - (1-k^2)K^2) = \frac{2}{k} (E-K)^2, \quad (0 < k < 1).$$

Thus  $EK(k)$  varies from  $\frac{\pi^2}{4}$  to  $\infty$  when  $k$  varies from 0 to 1. Thus (3.7) defines  $m$  as a strictly increasing function of  $k \in (0, 1)$  with range  $(\frac{\pi^2}{bT}, \infty)$  and hence has an inverse function. For fixed  $b, m, T$ , the value  $k \in (0, 1)$  is uniquely determined by (3.7). We also have  $\beta = \frac{T}{2K(k)}$  and  $\alpha = \beta\sqrt{b/2}$ . The above shows (iii).

The above calculation also shows that  $m > \frac{\pi^2}{bT}$  if  $u_\infty = \operatorname{dn}_{\alpha,\beta,k}$ . Thus  $u_\infty$  must be a constant when  $0 < m \leq \frac{\pi^2}{bT}$ . This shows (ii).

In the case we are given  $k \in (0, 1)$ ,  $T = 2K(k)$ ,  $b = 2$  and  $m = \mathcal{M}(\operatorname{dn}) = E(k)$ , we want to show that  $u_\infty(x) = \operatorname{dn}(x, k)$ . In this case  $m > \frac{\pi^2}{bT}$  since  $EK > \frac{\pi^2}{4}$ . Thus, by Lemma 2.1 (a),  $u_\infty = \operatorname{dn}_{\alpha,\beta,s}$  for some  $\alpha, \beta > 0$  and  $s \in (0, 1)$ , up to translation and phase. By the same Sturm-Liouville theory argument, the fundamental period of  $u_\infty$  is  $T = 2K(s)\beta$ . The same calculation leading to (3.7) shows

$$m = \frac{4}{bT} E(s)K(s).$$

Thus  $E(k)K(k) = E(s)K(s)$ . Using the monotonicity of  $EK(k)$  in  $k$ , we have  $k = s$ . Thus  $\alpha = \beta = 1$  and  $u_\infty(x) = \operatorname{dn}(x, k)$ . This gives (iv) and finishes the proof.  $\square$

### 3.2.2. The Defocusing Case in $P_T$ .

**Proposition 3.3.** *Assume  $b < 0$ . For all  $0 < m < \infty$ , the constrained minimization problems (3.1) and (3.3) have the same unique (up to phase shift) minimizers, which is the constant function  $u_{\min} \equiv \sqrt{\frac{2m}{T}}$ .*

*Proof.* This is a simple consequence of the fact that functions with constant modulus are the optimizers of the injection  $L^4(0, T) \hookrightarrow L^2(0, T)$ . More precisely, for every  $f \in L^4(0, T)$  we have by Hölder's inequality,

$$\|f\|_{L^2} \leq T^{1/4} \|f\|_{L^4},$$

with equality if and only if  $|f|$  is constant. Let  $\varphi_{m,0}$  be the constant function  $\varphi_{m,0} \equiv \sqrt{\frac{2m}{T}}$ . For any  $v \in H_{\text{loc}}^1 \cap P_T$  such that  $\mathcal{M}(v) = m$  and  $v \neq e^{i\theta} \varphi_{m,0}$  ( $\theta \in \mathbb{R}$ ) we have

$$\begin{aligned} 0 &= \|\partial_x \varphi_{m,0}\|_{L^2}^2 < \|\partial_x v\|_{L^2}^2, \\ \|\varphi_{m,0}\|_{L^4}^4 &= 4T^{-1} \mathcal{M}^2(\varphi_{m,0}) = 4T^{-1} \mathcal{M}^2(v) \leq \|v\|_{L^4}^4. \end{aligned}$$

As a consequence,  $\mathcal{E}(\varphi_{m,0}) < \mathcal{E}(v)$  and this proves the proposition.  $\square$

### 3.3. Minimization Among Half-Anti-Periodic Functions.

#### 3.3.1. The Focusing Case in $A_{T/2}$ .

**Proposition 3.4.** *Assume  $b > 0$ . For all  $m > 0$ , the minimization problems (3.4) and (3.5) in  $A_{T/2}$  satisfy the following properties.*

- (i) *The minimizers for (3.4) and (3.5) are the same.*
- (ii) *There exists a unique (up to translations and phase shift) minimizer of (3.4). It is the rescaled function  $\text{cn}_{\alpha,\beta,k} = \frac{1}{\alpha} \text{cn}\left(\frac{\cdot}{\beta}, k\right)$  where the parameters  $\alpha$ ,  $\beta$  and  $k$  are uniquely determined. Its fundamental period is  $T$ . The map from  $m \in (0, \infty)$  to  $k \in (0, 1)$  is one-to-one, onto and increasing.*
- (iii) *In particular, given  $k \in (0, 1)$ ,  $\text{cn} = \text{cn}(\cdot, k)$ , if  $b = 2k^2$ ,  $T = 4K(k)$ , and  $m = \mathcal{M}(\text{cn}) = 2(E - (1 - k^2)K)/k^2$ , then the unique (up to translations and phase shift) minimizer of (3.4) is  $\text{cn}$ .*

Before proving Proposition 3.4, we make the following crucial observation.

**Lemma 3.5.** *Let  $v \in H_{\text{loc}}^1 \cap A_{T/2}$ . Then there exists  $\tilde{v} \in H_{\text{loc}}^1 \cap A_{T/2}$  such that*

$$\tilde{v}(x) \in \mathbb{R}, \quad \|\tilde{v}\|_{L^2} = \|v\|_{L^2}, \quad \|\partial_x \tilde{v}\|_{L^2} = \|\partial_x v\|_{L^2}, \quad \|\tilde{v}\|_{L^4} \geq \|v\|_{L^4}.$$

*Proof of Lemma 3.5.* The proof relies on a combinatorial argument. Since  $v \in H_{\text{loc}}^1 \cap A_{T/2}$ , its Fourier series expansion contains only terms indexed by *odd* integers:

$$v(x) = \sum_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} v_j e^{ij \frac{2\pi}{T} x}.$$

We define  $\tilde{v}$  by its Fourier series expansion

$$\tilde{v}(x) = \sum_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} \tilde{v}_j e^{ij \frac{2\pi}{T} x}, \quad \tilde{v}_j := \sqrt{\frac{|v_j|^2 + |v_{-j}|^2}{2}}.$$

It is clear that  $\tilde{v}(x) \in \mathbb{R}$  for all  $x \in \mathbb{R}$ , and by Plancherel formula,

$$\|\tilde{v}\|_{L^2} = \|v\|_{L^2}, \quad \|\partial_x \tilde{v}\|_{L^2} = \|\partial_x v\|_{L^2},$$

so all we have to prove is that  $\|\tilde{v}\|_{L^4} \geq \|v\|_{L^4}$ . We have

$$|v(x)|^2 = \sum_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} |v_j|^2 + \sum_{\substack{n \in 2\mathbb{N} \\ n \geq 2}} w_n e^{in \frac{2\pi}{T} x} + \bar{w}_n e^{-in \frac{2\pi}{T} x},$$

where we have defined

$$w_n = \sum_{\substack{j > k, j+k=n \\ j, k \text{ odd}}} v_j \bar{v}_{-k} + v_k \bar{v}_{-j}.$$

Using the fact that for  $n \in \mathbb{N}$ ,  $n \neq 0$ , the term  $e^{in \frac{2\pi}{T} x}$  integrates to 0 due to periodicity,

$$\int_0^T e^{in \frac{2\pi}{T} x} dx = 0,$$

we compute

$$\frac{1}{T} \int_0^T |v|^4 dx = \left( \sum_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} |v_j|^2 \right)^2 + 2 \sum_{\substack{n \in 2\mathbb{N} \\ n \geq 2}} |w_n|^2.$$

The first part is just

$$\left( \sum_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} |v_j|^2 \right)^2 = \frac{1}{T^2} \|v\|_{L^2}^4 = \frac{1}{T^2} \|\tilde{v}\|_{L^2}^4.$$

For the second part, we observe that

$$w_n = \sum_{\substack{j > k, j+k=n \\ j, k \text{ odd}}} \begin{pmatrix} v_j \\ \bar{v}_{-j} \end{pmatrix} \cdot \begin{pmatrix} v_{-k} \\ \bar{v}_k \end{pmatrix}, \quad (3.8)$$

where the  $\cdot$  denotes the complex vector scalar product. Therefore,

$$\begin{aligned} |w_n| &\leq \sum_{\substack{j > k, j+k=n \\ j, k \text{ odd}}} \left| \begin{pmatrix} v_j \\ \bar{v}_{-j} \end{pmatrix} \right| \left| \begin{pmatrix} v_{-k} \\ \bar{v}_k \end{pmatrix} \right| = \sum_{\substack{j > k, j+k=n \\ j, k \text{ odd}}} \sqrt{2\tilde{v}_j^2} \sqrt{2\tilde{v}_k^2} \\ &= 2 \sum_{\substack{j > k, j+k=n \\ j, k \text{ odd}}} \tilde{v}_j \tilde{v}_k = \tilde{w}_n, \end{aligned}$$

where by  $\tilde{w}_n$  we denote the quantity defined similarly as in (3.8) for  $(\tilde{v}_j)$ . As a consequence,

$$\|v\|_{L^4} \leq \|\tilde{v}\|_{L^4}$$

and this finishes the proof of Lemma 3.5.  $\square$

*Proof of Proposition 3.4.* All functions are considered in  $A_{T/2}$ . Consider a minimizing sequence  $(u_n)$  for (3.5). By Lemma 3.5, the minimizing sequence can be chosen such that  $u_n(x) \in \mathbb{R}$  for all  $x \in \mathbb{R}$  and this readily implies the equivalence between (3.5) and (3.4), which is (i).

Using the same arguments as in the proof of Proposition 3.2, we infer that the minimizing sequence converges strongly in  $H_{\text{loc}}^1 \cap A_{T/2}$  to  $u_\infty \in H_{\text{loc}}^1 \cap A_{T/2}$  verifying for some  $a \in \mathbb{R}$  the Euler-Lagrange equation

$$\partial_{xx} u_\infty + b u_\infty^3 + a u_\infty = 0.$$

Then, since  $u_\infty$  is real and in  $A_{T/2}$ , we may assume  $\max u = u(0) > 0$  and, by Lemma 2.1 (b), there exists a set of parameters  $\alpha, \beta \in (0, \infty)$ ,  $k \in (0, 1)$  such that

$$u_\infty(x) = \frac{1}{\alpha} \text{cn} \left( \frac{x}{\beta}, k \right),$$

and the parameters  $\alpha, \beta, k$  are determined by  $T, a, b$  and  $\max u$ , with  $2k^2\alpha^2 = b\beta^2$ .

There exists an odd, positive integer  $n$  so that the fundamental period of  $u_\infty$  is  $4K(k)\beta = T/n$ . Since  $u_\infty$  is a minimizer for (3.4), the operator

$$-\partial_{xx} - a - 3b u_\infty^2$$

can have at most one negative eigenvalue in  $L_{\text{loc}}^2 \cap A_{T/2}$ . The function  $\partial_x u_\infty$  is in its kernel and has  $2n$  zeros in  $[0, T)$ . By Sturm-Liouville theory, there are at least  $n - 1$  eigenvalues (with eigenfunctions in  $A_{T/2}$ ) below 0. Hence, since  $n$  is odd,  $n = 1$  and  $4K(k)\beta = T$ .



The mass verifies, using  $2k^2\alpha^2 = b\beta^2$  and (2.11),

$$m = \frac{1}{2} \int_0^T |\text{cn}_{\alpha,\beta,k}(x)|^2 dx = \frac{\beta}{\alpha^2} \frac{1}{2} \int_0^{4K(k)} |\text{cn}(y,k)|^2 dy = \frac{4}{\beta b} (E(k) - (1-k^2)K(k)).$$

Using  $4K(k)\beta = T$ ,

$$m = M(k) := \frac{16}{bT} K(k)(E(k) - (1-k^2)K(k)). \quad (3.9)$$

Note all factors of  $M(k)$  are positive,  $\frac{\partial}{\partial k} K(k) > 0$  and

$$\frac{\partial}{\partial k} (E - (1-k^2)K) = \frac{E-K}{k} - \frac{E - (1-k^2)K}{k} + 2kK = kK > 0.$$

Thus (3.9) defines  $m$  as a strictly increasing function of  $k \in (0, 1)$  with range  $(0, \infty)$  and hence has an inverse function. For fixed  $T, b, m$ , the value  $k \in (0, 1)$  is uniquely determined by (3.9). We also have  $\beta = \frac{T}{4K(k)}$  and  $\alpha^2 = \frac{b\beta^2}{2k^2}$ . The above shows (ii).

In the case we are given  $k \in (0, 1)$ ,  $T = 4K(k)$ ,  $b = 2k^2$  and  $m = \mathcal{M}(\text{cn}(\cdot, k))$ , we want to show that  $u_\infty(x) = \text{cn}(x, k)$ . In this case, by Lemma 2.1 (b),  $u_\infty = \text{cn}_{\alpha,\beta,s}$  for some  $\alpha, \beta > 0$  and  $s \in (0, 1)$ , up to translation and phase. By the same Sturm-Liouville theory argument, the fundamental period of  $u_\infty$  is  $T = 4K(s)\beta$ . The same calculation leading to (3.9) shows

$$m = M(s).$$

Thus  $M(s) = M(k)$ . By the monotonicity of  $M(k)$  in  $k$ , we have  $k = s$ . Thus  $\alpha = \beta = 1$  and  $u_\infty(x) = \text{cn}(x, k)$ . This shows (iii) and concludes the proof.  $\square$

### 3.3.2. The Defocusing Case in $A_{T/2}$ .

**Proposition 3.6.** *Assume  $b < 0$ . There exists a unique (up to phase shift and complex conjugate) minimizer for (3.4). It is the plane wave  $u_{\min} \equiv \sqrt{\frac{2m}{T}} e^{\frac{2i\pi x}{T}}$ .*

*Proof.* Denote the supposed minimizer by  $w(x) = \sqrt{\frac{2m}{T}} e^{\pm \frac{2i\pi x}{T}}$ . Let  $v \in H_{\text{loc}}^1 \cap A_{2K}$  such that  $\mathcal{M}(v) = m$  and  $v \not\equiv e^{i\theta} w$  ( $\theta \in \mathbb{R}$ ). As in the proof of Proposition 3.3, we have

$$\|w\|_{L^4}^4 = 4T^{-1} \mathcal{M}^2(w) = 4T^{-1} \mathcal{M}^2(v) \leq \|v\|_{L^4}^4.$$

Since  $v \in A_{2K}$ ,  $v$  must have 0 mean value. Recall that in that case  $v$  verifies the Poincaré-Wirtinger inequality

$$\|v\|_{L^2} \leq \frac{T}{2\pi} \|v'\|_{L^2},$$

and that the optimizers of the Poincaré-Wirtinger inequality are of the form  $Ce^{\pm \frac{2i\pi}{T}x}$ ,  $C \in \mathbb{C}$ . This implies that

$$\|\partial_x w\|_{L^2}^2 = \frac{8\pi^2}{T^2} \mathcal{M}(w) = \frac{8\pi^2}{T^2} \mathcal{M}(v) < \|\partial_x v\|_{L^2}^2.$$

As a consequence,  $\mathcal{E}(w) < \mathcal{E}(v)$  and this proves the lemma.  $\square$

As far as (3.5) is concerned, we make the following conjecture

**Conjecture 3.7.** *Assume  $b < 0$ . The unique (up to translations and phase shift) minimizer of (3.5) is the rescaled function  $\text{sn}_{\alpha,\beta,k} = \frac{1}{\alpha} \text{sn}\left(\frac{\cdot}{\beta}, k\right)$  where the parameters  $\alpha$ ,  $\beta$  and  $k$  are uniquely determined.*

*In particular, given  $k \in (0, 1)$ ,  $\text{sn} = \text{sn}(\cdot, k)$ , if  $b = -2k^2$ ,  $T = 4K(k)$ , and  $m = \mathcal{M}(\text{sn})$ , then the unique (up to translations and to phase shift) minimizer of (3.5) is  $\text{sn}$ .*

This conjecture is supported by numerical evidence, see Observation 7.1. The main difficulty in proving the conjecture is to show that the minimizer is real up to a phase.

3.3.3. *The Defocusing Case in  $A_{T/2}^-$ .* In light of our uncertainty about whether  $\text{sn}$  solves (3.5), let us settle for the simple observation that it is the energy minimizer among odd, half-anti-periodic functions:

**Proposition 3.8.** *Assume  $b < 0$ . The unique (up to phase shift) minimizer of the problem*

$$\min \left\{ \mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H_{\text{loc}}^1 \cap A_{T/2}^- \right\}, \quad (3.10)$$

*is the rescaled function  $\text{sn}_{\alpha,\beta,k} = \frac{1}{\alpha} \text{sn}\left(\frac{\cdot}{\beta}, k\right)$  where the parameters  $\alpha$ ,  $\beta$  and  $k$  are uniquely determined. Its fundamental period is  $T$ . The map from  $m \in (0, \infty)$  to  $k \in (0, 1)$  is one-to-one, onto and increasing.*

*In particular, given  $k \in (0, 1)$ ,  $\text{sn} = \text{sn}(\cdot, k)$ , if  $b = -2k^2$ ,  $T = 4K(k)$ , and  $m = \mathcal{M}(\text{sn})$ , then the unique (up to phase shift) minimizer of (3.10) is  $\text{sn}$ .*

*Proof.* If  $u \in A_{T/2}^-$ , then  $0 = u(0) = u(T/2)$ , and since  $u$  is completely determined by its values on  $[0, T/2]$ , we may replace (3.10) by

$$\min \left\{ \int_0^{T/2} \left( |u_x|^2 - \frac{b}{2} |u|^4 \right) dx \mid \int_0^{T/2} |u(x)|^2 dx = m, u \in H_0^1([0, T/2]) \right\},$$

for which the map  $u \mapsto |u|$  is admissible, showing that minimizers are non-negative (up to phase), and in particular real-valued, hence a (rescaled)  $\text{sn}$  function by Lemma 2.2. The remaining statements follow as in the proof of Proposition 3.4. In particular, the mass verifies, using  $2k^2\alpha^2 = |b|\beta^2$ , (2.11), and  $4K(k)\beta = T$ ,

$$\begin{aligned} m &= \frac{1}{2} \int_0^T |\text{sn}_{\alpha,\beta,k}(x)|^2 dx = \frac{\beta}{\alpha^2} \frac{1}{2} \int_0^{4K(k)} |\text{sn}(y, k)|^2 dy \\ &= \frac{4}{\beta|b|} (K(k) - E(k)) = \frac{16}{|b|T} K(k)(K(k) - E(k)), \end{aligned}$$

which is a strictly increasing function of  $k \in (0, 1)$  with range  $(0, \infty)$  and hence has an inverse function.  $\square$

**3.4. Orbital Stability.** Recall that we say that a standing wave  $\psi(t, x) = e^{-iat}u(x)$  is orbitally stable for the flow of (1.1) in the function space  $X$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds: if  $\psi_0 \in X$  verifies

$$\|\psi_0 - u\|_X \leq \delta$$

then the solution  $\psi$  of (1.1) with initial data  $\psi(0, x) = \psi_0$  verifies for all  $t \in \mathbb{R}$  the estimate

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}} \|\psi(t, \cdot) - e^{i\theta} u(\cdot - y)\|_X < \varepsilon.$$

As an immediate corollary of the variational characterizations above, we have the following orbital stability statements:

**Corollary 3.9.** *The standing wave  $\psi(t, x) = e^{-iat}u(x)$  is a solution of (1.1), and is orbitally stable in  $X$  in the following cases. For Jacobi elliptic functions: for any  $k \in (0, 1)$ ,*

$$\begin{aligned} a = 1 + k^2, & & b = -2k^2, & & u = \operatorname{sn}(\cdot, k), & & X = H_{\text{loc}}^1 \cap A_{2K}^-; \\ a = 1 - 2k^2, & & b = 2k^2, & & u = \operatorname{cn}(\cdot, k), & & X = H_{\text{loc}}^1 \cap A_{2K}; \\ a = -(2 - k^2), & & b = 2, & & u = \operatorname{dn}(\cdot, k), & & X = H_{\text{loc}}^1 \cap P_{2K}. \end{aligned}$$

For constants and plane waves: ( $b \neq 0$ )

$$\begin{aligned} a = -\frac{2bm}{T}, & & -\infty < b \leq \frac{\pi^2}{Tm}, & & u = \sqrt{\frac{2m}{T}}, & & X = H_{\text{loc}}^1 \cap P_T; \\ a = \frac{4\pi^2}{T^2} - \frac{2bm}{T}, & & b < 0, & & u = e^{\pm \frac{2i\pi x}{T}} \sqrt{\frac{2m}{T}}, & & X = H_{\text{loc}}^1 \cap A_{T/2}. \end{aligned}$$

The proof of this corollary uses the variational characterizations from Propositions 3.2, 3.3, 3.4, 3.6, and 3.8. Note that for all the minimization problems considered we have the compactness of minimizing sequences. The proof follows the standard line introduced by Cazenave and Lions [8], we omit the details here.

*Remark 3.10.* The orbital stability of  $\operatorname{sn}$  [13] in  $H_{\text{loc}}^1 \cap A_{T/2}$  was proved using the Grillakis-Shatah-Strauss [18, 19] approach, which amounts to identifying the periodic wave as a *local* constrained minimizer in this subspace. So the above may be considered an alternate proof, using *global* variational information. In the case of  $\operatorname{sn}$ , without Conjecture 3.7, some additional spectral information in the subspace  $A_{T/2}^+$  is needed to obtain orbital stability in  $H_{\text{loc}}^1 \cap A_{T/2}$  (rather than just  $H_{\text{loc}}^1 \cap A_{T/2}^-$ ) – see Corollary 4.7 in the next section for this.

Orbital stability of  $\operatorname{cn}$  was obtained in [13] only for small amplitude  $\operatorname{cn}$ . We extend this result to all possible values of  $k \in (0, 1)$ .

*Remark 3.11.* Using the complete integrability of (1.1), Bottman, Deconinck and Nivala [5], and Gallay and Pelinovsky [15] showed that  $\operatorname{sn}$  is in fact a minimizer of a higher-order functional in  $H_{\text{loc}}^2 \cap P_{nT}$  for any  $n \in \mathbb{N}$ , and thus showed it is orbitally stable in these spaces.

#### 4. SPECTRAL STABILITY

Given a standing wave  $\psi(t, x) = e^{-iat}u(x)$  solution of (1.1), we consider the linearization of (1.1) around this solution: if  $\psi(t, x) = e^{-iat}(u(x) + h)$ , then  $h$  verifies

$$i\partial_t h - Lh + N(h) = 0,$$

where  $L$  denotes the linear part and  $N$  the nonlinear part. Assuming  $u$  is real-valued, we separate  $h$  into real and imaginary parts to get the equation

$$\partial_t \begin{pmatrix} \operatorname{Re}(h) \\ \operatorname{Im}(h) \end{pmatrix} = J\mathcal{L} \begin{pmatrix} \operatorname{Re}(h) \\ \operatorname{Im}(h) \end{pmatrix} + \begin{pmatrix} -\operatorname{Im}(N(h)) \\ \operatorname{Re}(N(h)) \end{pmatrix},$$

where

$$\mathcal{L} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{aligned} L_+ &= -\partial_{xx} - a - 3b u^2, \\ L_- &= -\partial_{xx} - a - b u^2. \end{aligned}$$

We call

$$J\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \quad (4.1)$$

the *linearized operator* of (1.1) about the standing wave  $e^{-iat}u(x)$ .

Now suppose  $u \in H_{\text{loc}}^1 \cap P_T$  is a (period  $T$ ) periodic wave, and consider its linearized operator  $J\mathcal{L}$  as an operator on the Hilbert space  $(P_T)^2$ , with domain  $(H_{\text{loc}}^2 \cap P_T)^2$ . The main structural properties of  $J\mathcal{L}$  are:

- since  $L_{\pm}$  are self-adjoint operators on  $P_T$ ,  $\mathcal{L}$  is self-adjoint on  $(P_T)^2$ , while  $J$  is skew-adjoint and unitary

$$\mathcal{L}^* = \mathcal{L}, \quad J^* = -J = J^{-1}, \quad (4.2)$$

- $J\mathcal{L}$  commutes with complex conjugation,

$$\overline{J\mathcal{L}f} = J\mathcal{L}\bar{f}, \quad (4.3)$$

- $J\mathcal{L}$  is antisymmetric under conjugation by the matrix

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(which corresponds to the operation of complex conjugation *before* complexification),

$$J\mathcal{L}C = -CJ\mathcal{L}. \quad (4.4)$$

At the *linear* level, the stability of the periodic wave is determined by the location of the spectrum  $\sigma(J\mathcal{L})$ , which in this periodic setting consists of isolated eigenvalues of finite multiplicity [29]. We first make the standard observation that as a result of (4.3) and (4.4), the spectrum of  $J\mathcal{L}$  is invariant under reflection about the real and imaginary axes:

$$\lambda \in \sigma(J\mathcal{L}) \implies \pm\lambda, \pm\bar{\lambda} \in \sigma(J\mathcal{L}).$$

Indeed, if  $J\mathcal{L}f = \lambda f$ , then

$$(4.3) \implies J\mathcal{L}\bar{f} = \bar{\lambda}\bar{f}, \quad (4.4) \implies J\mathcal{L}Cf = -\lambda Cf,$$

$$(4.3) \text{ and } (4.4) \implies J\mathcal{L}C\bar{f} = -\bar{\lambda}C\bar{f}.$$

We are interested in whether the entire spectrum of  $J\mathcal{L}$  lies on the imaginary axis, denoted  $\sigma(J\mathcal{L}|_{P_T}) \subset i\mathbb{R}$ , in which case we say the periodic wave  $u$  is *spectrally stable in  $P_T$* . Moreover, if  $S \subset P_T$  is an invariant subspace – more precisely,  $J\mathcal{L} : (H_{\text{loc}}^2 \cap S)^2 \rightarrow (S)^2$  – then we will say that the periodic wave  $u$  is *spectrally stable in  $S$*  if the entire  $(S)^2$  spectrum of  $J\mathcal{L}$  lies on the imaginary axis, denoted  $\sigma(J\mathcal{L}|_S) \subset i\mathbb{R}$ . In particular, for  $k \in (0, 1)$  and  $K = K(k)$ , since  $\text{sn}^2, \text{cn}^2, \text{dn}^2 \in P_{2K}^+$ , the corresponding linearized operators respect the decomposition (2.2), and we may consider  $\sigma(J\mathcal{L}|_S)$  for  $S = P_{2K}^{\pm}, A_{2K}^{\pm} \subset P_{4K}$ , with

$$\begin{aligned} \sigma(J\mathcal{L}|_{P_{4K}}) &= \sigma(J\mathcal{L}|_{P_{2K}}) \cup \sigma(J\mathcal{L}|_{A_{2K}}) \\ &= \sigma(J\mathcal{L}|_{P_{2K}^+}) \cup \sigma(J\mathcal{L}|_{P_{2K}^-}) \cup \sigma(J\mathcal{L}|_{A_{2K}^+}) \cup \sigma(J\mathcal{L}|_{A_{2K}^-}). \end{aligned} \quad (4.5)$$

Of course, spectral stability (which is purely linear) is a weaker notion than orbital stability (which is nonlinear). Indeed, the latter implies the former – see Proposition 4.10 and the remarks preceding it.

The main result of this section is the following.

**Theorem 4.1.** *Spectral stability in  $P_T$ ,  $T = 4K(k)$ , holds for:*

- $u = \text{sn}$ ,  $k \in (0, 1)$ ,
- $u = \text{cn}$  and  $k \in (0, k_c)$ , where  $k_c$  is the unique  $k \in (0, 1)$  so that  $K(k) = 2E(k)$ ,  $k_c \approx 0.908$ .

*Remark 4.2.* The function  $f(k) = K(k) - 2E(k)$  is strictly increasing in  $k \in (0, 1)$ , (since  $K(k)$  is increasing while  $E(k)$  is decreasing in  $k$ ), with  $f(0) = -\frac{\pi}{2}$  and  $f(1) = \infty$ .

*Remark 4.3.* Using Evans function techniques, it was proved in [21] that  $\sigma(J\mathcal{L}^{\text{cn}}) \subset i\mathbb{R}$  also for  $k \in [k_c, 1)$ . This fact is also supported by numerical evidence (see Section 7).

*Remark 4.4.* In the case of  $\text{sn}$ , the  $H_{\text{loc}}^2 \cap P_{nT}$  orbital stability obtained in [5, 15] (using integrability) immediately implies spectral stability in  $P_{nT}$ , and in particular in  $P_T$ . So our result for  $\text{sn}$  could be considered an alternate, elementary proof, not relying on the integrability.

*Remark 4.5.* The spectral stability of  $\text{dn}$  in  $P_{2K}$  (its own fundamental period) is an immediate consequence of its orbital stability in  $H_{\text{loc}}^1 \cap P_{2K}$ , see Proposition 4.10.

**4.1. Spectra of  $L_+$  and  $L_-$ .** We assume now that we are given  $k \in (0, 1)$  and we describe the spectrum of  $L_+$  and  $L_-$  in  $P_{4K}$  when  $\phi$  is  $\text{cn}$ ,  $\text{dn}$  or  $\text{sn}$ . When  $\phi = \text{sn}$ , we denote  $L_+$  by  $L_+^{\text{sn}}$ , and we use similar notations for  $L_-$  and  $\text{cn}$ ,  $\text{dn}$ . Due to the algebraic relationships between  $\text{cn}$ ,  $\text{dn}$  and  $\text{sn}$ , we have

$$\begin{aligned} L_+^{\text{sn}} &= -\partial_{xx} - (1 + k^2) + 6k^2 \text{sn}^2, \\ L_+^{\text{cn}} &= -\partial_{xx} - (1 - 2k^2) - 6k^2 \text{cn}^2 = L_+^{\text{sn}} - 3k^2, \\ L_+^{\text{dn}} &= -\partial_{xx} + (2 - k^2) - 6 \text{dn}^2 = L_+^{\text{sn}} - 3. \end{aligned}$$

Similarly for  $L_-$ , we obtain

$$\begin{aligned} L_-^{\text{sn}} &= -\partial_{xx} - (1 + k^2) + 2k^2 \text{sn}^2, \\ L_-^{\text{cn}} &= -\partial_{xx} - (1 - 2k^2) - 2k^2 \text{cn}^2 = L_-^{\text{sn}} + k^2, \\ L_-^{\text{dn}} &= -\partial_{xx} + (2 - k^2) - 2 \text{dn}^2 = L_-^{\text{sn}} + 1. \end{aligned}$$

As a consequence,  $L_{\pm}^{\text{sn}}$ ,  $L_{\pm}^{\text{cn}}$ , and  $L_{\pm}^{\text{dn}}$  share the same eigenvectors. Moreover, these operators enter in the framework of Schrödinger operators with periodic potentials and much can be said about their spectrum (see e.g. [10, 29]). Recall in particular that given a Schrödinger operator  $L = -\partial_{xx} + V$  with periodic potential  $V$  of period  $T$ , the eigenvalues  $\lambda_n$  of  $L$  on  $P_T$  satisfy

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots,$$

with corresponding eigenfunctions  $\psi_n$  such that  $\psi_0$  has no zeros,  $\psi_{2m+1}$  and  $\psi_{2m+2}$  have exactly  $2m+2$  zeros in  $[0, T)$  ([10, p. 39]). From the equations satisfied by  $\text{cn}$ ,  $\text{dn}$ ,  $\text{sn}$ , we directly infer that

$$L_-^{\text{sn}} \text{dn} = -\text{dn}, \quad L_-^{\text{sn}} \text{cn} = -k^2 \text{cn}, \quad L_-^{\text{sn}} \text{sn} = 0.$$

Taking the derivative with respect to  $x$  of the equations satisfied by  $\text{cn}$ ,  $\text{dn}$ ,  $\text{sn}$ , we obtain

$$L_+^{\text{sn}} \partial_x \text{sn} = 0, \quad L_+^{\text{sn}} \partial_x \text{cn} = 3k^2 \partial_x \text{cn}, \quad L_+^{\text{sn}} \partial_x \text{dn} = 3\partial_x \text{dn}.$$

Looking for eigenfunctions in the form  $\chi = 1 - A \operatorname{sn}^2$  for  $A \in \mathbb{R}$ , we find two other eigenfunctions:

$$L_+^{\operatorname{sn}} \chi_- = e_- \chi_-, \quad L_+^{\operatorname{sn}} \chi_+ = e_+ \chi_+,$$

where

$$\begin{aligned} \chi_{\pm} &= 1 - \left( k^2 + 1 \pm \sqrt{k^4 - k^2 + 1} \right) \operatorname{sn}^2, \\ \pm e_{\pm} &= \pm \left( k^2 + 1 \pm 2\sqrt{k^4 - k^2 + 1} \right) > 0. \end{aligned}$$

In the interval  $[0, 4K)$ ,  $\chi_-$  has no zero,  $\operatorname{sn}_x$  and  $\operatorname{cn}_x$  have two zeros each, while  $\operatorname{dn}_x$  and  $\chi_+$  have 4 zeros each. By Sturm-Liouville theory, they are the first 5 eigenvectors of  $L_+$  for each of  $\operatorname{sn}$ ,  $\operatorname{cn}$ , and  $\operatorname{dn}$ , and all other eigenfunctions have strictly greater eigenvalues. Similarly,  $\operatorname{dn} > 0$  has no zeros, while  $\operatorname{cn}$  and  $\operatorname{sn}$  have two each, so these are the first 3 eigenfunctions of  $L_-$  for each of  $\operatorname{sn}$ ,  $\operatorname{cn}$ , and  $\operatorname{dn}$ , and all other eigenfunctions have strictly greater eigenvalues.

The spectra of  $L_{\pm}^{\operatorname{sn}}$ ,  $L_{\pm}^{\operatorname{cn}}$ , and  $L_{\pm}^{\operatorname{dn}}$  are represented in Figure 4.1, where the eigenfunctions are also classified with respect to the subspaces of decomposition (2.2).

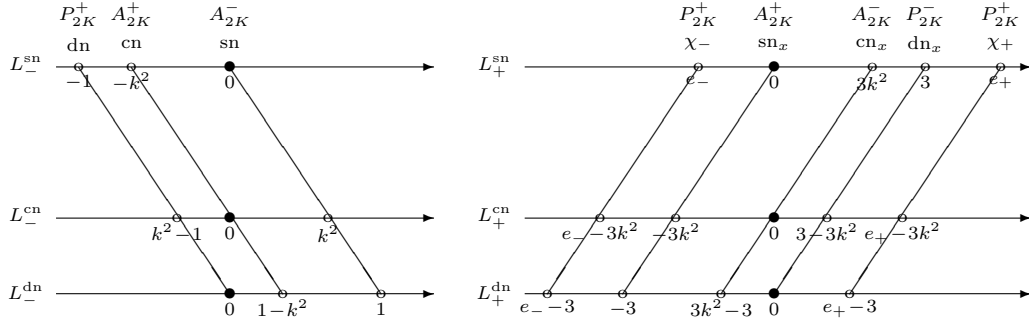


FIGURE 4.1. Eigenvalues for  $L_-$  and  $L_+$  in  $P_{4K}$ .

We may now recover the result of [13] that  $\operatorname{sn}$  is orbitally stable in  $H_{\operatorname{loc}}^1 \cap A_{2K}$ , using the following simple consequences of the spectral information above:

**Lemma 4.6.** *There exists  $\delta > 0$  such that the following coercivity properties hold.*

- (1)  $L_+^{\operatorname{sn}}|_{A_{2K}^-} > \delta$ ,
- (2)  $L_-^{\operatorname{sn}}|_{A_{2K}^- \cap \{\operatorname{sn}\}^\perp} > \delta$ ,
- (3)  $L_+^{\operatorname{sn}}|_{A_{2K}^+ \cap \{(\operatorname{sn})_x\}^\perp} > \delta$ ,
- (4)  $L_-^{\operatorname{sn}}|_{A_{2K}^+ \cap \{(\operatorname{sn})_x\}^\perp} > \delta$ .

*Proof.* The first three are immediate from figure 4.1 (note the first two also follow from the minimization property Proposition 3.8), while we see that in  $A_{2K}^+$ ,  $L_+^{\operatorname{sn}}|_{\{(\operatorname{sn})_x\}^\perp} > e_+$ , so since  $\operatorname{sn}^2(x) \leq 1$ ,

$$L_-^{\operatorname{sn}}|_{\{(\operatorname{sn})_x\}^\perp} = (L_+^{\operatorname{sn}} - 4k^2 \operatorname{sn}^2)|_{\{(\operatorname{sn})_x\}^\perp} > e_+ - 4k^2 > 0$$

where the last inequality is easily verified.  $\square$

**Corollary 4.7.** *For all  $k \in (0, 1)$ , the standing wave  $\psi(t, x) = e^{-i(1+k^2)t} \operatorname{sn}(x, k)$  is orbitally stable in  $H_{\operatorname{loc}}^1 \cap A_{2K}$ .*

*Proof.* Lemma 4.6 shows that  $\text{sn}$  is a non-degenerate (up to phase and translation) *local* minimizer of the energy with fixed mass and momentum. So the classical Cazenave-Lions [8] argument yields the orbital stability.  $\square$

Finally, we also record here the following computations concerning  $L_{\pm}^{\text{cn}}$ , used in analyzing the generalized kernel of  $J\mathcal{L}^{\text{cn}}$  in the next subsection:

**Lemma 4.8.** *Define  $\hat{E}(x, k) = E(\phi, k)|_{\sin \phi = \text{sn}(x, k)}$ . Let  $\phi_1$  and  $\xi_1$  be given by the following expressions.*

$$\phi_1 = \frac{\left(\hat{E}(x, k) - \frac{E}{K}x\right) \text{cn}_x - k^2 \text{cn}^3 + \frac{Kk^2 - E}{K} \text{cn}}{2(2k^2 - 1)\frac{E}{K} + 2(1 - k^2)},$$

$$\xi_1 = \frac{\left(\hat{E}(x, k) - \frac{E}{K}x\right) \text{cn} + \text{cn}_x}{-2(1 - k^2) + \frac{2E}{K}}.$$

The denominators are positive and we have

$$L_+^{\text{cn}}\phi_1 = \text{cn}, \quad L_-^{\text{cn}}\xi_1 = \text{cn}_x.$$

Note  $\hat{E}$  and  $\xi_1$  are odd while  $\phi_1$  is even. In particular  $(\phi_1, \text{cn}_x) = 0 = (\xi_1, \text{cn})$ . Moreover,  $L_+^{\text{cn}}(\frac{1}{2} \text{cn} - (1 - 2k^2)\phi_1) = \text{cn}_{xx}$ .

*Proof.* Recall that the elliptic integral of the second kind  $\hat{E}(x, k)$  is not periodic. In fact, it is asymptotically linear in  $x$  and verifies

$$\hat{E}(x + 2K, k) = \hat{E}(x, k) + 2E(k).$$

By (2.10),  $\partial_x \hat{E}(x, k) = \text{dn}^2(x, k)$ . Denote  $L_{\pm} = L_{\pm}^{\text{cn}}$  in this proof. Using (2.4) and (2.5), we have

$$\begin{aligned} L_+ \text{cn} &= -4k^2 \text{cn}^3, \\ L_+(x \text{cn}_x) &= 4k^2 \text{cn}^3 - 2(2k^2 - 1) \text{cn}, \\ L_+ \text{cn}^3 &= 6k^2 \text{cn}^5 - 8(2k^2 - 1) \text{cn}^3 - 6(1 - k^2) \text{cn}, \\ L_+(\hat{E}(x, k) \text{cn}_x) &= 6k^4 \text{cn}^5 - 4k^2(3k^2 - 2) \text{cn}^3 + 2(1 - 4k^2 + 3k^4) \text{cn}. \end{aligned}$$

Define

$$\tilde{\phi}_1 = \left(\hat{E}(x, k) - \frac{E(k)}{K(k)}x\right) \text{cn}_x - k^2 \text{cn}^3 + \frac{K(k)k^2 - E(k)}{K(k)} \text{cn}.$$

Then  $\tilde{\phi}_1$  is periodic (of period  $4K$ ) and verifies

$$L_+ \tilde{\phi}_1 = \left(2(2k^2 - 1)\frac{E(k)}{K(k)} + 2(1 - k^2)\right) \text{cn}.$$

The factor is positive if  $2k^2 \geq 1$ . If  $2k^2 < 1$ , it is greater than  $2(2k^2 - 1) + 2(1 - k^2) = 2k^2$ . Define,

$$\phi_1 = \left(2(2k^2 - 1)\frac{E(k)}{K(k)} + 2(1 - k^2)\right)^{-1} \tilde{\phi}_1.$$

Then

$$L_+ \phi_1 = \text{cn}.$$

As for  $L_-$ , we have

$$\begin{aligned} L_-(\text{cn}_x) &= 4k^2 \text{cn}^2 \text{cn}_x, \\ L_-(x \text{cn}) &= -2 \text{cn}_x, \\ L_-(\hat{E}(x, k) \text{cn}) &= -2(1 - k^2) \text{cn}_x - 4k^2 \text{cn}^2 \text{cn}_x. \end{aligned}$$

Define

$$\tilde{\xi}_1 = \left( \hat{E}(x, k) - \frac{E(k)}{K(k)} x \right) \text{cn} + \text{cn}_x.$$

Then  $\tilde{\xi}_1$  is periodic (of period  $4K$ ) and verifies

$$L_- \tilde{\xi}_1 = \left( -2(1 - k^2) + \frac{2E(k)}{K(k)} \right) \text{cn}_x.$$

The factor is positive by (2.11). Defining

$$\xi_1 = \left( -2(1 - k^2) + \frac{2E(k)}{K(k)} \right)^{-1} \tilde{\xi}_1$$

we get  $L_- \xi_1 = \text{cn}_x$ . The last statement of the lemma follows from (2.8).  $\square$

**4.2. Orthogonality Properties.** The following lemma records some standard properties of eigenvalues and eigenfunctions of the linearized operator  $J\mathcal{L}$ , which follow only from the structural properties (4.2) and (4.4):

**Lemma 4.9.** *The following properties hold.*

- (1) *(symplectic orthogonality of eigenfunctions) Let  $f = (f_1, f_2)^T$  and  $g = (g_1, g_2)^T$  be two eigenvectors of  $J\mathcal{L}$  corresponding to eigenvalues  $\lambda, \mu \in \mathbb{C}$ . Then (4.2) implies*

$$\lambda + \bar{\mu} \neq 0 \implies (f, Jg) = (f, \mathcal{L}g) = 0,$$

while (4.4) implies

$$\lambda - \bar{\mu} \neq 0 \implies (Cf, Jg) = (Cf, \mathcal{L}g) = 0,$$

so that

$$\lambda \pm \bar{\mu} \neq 0 \implies (f_1, g_2) = (f_2, g_1) = 0.$$

- (2) *(unstable eigenvalues have zero energy) If  $J\mathcal{L}f = \lambda f$ ,  $\lambda \notin i\mathbb{R}$ , then (4.2) implies*

$$(f, \mathcal{L}f) = 0.$$

*Proof.* We first prove (1). We have

$$\lambda (f, Jg) = (\lambda f, Jg) = (J\mathcal{L}f, Jg) = (\mathcal{L}f, g) = (f, \mathcal{L}g) = -(f, \mu Jg) = -\bar{\mu} (f, Jg),$$

so  $(\lambda + \bar{\mu})(f, Jg) = 0$  which gives the first statement. The second statement follows from the same argument with  $f$  replaced by  $Cf$ , while the third statement is a consequence of  $(f, Jg) = (Cf, Jg) = 0$ .

Item (2) is a special case of the first statement of (1), with  $g = f$ .  $\square$



**4.3. Spectral Stability of sn and cn.** Our goal in this section is to establish Theorem 4.1, i.e. to prove the spectral stability of sn in  $P_{4K}$  for all  $k \in (0, 1)$ , and the spectral stability of cn in  $P_{4K}$  for all  $k \in (0, k_c)$ .

We first recall the standard fact that

$$\text{orbital stability} \implies \text{spectral stability.}$$

Indeed, an eigenvalue  $\lambda = \alpha + i\beta$  of  $J\mathcal{L}$  with  $\alpha > 0$  produces a solution of the linearized equation whose magnitude grows at the exponential rate  $e^{\alpha t}$ , and this linear growing mode (together with its orthogonality properties from Lemma 4.9) can be used to contradict orbital stability. Rather than go through the nonlinear dynamics, however, we will give a simple direct proof of spectral stability in the symmetry subspaces where we have the orbital stability – that is, in  $P_{2K}$  for dn, and in  $A_{2K}$  for cn and sn – using just the spectral consequences for  $L_{\pm}$  implied by the (local) minimization properties of these elliptic functions:

**Proposition 4.10.** *For  $0 < k < 1$ ,  $K = K(k)$ , dn is spectrally stable in  $P_{2K}$  while cn and sn are spectrally stable in  $A_{2K}$ . Precisely, we have*

$$\sigma(J\mathcal{L}^{\text{dn}}|_{P_{2K}}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\text{cn}}|_{A_{2K}}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\text{sn}}|_{A_{2K}}) \subset i\mathbb{R}.$$

*Proof.* Begin with dn in  $P_{2K}$ . From Figure 4.1, we see  $L_-^{\text{dn}}|_{\text{dn}^{\perp}} > 0$ , and thus  $(L_-^{\text{dn}})^{\pm 1/2}$  exist on  $\text{dn}^{\perp}$ . It follows from the minimization property Proposition 3.2 that on  $\text{dn}^{\perp}$ ,  $L_+^{\text{dn}} \geq 0$  (otherwise there is a perturbation of dn lowering the energy while preserving the mass). Suppose  $J\mathcal{L}^{\text{dn}}f = \lambda f$ ,  $\lambda \notin i\mathbb{R}$ . Then  $L_-^{\text{dn}}L_+^{\text{dn}}f_1 = -\lambda^2 f_1$ . Since  $(\text{dn}, 0)^T$  is an eigenvector of  $J\mathcal{L}$  for the eigenvalue 0, Lemma 4.9 implies  $f_1 \perp \text{dn}$ . Therefore, we have

$$(L_-^{\text{dn}})^{1/2}L_+^{\text{dn}}(L_-^{\text{dn}})^{1/2} \left( (L_-^{\text{dn}})^{-1/2}f_1 \right) = -\lambda^2 \left( (L_-^{\text{dn}})^{-1/2}f_1 \right)$$

and on  $\text{dn}^{\perp}$ ,

$$L_+ \geq 0 \implies L_-^{1/2}L_+L_-^{1/2} \geq 0 \implies \lambda^2 \leq 0$$

contradicting  $\lambda \notin i\mathbb{R}$ .

Next, consider cn in  $A_{2K}$ . Again from Figure 4.1, we see  $L_-^{\text{cn}}|_{\text{cn}^{\perp}} > 0$ , while the minimization property Proposition 3.4 implies that  $L_+^{\text{cn}} \geq 0$  on  $\text{cn}^{\perp}$ , and so the spectral stability follows just as for dn above.

Finally, consider sn in  $A_{2K}$ . By Lemma 4.6,  $L_+^{\text{sn}} > 0$  on  $\{(\text{sn})_x\}^{\perp}$ , while  $L_-^{\text{sn}} \geq 0$  on  $\{(\text{sn})_x\}^{\perp}$ , and so the spectral stability follows from the same argument as above, with the roles of  $L_+$  and  $L_-$  reversed.  $\square$

Moreover, both sn and cn are spectrally stable in  $P_{2K}^-$ :

**Lemma 4.11.** *For  $0 < k < 1$ ,  $K = K(k)$ ,*

$$\sigma(J\mathcal{L}^{\text{cn}}|_{P_{2K}^-}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\text{sn}}|_{P_{2K}^-}) \subset i\mathbb{R}.$$

*Proof.* This is an immediate consequence of the positivity of  $\mathcal{L}^{\text{sn}}$  and  $\mathcal{L}^{\text{cn}}$  on  $P_{2K}^-$  (see Figure 4.1), and Lemma 4.9.  $\square$

So in light of (4.5), to prove Theorem 4.1, it remains only to show  $\sigma(J\mathcal{L}|_{P_{2K}^+}) \subset i\mathbb{R}$  for each of cn and sn.

This will follow from a simplified version of a general result for infinite dimensional Hamiltonian systems (see [20, 22, 23]) relating coercivity of the linearized

energy with the number of eigenvalues with negative Krein signature of the linearized operator  $J\mathcal{L}$  of the form (4.1):

**Lemma 4.12** (coercivity lemma). *Consider  $J\mathcal{L}$  on  $S \times S$  for some invariant subspace  $S \subset P_T$ , and suppose it has an eigenvalue whose eigenfunction  $\xi = (\xi_1, \xi_2)^T$  has negative (linearized) energy:*

$$J\mathcal{L}\xi = \mu\xi, \quad (\xi, \mathcal{L}\xi) < 0.$$

Then the following results hold.

(1) If  $L_+$  has a one-dimensional negative subspace (in  $S$ ):

$$L_+f = -\lambda f, \quad \lambda > 0, \quad L_+|_{f^\perp} > 0 \quad (4.6)$$

Then  $L_+|_{\xi_2^\perp} > 0$ .

(2) If  $L_-$  has a one-dimensional negative subspace (in  $S$ ):

$$L_-g = -\nu g, \quad \nu > 0, \quad L_-|_{g^\perp} > 0 \quad (4.7)$$

Then  $L_-|_{\xi_1^\perp} > 0$ .

(3) If both (4.6) and (4.7) hold, then  $\sigma(J\mathcal{L}|_{S \times S}) \subset i\mathbb{R}$ .

*Proof.* First note that by Lemma 4.9 (2),  $0 \neq \mu \in i\mathbb{R}$ , and writing  $\mu = i\gamma$ ,  $0 \neq \gamma \in \mathbb{R}$ , we have  $L_-\xi_2 = i\gamma\xi_1$ ,  $L_+\xi_1 = -i\gamma\xi_2$ .

Moreover,

$$(\xi_1, L_+\xi_1) = -\gamma(\xi_1, i\xi_2) = \gamma(i\xi_1, \xi_2) = (L_-\xi_2, \xi_2),$$

so by assumption  $(\xi_1, L_+\xi_1) = (\xi_2, L_-\xi_2) < 0$ .

We prove (1). For any  $h \perp \xi_2$ , decompose

$$h = \alpha f + h_+, \quad \xi_1 = \beta f + \xi_+, \quad h_+ \perp f, \quad \xi_+ \perp f,$$

where we may assume  $\alpha \geq 0$  and  $\beta \geq 0$ . We have

$$0 = i\gamma(h, \xi_2) = (h, -i\gamma\xi_2) = (h, L_+\xi_1) = -\lambda\alpha\beta + (h_+, L_+\xi_+).$$

Thus, using  $L_+|_{f^\perp} > 0$ ,  $L_+^{1/2} = (L_+|_{f^\perp})^{1/2}$  is well defined on  $f^\perp$  and

$$\begin{aligned} (\alpha\beta\lambda)^2 &= (h_+, L_+\xi_+)^2 = \left(L_+^{1/2}h_+, L_+^{1/2}\xi_+\right)^2 \\ &\leq (h_+, L_+h_+)(\xi_+, L_+\xi_+) \\ &= ((h, L_+h) + \alpha^2\lambda)((\xi_1, L_+\xi_1) + \beta^2\lambda) \end{aligned}$$

with both factors on the right  $> 0$ . Since  $(\xi_1, L_+\xi_1) < 0$ , we must have  $(h, L_+h) > 0$ .

Statement (2) follows in exactly the same way, with the roles of  $L_+$  and  $L_-$  reversed, the roles of  $\xi_1$  and  $\xi_2$  reversed, and with  $g$  and  $\nu$  replacing  $f$  and  $\lambda$ .

Finally, for (3), suppose  $J\mathcal{L}\eta = \zeta\eta$ . If  $\zeta \notin i\mathbb{R}$ , then by Lemma 4.9 (1),  $(\xi_1, \eta_2) = (\xi_2, \eta_1) = 0$ , and so by parts (1) and (2),

$$(\eta_1, L_+\eta_1) > 0, \quad (\eta_2, L_+\eta_2) > 0, \quad \implies (\eta, \mathcal{L}\eta) > 0,$$

contradicting Lemma 4.9 (2). Thus  $\zeta \in i\mathbb{R}$ .  $\square$

*Proof of Theorem 4.1.* Begin with sn in  $P_{2K}^+$ . From Figure 4.1 it is clear that in  $P_{2K}^+$ , condition (4.6) holds for  $L_+^{\text{sn}}$  and (4.7) holds for  $L_-^{\text{sn}}$ . Explicit computation yields

$$L_+^{\text{sn}}(\text{dn}^2 + k^2 \text{cn}^2) = -(1 - k^2)^2, \quad L_-^{\text{sn}}1 = -(\text{dn}^2 + k^2 \text{cn}^2),$$

which implies

$$J\mathcal{L}^{\text{sn}} \begin{pmatrix} \text{dn}^2 + k^2 \text{cn}^2 \\ i(1 - k^2) \end{pmatrix} = i(1 - k^2) \begin{pmatrix} \text{dn}^2 + k^2 \text{cn}^2 \\ i(1 - k^2) \end{pmatrix}.$$

Moreover,

$$\begin{aligned} & \left\langle \mathcal{L}^{\text{sn}} \begin{pmatrix} \text{dn}^2 + k^2 \text{cn}^2 \\ i(1 - k^2) \end{pmatrix}, \begin{pmatrix} \text{dn}^2 + k^2 \text{cn}^2 \\ i(1 - k^2) \end{pmatrix} \right\rangle \\ &= \langle L_+^{\text{sn}}(\text{dn}^2 + k^2 \text{cn}^2), \text{dn}^2 + k^2 \text{cn}^2 \rangle + (1 - k^2) \langle L_-^{\text{sn}} \mathbf{1}, \mathbf{1} \rangle \\ &= -((1 - k^2)^2 + (1 - k^2)) \langle \mathbf{1}, (\text{dn}^2 + k^2 \text{cn}^2) \rangle \\ &= -((1 - k^2)^2 + (1 - k^2))(4E(k) - 2(1 - k^2)K(k)) < 0, \end{aligned}$$

by (2.11). Hence all the conditions of Lemma 4.12 are verified for sn in  $P_{2K}^+$ , and so we conclude  $\sigma(J\mathcal{L}^{\text{sn}}|_{P_{2K}^+}) \subset i\mathbb{R}$ , as required.

Next we turn to cn. Again from Figure 4.1 it is clear that in  $P_{2K}^+$ , condition (4.6) holds for  $L_+^{\text{cn}}$  and (4.7) holds for  $L_-^{\text{cn}}$ . Explicit computation yields

$$L_+^{\text{cn}}(-\text{dn}^2 + k^2 \text{sn}^2) = 1, \quad L_-^{\text{cn}} \mathbf{1} = -\text{dn}^2 + k^2 \text{sn}^2,$$

which implies

$$J\mathcal{L}^{\text{cn}} \begin{pmatrix} -\text{dn}^2 + k^2 \text{sn}^2 \\ i \end{pmatrix} = i \begin{pmatrix} -\text{dn}^2 + k^2 \text{sn}^2 \\ i \end{pmatrix}.$$

Moreover, when  $k < k_c$ , we have

$$\left\langle \mathcal{L}^{\text{cn}} \begin{pmatrix} -\text{dn}^2 + k^2 \text{sn}^2 \\ i \end{pmatrix}, \begin{pmatrix} -\text{dn}^2 + k^2 \text{sn}^2 \\ i \end{pmatrix} \right\rangle = 2 \langle L_-^{\text{cn}} \mathbf{1}, \mathbf{1} \rangle = 4K(k) - 8E(k) < 0.$$

Hence the conditions of Lemma 4.12 are verified for cn in  $P_{2K}^+$  when  $k < k_c$ , yielding  $\sigma(J\mathcal{L}^{\text{cn}}|_{P_{2K}^+}) \subset i\mathbb{R}$ , as required.  $\square$

## 5. LINEAR INSTABILITY

Theorem 4.1 (and Proposition 4.10) give the spectral stability of the periodic waves dn, sn, and cn (at least for  $k < k_c$ ) against perturbations which are periodic with their fundamental period. It is also natural to ask if this stability is maintained against perturbations whose period is a *multiple* of the fundamental period. In light of Bloch-Floquet theory, this question is also relevant for stability against *localized* perturbations in  $L^2(\mathbb{R})$ .

**5.1. Theoretical Analysis.** It is a simple observation that dn immediately becomes unstable against perturbations with twice its fundamental period:

**Proposition 5.1.** *Both  $\sigma(J\mathcal{L}^{\text{dn}}|_{A_{2K}^+})$  and  $\sigma(J\mathcal{L}^{\text{dn}}|_{A_{2K}^-})$  contain a pair of non-zero real eigenvalues. In particular dn is linearly unstable against perturbations in  $P_{4K}$ .*

*Proof.* In each of  $A_{2K}^+$  and  $A_{2K}^-$ ,  $L_-^{\text{dn}} > 0$  while  $L_+^{\text{dn}}$  has a negative eigenvalue:  $L_+^{\text{dn}} f = -\lambda f$ ,  $\lambda > 0$ . So the self-adjoint operator  $(L_-^{\text{dn}})^{1/2} L_+^{\text{dn}} (L_-^{\text{dn}})^{1/2}$  has a negative direction,

$$\left( (L_-^{\text{dn}})^{-1/2} f, ((L_-^{\text{dn}})^{1/2} L_+^{\text{dn}} (L_-^{\text{dn}})^{1/2}) (L_-^{\text{dn}})^{-1/2} f \right) = -\lambda(f, f) < 0,$$

hence a negative eigenvalue  $(L_-^{\text{dn}})^{1/2}L_+^{\text{dn}}(L_-^{\text{dn}})^{1/2}g = -\mu^2g$ ,  $\mu > 0$ . Setting  $h := (L_-^{\text{dn}})^{-1/2}g$ ,  $h \in A_{2K}^+$  ( $A_{2K}^-$ ), we see

$$L_+^{\text{dn}}L_-^{\text{dn}}h = -\mu^2h \quad \implies \quad J\mathcal{L}^{\text{dn}}\begin{pmatrix} L_-^{\text{dn}}h \\ \pm\mu h \end{pmatrix} = \pm\mu\begin{pmatrix} L_-^{\text{dn}}h \\ \pm\mu h \end{pmatrix}.$$

Hence  $\mu, -\mu \in \mathbb{R}$  are eigenvalues of  $J\mathcal{L}$  in  $A_{2K}^+$  ( $A_{2K}^-$ ).  $\square$

*Remark 5.2.* The proof shows dn is unstable in  $P_{2nK}$  for every even  $n$  since  $h \in P_{2nK}$ . In fact, dn is unstable in any  $P_{2nK}$ ,  $n \geq 2$ . Indeed, we always have  $L_- \text{dn} = 0$ , thus by Sturm-Liouville Theory (see e.g. [10, Theorem 3.1.2]), 0 is always the first simple eigenvalue of  $L_-$  in  $P_{2nK}$ . Moreover,  $L_+ \text{dn}_x = 0$ , and  $\text{dn}_x$  has  $2n$  zeros in  $P_{2nK}$ . Hence there are at least  $2n - 2$  negative eigenvalues for  $L_+$  in  $P_{2nK}$ . With the above argument, this proves linear instability in  $P_{2nK}$  for any  $n \geq 2$ .

For sn, the  $H^2(\mathbb{R})$  orbital stability result of [5, 15] implies spectral stability against perturbations which are periodic with *any* multiple of the fundamental period.

Using formal perturbation theory, [30] showed that cn becomes unstable against perturbations which are periodic with period a sufficiently large multiple of the fundamental period. Our main goal in this section is to make this rigorous:

**Theorem 5.3.** *For  $0 < k < 1$ , there exists  $n_1 = n_1(k) \in \mathbb{N}$  such that cn is linearly unstable in  $P_{4nK}$  for  $n \geq n_1$ , i.e., the spectrum of  $J\mathcal{L}^{\text{cn}}$  as an operator on  $P_{4nK}$  contains an eigenvalue with positive real part.*

We will in fact prove a slightly more general result, which is the existence of a branch strictly contained in the first quadrant for the spectrum of  $J\mathcal{L}^{\text{cn}}$  considered as an operator on  $L^2(\mathbb{R})$ . Theorem 5.3 will be a consequence of a more general perturbation result applying to all real periodic waves (see Proposition 5.4), and in particular not relying on any integrable structure.

We start with some preliminaries. Let

$$J\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

with

$$L_- = -\partial_{xx} - a - bu^2, \quad L_+ = -\partial_{xx} - a - 3bu^2$$

where  $u$  a periodic solution to

$$u_{xx} + au + b|u|^2 u = 0. \quad (5.1)$$

We assume that  $u(x) \in \mathbb{R}$  and let  $T$  denote a period of  $u^2$ . The spectrum of  $J\mathcal{L}$  as an operator on  $L^2(\mathbb{R})$  can be analyzed using Bloch-Floquet decomposition. For  $\theta \in [0, 2\pi/T)$ , define

$$J\mathcal{L}^\theta = \begin{pmatrix} 0 & L_-^\theta \\ -L_+^\theta & 0 \end{pmatrix}$$

where  $L_\pm^\theta$  is the operator obtained when formally replacing  $\partial_x$  by  $\partial_x + i\theta = e^{-i\theta x}\partial_x(e^{i\theta x}\cdot)$  in the expression of  $L_\pm$ . If we let  $(M^\theta f)(x) = e^{i\theta x}f(x)$ , then  $L_\pm^\theta = M^{-\theta}L_\pm M^\theta$ . Then we have

$$\sigma(J\mathcal{L}|_{L^2(\mathbb{R})}) = \bigcup_{\theta \in [0, \frac{2\pi}{T}]} \sigma(J\mathcal{L}^\theta|_{P_T}). \quad (5.2)$$

In what follows, all operators are considered on  $P_T$  unless otherwise mentioned.

Let us consider the case  $\theta = \frac{\pi}{T}$ . Denote

$$D = \partial_x + i\frac{\pi}{T}.$$

Since  $u$  is a real valued periodic solution to (5.1), by Lemmas 2.1 and 2.2,  $u$  is a rescaled cn, dn or sn. In any case, the following holds:

$$\begin{aligned} \varphi = e^{-i\frac{\pi}{T}x}u, \quad \psi = D\varphi = e^{-i\frac{\pi}{T}x}u_x \in H_{\text{loc}}^1 \cap P_T \setminus \{0\} \\ \text{are such that } \ker(L_-^{\frac{\pi}{T}}) = \langle \varphi \rangle, \quad \ker(L_+^{\frac{\pi}{T}}) = \langle \psi \rangle. \end{aligned} \quad (5.3)$$

Note that for any  $f, g \in H_{\text{loc}}^1 \cap P_T$ , we can integrate by parts with  $D$ :

$$\int_0^T Df\bar{g} \, dx = - \int_0^T f\overline{Dg} \, dx.$$

Remark that

$$(\varphi, \psi) = \int_0^T \varphi\bar{\psi} \, dx = \int_0^T uu_x \, dx = 0.$$

Therefore there exist  $\varphi_1, \psi_1$  such that

$$L_-^{\frac{\pi}{T}}\psi_1 = \psi, \quad L_+^{\frac{\pi}{T}}\varphi_1 = \varphi, \quad \varphi_1 \perp \psi, \quad \psi_1 \perp \varphi.$$

The kernel of the operator  $J\mathcal{L}^{\frac{\pi}{T}}$  is generated by  $\begin{pmatrix} \psi \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$ . On top of that, the generalized kernel of  $J\mathcal{L}^{\frac{\pi}{T}}$  contains (at least)  $\begin{pmatrix} 0 \\ \psi_1 \end{pmatrix}$ ,  $\begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}$ .

Our goal is to analyze the spectrum of the operator  $J\mathcal{L}^{\frac{\pi}{T}-\varepsilon}$  when  $|\varepsilon|$  is small. In particular, we want to locate the eigenvalues generated by perturbation of the generalized kernel of  $J\mathcal{L}^{\frac{\pi}{T}}$ . For the sake of simplicity in notation, when  $\theta = \frac{\pi}{T}$ , we use a tilde to replace the exponent  $\frac{\pi}{T}$ . In particular, we write

$$J\mathcal{L}^{\frac{\pi}{T}} = J\tilde{\mathcal{L}}, \quad L_{\pm}^{\frac{\pi}{T}} = \tilde{L}_{\pm}.$$

**Proposition 5.4.** *Assume the condition (5.16) stated below. There exist  $\lambda_1 \in \mathbb{C}$  with  $\text{Re}(\lambda_1) > 0$ ,  $\text{Im}(\lambda_1) > 0$ ;  $b_0 \in \mathbb{C}$ ; and  $\varepsilon_0 > 0$ , such that for all  $0 \leq \varepsilon < \varepsilon_0$  there exist  $\lambda_2(\varepsilon) \in \mathbb{C}$ ,  $b_1(\varepsilon) \in \mathbb{C}$ ,  $v_2(\varepsilon), w_2(\varepsilon) \in H_{\text{loc}}^2 \cap P_T$ ,*

$$\begin{aligned} |b_1(\varepsilon)| + |\lambda_2(\varepsilon)| + \|v_2(\varepsilon)\|_{H_{\text{loc}}^2 \cap P_T} + \|w_2(\varepsilon)\|_{H_{\text{loc}}^2 \cap P_T} \lesssim 1 \\ v_2(\varepsilon) \perp \psi, \quad w_2(\varepsilon) \perp \varphi, \end{aligned} \quad (5.4)$$

verifying the following property. Set

$$v_0 = b_0\psi, \quad v_1 = b_1(\varepsilon)\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1, \quad (5.5)$$

$$w_0 = \varphi, \quad w_1 = (b_0\lambda_1 - 2i)\psi_1. \quad (5.6)$$

Here,  $\tilde{L}_+^{-1}$  is taken from  $\psi^\perp$  to  $\psi^\perp$ . Define

$$\begin{aligned} \lambda &= \varepsilon\lambda_1 + \varepsilon^2\lambda_2(\varepsilon), \\ v &= v_0 + \varepsilon v_1(\varepsilon) + \varepsilon^2 v_2(\varepsilon), \\ w &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2(\varepsilon). \end{aligned}$$

Then

$$J\mathcal{L}^{\frac{\pi}{T}-\varepsilon} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} v \\ w \end{pmatrix}.$$

Note that the orthogonality conditions in (5.4) are reasonable: The eigenvector is normalized by  $P_\varphi w = w_0 = \varphi$ , and hence  $w_2 \perp \varphi$ . To impose  $v_2 \perp \psi$ , we allow  $b_1(\varepsilon)\psi$  in  $v_1$  to be  $\varepsilon$ -dependent to absorb  $P_\psi(v - v_0)$ .

*Proof of Proposition 5.4.* Let us write the expansion of the operators in  $\varepsilon$ . We have

$$L_{\pm}^{\frac{\pi}{T}-\varepsilon} = L_{\pm}^{\frac{\pi}{T}} + 2i\varepsilon D + \varepsilon^2,$$

Therefore

$$J\mathcal{L}^{\frac{\pi}{T}-\varepsilon} = J\mathcal{L}^{\frac{\pi}{T}} + \varepsilon \begin{pmatrix} 0 & 2iD \\ -2iD & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We expand in  $\varepsilon$  the equation  $(J\mathcal{L}^{\frac{\pi}{T}-\varepsilon} - \lambda\mathcal{I}) \begin{pmatrix} v \\ w \end{pmatrix} = 0$  and show that it can be satisfied at each order of  $\varepsilon$ .

At order  $\mathcal{O}(1)$ , we have

$$J\tilde{\mathcal{L}} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

which is satisfied because  $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \in \ker(J\tilde{\mathcal{L}})$  by definition.

At order  $\mathcal{O}(\varepsilon)$ , we have

$$J\tilde{\mathcal{L}} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

which can be rewritten, using the expression of  $v_0$ ,  $w_0$ , and  $D\varphi = \psi$ , as

$$\tilde{L}_- w_1 = (b_0\lambda_1 - 2i)\psi, \quad (5.7)$$

$$\tilde{L}_+ v_1 = -2ib_0 D\psi - \lambda_1\varphi. \quad (5.8)$$

It is clear that the functions  $v_1(\varepsilon)$  and  $w_1$  defined in (5.5)-(5.6) satisfy (5.7)-(5.8).

At order  $\mathcal{O}(\varepsilon^2)$ , we consider the equation as a whole, involving also the higher orders of  $\varepsilon$ . We have

$$\begin{aligned} J\tilde{\mathcal{L}} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \\ + \varepsilon \left( \begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \right) \\ + \varepsilon^2 \begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = 0, \end{aligned}$$

in other words

$$\tilde{L}_- w_2 = W_2 + \varepsilon W_3 + \varepsilon^2 W_4 \quad (5.9)$$

$$\tilde{L}_+ v_2 = V_2 + \varepsilon V_3 + \varepsilon^2 V_4 \quad (5.10)$$

where

$$\begin{aligned} W_2 &= \lambda_1 v_1 - 2iDw_1 + \lambda_2 v_0 - w_0, & V_2 &= -2iDv_1 - \lambda_1 w_1 - v_0 - \lambda_2 w_0, \\ W_3 &= \lambda_1 v_2 - 2iDw_2 + \lambda_2 v_1 - w_1, & V_3 &= -2iDv_2 - \lambda_1 w_2 - v_1 - \lambda_2 w_1, \\ W_4 &= \lambda_2 v_2 - w_2, & V_4 &= -v_2 - \lambda_2 w_2. \end{aligned} \quad (5.11)$$

Note that  $V_2$  and  $W_2$  depend on  $b_0, \lambda_1$  and  $b_1, \lambda_2$ , whereas  $V_3, V_4$  and  $W_3, W_4$  also depend on  $v_2$  and  $w_2$ . Our strategy to solve the system (5.9)-(5.10) is divided into two steps. We first ensure that it can be solved at the main order by ensuring that the compatibility conditions

$$(W_2, \varphi) = (V_2, \psi) = 0 \quad (5.12)$$

are satisfied. This is achieved by making a suitable choice of  $b_0, \lambda_1$ . Then we solve for  $b_1, \lambda_2, v_2, w_2$  by using a Lyapunov-Schmidt argument.

We rewrite the compatibility conditions (5.12) in the following form, using the expressions for  $v_0$ ,  $w_0$ ,  $v_1$  and  $w_1$ , and the properties of  $\varphi$  and  $\psi$ :

$$\begin{aligned} (\varphi_1, \varphi) \lambda_1^2 + b_0 2i ((D\psi, \varphi_1) - (\psi_1, \psi)) \lambda_1 + ((\varphi, \varphi) - 4(\psi_1, \psi)) &= 0 \\ b_0 (\psi_1, \psi) \lambda_1^2 + 2i ((\varphi_1, D\psi) - (\psi_1, \psi)) \lambda_1 + b_0 \left( (\psi, \psi) - 4 \left( \tilde{L}_+^{-1} D\psi, D\psi \right) \right) &= 0 \end{aligned}$$

These equations do not depend on  $b_1$  or  $\lambda_2$  although  $W_2$  and  $V_2$  do. For a moment, we write these equations as

$$A_1 \lambda_1^2 + b_0 B \lambda_1 + C_1 = 0, \quad (5.13)$$

$$b_0 A_2 \lambda_1^2 + B \lambda_1 + b_0 C_2 = 0, \quad (5.14)$$

where

$$\begin{aligned} A_1 &:= (\varphi_1, \varphi) \in \mathbb{R}, \\ A_2 &:= (\psi_1, \psi) \in \mathbb{R}, \\ B &:= 2i ((D\psi, \varphi_1) - (\psi_1, \psi)) \in i\mathbb{R}, \\ C_1 &:= (\varphi, \varphi) - 4(\psi_1, \psi) \in \mathbb{R}, \\ C_2 &:= (\psi, \psi) - 4 \left( \tilde{L}_+^{-1} D\psi, D\psi \right) \in \mathbb{R}. \end{aligned}$$

Multiplying (5.13) by  $C_2 + A_2 \lambda_1^2$ , (5.14) by  $B \lambda_1$ , and subtracting gives

$$A_1 A_2 \lambda_1^4 + (A_1 C_2 + A_2 C_1 - B^2) \lambda_1^2 + C_1 C_2 = 0, \quad (5.15)$$

a quadratic equation in  $\lambda_1^2$  with real coefficients. If  $A_1 A_2 \neq 0$ , the roots of (5.15) are given by

$$\lambda_1^2 = \frac{-(A_1 C_2 + A_2 C_1 - B^2) \pm \sqrt{(A_1 C_2 + A_2 C_1 - B^2)^2 - 4A_1 A_2 C_1 C_2}}{2A_1 A_2}$$

We now assume the discriminant of this quadratic is negative:

$$(A_1 C_2 + A_2 C_1 - B^2)^2 - 4A_1 A_2 C_1 C_2 < 0 \quad (5.16)$$

which implies that  $A_1 A_2 \neq 0$ , and moreover guarantees the existence of a root  $\lambda_1$  of (5.15) strictly contained in the first quadrant:  $\Re \lambda_1 > 0$  and  $\Im \lambda_1 > 0$  (the other roots being  $-\lambda_1$ ,  $\pm \bar{\lambda}_1$ ). It follows from (5.16) that  $B \neq 0$ , and so we may solve (5.13) and set

$$b_0 := -\frac{(A_1 \lambda_1^2 + C_1)}{B \lambda_1},$$

so that both (5.13) and (5.14) are satisfied.

We now solve for  $b_1, \lambda_2, v_2, w_2$  using a Lyapunov-Schmidt argument. The first step is to solve, given  $(b_1, \lambda_2)$ , projected versions of (5.9)-(5.10),

$$\begin{aligned} \tilde{L}_- w_2 &= W_2 + P_{\varphi^\perp} [\varepsilon W_3 + \varepsilon^2 W_4] \\ \tilde{L}_+ v_2 &= V_2 + P_{\psi^\perp} [\varepsilon V_3 + \varepsilon^2 V_4] \end{aligned} \quad (5.17)$$

to obtain  $v_2 = v_2(b_1, \lambda_2) \in \psi^\perp$ ,  $w_2 = w_2(b_1, \lambda_2) \in \varphi^\perp$ :

**Lemma 5.5.** *Given any  $b_1 \in \mathbb{C}$ ,  $\lambda_2 \in \mathbb{C}$  with  $|b_1| + |\lambda_2| \leq M$ , there is a unique solution*

$$(v_2, w_2) = (v_2(b_1, \lambda_2), w_2(b_1, \lambda_2)) \in (H_{loc}^2 \cap P_T \cap \psi^\perp) \times (H_{loc}^2 \cap P_T \cap \varphi^\perp)$$

of (5.17), with  $\|v_2\|_{H^2} + \|w_2\|_{H^2} \leq C(M)$ .

*Proof.* By the expressions (5.11), we may rewrite system (5.17) as a linear system of  $v_2$  and  $w_2$ ,

$$\tilde{\mathcal{L}}_\varepsilon \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} S_v \\ S_w \end{pmatrix} + \varepsilon \begin{pmatrix} R_v \\ R_w \end{pmatrix},$$

where

$$\tilde{\mathcal{L}}_\varepsilon = \begin{pmatrix} \tilde{L}_+ + P_{\psi^\perp} 2i\varepsilon D + \varepsilon^2 & (\varepsilon\lambda_1 + \varepsilon^2\lambda_2)P_{\psi^\perp} \\ -(\varepsilon\lambda_1 + \varepsilon^2\lambda_2)P_{\varphi^\perp} & \tilde{L}_- + P_{\varphi^\perp} 2i\varepsilon D + \varepsilon^2 \end{pmatrix},$$

$$\begin{aligned} S_v &= P_{\psi^\perp}(-2iD[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] - \lambda_1(b_0\lambda_1 - 2i)\psi_1 - \lambda_2\varphi) \\ S_w &= P_{\varphi^\perp}(\lambda_1[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] - 2iD(b_0\lambda_1 - 2i)\psi_1 + \lambda_2b_0\psi) \\ R_v &= P_{\psi^\perp}(\lambda_2(2i - b_0\lambda_1)\psi_1 - [-2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1]) \\ R_w &= P_{\varphi^\perp}(\lambda_2[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] + (2i - b_0\lambda_1)\psi_1). \end{aligned} \quad (5.18)$$

Note that  $S_v, S_w, R_v$  and  $R_w$  do not contain  $v_2, w_2$  or  $\varepsilon$ . Recalling the definition

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{L}_+ & 0 \\ 0 & \tilde{L}_- \end{pmatrix},$$

it follows from (5.3) that

$$\tilde{\mathcal{L}}^{-1} : (P_T \cap \psi^\perp) \times (P_T \cap \varphi^\perp) \rightarrow (P_T \cap H_{loc}^2 \cap \psi^\perp) \times (P_T \cap H_{loc}^2 \cap \varphi^\perp)$$

is bounded, and hence so is  $\tilde{\mathcal{L}}_\varepsilon^{-1}$ , uniformly in  $\varepsilon$  for  $\varepsilon$  sufficiently small, with

$$\|\tilde{\mathcal{L}}_\varepsilon^{-1} - \tilde{\mathcal{L}}^{-1}\|_{(L^2 \times L^2 \rightarrow H^2 \times H^2)} \lesssim \varepsilon.$$

Thus

$$\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \tilde{\mathcal{L}}_\varepsilon^{-1} \left( \begin{pmatrix} S_v \\ S_w \end{pmatrix} + \varepsilon \begin{pmatrix} R_v \\ R_w \end{pmatrix} \right) = \begin{pmatrix} \tilde{L}_+^{-1} S_v \\ \tilde{L}_-^{-1} S_w \end{pmatrix} + O_{H^2 \times H^2}(\varepsilon) \quad (5.19)$$

gives  $(v_2(b_1, \lambda_2), w_2(b_1, \lambda_2))$  as desired.  $\square$

The second step is to plug  $(v_2(b_1, \lambda_2), w_2(b_1, \lambda_2))$  back into  $V_3, V_4, W_3, W_4$ , and solve, for  $(b_1, \lambda_2)$ , the remaining compatibility conditions

$$(V_3 + \varepsilon V_4, \psi) = (W_3 + \varepsilon W_4, \varphi) = 0 \quad (5.20)$$

which, together with (5.17), complete the solution of the eigenvalue problem. Using (5.19) and (5.11), we may write (5.20) as the system

$$\begin{aligned} 0 &= (-2iD[\tilde{L}_+^{-1}S_v + O(\varepsilon)] - \lambda_1[\tilde{L}_-^{-1}S_w + O(\varepsilon)] - v_1 - \lambda_2w_1 + \varepsilon V_4, \psi) \\ 0 &= (\lambda_1[\tilde{L}_+^{-1}S_v + O(\varepsilon)] - 2iD[\tilde{L}_-^{-1}S_w + O(\varepsilon)] + \lambda_2v_1 - w_1 + \varepsilon W_4, \varphi) \end{aligned}$$

and then by the expressions (5.5)-(5.6) and (5.18), we may further rewrite as

$$\Phi(b_1, \lambda_2, \varepsilon) = (M + O(\varepsilon)) \begin{pmatrix} b_1 \\ \lambda_2 \end{pmatrix} + F + O(\varepsilon) = 0 \quad (5.21)$$



where  $\Phi$  is a rational vector function of  $b_1, \lambda_2$  and  $\varepsilon$ ;  $F$  is a fixed (independent of  $(b_1, \lambda_2)$ ) vector with  $|F| \lesssim 1$ ; and  $M = \frac{\partial \Phi}{\partial (b_1, \lambda_2)}|_{\varepsilon=0}$  is the matrix

$$\begin{aligned} M &= \begin{pmatrix} (-4D\tilde{L}_+^{-1}D\psi - \lambda_1^2\psi_1 - \psi, \psi) & (2iD\varphi_1 - 2(\lambda_1 b_0 - i)\psi_1, \psi) \\ (-2i\lambda_1\tilde{L}_+^{-1}D\psi - 2i\lambda_1D\psi_1, \varphi) & (-\lambda_1\varphi_1 - 2ib_0D\psi_1 - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1, \varphi) \end{pmatrix} \\ &= \begin{pmatrix} 4(\tilde{L}_+^{-1}D\psi, D\psi) - \lambda_1^2(\psi_1, \psi) - (\psi, \psi) & -2i(\varphi_1, D\psi) - 2(\lambda_1 b_0 - i)(\psi_1, \psi) \\ -2i\lambda_1(D\psi, \varphi_1) + 2i\lambda_1(\psi_1, \psi) & -2\lambda_1(\varphi_1, \varphi) + 2ib_0(\psi_1, \psi) - 2ib_0(D\psi, \varphi_1) \end{pmatrix} \\ &= \begin{pmatrix} -C_2 - A_2\lambda_1^2 & -B - 2\lambda_1 b_0 A_2 \\ -\lambda_1 B & -b_0 B - 2\lambda_1 A_1 \end{pmatrix} = \begin{pmatrix} \frac{B}{b_0}\lambda_1 & -B - 2\lambda_1 b_0 A_2 \\ -\lambda_1 B & -b_0 B - 2\lambda_1 A_1 \end{pmatrix}. \end{aligned}$$

where in the last step we used (5.14). The determinant of  $M$  is, using (5.14) and (5.13) to eliminate  $b_0$ ,

$$\begin{aligned} \det M &= -2\lambda_1 B \left( B + \frac{\lambda_1}{b_0} A_1 + \lambda_1 b_0 A_2 \right) \\ &= -2\lambda_1 B \left( B - \frac{A_2 \lambda_1^2 + C_2}{B} A_1 - \frac{A_1 \lambda_1^2 + C_1}{B} A_2 \right) \\ &= -2\lambda_1 (B^2 - 2A_1 A_2 \lambda_1^2 - C_2 A_1 - C_1 A_2). \end{aligned}$$

Since  $A_1, A_2, C_1, C_2, B^2$  are real, and  $\lambda_1 A_1 A_2 \neq 0$ , we have  $\det M \neq 0$ , otherwise  $\lambda_1^2 \in \mathbb{R}$ .

Thus  $(b_1, \lambda_2)$  may be solved from (5.21) for  $\varepsilon$  sufficiently small by the implicit function theorem, providing the required solution to (5.20), and so completing the proof of Proposition 5.4.  $\square$

*Proof of Theorem 5.3.* We need only verify the assumptions of Proposition 5.4 for the case of  $u(x) = \text{cn}(x; k)$ ,  $T = 2K(k)$ . Since  $u = \text{cn} \in A_{2K}$ , we have  $u^2 = \text{cn}^2 \in P_T$ . Moreover, (5.3) holds (see Figure 4.1). It remains to verify the condition (5.16). The values of the coefficients for the equations of  $b_0$  and  $\lambda_1$  are given by the following formulas, obtained by using the equation verified by  $\text{cn}$  and the explicit expressions given by Lemma 4.8. Due to the complicated nature of the expressions, the dependence of  $E$  and  $K$  on  $k$  will be left implicit.

$$\begin{aligned} A_1 &= (\varphi_1, \varphi) = (\phi_1, u) = \frac{k^2 K(2E - K) + (E - K)^2}{2k^2(E(1 - 2k^2) - K(1 - k^2))}, \\ A_2 &= (\psi_1, \psi) = (\xi_1, u_x) = \frac{k^2 K(2E - K) + (E - K)^2}{2k^2(E - (1 - k^2)K)}, \\ B &= 2i((\varphi_1, D\psi) - (\psi_1, \psi)) = 2i((\phi_1, u_{xx}) - A_2) \\ &= -i \frac{2EK(k-1)(k+1)(K-E)}{(E(1-2k^2) - K(1-k^2))(E - (1-k^2)K)}, \\ C_1 &= (\varphi, \varphi) - 4(\psi_1, \psi) = (u, u) - 4A_2 = \frac{2K^2(k-1)(k+1)}{E - (1-k^2)K}, \\ C_2 &= (\psi, \psi) - 4\langle \tilde{L}_+^{-1}D\psi, D\psi \rangle = (u_x, u_x) - 4\langle L_+^{-1}u_{xx}, u_{xx} \rangle \\ &= \frac{2K^2(k-1)(k+1)}{E(1-2k^2) - K(1-k^2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & (A_1C_2 + A_2C_1 - B^2)^2 - 4A_1A_2C_1C_2 \\ &= -\frac{16K^4E^2(1-k)^3(1+k)^3(K-E)^2}{k^2(E - (1-k^2)K)^2(E(1-2k^2) - (1-k^2)K)^2} < 0, \end{aligned}$$

Thus Proposition (5.4) applies, providing an unstable eigenvalue of  $J(\mathcal{L}^{\text{cn}})^\theta$  for  $\theta = \frac{\pi}{2K} - \varepsilon$ , and all  $0 < \varepsilon \leq \varepsilon_0$ . It follows in particular that cn is unstable against perturbations with period  $4nK$ , where  $n$  is the smallest even integer  $\geq \frac{\pi}{K\varepsilon_0}$ . This concludes the proof of Theorem 5.3.  $\square$

**5.2. Numerical Spectra.** We have tested numerically the spectra of the different operators involved. To this aim, we used a fourth order centered finite difference discretization of the second derivative operator. Unless otherwise specified, we have used  $2^{10}$  grid points. The spectra are then obtained using the built in function of our scientific computing software (Scilab). Whenever the spectra can be theoretically described, the theoretical description and our numerical computations are in good agreement.

We start by the presentation of the spectra of  $J\mathcal{L}^{\text{pq}}$ , for pq = cn, dn, sn on  $P_{4K}$ .

**Observation 5.6.** *On  $P_{4K}$ , the spectrum of  $J\mathcal{L}^{\text{pq}}$  is such that*

- if pq = sn then  $\sigma(J\mathcal{L}^{\text{sn}}) \subset i\mathbb{R}$  for all  $k \in (0, 1)$ ,
- if pq = cn, then  $\sigma(J\mathcal{L}^{\text{cn}}) \subset i\mathbb{R}$  for all  $k \in (0, 1)$ , including when  $k > k_e$ ,
- if pq = dn, then  $J\mathcal{L}^{\text{dn}}$  admits two double eigenvalues  $\pm\lambda$  with  $\lambda > 0$  and the rest of the spectrum verifies  $(\sigma(J\mathcal{L}^{\text{dn}}) \setminus \{\pm\lambda\}) \subset i\mathbb{R}$  for all  $k \in (0, 1)$ .

The numerical observations for cn and dn at  $k = 0.95$  are represented in Figure 5.1.

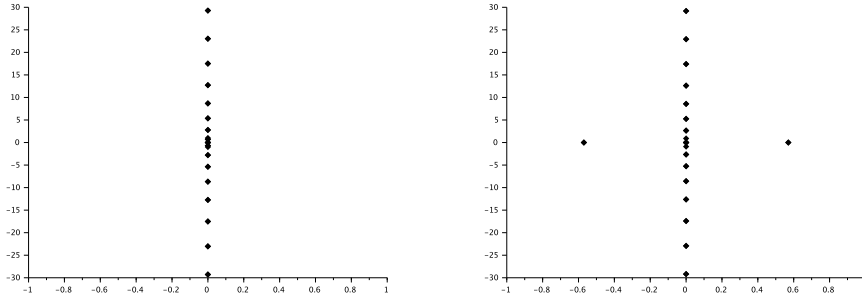


FIGURE 5.1.  $\sigma(J\mathcal{L}^{\text{cn}})$  (left) and  $\sigma(J\mathcal{L}^{\text{dn}})$  (right) on  $P_{4K}$  for  $k = 0.95$

We then compare the results of Theorem 5.3 with the numerical results. In Figure 5.2, we have drawn the numerical spectrum of  $J\mathcal{L}^{\text{cn}}$  as an operator on  $L^2(\mathbb{R})$ . To this aim, we have used the Bloch decomposition of the spectrum of  $J\mathcal{L}^{\text{cn}}$  given in (5.2): we computed the spectrum of  $J(\mathcal{L}^{\text{cn}})^\theta$  for  $\theta$  in a discretization of  $(0, \frac{\pi}{2K}]$  and we have interpolated between the values obtained to get the curve in plain (blue) line. In order to keep the computation time reasonable, we have dropped the number of space points from  $2^{10}$  to  $2^8$ . We then have drawn in dashed (red) the straight lines passing through the origin and the points whose coordinates are

given in the complex plane by  $\pm\lambda_1, \pm\bar{\lambda}_1$ ,  $\lambda_1$  given in the proof of Proposition 5.4. The picture shows that the dashed (red) lines are tangent to the plain (blue) curve, thus confirming  $\lambda_1$  as the first order in the expansion for the eigenvalue emerging from 0 performed in Proposition 5.4.

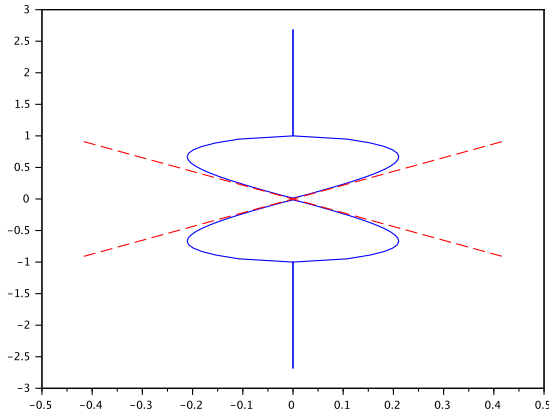


FIGURE 5.2.  $\sigma(J\mathcal{L}^{\text{cn}})$  on  $L^2(\mathbb{R})$  for  $k = 0.9$  (plain (blue) curve), first order asymptotic around 0 (dashed (red) lines)

Numerically, eigenvalues on the number 8 curve in Figure 5.2 are simple, and move from the origin toward the intersection points of the number 8 curve with the imaginary axis, when  $\theta$  is decreased from  $\pi/(2K)$  to  $0^+$ .

These eigenvalues are simple because we did the Block decomposition (5.2) in  $P_{2K}$  with  $\theta \in [0, 2\pi/T) = [0, \pi/K)$ , and  $\text{cn}$  is only in  $P_{2K}(-1)$ , not in  $P_{2K}$ . Thus it is in the kernel of  $L_+^\theta$  only for  $\theta = \pi/(2K)$ . The bifurcation occurs only near  $\theta = \pi/(2K)$ , not at  $\theta = 0$ .

In contrast, Rowlands [30] did the Block decomposition in  $P_{4K}$  with  $\theta \in [0, \pi/(2K))$ . We have  $\text{cn} \in P_{4K}$ , and  $\text{cn}$  is in the kernel of  $L_+^\theta$  only for  $\theta = 0$ . The bifurcation occurs only near  $\theta = 0$ .

These two approaches are essentially the same, and our approach does not give a new instability branch.

## 6. NUMERICS

We describe here the numerical experiments performed to understand better the nature of the Jacobi elliptic functions as constrained minimizers of some functionals. To this aim, we use a normalized gradient flow approach related to the minimization problem (3.3).

**6.1. Gradient Flow With Discrete Normalization.** It is relatively natural when dealing with constrained minimization problems like (3.3)-(3.4) to use the following construction. Define an increasing sequence of time  $0 = t_0 < \dots < t_n$  and take an initial data  $u_0$ . Between each time step, let  $u(t, x)$  evolve along the

gradient flow

$$\begin{cases} u_t = -\mathcal{E}'(u) = u_{xx} + b|u|^2u, & x \in \mathbb{R}, t_n < t < t_{n+1}, n \geq 0. \\ u(t_n, x) = u_n(x), \end{cases}$$

At each time step  $t_n$ , the function is renormalized so as to have the desired mass and momentum. The renormalization for the mass is obtained by a straightforward scaling:

$$u_{n+1}(x) := u(t_{n+1}, x) \sqrt{\frac{m}{\mathcal{M}(u(t_{n+1}, x))}}. \quad (6.1)$$

When there is no momentum, like in the minimization problems (3.1), (3.4), and only real-valued functions are considered, such approach to compute the minimizers was developed by Bao and Du [4].

However, dealing with complex valued solutions and with an additional momentum constraint as in problems (3.3), (3.5) turns out to make the problem more challenging and to our knowledge little is known about the strategies that one can use to deal with this situation (see [9] for an approach on a related problem).

To construct  $u_n$  in such a way that  $\mathcal{P}(u_n) = p$ , a simple scaling is not possible for at least two reasons. First of all, if  $p = 0$ , a scaling would obviously lead to failure of our strategy. Second, even if  $p \neq 0$ , as we are already using a scaling to get the correct mass, making a different scaling to obtain the momentum constraint will result into a modification of the mass. To overcome these difficulties, we propose the following approach.

Recall that, as noted in [4], the renormalizing step (6.1) is equivalent to solving exactly the following ordinary differential equation

$$u_t = \mu_n u, \quad t_n < t < t_{n+1}, \quad n \geq 0, \quad \mu_n = \frac{1}{t_{n+1} - t_n} \ln \left( \frac{\sqrt{2m}}{\|u(t_n)\|_{L^2}} \right). \quad (6.2)$$

Inspired by this remark, we consider the following problem, which we see as the equivalent of (6.2) for the momentum renormalization.

$$u_t = i\varpi_n u_x, \quad x \in \mathbb{R}, \quad t_n < t < t_{n+1}, \quad n \geq 0, \quad (6.3)$$

where we want to choose the values of  $\varpi_n$  in such a way that  $\mathcal{P}(u(t_{n+1})) = p$ . To this aim, we need to solve (6.3). Note that (6.3) is a partial differential equation, whereas (6.2) was only an ordinary differential equation. We make the following formal computations, which can be justified if the functions involved are regular enough. As we work with periodic functions, we consider the Fourier series representation of  $u$ , that is

$$u(t, x) = \sum_{j=-\infty}^{\infty} c_j(t) e^{i\frac{2\pi}{T} jx}$$

with the Fourier coefficients

$$c_j(t) = \frac{1}{T} \int_{-T/2}^{T/2} u(t, x) e^{-i\frac{2\pi}{T} jx} dx.$$

Then (6.3) becomes

$$\partial_t c_j = -\frac{2\pi}{T} j \varpi_n c_j, \quad j \in \mathbb{Z}, \quad t_n < t < t_{n+1}, \quad n \geq 0.$$

For each  $j \in \mathbb{Z}$  and for any  $t_n < t < t_{n+1}$  the solution is

$$c_j(t) = \exp\left(-\frac{2\pi}{T}j\varpi_n(t-t_n)\right)c_j(t_n),$$

and therefore the solution of (6.3) is

$$u(t, x) = \sum_{j=-\infty}^{\infty} \exp\left(-\frac{2\pi}{T}j\varpi_n(t-t_n)\right)c_j(t_n)e^{i\frac{2\pi}{T}jx}.$$

Using this Fourier series expansion of  $u$ , we have

$$\mathcal{P}(u(t_{n+1})) = -\sum_{j=-\infty}^{\infty} \pi j \exp\left(-\frac{4\pi}{T}j\varpi_n(t_{n+1}-t_n)\right)|c_j(t_n)|^2.$$

We determine implicitly the value of  $\varpi_n$ , by requiring that  $\varpi_n$  is such that

$$\mathcal{P}(u(t_{n+1})) = p.$$

In practice, it might not be so easy to compute  $\varpi_n$  and therefore we shall use the following approximation. We replace the exponential by its first order Maclaurin polynomial. We get

$$\mathcal{P}(u(t_{n+1})) = -\sum_{j=-\infty}^{\infty} \pi j \left(1 - \frac{4\pi}{T}j\varpi_n(t_{n+1}-t_n)\right)|c_j(t_n)|^2 + \mathcal{O}(\varpi_n^2(t_{n+1}-t_n)^2).$$

Therefore, an approximation for  $\varpi_n$  is given by  $\tilde{\varpi}_n$ , which is defined implicitly by

$$p = -\sum_{j=-\infty}^{\infty} \pi j \left(1 - \frac{4\pi}{T}j\tilde{\varpi}_n(t_{n+1}-t_n)\right)|c_j(t_n)|^2.$$

Solving for  $\tilde{\varpi}_n$ , we obtain

$$\tilde{\varpi}_n = \left(p + \sum_{j=-\infty}^{\infty} \pi j |c_j(t_n)|^2\right) \left((t_{n+1}-t_n) \frac{4\pi^2}{T} \sum_{j=-\infty}^{\infty} j^2 |c_j(t_n)|^2\right)^{-1}.$$

We can further simplify the expression of  $\tilde{\varpi}_n$  by remarking that

$$\mathcal{P}(u(t_n)) = -\sum_{j=-\infty}^{\infty} \pi j |c_j(t_n)|^2, \quad \int_{-T/2}^{T/2} |\partial_x u(t_n)|^2 dx = \frac{4\pi^2}{T} \sum_{j=-\infty}^{\infty} j^2 |c_j(t_n)|^2.$$

This gives

$$\tilde{\varpi}_n = \frac{p - \mathcal{P}(u(t_n))}{(t_{n+1}-t_n) \|\partial_x u(t_n)\|_{L^2}^2}.$$

This is the value we will use in practice.

**6.2. Discretization.** Let us now further discretize our problem. We first present a semi-implicit time discretization, given by the following scheme.

$$\begin{aligned} \frac{\tilde{u}_{n+1} - u_n}{\delta t} &= \partial_{xx} \tilde{u}_{n+1} + b|u_n|^2 \tilde{u}_{n+1}, \quad \tilde{u}_{n+1} \in P_T, \\ \hat{u}_{n+1} &= \sum_{j=-\infty}^{\infty} c_j(\tilde{u}_{n+1}) \left(1 - \frac{2\pi}{T} \delta t \tilde{\varpi}_n j\right) e^{i\frac{2\pi}{T}jx}, \\ u_{n+1} &= \hat{u}_{n+1} \sqrt{\frac{m}{\mathcal{M}(\hat{u}_{n+1})}}, \end{aligned}$$

where  $\tilde{\omega}_n$  is given by

$$\tilde{\omega}_n = \frac{p - \mathcal{P}(u_n)}{\delta t \|\partial_x u_n\|_{L^2}^2},$$

and  $(c_j(\tilde{u}_{n+1}))$  are the Fourier coefficients of  $\tilde{u}_{n+1}$ . Note that the system is linear.

*Remark 6.1.* If  $p = 0$ , at the end of each step,  $u_{n+1}$  has the desired mass and momentum. If  $p \neq 0$ , then  $u_{n+1}$  only has the desired mass and it is unclear if the algorithm will still give convergence toward the desired mass-momentum constraint minimizer. We plan to investigate this question in further works.

Finally, we present the fully discretized problem. We discretize the space interval  $[-\frac{T}{2}, \frac{T}{2}]$  by setting

$$x^0 = -\frac{T}{2}, \quad x^l = x^0 + l\delta x, \quad \delta x = \frac{T}{L}, \quad L \in 2\mathbb{N}.$$

We denote by  $u_n^l$  the numerical approximation of  $u(t_n, x^l)$ . Using the (backward Euler) semi-implicit scheme for time discretization and second-order centered finite difference for spatial derivatives, we obtain the following scheme.

$$\frac{\tilde{u}_{n+1}^l - u_n^l}{\delta t} = \frac{\tilde{u}_{n+1}^{l-1} - 2\tilde{u}_{n+1}^l + \tilde{u}_{n+1}^{l+1}}{\delta x^2} + b|u_n^l|^2 \tilde{u}_{n+1}^l, \quad u_{n+1}^0 = u_{n+1}^L, \quad (6.4)$$

$$\hat{u}_{n+1}^l = \sum_{j=-L/2}^{L/2} c_j(\tilde{u}_{n+1}) \left(1 - \frac{2\pi}{T} \delta t \tilde{\omega}_n j\right) e^{i \frac{2\pi}{L} j l \delta x}, \quad (6.5)$$

$$\tilde{u}_{n+1}^l = \hat{u}_{n+1}^l \sqrt{\frac{m}{\mathcal{M}(\hat{u}_{n+1})}}, \quad (6.6)$$

where  $c_j(\tilde{u}_{n+1}) = \frac{1}{L+1} \sum_{l=0}^L \tilde{u}_{n+1}^l e^{i \frac{2\pi}{L} j l \delta x}$ .

As the system (6.4) is linear, we can solve it using a Thomas algorithm for tridiagonal matrix modified to take into account the periodic boundary conditions. The discrete Fourier transform and its inverse are computed using the built in Fast Fourier Transform algorithm.

We have not gone further in the analysis of the scheme presented above. As shown in the next section, the outcome of the numerical experiments are in good agreement with the theoretical results. We plan to further analyze and generalize our approach in future works.

## 7. NUMERICAL SOLUTIONS OF MINIMIZATION PROBLEMS

Before presenting the numerical experiments, we introduce some notation for particular plane waves. Define

$$\varphi_{\mu, \rho} = \sqrt{\frac{2\mu}{T}} e^{-i \frac{\rho}{\mu} x}, \quad \text{the plane wave with } \mathcal{M}(\varphi_{\mu, \rho}) = \mu \text{ and } \mathcal{P}(\varphi_{\mu, \rho}) = \rho.$$

In the numerical experiments, we have chosen to fix  $k = 0.9$ . The period will be either  $T = 2K(k)$  or  $T = 4K(k)$ . We use  $2^{10}$  grid points for the interval  $[-\frac{T}{2}, \frac{T}{2}]$ . The time step will be set to 1. We decided to run the algorithm until a maximal difference of  $10^{-3}$  between the absolute values of the moduli of  $u_j^l$  and the expected minimizer has been reached.

We made the tests with the following initial data:

$$(a) u_0(x) = 5, \quad (b) u_0(x) = \exp(2i\pi x/T), \quad (c) u_0(x) = 1 + \cos(2i\pi x/T) + i. \quad (7.1)$$

Depending on the expected profile, we may have shifted  $u_j$  so that a minimum or a maximum of its modulus is at the boundary. Since the problem is translation invariant, this causes no loss of generality.

Since the initial data  $u_0$  in (7.1) do not match the required mass/momentum,  $u_1$  are very different from  $u_0$ . Thus (7.1) is a random choice, and this shows up in the rapid drop from  $t_0$  to  $t_1$  in Figure 7.1. The idea is to show that the choice of initial data is not important for the algorithm and that no matter from where the algorithm is starting, it converges to the supposed minimizer (unless the initial data has some symmetry preserved by the algorithm).

**7.1. Minimization Among Periodic Functions.** Minimization among periodic functions is completely covered by the theoretical results Propositions 3.2 and 3.3. We have performed different tests using the scheme described in (6.4)-(6.6) and we have found that the numerical results are in good agreement with the theoretical ones.

**7.1.1. The Focusing Case.** In all the experiments performed in this case, we have tested the scheme with and without the momentum renormalization step (6.6) and we have obtained the same result each time. This confirms that in the periodic case the momentum constraint plays no role (see (i) in Proposition 3.2, and Proposition 3.3). In what follows, we present only the results obtained using the full scheme with renormalization of mass and momentum.

We fix  $T = 2K(k)$  and  $b = 2$ . We first perform an experiment to verify the agreement with case (ii) in Proposition 3.2. Let  $m = \frac{\pi^2}{8K} < \frac{\pi^2}{bT}$ . With each initial data in (7.1), we observe convergence towards the constant solution, hereby confirming case (ii) of Proposition 3.2. The results are presented in Figure 7.1 for initial data (c) of (7.1). The requested precision is achieved after 12 time steps.

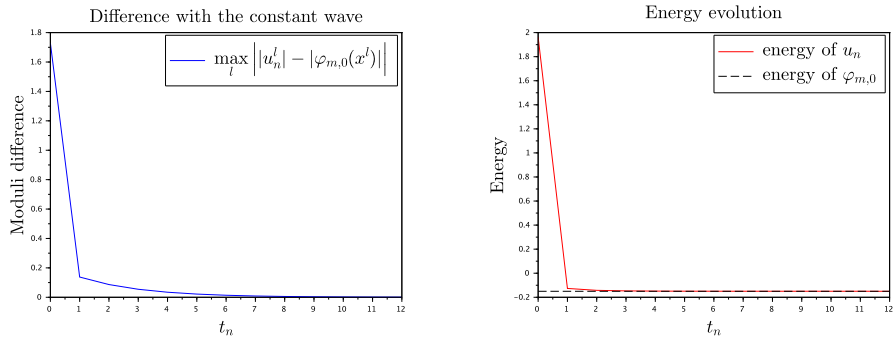
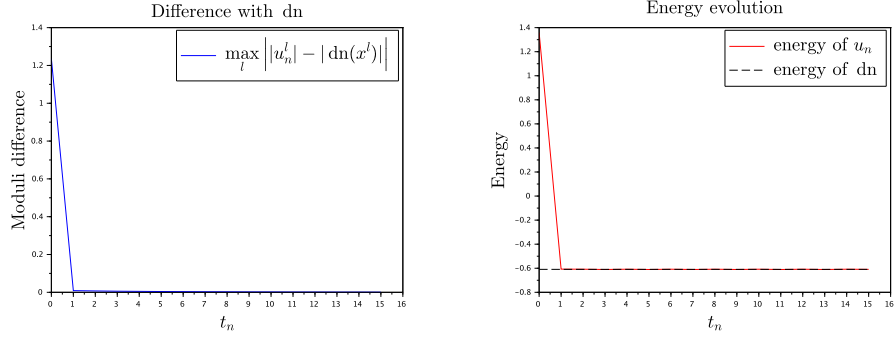


FIGURE 7.1. For  $m = \frac{\pi^2}{8K} < \frac{\pi^2}{bT}$ , focusing, periodic case

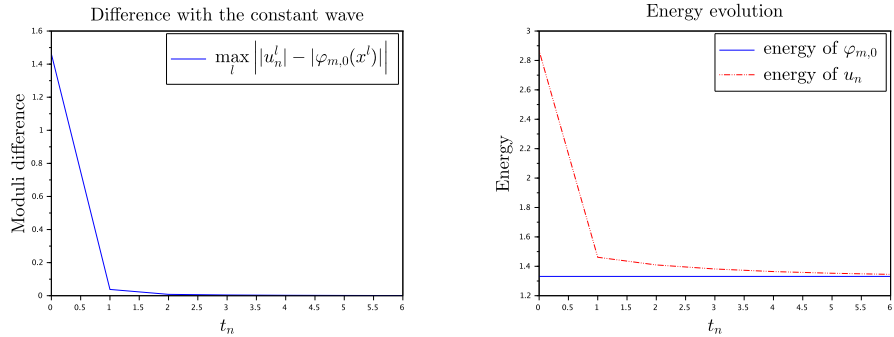
The second experiment that we perform is aimed at testing case (iv) of Proposition 3.2. Let  $m = \mathcal{M}(\text{dn}) = E(k)$ . Once again we observe a good agreement between the theoretical prediction and the numerical experiment. The results are


 FIGURE 7.2. For  $m = \mathcal{M}(\text{dn}) = E(k)$ , focusing, periodic case

presented in Figure 7.2 for initial data (c) of (7.1). The requested precision is achieved after 14 time steps.

All the other experiments that we have performed show a good agreement with the theoretical results in the focusing case for minimization among periodic functions. To avoid repetition, we give no further details here.

**7.1.2. The Defocusing Case.** We now present the experiment in the defocusing case. We have used  $b = -2k^2$  and  $T = 4K$ . We have tested the algorithm with and without the momentum renormalization step (6.6), obtaining the same results. The results are presented in Figure 7.3 for initial data (c) of (7.1) and mass constraint  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$ . The requested precision is achieved after 6 time steps.


 FIGURE 7.3. For  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$ , defocusing, periodic case

**7.2. Minimization Among Half-Anti-Periodic Functions.** We will in that case add an additional step in the algorithm in which we keep only the anti-periodic part of the function. This way it will not matter whether or not our initial data has the right anti-periodicity, since anti-periodicity will be forced at each iteration of the algorithm.



7.2.1. *The Focusing Case.* We compare in this section the numerical results with Proposition 3.4. We have used  $b = 2k^2$  and  $T = 4K$ . The tests performed show a good agreement between the numerics and the theoretical result. We present in Figure 7.4 the result for initial data (c) of (7.1) and mass constraint  $m = \mathcal{M}(\text{cn}) = 2(E - (1 - k^2)K)/k^2$

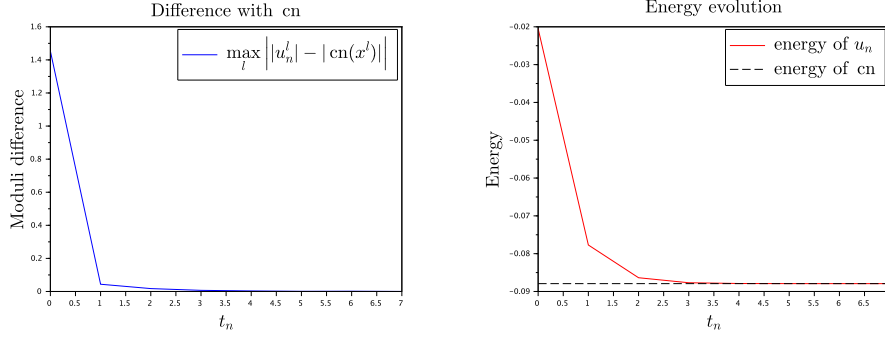


FIGURE 7.4. For  $m = \mathcal{M}(\text{cn}) = 2(E - (1 - k^2)K)/k^2$ , focusing, anti-periodic case

7.2.2. *The Defocusing Case.* We finally turn out to the defocusing case, still imposing anti-periodicity. We have used  $b = -2k^2$  and  $T = 4K$ .

We have tested the algorithm without the momentum renormalization step (6.6) and confirmed the theoretical result Proposition 3.6, which states that a plane wave is the minimizer. We present the result in Figure 7.5 for initial data (c) of (7.1) and mass constraint  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$ . Note a plateau in the two graphs of Figure 7.5. This is due to the fact that the sequence remains for some time close to sn (which is the expected minimizer if we impose in addition the momentum constraint), before eventually converging to the plane wave minimizer.

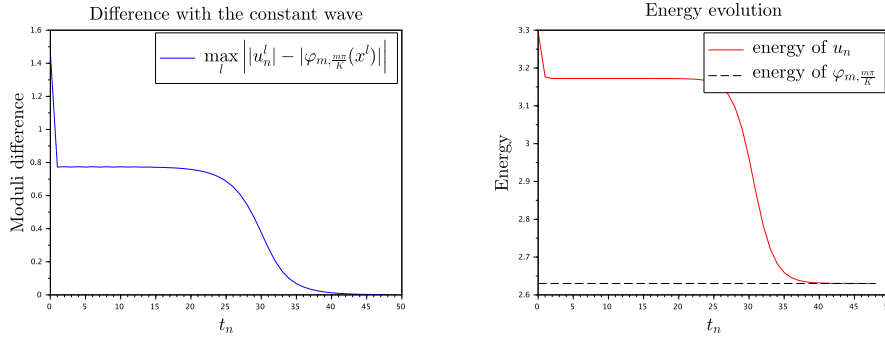


FIGURE 7.5. For  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$ , defocusing, anti-periodic case without momentum constraint

Finally, we run the full algorithm with mass and momentum renormalization for mass constraint  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$  and 0 momentum constraint. No theoretical

result is available in this case. We made the following observation, which confirms Conjecture 3.7.

**Observation 7.1.** *The function  $\text{sn}$  is a minimizer for problem (3.5) with  $m = \mathcal{M}(\text{sn})$ .*

We present in Figure 7.6 the result of the experiment with full algorithm for initial data (c) of (7.1) and mass constraint  $m = \mathcal{M}(\text{sn}) = \frac{2(K-E)}{k^2}$ .

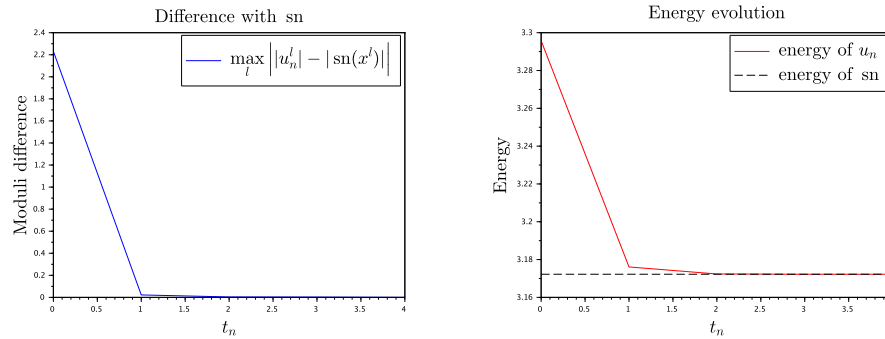


FIGURE 7.6. For  $m = \mathcal{M}(\text{sn}) = 2(E(k) - K)/k^2$ , defocusing, anti-periodic case with momentum constraint

## REFERENCES

- [1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.
- [2] J. Angulo Pava. Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations. *J. Differential Equations*, 235(1):1–30, 2007.
- [3] J. Angulo Pava. *Nonlinear dispersive equations: Existence and stability of solitary and periodic travelling wave solutions*, volume 156 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009.
- [4] W. Bao and Q. Du. Computing the ground state solution of Bose-Einstein condensates by a normalized gradient flow. *SIAM J. Sci. Comput.*, 25(5):1674–1697, 2004.
- [5] N. Bottman, B. Deconinck, and M. Nivala. Elliptic solutions of the defocusing NLS equation are stable. *J. Phys. A*, 44(28):285201, 24, 2011.
- [6] J. Bourgain. *Global solutions of nonlinear Schrödinger equations*, volume 46 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.
- [7] T. Cazenave. *Semilinear Schrödinger equations*. New York University – Courant Institute, New York, 2003.
- [8] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.
- [9] D. Chiron and C. Scheid. Travelling waves for the Nonlinear Schrödinger Equation with general nonlinearity in dimension two. *Journal of Nonlinear Science*, 26(1):171–231, 2016.
- [10] M. S. P. Eastham. *The spectral theory of periodic differential equations*. Texts in Mathematics (Edinburgh). Scottish Academic Press, Edinburgh; Hafner Press, New York, 1973.
- [11] G. Fibich. *The nonlinear Schrödinger equation*, volume 192 of *Applied Mathematical Sciences*. Springer, Cham, 2015.
- [12] T. Gallay. *Existence et stabilité des fronts dans l'équation de Ginzburg-Landau à une dimension*. PhD thesis, Université de Genève, 1994.

- [13] T. Gally and M. Hărăguș. Orbital stability of periodic waves for the nonlinear Schrödinger equation. *J. Dynam. Differential Equations*, 19(4):825–865, 2007.
- [14] T. Gally and M. Hărăguș. Stability of small periodic waves for the nonlinear Schrödinger equation. *J. Differential Equations*, 234(2):544–581, 2007.
- [15] T. Gally and D. Pelinovsky. Orbital stability in the cubic defocusing NLS equation: I. Cnoidal periodic waves. *J. Differential Equations*, 258(10):3607–3638, 2015.
- [16] A. Geyer and D. E. Pelinovsky. Spectral stability of periodic waves in the generalized reduced Ostrovsky equation. *ArXiv e-prints*, June 2016.
- [17] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015.
- [18] M. Grillakis, J. Shatah, and W. A. Strauss. Stability theory of solitary waves in the presence of symmetry I. *J. Funct. Anal.*, 74(1):160–197, 1987.
- [19] M. Grillakis, J. Shatah, and W. A. Strauss. Stability theory of solitary waves in the presence of symmetry II. *J. Funct. Anal.*, 94(2):308–348, 1990.
- [20] M. Hărăguș and T. Kapitula. On the spectra of periodic waves for infinite-dimensional Hamiltonian systems. *Phys. D*, 237(20):2649–2671, 2008.
- [21] T. Ivey and S. Laforune. Spectral stability analysis for periodic traveling wave solutions of NLS and CGL perturbations. *Phys. D*, 237(13):1750–1772, 2008.
- [22] T. Kapitula, P. G. Kevrekidis, and B. Sandstede. Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems. *Phys. D*, 195(3-4):263–282, 2004.
- [23] T. Kapitula, P. G. Kevrekidis, and B. Sandstede. Addendum: “Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems” [Phys. D **195** (2004), no. 3-4, 263–282; mr2089513]. *Phys. D*, 201(1-2):199–201, 2005.
- [24] D. F. Lawden. *Elliptic functions and applications*, volume 80 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.
- [25] S. Le Coz, D. Li, and T.-P. Tsai. Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(6):1251–1282, 2015.
- [26] S. Le Coz and T.-P. Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity*, 27(11):2689–2709, 2014.
- [27] Y. C. Ma and M. J. Ablowitz. The periodic cubic Schrödinger equation. *Stud. Appl. Math.*, 65(2):113–158, 1981.
- [28] J. L. Marzuola and D. E. Pelinovsky. Ground State on the Dumbbell Graph. *Appl. Math. Res. Express. AMRX*, 2016(1):98–145, 2016.
- [29] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
- [30] G. Rowlands. On the stability of solutions of the non-linear Schrödinger equation. *IMA Journal of Applied Mathematics*, 13(3):367–377, 1974.
- [31] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

(Stephen Gustafson and Tai-Peng Tsai) DEPARTMENT OF MATHEMATICS,  
 UNIVERSITY OF BRITISH COLUMBIA,  
 VANCOUVER BC  
 CANADA V6T 1Z2  
*E-mail address*, Stephen Gustafson: [gustaf@math.ubc.ca](mailto:gustaf@math.ubc.ca)  
*E-mail address*, Tai-Peng Tsai: [ttsai@math.ubc.ca](mailto:ttsai@math.ubc.ca)

(Stefan Le Coz) INSTITUT DE MATHÉMATIQUES DE TOULOUSE,  
 UNIVERSITÉ PAUL SABATIER  
 118 ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX 9  
 FRANCE  
*E-mail address*, Stefan Le Coz: [slecoz@math.univ-toulouse.fr](mailto:slecoz@math.univ-toulouse.fr)