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A BOUNDARY THIN OBSTACLE PROBLEM FOR A WAVE EQUATION

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0. Introduction.

In this paper we shall discuss a hyperbolic variational inequality associated with the following initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \text{ in } \Omega \times (0, T), \quad (0-1)$$

$$u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x) \text{ in } \Omega, \quad (0-2)$$

$$u = 0 \text{ on } \Gamma_1 \times (0, T), \quad (0-3)$$

$$u(x, t) \geq \phi(x) \text{ on } \Gamma_2 \times (0, T), \quad (0-4)$$

$$\frac{\partial u}{\partial \nu} \geq 0, \quad (u - \phi) \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2 \times (0, T). \quad (0-5)$$

Here Ω is a bounded open subset of R^n , $n \geq 2$, with smooth boundary $\partial\Omega \in C^2$. We assume that Γ_1 and Γ_2 are disjoint, nonempty open subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_1 \cup \bar{\Gamma}_2 = \bar{\Gamma}_1 \cup \Gamma_2$ and $\partial\Gamma_1 = \partial\Gamma_2 \in C^2$. The above functions $f(x, t)$, $u_0(x)$, $u_1(x)$ and $\phi(x)$ are given, and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$. We assume that $\phi \leq 0$ and $u_0 \geq \phi$ on Γ_2 . We can put the above problem in the form of a variational inequality: find $u(x, t)$ which satisfies (0-2), (0-3), (0-4) and

$$< \frac{\partial^2 u}{\partial t^2}, w - u > + a(u, w - u) \geq < f, w - u > \quad (0-6)$$

for every $w \in G$, for all $t \in (0, T)$. Here we use the notation:

$$\begin{aligned} \langle u, w \rangle &= \int_{\Omega} u w dx, \\ a(u, w) &= \int_{\Omega} \nabla u \cdot \nabla w dx, \\ G &= \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_1 \text{ and } w \geq \phi \text{ on } \Gamma_2\}. \end{aligned}$$

If a solution u of this new problem is sufficiently smooth, one can show that u also satisfies (0-1) and (0-5) so that u is a solution of the original problem (0-1) through (0-5). The corresponding stationary variational inequality: find $u \in G$ which satisfies

$$a(u, w - u) \geq \langle f, w - u \rangle, \text{ for all } w \in G, \quad (0-7)$$

is called the boundary thin obstacle problem (or the Signorini problem), and its mathematical theory has been well-established: see [6] and references therein. However, to the author's knowledge, nothing has been known about the solution of (0-1) through (0-5). The purpose of this paper is to establish the existence of a (weak) solution of (0-1) through (0-5). This problem is an example of a hyperbolic variational inequality where a constraint is imposed on the unknown function rather than on the time derivative of the unknown function. Hyperbolic variational inequalities have been extensively investigated in the case where a constraint is imposed on the time derivative of the unknown function; see [3] and [11] among others. On the other hand, when a constraint is imposed on the unknown function, existence theorems seem to be known mainly for special problems of space dimension one; see [1], [2], [5], [8], [12], [13] and [14]. In the case of several space dimensions, [10] and this paper seem to be the only known works. [10] resolved the above problem by microlocal techniques when Ω is a half-space; but the same method does not extend to a general domain, as is explained in [10].

In this paper, the basic procedure to prove the existence of a solution consists of a penalty method [11], a multiplier technique [9] and compensated compactness [4]. Finally, we note that the questions of uniqueness and energy conservation for solutions of our problem are still open.

1. Notation and Preliminaries.

$$\partial_t = \frac{\partial}{\partial t}, \partial_{tt} = \frac{\partial^2}{\partial t^2}, \nabla g = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right).$$

When E is an open subset of R^N , $N \geq 1$, $\mathcal{D}'(E)$ denotes the space of distributions over E and $C_0^m(E) = \{w \in C^m(E) : \text{supp } w \text{ is a compact subset of } E\}$. We assume that $\partial\Omega \in C^2$ and $\partial\Gamma_1 = \partial\Gamma_2 \in C^2$. For $m = \text{nonnegative integer}$,

$$H^m(\Omega) = \{w \in L^2(\Omega) : \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} w \in L^2(\Omega), 0 \leq \alpha_1 + \dots + \alpha_n \leq m\},$$

$$H_0^m(\Omega) = \text{the completion of } C_0^\infty(\Omega) \text{ in } H^m(\Omega),$$

$$H^{-m}(\Omega) = \text{the dual of } H_0^m(\Omega),$$

$$H_{\Gamma_1}^1(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_1\}.$$

For s =nonnegative real, $H^s(\Omega)$ is defined by interpolation and $H^{-s}(\Omega)$ =the dual of $H_0^s(\Omega)$. We denote by \langle, \rangle the duality pairing between $H^{-\alpha}(\Omega)$ and $H^\alpha(\Omega)$, $0 \leq \alpha \leq \frac{1}{2}$. Then, for $g \in L^2(\Omega)$, $h \in H^\alpha(\Omega)$, $0 \leq \alpha \leq \frac{1}{2}$, we have $\langle g, h \rangle = \int_\Omega g h dx$.

As above, the bilinear form $a(\cdot, \cdot)$ is defined by $a(g, h) = \int_\Omega \nabla g \cdot \nabla h dx$ and

$$G = \{w \in H_{\Gamma_1}^1(\Omega) : w \geq \phi \text{ on } \Gamma_2\},$$

where $\phi(x) \in C^1(\bar{\Omega})$ and $\phi(x) \leq 0$ on Γ_2 .

Next we define the distance function d by $d(x) = \text{dist}(x, \partial\Omega)$ and set

$$V_\delta = \{x \in \bar{\Omega} : d(x) < \delta\}.$$

Let δ_0 be a positive number such that $1/\delta_0$ bounds the principal curvatures of $\partial\Omega$. Since Ω is bounded and $\partial\Omega \in C^2$, it is known that $d(x) \in C^2(V_{\delta_0})$; see [7]. We also set

$$U_\delta = \{x \in \Omega : d(x) < \delta\},$$

$$S_\delta = \{x \in \Omega : d(x) = \delta\}.$$

Then, S_δ is a C^2 -surface for $0 \leq \delta < \delta_0$. We shall use the following lemmas later.

LEMMA 1.1. *There is a function $h(x) \in C^1(\bar{\Omega})^n$ such that $h(x)$ =the outward unit normal vector at x on S_δ if $d(x) = \delta \leq \delta_0/2$ and $h(x) = 0$ if $d(x) \geq \delta_0$.*

PROOF: For each $x \in S_\delta$, $0 \leq \delta < \delta_0$, there is a unique point $y(x) \in \partial\Omega$ such that $\text{dist}(x, y(x)) = \delta$ and the outward normal vector at $y(x)$ on $\partial\Omega$ is also outward normal at x on S_δ . Furthermore, $y(x) \in C^1(V_{\delta_0})$. Next we denote by $\nu(y)$ the outward unit normal vector at y on $\partial\Omega$. Then, $\nu(y)$ is a C^1 function of y . Let us choose a function $\rho(t) \in C^1(R)$ such that $\rho(t) = 1$ for $t \leq \delta_0/2$ and $\rho(t) = 0$ for $t > \frac{2}{3}\delta_0$. We then set $h(x) = \nu(y(x))\rho(d(x))$, which achieves our goal.

The next lemma is known and stated without proof.

LEMMA 1.2. *Let $g \in L^1(\Omega)$. Then, for $0 < \delta^* < \delta_0$,*

$$\int_{U_{\delta^*}} g dx = \int_0^{\delta^*} \int_{S_\delta} g d\sigma_\delta d\delta \quad (1-1)$$

holds where $d\sigma_\delta$ is the surface element on S_δ .

Next we write a well-known consequence of the Div-Curl lemma of compensated compactness.

LEMMA 1.3. *Let $\{u_k\}$ be a sequence of functions such that*

$$\|u_k\|_{L^2(0,T;H^1(\Omega))} \leq M, \quad (1-2)$$

$$\|\partial_t u_k\|_{L^2(0,T;L^2(\Omega))} \leq M, \quad (1-3)$$

where M is a positive constant independent of k and each u_k satisfies

$$\frac{\partial^2 u_k}{\partial t^2} - \Delta u_k = g_k \quad (1-4)$$

where $\{g_k\}$ is a bounded sequence in $L^2(\Omega \times (0, T))$, and, as $k \rightarrow \infty$,

$$u_k \rightarrow u \text{ weakly in } L^2(0, T; H^1(\Omega)). \quad (1-5)$$

Then, as $k \rightarrow \infty$

$$\left(\frac{\partial u_k}{\partial t}\right)^2 - |\nabla u_k|^2 \rightarrow \left(\frac{\partial u}{\partial t}\right)^2 - |\nabla u|^2 \quad (1-6)$$

in $\mathcal{D}'(\Omega \times (0, T))$.

PROOF: Set $v_{k_i} = \frac{\partial u}{\partial x_i}$, $w_{k_i} = \frac{\partial u}{\partial x_i}$, $i = 1, \dots, n$, and $v_{k(n+1)} = -\frac{\partial u}{\partial t}$, $w_{k(n+1)} = \frac{\partial u}{\partial t}$. Then, apply Corollary 4.3 of [4].

Finally, we state

LEMMA 1.4. *Let $\{u_k\}$ be a sequence of functions such that as $k \rightarrow \infty$,*

$$u_k \rightarrow u \text{ weak}^* \text{ in } L^\infty(0, T; H^\beta(\Omega)) \quad (1-7)$$

$$\partial_i u_k \rightarrow \partial_i u \text{ weakly in } L^2(0, T; H^\alpha(\Omega)) \quad (1-8)$$

where $-1 \leq \alpha < \beta \leq 1$. Then, we have

$$u_k \rightarrow u \text{ in } C([0, T]; H^r(\Omega)) \quad (1-9)$$

for any $r < \beta$.

PROOF: A special case was proved in [8]. The same argument can still be employed. Without loss of generality, we may assume $\alpha \neq -\frac{1}{2}$, $\beta \neq -\frac{1}{2}$, because we can take smaller α or β if necessary.

Let $r = \theta\alpha + (1 - \theta)\beta$, $0 < \theta < 1$, $r \neq -\frac{1}{2}$, $\frac{1}{2}$. There is $E \subset [0, T]$ such that $\text{meas}(E) = 0$ and for all $t_1, t_2 \in [0, T] \setminus E$, $t_1 < t_2$,

$$\begin{aligned}
& \|u_k(t_2) - u_k(t_1)\|_{H^r(\Omega)} \leq \\
& \leq M_\theta \|u_k(t_2) - u_k(t_1)\|_{H^\alpha(\Omega)}^\theta \|u_k(t_2) - u_k(t_1)\|_{H^\beta(\Omega)}^{1-\theta}, \\
& \leq M_\theta \left(\int_{t_1}^{t_2} \|\partial_t u_k(t)\|_{H^\alpha(\Omega)} dt \right)^\theta \leq M_\theta (t_2 - t_1)^{\theta/2}, \tag{1-10}
\end{aligned}$$

holds for all k and $0 < \theta < 1$ such that $\theta\alpha + (1-\theta)\beta \neq \frac{1}{2}, -\frac{1}{2}$, where M_θ denotes positive constants independent of t_1, t_2 and k . Consequently, each u_k belongs to $C([0, T]; H^r(\Omega))$ for every $\alpha < r < \beta$ possibly after a modification on E . Since the embedding $H^{r_2}(\Omega) \subset H^{r_1}(\Omega)$ is compact for every $r_1 < r_2$, we can use the Ascoli theorem to arrive at the conclusion.

2. Approximate Problem.

In this section we shall consider an approximate problem with a penalty parameter $k > 0$. The problem is: find a function $u_k(x, t)$ such that

$$\partial_{tt} u_k(x, t) \in L^\infty(0, T; L^2(\Omega)), \partial_t u_k \in L^\infty(0, T; H_{\Gamma_1}^1(\Omega)) \tag{2-1}$$

$$u_k(x, 0) = u_{0k}(x), \partial_t u_k(x, 0) = u_{1k}(x) \tag{2-2}$$

$$\langle \partial_{tt} u_k, w \rangle + a(u_k, w) = \langle f_k, w \rangle + \int_{\Gamma_2} (k(u_k - \phi)^- - \frac{1}{k} \partial_t u_k) w d\sigma \tag{2-3}$$

holds for all $w \in H_{\Gamma_1}^1(\Omega)$, for almost all $t \in (0, T)$. Here $d\sigma$ is the surface element on $\partial\Omega$ and $g^- = \max(0, -g)$.

Our strategy consists of three steps.

- (Step 1) Establish the existence of solution of (2-1), (2-2) and (2-3).
- (Step 2) Obtain a priori estimates independent of $k > 0$.
- (Step 3) Pass $k \rightarrow \infty$ to arrive at a solution of the original problem.

In this section, the first two steps will be carried out.

2.1 Existence of solution

Throughout this subsection, we shall suppress the subscript k .

PROPOSITION 2.1. *Let $u_0, u_1 \in C_0^\infty(\Omega)$, $f \in C^1([0, T]; L^2(\Omega))$ and $\phi \in C^1(\bar{\Omega})$ with $\phi \leq 0$ on Γ_2 . Then there is a solution of (2-1), (2-2) and (2-3).*

PROOF: We set $u = v + u_0 + tu_1$. Then,

$$v(x, 0) \equiv 0, \partial_t v(x, 0) \equiv 0, \tag{2-4}$$

and (2-3) is equivalent to

$$\begin{aligned}
& \langle \partial_{tt} v, w \rangle + a(v, w) = \langle f, w \rangle \\
& + \langle \Delta(u_0 + tu_1), w \rangle + k \int_{\Gamma_2} (v - \phi)^- w d\sigma \\
& - \frac{1}{k} \int_{\Gamma_2} (\partial_t v) w d\sigma, \tag{2-5}
\end{aligned}$$

for all $w \in H_{\Gamma_1}^1(\Omega)$, for almost all $t \in (0, T)$. We shall employ the Galerkin approximation procedure. Let $\{w_j\}_{j=1}^\infty$ be a basis for $H_{\Gamma_1}^1(\Omega)$ where each $w_j \in C^\infty(\bar{\Omega}) \cap H_{\Gamma_1}^1(\Omega)$. Let us set

$$v_m = \sum_{j=1}^m a_{mj}(t) w_j(x) \quad (2-6)$$

where each $a_{mj}(t)$ is determined from

$$a_{mj}(0) = \dot{a}_{mj}(0) = 0, \quad (2-7)$$

and

$$\begin{aligned} & \langle \partial_{tt} v_m, w_j \rangle + a(v_m, w_j) = \langle f, w_j \rangle \\ & + \langle \Delta(u_0 + t u_1), w_j \rangle + k \int_{\Gamma_2} (v_m - \phi)^- w_j d\sigma \\ & - \frac{1}{k} \int_{\Gamma_2} (\partial_t v_m) w_j d\sigma, \quad j = 1, \dots, m. \end{aligned} \quad (2-8)$$

For each m , there is a unique set of functions $a_{mj}(t) \in C^2([0, T])$, $j = 1, \dots, m$, which satisfy (2-7) and (2-8). We next obtain estimates independent of m . Let us substitute $\partial_t v_m$ for w_j in (2-8):

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t v_m\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} a(v_m, v_m) \\ & + \frac{1}{2} k \frac{d}{dt} \|(v_m - \phi)^-\|_{L^2(\Gamma_2)}^2 \\ & = \langle f, \partial_t v_m \rangle + \langle \Delta(u_0 + t u_1), \partial_t v_m \rangle - \frac{1}{k} \|\partial_t v_m\|_{L^2(\Gamma_2)}^2. \end{aligned} \quad (2-9)$$

Note that $v_m - \phi \in C^1([0, T] \times \bar{\Omega})$ and hence, $(v_m - \phi)^-$ is Lipschitz continuous and

$$\frac{d}{dt} \|(v_m - \phi)^-\|_{L^2(\Gamma_2)}^2 \in L^\infty(0, T).$$

It follows from (2-7) and (2-9) that

$$\|\partial_t v_m\|_{L^2(\Omega)} \leq M, \text{ for all } t \in [0, T] \quad (2-10)$$

$$\|v_m\|_{H_{\Gamma_1}^1(\Omega)} \leq M, \text{ for all } t \in [0, T] \quad (2-11)$$

$$\|(v_m - \phi)^-\|_{L^2(\Gamma_2)} \leq M, \text{ for all } t \in [0, T] \quad (2-12)$$

$$\int_0^T \|\partial_t v_m\|_{L^2(\Gamma_2)}^2 dt \leq M, \quad (2-13)$$

where M denotes positive constants independent of m . Next we differentiate (2-8) with respect to t and substitute $\partial_{tt} v_m$ for w_j :

$$\begin{aligned}
& \langle \partial_{tt}v_m, \tilde{\partial}_{tt}v_m \rangle + a(\partial_t v_m, \partial_{tt}v_m) \\
& = \langle \partial_t f, \partial_{tt}v_m \rangle + \langle \Delta u_1, \partial_{tt}v_m \rangle \\
& + k \int_{\Gamma_2} \partial_t \{(v_m - \phi)^-\} \partial_{tt}v_m d\sigma \\
& - \frac{1}{k} \int_{\Gamma_2} (\partial_{tt}v_m)^2 d\sigma,
\end{aligned} \tag{2-14}$$

which yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_{tt}v_m\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} a(\partial_t v_m, \partial_{tt}v_m) \\
& \leq \|\partial_t f\|_{L^2(\Omega)} \|\partial_{tt}v_m\|_{L^2(\Omega)} + \|\Delta u_1\|_{L^2(\Omega)} \|\partial_{tt}v_m\|_{L^2(\Omega)} \\
& + \frac{1}{2} k^2 \|\partial_t \{(v_m - \phi)^-\}\|_{L^2(\Gamma_2)}^2 - \frac{1}{2k} \|\partial_{tt}v_m\|_{L^2(\Gamma_2)}^2.
\end{aligned} \tag{2-15}$$

We consider (2-8) at $t = 0$ and substitute $\partial_{tt}v_m(0)$ for w_j :

$$\|\partial_{tt}v_m(0)\|_{L^2(\Omega)}^2 = \langle f(0), \partial_{tt}v_m(0) \rangle + \langle \Delta u_0, \partial_{tt}v_m(0) \rangle. \tag{2-16}$$

We have used the fact that $v_m(0) = \partial_t v_m(0) = 0$. Now (2-16) implies

$$\|\partial_{tt}v_m(0)\|_{L^2(\Omega)} \leq M, \tag{2-17}$$

M being a positive constant independent of m . Combining (2-13), (2-15) and (2-17), we derive

$$\|\partial_{tt}v_m\|_{L^2(\Omega)} \leq M, \text{ for all } t \in [0, T], \tag{2-18}$$

$$\|\partial_t v_m\|_{H_{\Gamma_1}^1(\Omega)} \leq M, \text{ for all } t \in [0, T], \tag{2-19}$$

where M denotes positive constants independent of m . By means of (2-11), (2-18) and (2-19), we can extract a subsequence still denoted by $\{v_m\}$ such that

$$v_m \rightarrow v \text{ weak* in } L^\infty(0, T; H_{\Gamma_1}^1(\Omega)) \tag{2-20}$$

$$\partial_t v_m \rightarrow \partial_t v \text{ weak* in } L^\infty(0, T; H_{\Gamma_1}^1(\Omega)) \tag{2-21}$$

$$\partial_{tt}v_m \rightarrow \partial_{tt}v \text{ weak* in } L^\infty(0, T; L^2(\Omega)) \tag{2-22}$$

for some function v . By virtue of Lemma 1.4, we have

$$(v_m - \phi)^- \rightarrow (v - \phi)^- \text{ in } C([0, T]; L^2(\partial\Omega)). \tag{2-23}$$

We then make use of (2-20) through (2-23) to derive (2-4) and (2-5) through a standard procedure.

Remark 2.2. The purpose of the term $\frac{1}{k}\partial_t u_k$ in (2-3) is to obtain approximate solutions which are regular enough to justify manipulations in using a multiplier technique in the next subsection. The utility of that term is seen in (2-15).

2.2 A priori estimates independent of k .

We still suppress the subscript k and let $u(x, t)$ be a solution constructed in Proposition 2.1 for a fixed $k > 0$. We can substitute $\partial_t u$ for w in (2-3):

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} a(u, u) = \langle f, \partial_t u \rangle \\ & - \frac{1}{2} k \frac{d}{dt} \int_{\Gamma_2} \{(u - \phi)^-\}^2 d\sigma - \frac{1}{k} \int_{\Gamma_2} (\partial_t u)^2 d\sigma, \end{aligned} \quad (2-24)$$

for almost all $t \in (0, T)$. We now derive from (2-24)

$$\|\partial_t u\|_{L^2(\Omega)} \leq M, \text{ for all } t \in [0, T] \quad (2-25)$$

$$\|u\|_{H_{\Gamma_1}^1(\Omega)} \leq M, \text{ for all } t \in [0, T] \quad (2-26)$$

$$\|(u - \phi)^-\|_{L^2(\Gamma_2)} \leq M/\sqrt{k}, \text{ for all } t \in [0, T], \quad (2-27)$$

$$\int_0^T \|\partial_t u\|_{L^2(\Gamma_2)}^2 dt \leq M k, \quad (2-28)$$

where M denotes positive constants which are independent of k , and depend only on the number L such that $\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega \times (0, T))} \leq L$.

Next we shall adapt a multiplier technique to derive some uniform estimates in a neighborhood of $\partial\Omega$. Let us set

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$$

for $0 < \delta \leq \delta_0$. Obviously, $\partial\Omega_\delta = S_\delta$, which was defined in Section 1. We need to observe that solution u of (2-1), (2-2) and (2-3) has sufficient regularity to justify manipulations in using a multiplier technique. First of all, (2-3) implies that

$$\partial_{tt} u - \Delta u = f \quad (2-29)$$

holds in $\mathcal{D}'(\Omega \times (0, T))$. By virtue of (2-1) and the assumption that $f \in C^1([0, T]; L^2(\Omega))$, we infer that

$$\Delta u \in L^\infty(0, T; L^2(\Omega)). \quad (2-30)$$

In fact, (2-29) holds in $L^\infty(0, T; L^2(\Omega))$. Now (2-30) implies

$$u \in L^\infty(0, T; H^2(\Omega_\delta)), \quad (2-31)$$

for each $\delta > 0$. Because of the mixed boundary condition on $\partial\Omega$, one cannot claim that $u \in L^\infty(0, T; H^2(\Omega))$. We also note that (2-1) implies

$$u \in C([0, T]; H_{\Gamma_1}^1(\Omega)), \partial_t u \in C([0, T]; L^2(\Omega)), \quad (2-32)$$

possibly after a modification on a set of measure zero in t .

We now proceed to obtain the estimates. Using the function $h(x)$ constructed in Lemma 1.1, we multiply both sides of (2-29) by $(h \cdot \nabla)u$ and integrate over $\Omega_\delta \times (0, T)$, $0 < \delta \leq \delta_0/2$:

$$\begin{aligned} & \int_0^T \int_{\Omega_\delta} (\partial_t u)((h \cdot \nabla)u) dx dt \\ &= \int_{\Omega_\delta} (\partial_t u(T))((h \cdot \nabla)u(T)) dx - \int_{\Omega_\delta} u_1((h \cdot \nabla)u_0) dx \\ & - \int_0^T \int_{\Omega_\delta} (\partial_t u)((h \cdot \nabla)\partial_t u) dx dt = \end{aligned}$$

(applying the divergence theorem to the last integral)

$$\begin{aligned} &= \int_{\Omega_\delta} (\partial_t u(T))((h \cdot \nabla)u(T)) dx - \int_{\Omega_\delta} u_1((h \cdot \nabla)u_0) dx \\ & - \frac{1}{2} \int_0^T \int_{S_\delta} (\partial_t u)^2 d\sigma_\delta dt + \frac{1}{2} \int_0^T \int_{\Omega_\delta} (\nabla \cdot h)(\partial_t u)^2 dx dt. \end{aligned} \quad (2-33)$$

$$\begin{aligned} & \int_0^T \int_{\Omega_\delta} (-\Delta u)((h \cdot \nabla)u) dx dt \\ &= - \int_0^T \int_{S_\delta} \left(\frac{\partial u}{\partial \nu} \right)^2 d\sigma_\delta dt + \int_0^T \int_{\Omega_\delta} \nabla u \cdot \nabla \{ (h \cdot \nabla)u \} dx dt \\ &= - \int_0^T \int_{S_\delta} \left(\frac{\partial u}{\partial \nu} \right)^2 d\sigma_\delta dt + \sum_{i,j=1}^n \int_0^T \int_{\Omega_\delta} \frac{\partial u}{\partial x_i} \frac{\partial h_j}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt \\ & - \int_0^T \int_{\Omega_\delta} \frac{1}{2} (\nabla \cdot h) |\nabla u|^2 dx dt + \frac{1}{2} \int_0^T \int_{S_\delta} |\nabla u|^2 d\sigma_\delta dt, \end{aligned} \quad (2-34)$$

where $\frac{\partial}{\partial \nu}$ is the unit outward normal derivative on S_δ , i.e., $\frac{\partial}{\partial \nu} = h \cdot \nabla$ on S_δ . Combining (2-33) and (2-34), we get

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_{S_\delta} (\partial_t u)^2 d\sigma_\delta dt - \frac{1}{2} \int_0^T \int_{S_\delta} |\nabla u|^2 d\sigma_\delta dt \\ & + \int_0^T \int_{S_\delta} \left(\frac{\partial u}{\partial \nu} \right)^2 d\sigma_\delta dt = C(\delta), \end{aligned} \quad (2-35)$$

where

$$\begin{aligned} C(\delta) &= - \int_0^T \int_{\Omega_\delta} f((h \cdot \nabla)u) dx dt \\ & + \text{obvious terms in (2-33) and (2-34)}. \end{aligned} \quad (2-36)$$

Recalling (2-25) and (2-26), it is evident that

$$|C(\delta)| \leq M, \text{ for all } 0 < \delta \leq \delta_0/2, \quad (2-37)$$

where M is a positive constant which is independent of k and depends only on the number L such that

$$\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega \times (0,T))} \leq L.$$

We next choose any $\rho(s) \in C(R)$ such that for all s , $0 \leq |\rho(s)| \leq 1$, and set $\psi(x) = \rho(\text{dist}(x, \partial\Omega))$. Then, $\psi(x)$ is a continuous function and is constant on each S_δ , $0 < \delta \leq \delta_0$. Hence, it follows from (2-35) that

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_0^{\delta^*} \int_{S_\delta} \psi(x) (\partial_t u)^2 d\sigma_\delta d\delta dt \\ & - \frac{1}{2} \int_0^T \int_0^{\delta^*} \int_{S_\delta} \psi(x) |\nabla u|^2 d\sigma_\delta d\delta dt \\ & + \int_0^T \int_0^{\delta^*} \int_{S_\delta} \psi(x) ((h \cdot \nabla) u)^2 d\sigma_\delta d\delta dt = \end{aligned}$$

(using Lemma 1.2)

$$\begin{aligned} & = \frac{1}{2} \int_0^T \int_{U_{\delta^*}} \psi(x) (\partial_t u)^2 dx dt - \frac{1}{2} \int_0^T \int_{U_{\delta^*}} \psi(x) |\nabla u|^2 dx dt \\ & + \int_0^T \int_{U_{\delta^*}} \psi(x) ((h \cdot \nabla) u)^2 dx dt = 0(\delta^*), \end{aligned} \quad (2-38)$$

for each $0 < \delta^* \leq \delta_0/2$, where $0(\delta^*)$ is a number such that

$$|0(\delta^*)| \leq M\delta^*, \quad (2-39)$$

where M is the same positive number as in (2-37). (2-38) will be used in the next section.

3. Convergence of approximate solutions.

We assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(\Omega \times (0,T))$, $\phi \in C^1(\bar{\Omega})$ with $\phi \leq 0$ on Γ_2 , and define a solution of (0-1) through (0-5) by

Definition 3.1 A function $u(x,t)$ is a solution of (0-1) through (0-5) if

$$u \in L^\infty(0,T; H_{\Gamma_1}^1(\Omega)), \partial_t u \in L^\infty(0,T; L^2(\Omega)) \cap C([0,T]; H^{-1/2}(\Omega)), \quad (3-1)$$

$$u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), \quad (3-2)$$

$$u(t) \in G, \text{ for almost all } t \in (0,T) \quad (3-3)$$

$$\langle \partial_t u(T), w(T) - u(T) \rangle - \langle u_1, w(0) - u_0 \rangle$$

$$\begin{aligned}
& - \int_0^T < \partial_t u, \partial_t w - \partial_t u > dt + \int_0^T a(u, w - u) dt \\
& \geq \int_0^T < f, w - u > dt,
\end{aligned} \tag{3-4}$$

for every w such that $w \in L^\infty(0, T; H_{\Gamma_1}^1(\Omega))$, $\partial_t w \in L^\infty(0, T; L^2(\Omega))$, $w(t) \in G$, for almost all $t \in (0, T)$.

We shall justify this definition. Suppose that u is a solution according to the above definition. By taking $w = u \pm \xi$, $\xi \in C_0^\infty(\Omega \times (0, T))$, in (3-4), we find that (0-1) is satisfied in $\mathcal{D}'(\Omega \times (0, T))$. (0-3) and (0-4) are satisfied in the sense $u(t) \in G$, for almost all $t \in (0, T)$. Now we assume further regularity on u :

$$\partial_{tt} u \in L^2(0, T; L^2(\Omega \setminus E_\delta)) \tag{3-5}$$

$$u \in L^2(0, T; H^2(\Omega \setminus E_\delta)) \tag{3-6}$$

for each $\delta > 0$, where we set

$$E_\delta = \{x \in \bar{\Omega} : \text{dist}(x, \partial\Gamma_1) \leq \delta\}.$$

Recall that $\partial\Gamma_1 = \partial\Gamma_2$ is an $(n-2)$ -dimensional, C^2 -manifold by assumption. Then, (0-1) holds in a stronger sense:

$$\int_0^T < \partial_{tt} u, w > dt - \int_0^T < \Delta u, w > dt = \int_0^T < f, w > dt \tag{3-7}$$

holds for every $w \in L^2(\Omega \times (0, T))$ such that $\text{supp } w(x, t) \cap E_\delta$ is empty, for almost all t , for some $\delta > 0$.

Next choose any nonnegative function $\xi(x) \in C_0^1(\Gamma_2)$. Then, we can extend $\xi(x)$ so that $\xi(x) \in C^1(\bar{\Omega})$ and $\text{supp } \xi \cap \{E_\delta \cup \Gamma_1\}$ is empty for some $\delta > 0$. Choose any nonnegative function $\eta(t) \in C_0^1((0, T))$ and take $w = u + \xi(x)\eta(t)$ in (3-4). By means of integration by parts and the divergence theorem (this procedure is justified by (3-5) and (3-6)), we find

$$\begin{aligned}
& \int_0^T < \partial_{tt} u, \xi(x)\eta(t) > dt - \int_0^T < \Delta u, \xi(x)\eta(t) > dt \\
& + \int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \nu} \right) \xi(x)\eta(t) d\sigma dt \geq \int_0^T < f, \xi(x)\eta(t) > dt.
\end{aligned} \tag{3-8}$$

It follows from (3-7) and (3-8) that

$$\int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \nu} \right) \xi(x)\eta(t) d\sigma dt \geq 0, \tag{3-9}$$

which is true for all nonnegative functions $\xi(x) \in C_0^1(\Gamma_2)$ and $\eta(t) \in C_0^1((0, T))$. Hence we can conclude that

$$\frac{\partial u}{\partial \nu} \geq 0, \text{ almost everywhere in } \Gamma_2 \times (0, T). \quad (3-10)$$

It remains to show the second part of (0-5). Let us choose a function $\xi_\delta(x) \in C^1(\bar{\Omega})$ such that $0 \leq \xi_\delta(x) \leq 1$, for all $x \in \bar{\Omega}$, $\xi_\delta \equiv 1$ on $\Gamma_2 \setminus E_\delta$ and $\text{supp } \xi_\delta \cap \{E_{\delta/2} \cup \Gamma_1\}$ is empty. We also choose any $\eta(t) \in C_0^1((0, T))$ such that $-1 \leq \eta(t) \leq 1$ for all t . Then, we set

$$w = u + \xi_\delta(x)\eta(t)(u - \phi)$$

in (3.4). As above, we use integration by parts, the divergence theorem and (3-7) to arrive at

$$\int_0^T \int_{\Gamma_2} \xi_\delta(x)\eta(t) \left(\frac{\partial u}{\partial \nu} \right) (u - \phi) d\sigma dt = 0. \quad (3-11)$$

Here, we have the equality, because $-\eta(t)$ can be also chosen in place of $\eta(t)$. Since $u - \phi \geq 0$ almost everywhere in $\Gamma_2 \times (0, T)$ and $\delta > 0$ is arbitrary, (3-10) and (3-11) imply

$$\left(\frac{\partial u}{\partial \nu} \right) (u - \phi) = 0, \text{ almost everywhere in } \Gamma_2 \times (0, T). \quad (3-12)$$

This completes the justification of the above definition of a solution.

We now proceed to present the main result.

THEOREM 3.2. *For $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(\Omega \times (0, T))$, there is a solution of (0-1) through (0-5).*

PROOF: Choose sequences $\{u_{0k}\}, \{u_{1k}\}$ in $C_0^\infty(\Omega)$ and f_k in $C^1([0, T]; L^2(\Omega))$ such that as $k \rightarrow \infty$,

$$u_{0k} \rightarrow u_0 \text{ in } H_0^1(\Omega), \quad (3-13)$$

$$u_{1k} \rightarrow u_1 \text{ in } L^2(\Omega), \quad (3-14)$$

$$f_k \rightarrow f \text{ in } L^2(\Omega \times (0, T)). \quad (3-15)$$

Using these, for each k , let u_k be a solution constructed in Proposition 2.1. Then, each u_k satisfies

$$\begin{aligned} & \int_0^T \langle \partial_{tt} u_k, w - u_k \rangle dt + \int_0^T a(u_k, w - u_k) dt \\ &= \int_0^T \langle f, w - u_k \rangle dt + k \int_0^T \int_{\Gamma_2} (u_k - \phi)^-(w - u_k) d\sigma dt \\ & - \frac{1}{k} \int_0^T \int_{\Gamma_2} (\partial_t u_k)(w - u_k) d\sigma dt, \end{aligned} \quad (3-16)$$

for every $w(x, t)$ such that $w \in L^\infty(0, T; H_{\Gamma_1}^1(\Omega))$, $\partial_t w \in L^\infty(0, T; L^2(\Omega))$ and $w(t) \in G$, for almost all $t \in (0, T)$. Using integration by parts and

$$\begin{aligned} & \int_0^T \int_{\Gamma_2} (u_k - \phi)^-(w - u_k) d\sigma dt \\ &= \int_0^T \int_{\Gamma_2} (u_k - \phi)^-(w - \phi) d\sigma dt - \int_0^T \int_{\Gamma_2} (u_k - \phi)^-(u_k - \phi) d\sigma dt \\ & \geq 0, \end{aligned} \quad 12 \quad (3-17)$$

we obtain

$$\begin{aligned}
& \langle \partial_t u_k(T), w(T) - u_k(T) \rangle - \langle u_{1k}, w(0) - u_{0k} \rangle \\
& - \int_0^T \langle \partial_t u_k, \partial_t w \rangle dt + \int_0^T \int_{\Omega} (\partial_t u_k)^2 dx dt \\
& + \int_0^T a(u_k, w) dt - \int_0^T \int_{\Omega} |\nabla u_k|^2 dx dt \\
& \geq \int_0^T \langle f, w - u_k \rangle dt - \frac{1}{k} \int_0^T \int_{\Gamma_2} (\partial_t u_k) w d\sigma dt \\
& + \frac{1}{2k} \int_{\Gamma_2} (u_k(T))^2 d\sigma.
\end{aligned} \tag{3-18}$$

By virtue of (2-25) and (2-26), we can extract a subsequence still denoted by $\{u_k\}$ such that as $k \rightarrow \infty$,

$$u_k \rightarrow u \text{ weak* in } L^\infty(0, T; H_{\Gamma_1}^1(\Omega)), \tag{3-19}$$

$$\partial_t u_k \rightarrow \partial_t u \text{ weak* in } L^\infty(0, T; L^2(\Omega)). \tag{3-20}$$

Since each u_k satisfies (2-29), we infer from (3-15) and (3-19) that

$$\partial_{tt} u_k \rightarrow \partial_{tt} u \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \tag{3-21}$$

Hence, by Lemma 1.4, we find that

$$\partial_t u_k \rightarrow \partial_t u \text{ in } C([0, T]; H^{-\frac{1}{2}}(\Omega)), \tag{3-22}$$

$$u_k \rightarrow u \text{ in } C([0, T]; H^{\frac{1}{2}}(\Omega)), \tag{3-23}$$

so that

$$\langle \partial_t u_k(T), w(T) - u_k(T) \rangle \rightarrow \langle \partial_t u(T), w(T) - u(T) \rangle \tag{3-24}$$

as $k \rightarrow \infty$, for each w . In the meantime, we notice that

$$\begin{aligned}
& \frac{1}{k} \left| \int_0^T \int_{\Gamma_2} (\partial_t u_k) w d\sigma dt \right| \\
& \leq \frac{1}{k} \left(\int_0^T \int_{\Gamma_2} (\partial_t u_k)^2 d\sigma dt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Gamma_2} w^2 d\sigma dt \right)^{\frac{1}{2}}, \\
& \text{(using (2-28))}, \\
& \leq M \left(\int_0^T \int_{\Gamma_2} w^2 d\sigma dt \right)^{\frac{1}{2}} / \sqrt{k} \rightarrow 0,
\end{aligned} \tag{3-25}$$

as $k \rightarrow \infty$, for each w . Next we shall show that

$$\int_0^T \int_{\Omega} ((\partial_t u)^2 - |\nabla u|^2) dx dt \geq \limsup_{k \rightarrow \infty} \int_0^T \int_{\Omega} ((\partial_t u_k)^2 - |\nabla u_k|^2) dx dt \tag{3-26}$$

It is convenient to introduce the notation:

$$\begin{aligned} A_k &= \frac{1}{2}(\partial_t u_k)^2 - \frac{1}{2}|\nabla u_k|^2 + ((h \cdot \nabla)u_k)^2, \\ B_k &= ((h \cdot \nabla)u_k)^2, \\ A &= \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2 + ((h \cdot \nabla)u)^2, \\ B &= ((h \cdot \nabla)u)^2. \end{aligned}$$

First of all, by Lemma 1.3, as $k \rightarrow \infty$,

$$A_k - B_k \rightarrow A - B \text{ in } \mathcal{D}'(\Omega \times (0, T)). \quad (3-27)$$

Since the sequence $\{A_k - B_k\}$ is bounded in $L^\infty(0, T; L^1(\Omega))$, we observe that as $k \rightarrow \infty$,

$$\int_0^T \int_\Omega \psi(x)(A_k - B_k) dx dt \rightarrow \int_0^T \int_\Omega \psi(x)(A - B) dx dt \quad (3-28)$$

for each $\psi(x) \in C_0(\Omega)$. Let us define a function $\rho_\delta(s) \in C(R)$ such that $\rho_\delta(s)$ is nondecreasing in s and

$$\rho_\delta(s) = \begin{cases} 0 & \text{for all } s \leq \delta/2 \\ 1 & \text{for all } s \geq \delta. \end{cases} \quad (3-29)$$

We then set $\psi_\delta(x) = \rho_\delta(\text{dist}(x, \partial\Omega))$. Fix any $0 < \delta \leq \delta_0/2$. According to (3-28),

$$\int_0^T \int_\Omega \psi_\delta(x)(A_k - B_k) dx dt \rightarrow \int_0^T \int_\Omega \psi_\delta(x)(A - B) dx dt \quad (3-30)$$

as $k \rightarrow \infty$. Furthermore, with the aid of (2-38), we notice that

$$\int_0^T \int_\Omega (\psi_\delta(x) - 1)A_k dx dt = \int_0^T \int_{U_\delta} (\psi_\delta(x) - 1)A_k dx dt = 0_1(\delta), \quad (3-31)$$

where we can estimate $0_1(\delta)$ by using the number M in (2-37), and (3-13), (3-14) and (3-15),

$$|0_1(\delta)| \leq M\delta, \text{ for all } k. \quad (3-32)$$

Meanwhile we also have

$$\int_0^T \int_\Omega B_k dx dt \geq \int_0^T \int_\Omega \psi_\delta(x)B_k dx dt, \quad (3-33)$$

for each k , since B_k is nonnegative and integrable, and $0 \leq \psi_\delta(x) \leq 1$, for all x . Since $A - B \in L^\infty(0, T; L^1(\Omega))$, it is apparent that

$$\begin{aligned} \int_0^T \int_\Omega (1 - \psi_\delta(x))(A - B) dx dt &= \int_0^T \int_{U_\delta} (1 - \psi_\delta(x))(A - B) dx dt \\ &= 0_2(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0. \end{aligned} \quad (3-34)$$

Choose any $\epsilon > 0$. Then, by (3-32) and (3-34), there is $0 < \bar{\delta}(\epsilon) \leq \delta_0/2$ such that

$$|0_1(\delta)| + |0_2(\delta)| \leq \epsilon, \quad (3-35)$$

for all $0 < \delta \leq \tilde{\delta}(\epsilon)$ and all k . Using (3-30) and (3-34), we have for each $0 < \delta \leq \delta_0/2$,

$$\begin{aligned}
& \int_0^T \int_{\Omega} (A - B) dx dt = \int_0^T \int_{\Omega} \psi_{\delta}(x)(A - B) dx dt + 0_2(\delta) \\
& = 0_2(\delta) + \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \psi_{\delta}(x)(A_k - B_k) dx dt \\
& = 0_2(\delta) + \lim_{k \rightarrow \infty} \left\{ \int_0^T \int_{\Omega} (A_k - B_k) dx dt \right. \\
& \quad + \int_0^T \int_{\Omega} (\psi_{\delta}(x) - 1) A_k dx dt - \int_0^T \int_{\Omega} \psi_{\delta}(x) B_k dx dt \\
& \quad \left. + \int_0^T \int_{\Omega} B_k dx dt \right\}. \tag{3-36}
\end{aligned}$$

In particular, for $0 < \delta \leq \tilde{\delta}(\epsilon)$, we use (3-31), (3-33) and (3-35) to arrive at

$$\int_0^T \int_{\Omega} (A - B) dx dt \geq -\epsilon + \limsup_{k \rightarrow \infty} \int_0^T \int_{\Omega} (A_k - B_k) dx dt. \tag{3.37}$$

Since ϵ is arbitrary, it follows that

$$\int_0^T \int_{\Omega} (A - B) dx dt \geq \limsup_{k \rightarrow \infty} \int_0^T \int_{\Omega} (A_k - B_k) dx dt, \tag{3-38}$$

which shows (3-26). Combining (3-18), (3-19), (3-20), (3-24), (3-25) and (3-26), we conclude that (3-4) is satisfied. Finally, (3.23) implies

$$(u_k - \phi)^- \rightarrow (u - \phi)^- \text{ in } C([0, T]; L^2(\Gamma_2)). \tag{3-39}$$

But, each u_k satisfies (2-27) and thus

$$u \geq \phi, \text{ almost everywhere in } \Gamma_2, \text{ for each } t, \tag{3-40}$$

from which (3-3) follows. Now the proof of Theorem 3.2 is complete.

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