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To cite this version:

Mehrdad Lakestani, Mehdi Dehghan. The solution of a second-order nonlinear differential equation with Neumann boundary conditions using semi-orthogonal B-spline wavelets. International Journal of Computer Mathematics, 2006, 83 (8-9), pp.685-694. 10.1080/00207160601025656 . hal-01330587

HAL Id: hal-01330587
https://hal.science/hal-01330587
Submitted on 13 Jun 2016

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The solution of a second-order nonlinear differential equation with Neumann boundary conditions using semi-orthogonal B-spline wavelets

MEHRDAD LAKESTANI†‡ and MEHDI DEHGHAN*¶

†Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
‡Department of Applied Mathematics, Amirkabir University of Technology,
No. 424 Hafez Avenue, Tehran, Iran

A numerical technique for solving a second-order nonlinear Neumann problem is presented. The authors’ approach is based on semi-orthogonal B-spline wavelets. Two test problems are presented and numerical results are tabulated to show the efficiency of the proposed technique for the studied problem.

Keywords: Nonlinear Neumann problem; Semi-orthogonal B-spline wavelets

1. Introduction

In this paper we solve the second-order nonlinear Neumann problem of the form

\[-\ddot{x}(t) = f(t, x(t)), \quad t \in [0, 1],\]
\[
\dot{x}(0) = A, \quad \dot{x}(1) = B.
\]

The existence of a solution to equation (1) with Neumann boundary conditions is studied in [5] using the quasi-linearization method. In the present paper we apply compactly supported linear semi-orthogonal (SO) B-spline wavelets, specially constructed for the bounded interval, to solve the nonlinear second-order Neumann problem of the form (1).

In recent years, the application of methods based on wavelets has influenced many areas of applied mathematics. In areas such as numerical analysis and differential equations, wavelets are recognized as a powerful tool. Another area in which the wavelet is gaining considerable attention is the study of integral equations. Wavelets can be separated into two distinct types, orthogonal and semi-orthogonal [1]. Publications on integral equation methods have shown a marked preference for orthogonal wavelets [2]. This is probably because the original wavelets,
which were widely used for signal processing, were primarily orthogonal. In signal processing
applications, unlike integral equation methods, the wavelet itself is never constructed since
only its scaling function and coefficients are needed. However, orthogonal wavelets either have
infinite support or an asymmetric, and in some cases fractal, nature. These properties can make
them a poor choice for characterization of a function. In contrast, semi-orthogonal wavelets
have finite support, both even and odd symmetry and simple analytical expressions—ideal
attributes for a basis function [3].

The use of SO compactly supported spline wavelets is justified by their interesting proper-
ties [3]. Our method consists of reducing expressions (1) and (2) to a set of algebraic equations
by expanding the unknown function as linear B-spline wavelets with unknown coefficients.
The properties of linear B-spline scaling functions and wavelets are then utilized to evaluate
the unknown coefficients.

This paper is organized as follows. In section 2, we describe the formulation of the B-spline
scaling functions and wavelets on [0,1], and derive the operational matrices of derivatives
required for our subsequent development. In section 3, the proposed method is used to approxi-
mate the solution of the second-order nonlinear Neumann problem. In section 4, we report
our computational results and demonstrate the accuracy of the proposed numerical scheme by
presenting numerical examples. Section 5 ends the paper with a brief conclusion.

2. B-spline scaling functions and wavelets on [0, 1]

When semi-orthogonal wavelets are constructed from B-splines of order \( m \), the lowest octave
level \( j = j_0 \) is determined in [4] by

\[
2^{j_0} \geq 2m - 1,
\]

(3)
to give a minimum of one complete wavelet on the interval [0, 1]. In this paper we will use a
wavelet generated by a linear spline (\( m = 2 \)) cardinal B-spline function. From equation (2),
the second-order B-spline lowest level, which must be an integer, is determined to be \( j_0 = 2 \). This
constrains all octave levels to \( j \geq 2 \).

As is the case with all semi-orthogonal wavelets, the second-order B-splines also serve as
scaling functions. The second-order B-splines scaling functions are given by

\[
\phi_{j,k}(t) = \begin{cases} 
  t_j - k & \text{for } k \leq t_j \leq k + 1, \\
  2 - (t_j - k) & \text{for } k + 1 \leq t_j \leq k + 2, \quad k = 0, \ldots, 2^j - 2, 0 \\
  0 & \text{otherwise,}
\end{cases}
\]

(4)

with the left- and right-hand side boundary scaling functions

\[
\phi_{j,k}(t) = \begin{cases} 
  2 - (t_j - k) & \text{for } 0 \leq t_j \leq 1, \quad k = -1, \\
  0 & \text{otherwise,}
\end{cases}
\]

(5)

and

\[
\phi_{j,k}(t) = \begin{cases} 
  t_j - k & \text{for } \leq t_j \leq k + 1, \quad k = 2^j - 1, \\
  0 & \text{otherwise,}
\end{cases}
\]

(6)
respectively. The actual coordinate position \( t \) is related to \( t_j \), according to \( t_j = 2^j t \). The second-order B-spline wavelets are given by

\[
\psi_{j,k}(t) = \frac{1}{6} \begin{cases} 
    t_j - k, & \text{for } k \leq t_j \leq k + 1/2, \\
    4 - 7(t_j - k), & \text{for } k + 1/2 \leq t_j \leq k + 1, \\
    -19 + 16(t_j - k), & \text{for } k + 1 \leq t_j \leq k + 3/2, \\
    29 - 16(t_j - k), & \text{for } k + 3/2 \leq t_j \leq k + 2, \\
    -17 + 7(t_j - k), & \text{for } k + 2 \leq t_j \leq k + 5/2, \\
    3 - (t_j - k), & \text{for } k + 5/2 \leq t_j \leq k + 3, \\
    0 & \text{otherwise},
\end{cases}
\]

with the left- and right- side boundary wavelets

\[
\psi_{j,k}(t) = \frac{1}{6} \begin{cases} 
    -6 + 23t_j, & \text{for } 0 \leq t_j \leq 1/2, \\
    14 - 17t_j, & \text{for } 1/2 \leq t_j \leq 1, \\
    -10 + 7t_j, & \text{for } 1 \leq t_j \leq 3/2, \quad k = -1, \\
    2 - t_j, & \text{for } 3/2 \leq t_j \leq 2, \\
    0 & \text{otherwise},
\end{cases}
\]

\[
\psi_{j,k}(t) = \frac{1}{6} \begin{cases} 
    2 - (k + 2 - t_j), & \text{for } k \leq t_j \leq k + 1/2, \\
    -10 + 7(k + 2 - t_j), & \text{for } k + 1/2 \leq t_j \leq k + 1, \\
    14 - 17(k + 2 - t_j), & \text{for } k + 1 \leq t_j \leq k + 3/2, \quad k = 2^j - 2, \\
    -6 + 23(k + 2 - t_j), & \text{for } k + 3/2 \leq t_j \leq k + 2, \\
    0 & \text{otherwise},
\end{cases}
\]

respectively. From expressions (4)–(9) we get

\[
\begin{align*}
\phi_{i,j}(t) & = \frac{1}{2} \phi_{i+1,2j}(t) \\
& \quad + \phi_{i+1,2j+1}(t) + \frac{1}{2} \phi_{i+1,2j+2}(t), \quad i = 2, 3, \ldots, \quad j = 0, 1, \ldots, 2^i - 2, \\
\phi_{i,-1}(t) & = \phi_{i+1,-1}(t) + \frac{1}{2} \phi_{i+1,0}(t), \quad i = 2, 3, \ldots \\
\phi_{i,2-1}(t) & = \phi_{i+1,2+1-1}(t) + \frac{1}{2} \phi_{i+1,2+1-2}(t), \quad i = 2, 3, \ldots
\end{align*}
\]
\[
\psi_{i,j}(t) = \frac{1}{12} \phi_{i+1,2j}(t) - \frac{1}{2} \phi_{i+1,2j+1}(t) + \frac{5}{6} \phi_{i+1,2j+2}(t) - \frac{1}{2} \phi_{i+1,2j+3}(t) + \frac{1}{12} \phi_{i+1,2j+4}(t), \quad i = 2, 3 \ldots j = 0, 1, 2, \ldots, 2^i - 3,
\]

\[
\psi_{i,-1}(t) = -\phi_{i+1,-1}(t) + \frac{11}{12} \phi_{i+1,0}(t) - \frac{1}{2} \phi_{i+1,1}(t) + \frac{1}{12} \phi_{i+1,2}(t), \quad i = 2, 3 \ldots
\]

\[
\psi_{i,2^{-2}}(t) = -\phi_{i+1,2^{-2}+1}(t) + \frac{11}{12} \phi_{i+1,2^{-2}+2}(t) - \frac{1}{2} \phi_{i+1,2^{-2}+3}(t)
\]

\[
- \frac{1}{2} \phi_{i+1,2^i+1}(t) + \frac{1}{12} \phi_{i+1,2^i+2}(t), \quad i = 2, 3 \ldots
\]

\[
(11)
\]

### 2.1 Function approximation

A function \( f(t) \) defined over \([0,1]\) may be represented by B-spline scaling functions as

\[
f(t) = \sum_{k=-1}^{2M+1-1} s_k \phi_{M+1,k} = S^T \Phi_{M+1},
\]

where

\[
S = [s_{-1}, s_0, \ldots, s_{2^{M+1}-1}]^T
\]

\[
\Phi_{M+1} = [\phi_{M+1,-1}, \phi_{M+1,0}, \ldots, \phi_{M+1,2^{M+1}-1}]^T
\]

with

\[
s_k = \int_0^1 f(t) \tilde{\phi}_{M+1,k}(t) \, dt, \quad k = -1, 0, \ldots, 2^{M+1} - 1,
\]

in which \( \tilde{\phi}_{M+1,k}(t) \) are dual functions of \( \phi_{M+1,k}(t) \). These can be obtained by linear combinations of \( \phi_{M+1,k}(t), k = -1, \ldots, 2^{M+1} - 1 \), as follows. Let \( \tilde{\Phi}_{M+1} \) be the dual functions of \( \Phi_{M+1} \) given by

\[
\tilde{\Phi}_{M+1} = [\tilde{\phi}_{M+1,-1}, \tilde{\phi}_{M+1,0}, \ldots, \tilde{\phi}_{M+1,2^{M+1}-1}]^T.
\]

Equations (12) and (13) give

\[
\int_0^1 \tilde{\Phi}_{M+1} \Phi_{M+1}^T \, dt = I_1,
\]

where \( I_1 \) is the \( (2^{M+1} + 1) \times (2^{M+1} + 1) \) identity matrix. Using expressions (4)–(6) and (12) we have

\[
\int_0^1 \Phi_{M+1} \Phi_{M+1}^T \, dt = P_{M+1} = \frac{1}{2^{M-1}}
\]

\[
\begin{bmatrix}
12 & 1 & 0 & \cdots & 0 \\
24 & 1 & 1 & \cdots & 0 \\
24 & 6 & 24 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \frac{1}{24} & 1 & 1 \\
0 & \cdots & 0 & \frac{24}{24} & 12 \\
\end{bmatrix},
\]

\[
(15)
\]
where $P_{M+1}$ is a symmetric $(2^{M+1} + 1) \times (2^{M+1} + 1)$ matrix. From (14) and (15) we get

$$\Phi_{M+1} = (P_{M+1})^{-1} \Phi_{M+1}.$$  

Furthermore, a function $f(x)$ defined over $[0,1]$ may be represented by B-spline wavelets as

$$f(t) = \sum_{k=-1}^{3} c_k \phi_{2,k}(t) + \sum_{i=2}^{M} \sum_{j=-1}^{2^{i-2}} d_{i,j} \psi_{i,j}(t) = \alpha^T \Psi(t),$$  \hspace{1cm} (16)

where $\phi_{2,k}$ and $\psi_{i,j}$ are scaling and wavelets functions, respectively, and $\alpha$ and $\Psi$ are $(2^{(M+1)} + 1) \times 1$ vectors given by

$$\alpha = [c_{-1}, c_0, \ldots, c_3, d_{2,-1}, \ldots, d_{2,2}, d_{3,-1}, \ldots, d_{3,6}, \ldots, d_{M,-1}, \ldots, d_{M,2^{M-2}}]^T,$$

$$\Psi = [\phi_{2,-1}, \phi_{2,0}, \ldots, \phi_{2,3}, \psi_{2,-1}, \ldots, \psi_{2,2}, \psi_{3,-1}, \ldots, \psi_{3,6}, \ldots, \psi_{M,-1}, \ldots, \psi_{M,2^{M-2}}]^T.$$  \hspace{1cm} (17, 18)

### 2.2 The operational matrices of derivatives

The differentiation of vectors $\Phi_{M+1}$ and $\Psi$ in (12) and (18) can be expressed as

$$\Phi'_{M+1} = D_{\Phi} \Phi_{M+1},$$  \hspace{1cm} (19)

$$\Psi' = D_{\Psi} \Psi,$$  \hspace{1cm} (20)

where $D_{\Phi}$ and $D_{\Psi}$ are $(2^{M+1} + 1) \times (2^{M+1} + 1)$ operational matrices of derivatives for B-spline scaling functions and wavelets, respectively. The matrix $D_{\Phi}$ can be obtained by considering

$$D_{\Phi} = \int_0^1 \Phi'_{M+1}(t) \Phi_{M+1}(t) dt = \left( \int_0^1 \Phi'_{M+1}(t) \Phi_{M+1}(t) dt \right) (P_{M+1})^{-1},$$  \hspace{1cm} (21)

where

$$E = \int_0^1 \Phi_{M+1}(t) \Phi_{M+1}(t) dt.$$  \hspace{1cm} (22)

In (22), $E$ is a $(2^{M+1} + 1) \times (2^{M+1} + 1)$ matrix given by

$$E = \begin{bmatrix}
\int_0^1 \phi_{M+1,-1}^\prime(t) \phi_{M+1,-1}(t) dt & \cdots & \int_0^1 \phi_{M+1,-1}^\prime(t) \phi_{M+1,2M+1-1}(t) dt \\
\vdots & \ddots & \vdots \\
\int_0^1 \phi_{M+1,2M+1-1}^\prime(t) \phi_{M+1,-1}(t) dt & \cdots & \int_0^1 \phi_{M+1,2M+1-1}^\prime(t) \phi_{M+1,2M+1-1}(t) dt
\end{bmatrix}.$$  

For any entries of $E_{j,k}$ we have

$$E_{j,k} = \int_0^1 \phi_{M+1,j}^\prime(t) \phi_{M+1,k}(t) dt = \int_{(j+1)/2^{M+1}}^{(j+2)/2^{M+1}} 2^{M+1} \phi_{M+1,k}(t) dt + \int_{(j+2)/2^{M+1}}^{(j+1)/2^{M+1}} 2^{M+1} \phi_{M+1,k}(t) dt.$$  \hspace{1cm} (23)
From (23) we get

\[
E = \begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2}
\end{bmatrix}.
\]

The matrix \( D \psi \) can be obtained by considering

\[
\Psi = G \Phi_{M+1},
\]  

(24)

where \( G \) is a \((2^{M+1} + 1) \times (2^{M+1} + 1)\) matrix, which can be calculated as follows. Let

\[
\Phi_j = [\phi_{j,-1}, \phi_{j,0}, \ldots, \phi_{j,2j-1}]^T, 
\]  

(25)

\[
\Psi_j = [\psi_{j,-1}, \psi_{j,0}, \ldots, \psi_{j,2j-2}]^T.
\]  

(26)

Equations (10) and (25) give

\[
\Phi_j = \beta_j \Phi_{j+1},
\]  

(27)

with

\[
\beta_j = \begin{bmatrix}
1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2}
\end{bmatrix},
\]

where \( \beta_j, j = 2, 3, \ldots, \) is a \((2^j + 1) \times (2^{j+1} + 1)\) matrix. From (11) and (20) we have

\[
\Psi_j = L_j \Phi_{j+1},
\]  

(28)
\[ L_j = \begin{bmatrix}
-1 & \frac{11}{12} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\
\end{bmatrix}, \]

where \( L_j, j = 2, 3, \ldots \), is a \( 2^j \times (2^{j+1} + 1) \) matrix. Using expressions (24), (27) and (28), we get

\[ G = \begin{bmatrix}
\beta_2 \times \beta_3 \times \cdots \times \beta_M \\
L_2 \times \beta_3 \times \cdots \times \beta_M \\
\vdots \\
L_{M-2} \times \beta_{M-1} \times \beta_M \\
L_{M-1} \beta_M \\
\ldots \\
L_M \\
\end{bmatrix}. \quad (29)\]

Using expressions (19), (21), (24) and (29) we have

\[ \Psi' = G \Phi_{M+1}' = GD_\Phi \Phi_{M+1} = GE(P_{M+1})^{-1} \Phi_{M+1} = GE(P_{M+1})^{-1} G^{-1} \Psi = D_\Psi \Psi, \quad (30)\]

where

\[ D_\Psi = GE(P_{M+1})^{-1} G^{-1}. \]

3. The second-order nonlinear differential equation

In this section we solve a second-order nonlinear Neumann problem of the form (1) by using B-spline wavelets. For this purpose, we use equation (16) to approximate \( x(t) \) and \( f(t, x(t)) \).
as
\[ x(t) = C^T \Psi(t), \quad (31) \]
\[ f(t, x(t)) = F^T \Psi(t), \quad (32) \]

where \( \Psi(t) \) is defined in (18) and \( C \) and \( F \) are \((2^{M+1} + 1) \times 1\) unknown vectors defined similar to \( \alpha \) in (17). Using (30) and (31) gives
\[ \dot{x}(t) = C^T \Psi'(t) = C^T D \Psi(t), \quad (33) \]
\[ \ddot{x}(t) = C^T D^2 \Psi(t). \quad (34) \]

From expressions (1), (32) and (33) we have
\[ -C^T D^2 w / \Psi(t) = F^T \Psi(T). \quad (35) \]

The entries of vector \( \Psi(t) \) are independent, so using (35) we get
\[ -C^T D^2 w = F^T. \quad (36) \]
Expressions (31) and (32) yield
\[ f(t, C^T \Psi(t)) = F^T \Psi(t). \quad (37) \]

Also using boundary values in expressions (2) and (33) we have
\[ C^T D w / \Psi(0) = A, \quad (38) \]
\[ C^T D w / \Psi(1) = B. \quad (39) \]

To find the solution in (1) we first collocate (37) in \( t_i = (2i - 1)/(2^{M+1} - 2), i = 1, 2, \ldots, 2^{M+1} - 1 \). The resulting equation generates \( 2^{M+1} - 1 \) algebraic equations. The total unknowns for vector \( C \) in (31) and \( F \) in (32) are \( 2 \times (2^{M+1} + 1) \). These unknowns can be obtained by using expressions (36)–(39).

4. Numerical examples

To support our theoretical discussion, we applied the method presented in this paper to several examples. To the best of our knowledge, the problem studied in this paper has not been investigated previously. Thus we could not compare the results obtained for our four test problems with others.

4.1 Example 1

Consider the second-order linear Neumann problem
\[ -\ddot{x}(t) = (2 - 4t^2) x(t), \quad t \in [0, 1], \]
\[ \dot{x}(0) = 0, \quad \dot{x}(1) = -2/e. \]

The exact solution of this problem is \( e^{-t^2} \). Table 1 shows the absolute values of errors of the solution at some points using the method proposed in this paper.
4.2 Example 2

As the second test problem, consider the nonlinear Neumann problem

\[-\ddot{x}(t) = 2x^3(t), \quad t \in [0, 1],\]

\[\dot{x}(0) = -1, \quad \dot{x}(1) = -1/4.\]

The exact solution of this problem is $1/(t + 1)$. Table 2 shows the absolute values of errors of the solution at some points using the technique developed in this work.

4.3 Example 3

Consider the Neumann problem

\[-\ddot{x}(t) = 4x(t) - 2, \quad t \in [0, 1],\]

\[\dot{x}(0) = 0, \quad \dot{x}(1) = \sin(2).\]

The exact solution of this problem is $\sin^2(t)$. Table 3 shows the absolute values of errors of the solution at some points using the method proposed in this paper.
4.4 Example 4

Consider the Neumann problem

\[ \dddot{x}(t) = x(t), \quad t \in [0, 1], \]
\[ \ddot{x}(0) = 0, \quad \dot{x}(1) = \sinh(1). \]

The exact solution of this problem is \( \cosh(t) \). Table 4 shows the absolute values of errors of the solution at some points using the method proposed in this paper.

5. Conclusion

In this paper a second-order differential equation with Neumann boundary conditions was investigated. The semi-orthogonal B-spline wavelet was proposed for solving this problem. The proposed technique was tested on some examples. The results obtained show the efficiency of the method presented.

References