Revealing some unexpected dependence properties of linear combinations of stable random variables using symmetric covaration

Ludovic d'Estampes, Bernard Garel, Dag Tjøstheim

REVEALING SOME UNEXPECTED DEPENDENCE PROPERTIES OF LINEAR COMBINATIONS OF STABLE RANDOM VARIABLES USING SYMMETRIC COVARIATION

Ludovic d'Estampes[a], Bernard Garel[a] and Dag Tjøstheim[b]

[a]Stochastic Models, ENSEEIHT, B.P.7122,

2, rue Camichel 31071 Toulouse Cedex 7, France.

garel@enseeiht.fr

[b]Department of Mathematics, University of Bergen,

Johs. Brunsgt. 12, N-5008 Bergen, Norway.

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ABSTRACT

The covariation is one of the possible dependence measures for variables where distribution is symmetric alpha-stable with parameter alpha between one and two. We introduce a symmetrized and normalized version of the covariation which enables us to reveal some unexpected dependence properties of stable variables.

1. INTRODUCTION

In many practical situations of interest in statistics, the observed data cannot be adequately described by a Gaussian distribution. Usually in such cases a distribution with heavier tails is needed, particularly in the fields of telecommunications and finance. From a theoretical point of view, the class of stable distributions is the most satisfying heavy-tail generalization. It contains the Gaussian distribution as a special case, and it retains at least some of its properties. For example, it is infinitely divisible, linear combinations of stable variables are still stable and the conditional expectation, granted that it exists, of a stable variable X with respect to another variable Y is still a linear function of Y. Moreover an extensive asymptotic theory has been developed (see Samorodnitsky and Taqqu (1)).

A main problem with non-Gaussian models has been the complexity of calculations, but recent developments of data processing techniques have made both model fitting with stable distributions and simulation of stable variables practical. On the other hand, because of the infinite variance of these distributions, the much applied correlation tool for assessing dependence cannot be used. So, other dependence measures are needed using moments of orders less than two.

Samorodnitsky and Taqqu (1) in chapter 4 of their book discuss several alternatives that have been used. Most of the interest seems to center around the so-called covariation measure. This measure was treated in the early papers by Miller (2) and Cambanis and Miller (3). Since then there has been a number of papers with many of the results summed up in Nikias and Shao (4). A recent alternative contribution is Pinkse (5) which introduced a non parametric serial dependence measure with the help of the characteristic function. General measures of dependence and resulting tests of independence are surveyed by Tjøstheim (6). Further developments can be found in Hong (7) and Hong and White (8).

In this paper we will focus on dependence properties of linear combinations of stable variables. Our main tool is introduced in section 2. It is a normalized and symmetrized version of the covariance measure. We use it to reveal some unexpected dependence properties in section 3. For instance, it will be shown that linear combinations exist that display both positive and negative dependence at the same time. Finally, in section 4, we present some finite sample results for the new symmetrized covariation measure and compare it to some other standard measures.

2. COVARIATION AND SYMMETRIC COVARIATION

We start by giving some basic properties of symmetric alpha-stable distributions.

Definition 1 *A vector $\underline{X} = (X_1, \ldots, X_n)$ has a symmetric alpha-stable (SαS) distribution if and only if (iff) its characteristic function φ is given by*

$$\forall (t_1, \ldots, t_n) \in \mathbb{R}^n, \ \varphi_{(X_1, \ldots, X_n)}(t_1, \ldots, t_n) = \exp \left\{ - \int_{S^{n-1}} |\langle \underline{t}, \underline{x} \rangle|^\alpha \, d\mu_{S^{n-1}}(\underline{x}) \right\}$$

$$\text{where } \begin{cases} \mu_{S^{n-1}} \text{ is a symmetric measure, called the spectral measure, on the} \\ \text{Borel sets of the unit sphere } S^{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \Big/ \sum_{i=1}^{n} x_i^2 = 1 \right\}, \\ \langle \cdot, \cdot \rangle \text{ represents the usual scalar product on } \mathbb{R}^n, \\ \alpha \in]0; 2]\,; \underline{t} = (t_1, \ldots, t_n)\,; \underline{x} = (x_1, \ldots, x_n)\,. \end{cases}$$

Definition 2 *A real random variable* X *has a* $S\alpha S$ *distribution, with parameters* (γ, α) *where* $\gamma > 0$ *is called the dispersion of* X, *iff its characteristic function is given by*

$$\forall t \in \mathbb{R}, \ \varphi_X(t) = \exp\{-\gamma |t|^\alpha\}.$$

Remarks

1. If $\underline{X}$ is a $S\alpha S$ vector, then every component X_i is a $S\alpha S$ random variable with parameters (γ_{X_i}, α). We have

$$\gamma_{X_i} = \int_{S^{n-1}} |x_i|^\alpha \, d\mu_{S^{n-1}}(\underline{x}) \quad \text{and} \quad \varphi_{X_i}(t_i) = \varphi_{(X_1, \ldots, X_n)}(0, \ldots, 0, t_i, 0 \ldots, 0).$$

2. The random vector $\underline{X}$ is a $S\alpha S$ random vector iff all linear combinations $\sum_{j=1}^{n} a_j X_j$ is a $S\alpha S$ random variable.

3. If X_1 and X_2 are two independent $S\alpha S$ random variables, then (X_1, X_2) is a $S\alpha S$ random vector.

The following proposition follows directly from point 2 of the preceding remarks.

Proposition 3 *Let* $\underline{X}$ *be a* $S\alpha S$ *random vector and* M *a* $m \times n$ *real matrix. If* $\underline{Y} = M\underline{X}$, *then* $\underline{Y}$ *is a* $S\alpha S$ *random vector.*

For the rest of the paper, we assume that α is greater than 1. The covariation generalizes the covariance to the situation where there are no second moments.

Definition 4 *Let* (Y_1, Y_2) *be a* $S\alpha S$ *random vector, the covariation between* Y_1 *and* Y_2 *is equal to*

$$[Y_1, Y_2]_\alpha = \int_{S^1} y_1 y_2^{\langle \alpha-1 \rangle} d\mu_{S^1}(y_1, y_2)$$

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where μ_{S^1} is the spectral measure on the unit sphere S^1 and where we have used the notation

$$u^{\langle v \rangle} = sign\,(u) \cdot |u|^v\,.$$

*Moreover, the **coefficient of covariation** of Y_1 on Y_2, is the quantity:*

$$\{Y_1, Y_2\}_\alpha = \frac{[Y_1, Y_2]_\alpha}{[Y_2, Y_2]_\alpha}\ .$$

For $\alpha = 2$, $[Y_1, Y_2]_2 = \frac{1}{2} \operatorname{Cov}(Y_1, Y_2)$. Note that $[Y, Y]_\alpha = \gamma_Y$.

Properties of the covariation are given in Samorodnitsky and Taqqu (1),pp. 63-64. Note that if X and Y are two independent $S\alpha S$ random variables then $[X, Y]_\alpha = 0$. Furthermore, for $(a, b) \in \mathbb{R}^2$, $[aX, bY]_\alpha = ab^\alpha [X, Y]_\alpha$, from which it follows that for $c \neq 0$,

$$\{cX, X\}_\alpha = \frac{[cX, X]_\alpha}{[X, X]_\alpha} = c \quad \text{and} \quad \{X, cX\}_\alpha = \frac{[X, cX]_\alpha}{[cX, cX]_\alpha} = \frac{1}{c}.$$

The two preceding equalities demonstrate the well-known fact that the coefficient of covariation is not symmetric and is not bounded. Using theorem 4.1.2 of Samorodnitsky and Taqqu (1),p. 175, we have for the conditional expectation

$$\mathbb{E}\,(Y_2\,|\,Y_1) = \{Y_2, Y_1\}_\alpha\,Y_1, \tag{2.1}$$

so that in general $\mathbb{E}\,(Y_2\,|\,Y_1) \neq \mathbb{E}\,(Y_1\,|\,Y_2)$. In the Gaussian case ($\alpha = 2$) we also have $\mathbb{E}\,(Y_2\,|\,Y_1) \neq \mathbb{E}\,(Y_1\,|\,Y_2)$ unless $\operatorname{Var}(Y_1) = \operatorname{Var}(Y_2)$. But the dependence is unambiguous in the sense that $\rho(Y_1, Y_2) = \rho(Y_2, Y_1)$ where ρ denotes ordinary correlation and the regression line $\mathbb{E}\,(Y_1\,|\,Y_2)$ has a positive slope iff $\mathbb{E}\,(Y_2\,|\,Y_1)$ has a positive slope. It will be shown in section 3 that for stable variables with $\alpha < 2$, the dependence is ambiguous. It is possible to find examples where the slope of $\mathbb{E}\,(Y_2\,|\,Y_1)$ is positive ($\{Y_2, Y_1\}_\alpha > 0$), whereas the slope of $\mathbb{E}\,(Y_1\,|\,Y_2)$ is negative ($\{Y_1, Y_2\}_\alpha < 0$). An example is given in the following scatter diagram.

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Figure 1: Plot of 50 realizations of a bivariate $S\alpha S$ random vector with $\alpha = 1.5$. The theoretical regression line $\mathbb{E}\left(Y_1 \mid Y_2\right)$ and $\mathbb{E}\left(Y_2 \mid Y_1\right)$ are drawn in. This example comes from the system (3.1) with the following parameters: $\gamma_{X_1} = 3; \gamma_{X_2} = 2; a_1 = 2^{(4/3)}; a_2 = 1; a_3 = 1; a_4 = -3^{(4/3)}$.
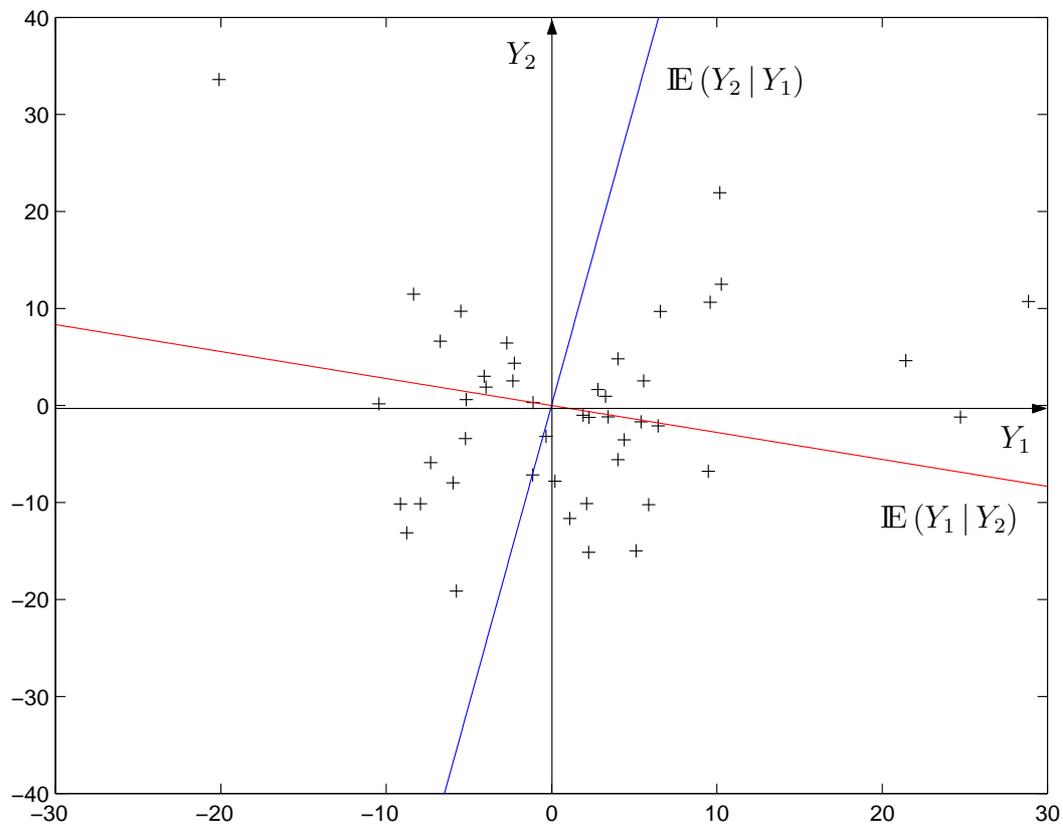
An important instrument needed to derive the results of the next section is the symmetric coefficient of covariation:

Definition 5 *Let* (Y_1, Y_2) *be a $S\alpha S$ random vector. The **symmetric coefficient of covariation** between* Y_1 *and* Y_2 *is equal to*

$$\mathrm{Corr}_\alpha\left(Y_1, Y_2\right) = \{Y_1, Y_2\}_\alpha \{Y_2, Y_1\}_\alpha = \frac{[Y_1, Y_2]_\alpha \, [Y_2, Y_1]_\alpha}{[Y_1, Y_1]_\alpha \, [Y_2, Y_2]_\alpha}. \tag{2.2}$$

In the Gaussian case (for $\alpha = 2$), $\mathrm{Corr}_\alpha(Y_1, Y_2) = \rho^2(Y_1, Y_2)$, so that the symmetric coefficient of covariation is only measuring the magnitude of the dependence not its direction. One would perhaps expect that $\mathrm{Corr}_\alpha(Y_1, Y_2) \geqslant 0$ also for $\alpha < 2$, but it turns out that we may have $\mathrm{Corr}_\alpha(Y_1, Y_2) < 0$, meaning that $[Y_1, Y_2]_\alpha$ and $[Y_2, Y_1]_\alpha$ may have different signs, one indicating positive dependence, the other negative. Hence $\mathrm{Corr}_\alpha(Y_1, Y_2)$ does not only have the capability of measuring the strength of what we may call concordant dependence, but also the degree of disconcordant dependence between two variables Y_1 and Y_2.

Before demonstrating this we show that $\mathrm{Corr}_\alpha(Y_1, Y_2)$ is bounded, taking values between -1 and 1.

Proposition 6 *Let (Y_1, Y_2) be a $S\alpha S$ random vector. We have*

1- $|\mathrm{Corr}_\alpha(Y_1, Y_2)| \leqslant 1$.

2- *If Y_1 and Y_2 are two independent $S\alpha S$ random variables, then* $\mathrm{Corr}_\alpha(Y_1, Y_2) = 0$.

PROOF

1- We know that (see Samorodnitsky and Taqqu (1), Property 2.8.4, p.96)

$$|[Y_1, Y_2]_\alpha| \leqslant \|Y_1\|_\alpha \|Y_2\|_\alpha^{\alpha-1} \quad,$$
$$\text{and} \quad |[Y_2, Y_1]_\alpha| \leqslant \|Y_2\|_\alpha \|Y_1\|_\alpha^{\alpha-1} \quad \text{where} \quad \|U\|_\alpha = ([U, U]_\alpha)^{1/\alpha}.$$

So $|[Y_1, Y_2]_\alpha| \times |[Y_2, Y_1]_\alpha| \leqslant \|Y_1\|_\alpha \|Y_2\|_\alpha^{\alpha-1} \times \|Y_2\|_\alpha \|Y_1\|_\alpha^{\alpha-1}$
which entails that $\left| \dfrac{[Y_1, Y_2]_\alpha [Y_2, Y_1]_\alpha}{[Y_1, Y_1]_\alpha [Y_2, Y_2]_\alpha} \right| \leqslant 1$.

2- We know that $[Y_1, Y_2]_\alpha$ is equal to zero. Because the denominator is different from 0, we obtain $\mathrm{Corr}_\alpha(Y_1, Y_2) = \dfrac{[Y_1, Y_2]_\alpha [Y_2, Y_1]_\alpha}{[Y_1, Y_1]_\alpha [Y_2, Y_2]_\alpha} = 0$.

■

A value of $\mathrm{Corr}_\alpha(Y_1, Y_2) = -1$ will indicate a very high degree of disconcordance in the dependence relationship between Y_1 and Y_2. It will be seen next that such a value is possible as a limiting case.

3. SOME UNEXPECTED DEPENDENCE PROPERTIES OF LINEAR COMBINATIONS OF STABLE VARIABLES

Let X_1 and X_2 be independent $S\alpha S$ variables with $1 < \alpha \leqslant 2$ and let

$$Y_1 = a_1 X_1 + a_2 X_2 \quad \text{and} \quad Y_2 = a_3 X_1 + a_4 X_2 \ . \tag{3.1}$$

In the Gaussian case ($\alpha = 2$), we have

$$\mathbb{E}\left(Y_1 Y_2\right) = a_1 a_3\, \mathbb{E}\left(X_1^2\right) + a_2 a_4\, \mathbb{E}\left(X_2^2\right) + \left(a_1 a_4 + a_2 a_3\right) \mathbb{E}\left(X_1 X_2\right).$$

With no restriction we assume that $\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_2^2\right)$. It is seen that Y_1 and Y_2 are independent iff $a_1 a_3 = -a_2 a_4$. In this case, $\mathbb{E}\left(Y_1 \mid Y_2\right) = \mathbb{E}\left(Y_2 \mid Y_1\right) = 0$. However, as will be seen in the next proposition, such linear combinations in general yields $\mathrm{Corr}_\alpha\left(Y_1, Y_2\right) < 0$. By (2.1) and (2.2), this means that if Y_2 increases with Y_1 in the regression of Y_2 on Y_1, then Y_1 decreases with increasing Y_2 in the regression of Y_1 on Y_2. Again, mainly to ease notation, we assume $[X_1, X_1]_\alpha = \gamma_{X_1} = \gamma_{X_2} = [X_2, X_2]_\alpha$, which corresponds to $\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_2^2\right)$ in the Gaussian case.

Proposition 7 *Consider the system (3.1) for $1 < \alpha \leqslant 2$. Then*

$$\mathrm{Corr}_\alpha\left(Y_1, Y_2\right) = \frac{a_1 a_3\, \mathrm{sign}\left(a_1 a_3\right) \left|a_1 a_3\right|^{\alpha-1} + a_2 a_4\, \mathrm{sign}\left(a_2 a_4\right) \left|a_2 a_4\right|^{\alpha-1}}{\left(\left|a_3\right|^\alpha + \left|a_4\right|^\alpha\right)\left(\left|a_1\right|^\alpha + \left|a_2\right|^\alpha\right)}$$

$$+ \frac{a_1 a_4\, \mathrm{sign}\left(a_2 a_3\right) \left|a_2 a_3\right|^{\alpha-1} + a_2 a_3\, \mathrm{sign}\left(a_1 a_4\right) \left|a_1 a_4\right|^{\alpha-1}}{\left(\left|a_3\right|^\alpha + \left|a_4\right|^\alpha\right)\left(\left|a_1\right|^\alpha + \left|a_2\right|^\alpha\right)} \ . \tag{3.2}$$

If $a_1 a_3 = -a_2 a_4$, this simplifies to

$$\mathrm{Corr}_\alpha\left(Y_1, Y_2\right) = -\frac{\left(a_1 a_3\right)^2 \left\{\left|a_2 a_3\right|^{\frac{\alpha}{2}-1} - \left|a_1 a_4\right|^{\frac{\alpha}{2}-1}\right\}^2}{\left(\left|a_1\right|^\alpha + \left|a_2\right|^\alpha\right)\left(\left|a_3\right|^\alpha + \left|a_4\right|^\alpha\right)} \tag{3.3}$$

which is negative.

PROOF

Using independence between X_1 and X_2 in (3.1), we have

$$\mathrm{Corr}_\alpha\left(Y_1, Y_2\right) = \frac{a_1 a_3^{\langle\alpha-1\rangle} + a_2 a_4^{\langle\alpha-1\rangle}}{\left|a_3\right|^\alpha + \left|a_4\right|^\alpha} \times \frac{a_3 a_1^{\langle\alpha-1\rangle} + a_4 a_2^{\langle\alpha-1\rangle}}{\left|a_1\right|^\alpha + \left|a_2\right|^\alpha} \ .$$

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Using the definition of $u^{\langle v \rangle}$ (Definition 4, page 3), we obtain (3.2). Inserting $a_1 a_3 = -a_2 a_4$, we have:

$$
\begin{aligned}
a_1 a_3 \operatorname{sign}(a_1 a_3) |a_1 a_3|^{\alpha-1} + a_2 a_4 \operatorname{sign}(a_2 a_4) |a_2 a_4|^{\alpha-1} &= 2 |a_1|^\alpha |a_3|^\alpha \\
&= 2 |a_1|^{\frac{\alpha}{2}} |a_2|^{\frac{\alpha}{2}} |a_3|^{\frac{\alpha}{2}} |a_4|^{\frac{\alpha}{2}} .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
a_1 a_4 \operatorname{sign}(a_2 a_3) |a_2 a_3|^{\alpha-1} &= \operatorname{sign}(a_1 a_2 a_3 a_4) |a_1| |a_4| |a_2|^{\alpha-1} |a_3|^{\alpha-1} \\
&= -|a_1| |a_4| |a_2|^{\alpha-1} |a_3|^{\alpha-1} ,
\end{aligned}
$$

where $\operatorname{sign}(a_1 a_2 a_3 a_4) = -1$ because $a_1 a_3 = -a_2 a_4$. Similarly, we have

$$
a_2 a_3 \operatorname{sign}(a_1 a_4) |a_1 a_4|^{\alpha-1} = -|a_2| |a_3| |a_1|^{\alpha-1} |a_4|^{\alpha-1} .
$$

Inserted in (3.2), this yields

$$
\begin{aligned}
\operatorname{Corr}_\alpha(Y_1, Y_2) &= \frac{2 |a_1|^{\frac{\alpha}{2}} |a_2|^{\frac{\alpha}{2}} |a_3|^{\frac{\alpha}{2}} |a_4|^{\frac{\alpha}{2}}}{(|a_3|^\alpha + |a_4|^\alpha)(|a_1|^\alpha + |a_2|^\alpha)} \\
&\quad - \frac{|a_1| |a_4| |a_2|^{\alpha-1} |a_3|^{\alpha-1} + |a_2| |a_3| |a_1|^{\alpha-1} |a_4|^{\alpha-1}}{(|a_3|^\alpha + |a_4|^\alpha)(|a_1|^\alpha + |a_2|^\alpha)} , \\
\operatorname{Corr}_\alpha(Y_1, Y_2) &= \frac{-|a_1 a_2 a_3 a_4| \left(|a_2 a_3|^{\alpha-2} - 2 |a_2 a_3|^{\frac{\alpha}{2}-1} |a_1 a_4|^{\frac{\alpha}{2}-1} + |a_1 a_4|^{\alpha-2} \right)}{(|a_3|^\alpha + |a_4|^\alpha)(|a_1|^\alpha + |a_2|^\alpha)}
\end{aligned}
$$

from which (3.3) follows by setting $|a_1 a_3| = |a_2 a_4|$.

∎

It is seen from (3.3) that once $a_1 a_3 = -a_2 a_4$ holds, $\operatorname{Corr}_\alpha(Y_1, Y_2) = 0$ if $\alpha = 2$ irrespective of the values of a_1, a_2, a_3, a_4. This is of course the Gaussian independence case. We have $\operatorname{Corr}_\alpha = 0$ also if one of the a_i-s is equal to zero or if $|a_2 a_3| = |a_1 a_4|$. The latter case does not imply independence for $\alpha < 2$ (cf Samorodnitsky and Taqqu (1), Proposition 2.9.5, p.102). In fact it follows from this proposition that for the system (3.1) we cannot have independence between Y_1 and Y_2 for $\alpha < 2$ if $a_1 a_2 a_3 a_4 \neq 0$.

It is possible to obtain a maximum case of ambiguity in the dependence relationship between Y_1 and Y_2 in that $\operatorname{Corr}_\alpha(Y_1, Y_2) \to -1$ as $\alpha \to 1$, $a_1 = a_4 \to 0$ and $|a_3| = |a_2|$.

In this limiting case, the regression lines $\mathbb{E}(Y_2 \,|\, Y_1)$ and $\mathbb{E}(Y_1 \,|\, Y_2)$ are orthogonal to each other.

It is fully possible to obtain $\text{Corr}_\alpha(Y_1, Y_2) < 0$ also when the condition $a_1 a_3 = -a_2 a_4$ does not hold. One can also say something about how often this will occur. To see this, rewrite $\text{Corr}_\alpha(Y_1, Y_2)$ as

$$
\begin{aligned}
\text{Corr}_\alpha(Y_1, Y_2) \;=\; & \left(\frac{|a_1|\,|a_3|^{\alpha-1}\,\text{sign}(a_1 a_3) + |a_2|\,|a_4|^{\alpha-1}\,\text{sign}(a_2 a_4)}{|a_3|^\alpha + |a_4|^\alpha} \right) \\
& \times \left(\frac{|a_3|\,|a_1|^{\alpha-1}\,\text{sign}(a_1 a_3) + |a_4|\,|a_2|^{\alpha-1}\,\text{sign}(a_2 a_4)}{|a_1|^\alpha + |a_2|^\alpha} \right).
\end{aligned}
$$

Next, note that there are four possible combinations for the sign $(a_1 a_3)$ and sign $(a_2 a_4)$.

(i) $\text{sign}(a_1 a_3) = \text{sign}(a_2 a_4) = +1$,

(ii) $\text{sign}(a_1 a_3) = \text{sign}(a_2 a_4) = -1$,

(iii) $\text{sign}(a_1 a_3) = +1$; $\text{sign}(a_2 a_4) = -1$,

(iv) $\text{sign}(a_1 a_3) = -1$; $\text{sign}(a_2 a_4) = +1$.

In situations **(i)** and **(ii)**, we have

$$
\text{Corr}_\alpha(Y_1, Y_2) = \frac{\left(|a_1|\,|a_3|^{\alpha-1} + |a_2|\,|a_4|^{\alpha-1}\right)\left(|a_3|\,|a_1|^{\alpha-1} + |a_4|\,|a_2|^{\alpha-1}\right)}{\left(|a_3|^\alpha + |a_4|^\alpha\right)\left(|a_1|^\alpha + |a_2|^\alpha\right)}
$$

which is positive, whereas in situations **(iii)** and **(iv)**, we have

$$
\text{Corr}_\alpha(Y_1, Y_2) = \frac{\left(|a_1|\,|a_3|^{\alpha-1} - |a_2|\,|a_4|^{\alpha-1}\right)\left(|a_3|\,|a_1|^{\alpha-1} - |a_4|\,|a_2|^{\alpha-1}\right)}{\left(|a_3|^\alpha + |a_4|^\alpha\right)\left(|a_1|^\alpha + |a_2|^\alpha\right)}.
$$

It is seen that the last expression can be negative as soon as $\alpha < 2$.

If we let $\alpha \to 1$, possibilities **(i)** and **(ii)** mean that $\text{Corr}_\alpha(Y_1, Y_2) \to 1$. On the other hand, possibilities **(iii)** and **(iv)** both imply

$$
\text{Corr}_\alpha(Y_1, Y_2) \longrightarrow \frac{(|a_1| - |a_2|)\,(|a_3| - |a_4|)}{(|a_3| + |a_4|)\,(|a_1| + |a_2|)}.
$$

Thus if the coefficients are drawn at random from uniform distributions on $[-1; 1]$, say, then in the long run 50% will give $\text{Corr}_\alpha(Y_1, Y_2) \approx 1$, which is the greatest possible value of the

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symmetric coefficient of covariation. The remaining 50% will be approximately symmetrically distributed around zero, i.e. about 25% will result in an ambiguous dependence relationship. This is confirmed in the four simulation plots in Figure 2.

Figure 2: Histogram of 5000 symmetric coefficients of covariation for system (3.1) with different values of α. The a_i-s have been drawn independently from uniform distribution on $[-1, 1]$.
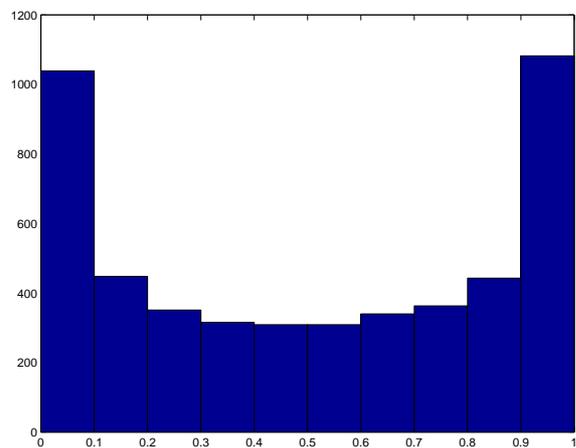
(a) $\alpha = 2$
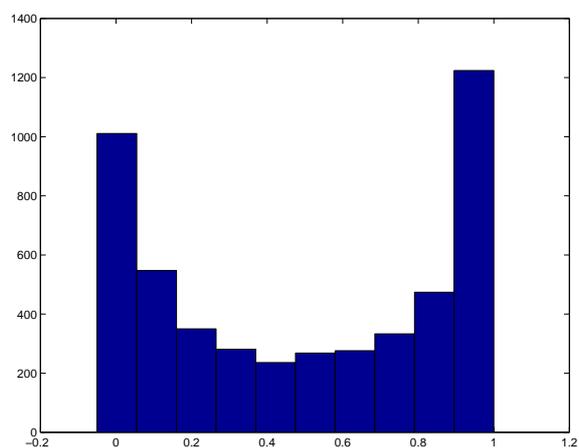
(b) $\alpha = 1.5$
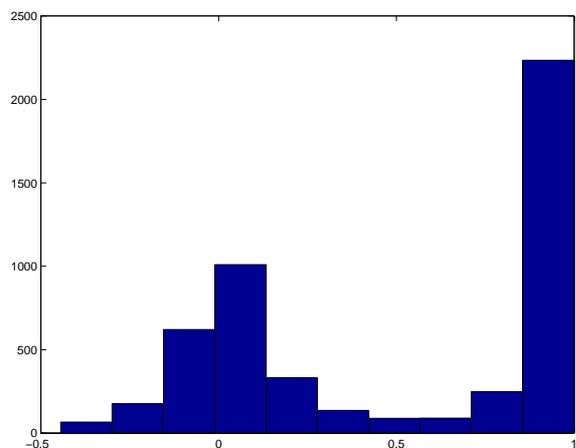
(c) $\alpha = 1.1$

(d) $\alpha = 1.01$

From the above expressions, it is easy to find values a_1, a_2, a_3, a_4 for which $\mathrm{Corr}_\alpha (Y_1, Y_2) = 0$. Moreover, we can also find values where $\{Y_1, Y_2\}_\alpha = 0$ and $\{Y_2, Y_1\}_\alpha \neq 0$. For $\alpha < 2$

neither of these examples would in general imply independence due to Samorodnitsky and Taqqu (1), Proposition 2.9.5, p.102.

One can also ask the question as to when $\mathrm{Corr}_\alpha\,(Y_1, Y_2) = 1$.

Proposition 8 *Consider the system (3.1) for $1 < \alpha \leqslant 2$. Then*
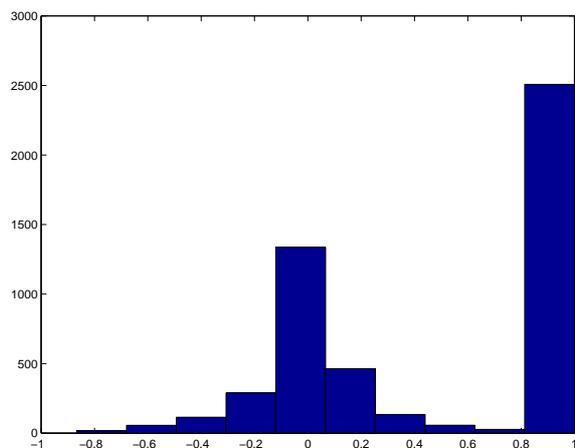
$$\mathrm{Corr}_\alpha\,(Y_1, Y_2) = 1 \Longleftrightarrow a_1 a_4 = a_2 a_3 \quad .$$

PROOF

By (3.2), we have

$$\mathrm{Corr}_\alpha\,(Y_1, Y_2) = \frac{|a_1 a_3|^\alpha + |a_2 a_4|^\alpha + \mathrm{sign}\,(a_1 a_2 a_3 a_4)\left(|a_1 a_4|\,|a_2 a_3|^{\alpha-1} + |a_2 a_3|\,|a_1 a_4|^{\alpha-1}\right)}{\left(|a_3|^\alpha + |a_4|^\alpha\right)\left(|a_1|^\alpha + |a_2|^\alpha\right)}.$$

Hence, $\mathrm{Corr}_\alpha\,(Y_1, Y_2) = 1$ iff

$$\mathrm{sign}\,(a_1 a_2 a_3 a_4)\left(|a_1 a_4|\,|a_2 a_3|^{\alpha-1} + |a_2 a_3|\,|a_1 a_4|^{\alpha-1}\right) = |a_1 a_4|^\alpha + |a_2 a_3|^\alpha \quad . \qquad (3.4)$$

At this point, there are two cases.

First case, $\mathrm{sign}(a_1 a_2 a_3 a_4) = 0$; then first we can take $a_1 = 0$. Inserting in the preceding equation, we have

$$|a_2 a_3|^\alpha = 0 \Rightarrow a_2 a_3 = 0.$$

In the same way, we have the three following implications

$$a_3 = 0 \Rightarrow a_1 a_4 = 0,$$
$$a_2 = 0 \Rightarrow a_1 a_4 = 0,$$
$$a_4 = 0 \Rightarrow a_2 a_3 = 0, \quad \text{and so, we have } \mathrm{Corr}_\alpha\,(Y_1, Y_2) = 1 \Leftrightarrow a_1 a_4 = a_2 a_3.$$

Second case, $\mathrm{sign}(a_1 a_2 a_3 a_4) \neq 0$; then equation (3.4) is equivalent to

$$\left[|a_1 a_4| - \mathrm{sign}\,(a_1 a_2 a_3 a_4)\,|a_2 a_3|\right]\left[|a_1 a_4|^{\alpha-1} - \mathrm{sign}\,(a_1 a_2 a_3 a_4)\,|a_2 a_3|^{\alpha-1}\right] = 0 \quad .$$

If $\mathrm{sign}(a_1 a_2 a_3 a_4) = -1$, then there is no solution.

If $\mathrm{sign}(a_1 a_2 a_3 a_4) = +1$, then $a_1 a_4 = a_2 a_3$.

∎

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In the Gaussian case, $\rho^2 (Y_1, Y_2) = \text{Corr}_\alpha (Y_1, Y_2) = 1$ is equivalent to $Y_1 = kY_2$ for some constant k. In the general case for $1 < \alpha < 2$, it is easily seen from proposition 8 that if $X_1 \neq 0$ then $\text{Corr}_\alpha (Y_1, Y_2) = 1$ is equivalent to $a_4 Y_1 = a_2 Y_2$, again Y_1 being a multiple of Y_2.

4. A COMPARISON OF DEPENDENCE MEASURES

We close this paper by examining some finite sample properties of the symmetric covariation $\text{Corr}_\alpha (Y_1, Y_2)$ and comparing to other dependence measures for stable variables, namely the covariation $\{Y_1, Y_2\}_\alpha$ and two rank-based measures.

4.1. Other dependence measures

We consider two rank-based measures of dependence: the Spearman and the van der Waerden rank correlation coefficients.

Definition 9 *Let $(X_1, \ldots, X_n)$ be n real random variables. The associated vector of ranks, $(R_1, \ldots, R_n)$ is defined by*

$$\forall i \in [\![1; n]\!], \ R_i = 1 + \sum_{j=1}^{n} \mathbb{1}_{]0;+\infty[} (X_i - X_j) \quad .$$

Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ is a sample. If we associate to X (resp. Y) the vector of ranks $(R_i)_{i=1,\ldots,n}$ (resp. $(S_i)_{i=1,\ldots,n}$), we calculate two measures of dependence.

Definition 10 *With the preceding notations, the **Spearman correlation coefficient** r_S and the **van der Waerden correlation coefficient** r_W are given by*

$$
\begin{aligned}
r_S &= 1 - \frac{6}{n (n^2 - 1)} \sum_{i=1}^{n} (R_i - S_i)^2, \\[2em]
r_W &= \frac{\displaystyle\sum_{i=1}^{n} \Phi^{-1} \left(\frac{R_i}{n+1} \right) \Phi^{-1} \left(\frac{S_i}{n+1} \right)}{\displaystyle\sum_{i=1}^{n} \left[\Phi^{-1} \left(\frac{i}{n+1} \right) \right]^2},
\end{aligned}
$$

where Φ^{-1} is the inverse cumulative distribution function of a standard normal variable.

These coefficients exist independently of the existence of any moments. So it is tempting to use them as a measure of dependence for alpha stable random variables. In particular, we have the following result.

Proposition 11 *Under the hypothesis of independence of X and Y, we have the asymptotic result*

$$\sqrt{n-1}\ r_S \quad \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0;1\right),$$

$$\sqrt{n-1}\ r_W \quad \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0;1\right),$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution.

4.2. Simulations

We will compare the new measure $\mathrm{Corr}_\alpha\left(Y_1, Y_2\right)$ to the other measures by simulations. We recall an important property of the coefficient of covariation.

Proposition 12 *Let (Y_1, Y_2) be a $S\alpha S$ random vector with $\alpha > 1$, we have for all $1 \leqslant p < \alpha$,*

$$\{Y_1, Y_2\}_\alpha = \frac{[Y_1, Y_2]_\alpha}{[Y_2, Y_2]_\alpha} = \frac{\mathbb{E}\left[Y_1 Y_2^{\langle p-1\rangle}\right]}{\mathbb{E}\,|Y_2|^p}\ .$$

Samorodnitsky and Taqqu (1), pp. 94-95, gave the proof of this result for $1 < p < \alpha$. Following the same lines as Samorodnitsky and Taqqu, the property is easily obtained for $p = 1$. Note that this last result is used by several authors, such as Nikias and Shao (4), p.47, equation (4.4), or Gallagher (9), p.382, equation (1.5).

Taking $p = 1$ in the preceding equation leads to a very convenient estimator of $\{Y_1, Y_2\}_\alpha$.

Let $(Y_{1,i})_{1\leqslant i\leqslant n}$ and $(Y_{2,i})_{1\leqslant i\leqslant n}$ be two vectors of realizations of Y_1 and Y_2. To estimate $\mathrm{Corr}_\alpha\left(Y_1, Y_2\right)$, we used the following quantity

$$\widehat{\mathrm{Corr}_\alpha}\left(Y_1, Y_2\right) = \left(\frac{\sum_{i=1}^{n} Y_{1,i}\ \mathrm{sign}\left(Y_{2,i}\right)}{\sum_{i=1}^{n} |Y_{2,i}|}\right)\left(\frac{\sum_{i=1}^{n} Y_{2,i}\ \mathrm{sign}\left(Y_{1,i}\right)}{\sum_{i=1}^{n} |Y_{1,i}|}\right)$$

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which is the product of the estimator of the covariation $\{Y_1, Y_2\}_\alpha$ by the estimator of the covariation $\{Y_2, Y_1\}_\alpha$.

The algorithm of Chambers et al. (10) is used to simulate $S\alpha S$ independent random variables.

We simulated 500 realizations of X_1 and X_2, with the same dispersion $\gamma_{X_1} = \gamma_{X_2} = 1$. Then, we calculated the different measures of dependence between Y_1 and Y_2 generated by (3.1). We made 100 replications. In Tables 1 and 2 there are three types of cells:

1. with a single value in the cell, it represents the theoretical value of the dependence measure (respectively the coefficient of covariation of Y_1 on Y_2, the coefficient of covariation of Y_2 on Y_1 and the symmetric coefficient of covariation between Y_1 and Y_2);

2. with two values; these two numbers represent the mean (upper value in the cell) and the standard deviation (lower value in the cell) of the 100-sample dependence measures (respectively the Spearman correlation coefficient and the van der Waerden correlation coefficient);

3. with three values; they represent the mean (upper value in the cell), the standard deviation (center value in the cell) and the squared error (lower value in the cell) of the 100-sample dependence measures (respectively the coefficient of covariation of Y_1 on Y_2, the coefficient of covariation of Y_2 on Y_1 and the symmetric coefficient of covariation between Y_1 and Y_2).

We only study two sets of the four parameters a_1, a_2, a_3 and a_4. For the first case, shown in Table 1, we have independence between Y_1 and Y_2. In the second case, shown in Table 2, the parameters were drawn at random from uniform distributions, resulting in $a_1 = 0.09, a_2 = 0.53, a_3 = -0.77, a_4 = -0.88$.

We recall that for model (3.1), we have

$$\{Y_1, Y_2\}_\alpha = \frac{a_1 a_3^{\langle \alpha - 1 \rangle} + a_2 a_4^{\langle \alpha - 1 \rangle}}{|a_3|^\alpha + |a_4|^\alpha}.$$

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In Table 1, we see that the sample coefficient of covariation has a much bigger dispersion than the other measures when α goes to 1. Both the bias and standard error vary with α. If one looks at the squared error, none of the criteria dominates all of the others.

For the Table 2 we remark:

1. The absolute theoretical value of $\{Y_1, Y_2\}_\alpha$ decreases as α goes to 1 whereas the absolute theoretical value of $\{Y_2, Y_1\}_\alpha$ increases. It seems to be another drawback of this measure of dependence.

2. In the Gaussian case, the dependence between Y_1 and Y_2 is negative. The direction of dependence is detected by the two rank-based coefficients of correlation and $\{Y_1, Y_2\}_\alpha$ but not of course by the symmetric coefficient of covariation.

3. Even the distribution-free, rank-based coefficients of correlation vary when alpha tends to 1. This means that there is a possible change in the dependence structure for alpha close to 1.

We made a number of additional simulations, with different sets of parameters (a_1, a_2, a_3, a_4) and different dispersions $(\gamma_{X_1}, \gamma_{X_2})$. They gave similar results: the rank-based coefficients of correlation and the symmetric coefficient of covariation appear to be more precise in terms of squared error.

Table I: Theoretical and estimated values of dependence measures, and rank-based coefficients of correlation for: $a_1 = 1; a_2 = 0; a_3 = 0; a_4 = 1$.

α	2	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1
$\{Y_1, Y_2\}_\alpha$	0	0	0	0	0	0	0	0	0	0
$\{\widehat{Y_1, Y_2}\}_\alpha$	0.0003 0.0575 0.0033	−0.0029 0.0628 0.0040	0.0092 0.0767 0.0060	−0.0013 0.1369 0.0187	0.0095 0.1129 0.0128	0.0279 0.2066 0.0434	−0.0090 0.3002 0.0902	−0.0288 0.3982 0.1594	−0.2753 3.5253 12.5036	0.0537 0.5525 0.3081
$\{Y_2, Y_1\}_\alpha$	0	0	0	0	0	0	0	0	0	0
$\{\widehat{Y_2, Y_1}\}_\alpha$	0.0046 0.0571 0.0033	−0.0098 0.0628 0.0040	0.0049 0.0938 0.0088	−0.0020 0.1027 0.0106	−0.0469 0.3834 0.1492	0.0064 0.1850 0.0343	−0.0169 0.1561 0.0246	−0.0483 1.1559 1.3383	−0.1333 0.9459 0.9125	−0.0061 0.8835 0.7806
Corr_α	0	0	0	0	0	0	0	0	0	0
$\widehat{\mathrm{Corr}_\alpha}$	0.0021 0.0029 0.0001	0.0021 0.0039 0.0001	0.0024 0.0070 0.0001	0.0021 0.0100 0.0001	0.0016 0.0107 0.0001	0.0045 0.0195 0.0004	0.0049 0.0213 0.0005	−0.0014 0.0301 0.0009	0.0013 0.0702 0.0049	−0.0041 0.0898 0.0081
r_S	0.0006 0.0446	−0.0048 0.0460	0.0058 0.0525	0.0017 0.0423	0.0046 0.0447	0.0009 0.0451	−0.0014 0.0448	−0.0044 0.0368	−0.0084 0.0443	0.0005 0.0391
r_W	−0.0010 0.0471	−0.0036 0.0438	0.0053 0.0500	−0.0007 0.0422	0.0029 0.0433	−0.0010 0.0456	−0.0019 0.0469	−0.0036 0.0375	−0.0094 0.0460	0.0009 0.0393

Table II: Theoretical and estimated values of dependence measures, and rank-based coefficients of correlation for: $a_1 = 0.09; a_2 = 0.53; a_3 = -0.77; a_4 = -0.88$.

α	2	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1
$\{Y_1, Y_2\}_\alpha$	−0.3918	−0.3902	−0.3886	−0.3870	−0.3854	−0.3838	−0.3822	−0.3806	−0.3790	−0.3774
$\{\widehat{Y_1, Y_2}\}_\alpha$	−0.3909 0.0125 0.0002	−0.3921 0.0186 0.0004	−0.3891 0.0192 0.0004	−0.3866 0.0201 0.0004	−0.3890 0.0267 0.0007	−0.3806 0.0284 0.0008	−0.3814 0.0377 0.0014	−0.3766 0.0447 0.0020	−0.3747 0.0586 0.0034	−0.3647 0.0743 0.0057
$\{Y_2, Y_1\}_\alpha$	−1.8536	−1.8899	−1.9326	−1.9830	−2.0421	−2.1113	−2.1921	−2.2858	−2.3943	−2.5189
$\{\widehat{Y_2, Y_1}\}_\alpha$	−1.8620 0.0584 0.0035	−1.8870 0.0815 0.0066	−1.9343 0.1090 0.0119	−1.9705 0.0928 0.0088	−2.0265 0.1225 0.0153	−2.1182 0.2159 0.0467	−2.1845 0.2170 0.0472	−2.2987 0.4982 0.2484	−2.3994 0.5074 0.2575	−2.6258 0.8516 0.7367
Corr_α	0.7262	0.7374	0.7510	0.7674	0.7870	0.8103	0.8378	0.8700	0.9074	0.9506
$\widehat{\mathrm{Corr}_\alpha}$	0.7278 0.0302 0.0009	0.7392 0.0347 0.0012	0.7513 0.0320 0.0010	0.7607 0.0322 0.0011	0.7859 0.0353 0.0012	0.8012 0.0338 0.0012	0.8261 0.0335 0.0013	0.8472 0.0371 0.0019	0.8736 0.0402 0.0028	0.9035 0.0381 0.0037
r_S	−0.8412 0.0139	−0.8412 0.0145	−0.8408 0.0151	−0.8420 0.0149	−0.8457 0.0165	−0.8473 0.0166	−0.8494 0.0172	−0.8571 0.0161	−0.8607 0.0184	−0.8665 0.0164
r_W	−0.8505 0.0120	−0.8520 0.0133	−0.8530 0.0146	−0.8548 0.0138	−0.8598 0.0145	−0.8608 0.0163	−0.8650 0.0160	−0.8726 0.0153	−0.8780 0.0159	−0.8840 0.0147

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