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► **To cite this version:**

Alexei Panchishkin. Graded structures and differential operators on nearly holomorphic and quasimodular forms on classical groups. 2016. hal-01329432v1

HAL Id: hal-01329432

<https://hal.science/hal-01329432v1>

Preprint submitted on 9 Jun 2016 (v1), last revised 3 Oct 2016 (v2)

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Graded structures and differential operators on nearly holomorphic and quasimodular forms on classical groups

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Abstract

We wish to use graded structures [KrVu87], [Vu01] on differential operators and quasimodular forms on classical groups and show that these structures provide a tool to construct p -adic measures and p -adic L -functions on the corresponding non-archimedean weight spaces.

An approach to constructions of automorphic L -functions on unitary groups and their p -adic avatars is presented. For an algebraic group G over a number field K these L functions are certain Euler products $L(s, \pi, r, \chi)$. In particular, our constructions cover the L -functions in [Shi00] via the doubling method of Piatetski-Shapiro and Rallis.

A p -adic analogue of $L(s, \pi, r, \chi)$ is a p -adic analytic function $L_p(s, \pi, r, \chi)$ of p -adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers defined through the normalized critical values $L^*(s, \pi, r, \chi)$ of the corresponding complex analytic L -function. We present a method using arithmetic nearly-holomorphic forms and general quasi-modular forms, related to algebraic automorphic forms. It gives a technique of constructing p -adic zeta-functions via quasi-modular forms and their Fourier coefficients.

Key words : graded structures, automorphic forms, classical groups, p -adic L -functions, differential operators, non-archimedean weight spaces, quasi-modular forms, Fourier coefficients.

2010 Mathematics Subject Classification

11F67 Special values of automorphic L -series, periods of modular forms, cohomology, modular symbols, 11F85 p -adic theory, local fields, 11F33 Congruences for modular and p -adic modular forms [See also 14G20, 22E50] 16W50, Graded rings and modules 16E45 , Differential graded algebras and applications

Introduction

Let p be a prime number. In the present paper we wish to use graded structures [KrVu87], [Kr80], [Vu01] on the rings and modules of differential operators and quasimodular forms on classical groups in order to construct p -adic measures and p -adic L -functions on certain non-archimedean weight spaces.

Arithmetical modular forms belong traditionally to the world of arithmetic, but also to the worlds of geometry, algebra and analysis. On the other hand, it is very useful, to attach zeta-functions (or L -functions) to mathematical objects of various nature as certain generating functions (or as certain Euler products).

Firstly, such L -functions give a tool to link these objects to each other (expressing a general form of functoriality), and secondly, this approach allows in favorable cases to obtain answers to fundamental questions about these objects; such answers are often expressed in the form of a number (complex or p -adic).

General graded structures [KrVu87], [Kr80], [Vu01] can be used on the rings and modules of differential operators and quasimodular forms on classical groups. Our purpose is to use them in order to construct p -adic measures and p -adic L -functions on the corresponding non-archimedean weight spaces.

In the definition of M. Krasner, without the commutativity hypothesis, the structure of a graded group is determined by the underlying structure of the abstract group, and only the homogeneous part is equipped with a partial composition, induced by that of the group.

Moreover, M. Krasner introduces useful notions of graded ring and module, and their homogeneous parts called *annéïde et moduloïde* where the suffixes *oïde*, *eïde* mean that the composition structure is partial in principle. In our situation the underlying abstract group is a certain p -adic analytic group Y of characters with values in \mathbb{C}_p^* which contains a subset Y^{class} of algebraic "motivic" points. Here $\bar{\mathbb{Q}}_p$ be an algebraic closure of \mathbb{Q}_p , $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the Tate field, the completion of $\bar{\mathbb{Q}}_p$, which is a fundamental object in p -adic analysis, and thanks to Krasner we know that \mathbb{C}_p is algebraically closed [Am75], Théorème 2.7.1 (Lemme de Krasner), [Kr74]. This famous result allows to develop analytic functions and analytic spaces over \mathbb{C}_p [[Kr74], Tate, Berkovich...]. and we fix an embedding $incl_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, so that elements of Y^{class} are identified with some characters with values in $\bar{\mathbb{Q}}^*$.

Methods of construction of p -adic L -functions:

- 1) p -adic interpolation (starting from Kubota-Leopoldt in 1964);
- 2) abstract Kummer congruences using a family of test elements (e.g. binomial coefficients as functions with values in \mathbb{Z}_p , or certain characters with values in $\bar{\mathbb{Q}}_p^*$, such as elements of Y^{class});
- 3) the use of p -adic Cauchy integral (Shnirelman integral), e.g. a theorem of Amice-Fresnel, see [Ko80], p.120. Let $f(z) = \sum_n a_n z^n \in \mathbb{C}_p[[z]]$ have the property that the coefficients a_n can be p -adically interpolated i.e., there exists a continuous function $\phi : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ such that $\phi(n) = a_n$. Then f (whose disc of convergence must be the open unit disc $D_0(1^-)$) is the restriction to $D_0(1^-)$ of a Krasner analytic function \tilde{f} on the complement of $D_1(1^-)$.

Acknowledgement

Many thanks to Mirjana Vuković for soliciting a paper for this collection.

My special thanks go to Khoai Ha Huy, Siegfried Boecherer and Vladimir Berkovich for fruitful discussions during a visit of the author to Vietnam in April 2015.

1 Automorphic L -functions and their p -adic analogues

Our main objects in this paper are automorphic L -functions and their p -adic analogues.

For an algebraic group G over a number field K these L functions are defined as certain Euler products. More precisely, we apply our constructions for the L -functions studied in Shimura's book [Shi00].

EXAMPLE 1.1 ($G = \mathrm{GL}(2)$, $K = \mathbb{Q}$, $L_f(s) = \sum_{n \geq 1} a_n n^{-s}$, $s \in \mathbb{C}$) . Here $f(z) = \sum_{n \geq 1} a_n q^n$ is a modular form on the upper-half plane

$$H = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\} = \mathrm{SL}(2)/\mathrm{SO}(2), q = e^{2\pi iz}.$$

An Euler product has the form

$$L_f(s) = \prod_{p \text{ primes}} (1 - a_p p^{-s} + \psi_f(p) p^{k-1-2s})^{-1}$$

where k is the weight and ψ_f the Dirichlet character of f . It is defined iff the automorphic representation π_f attached to f is irreducible.

Recall that π_f is generated by the lift \tilde{f} of f to the group $G(\mathbb{A})$, where \mathbb{A} is the ring of adèles

$$\mathbb{A} = \{x = (x_\infty, x_p)_p \mid x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p, \\ \text{such that } x_p \in \mathbb{Z}_p \text{ for all but a finite number of } p\}.$$

1.1 A p -adic analogue of $L_f(s)$ (Manin-Mazur)

It is a p -adic analytic function $L_{f,p}(s, \chi)$ of p -adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers

$$L_f^*(s, \chi) / \omega^\pm \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\mathbb{Q}}_p \text{ (the Tate field)}$$

for $1 \leq s \leq k-1$, ω^\pm are periods of f where the complex analytic L function of f is defined for all $s \in \mathbb{C}$ so that in the absolutely convergent case $\text{Re}(s) > (k+1)/2$,

$$L_f^*(s, \chi) = (2\pi)^{-s} \Gamma(s) \sum_{n \geq 1} \chi(n) a_n n^{-s}$$

which extends to holomorphic function with a functional equation. According to Manin and Shimura, this number is algebraic if the period ω^\pm is chosen according to the parity $\chi(-1)(-1)^{-s} = \pm 1$.

1.2 Constructions of p -adic avatars

In the general case of an irreducible automorphic representation of the adelic group $G(\mathbb{A}_K)$ there is an L -function

$$L(s, \pi, r, \chi) = \prod_{\mathfrak{p}_v \text{ primes in } K} \prod_{j=1}^m (1 - \beta_{j, \mathfrak{p}_v} N_{\mathfrak{p}_v}^{-s})^{-1}$$

where

$$\prod_{j=1}^m (1 - \beta_{j, \mathfrak{p}} X) = \det(1_m - r(\text{diag}(\alpha_{i, \mathfrak{p}})_i X)),$$

$\alpha_{i, \mathfrak{p}}$ are the Satake parameters of $\pi = \otimes_v \pi_v$ $v \in \Sigma_K$ (places in K), $\mathfrak{p} = \mathfrak{p}_v$. Here $h_v = \text{diag}(\alpha_{i, \mathfrak{p}})_i$ live in the Langlands group ${}^L G(\mathbb{C})$, $r : {}^L G(\mathbb{C}) \rightarrow \text{GL}_m(\mathbb{C})$ denotes a finite dimensional representation, and $\chi : \mathbb{A}_K^* / K^* \rightarrow \mathbb{C}^*$ is a character of finite order. Constructions admit extension to rather general automorphic representations on Shimura varieties via the following tools:

- Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms)
- Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or through the
- CM-values (special points on Shimura varieties),

In order to explain a general setting of the theory of p -adic L -functions let us consider a p -adic analytic space Y and a subset Y^{class} of points such that, for each $y \in Y^{class}$ there is a motive M_y , such that 0 is a critical value of the L -function $L(s, M_y)$. In our situation, we start with a CM field K over \mathbb{Q} , a quadratic extension of a totally real field K^+ , and an n -dimensional hermitian vector space V/K . Then Y is the space of pairs (λ, χ) , where λ runs through the set of ordinary p -adic modular forms on the Shimura variety $Sh(V)$ attached to $U(V)$ and χ runs through p -adic Hecke characters of K ; both λ and χ are assumed to be unramified outside a finite set S of primes of K , including those dividing p , and of bounded level at primes not dividing p .

1.3 Accessible cases: symplectic and unitary groups

- $G = \mathrm{GL}_1$ over \mathbb{Q} (Kubota-Leopoldt-Mazur) for the Dirichlet L -function $L(s, \chi)$.
- $G = \mathrm{GL}_1$ over a totally real field F (Deligne-Ribet, using algebraicity result by Klingen).
- $G = \mathrm{GL}_1$ over a CM-field K , i.e. a totally imaginary extension of a totally real field F (N.Katz, Manin-Vishik).
- the Siegel modular case $G = \mathrm{GSp}_n$ (the Siegel modular case, $F = \mathbb{Q}$).
- General symplectic and unitary groups over a CM-field K .

2 Automorphic L -functions attached to symplectic and unitary groups

Let us briefly describe the L -functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [Shi00], with critical values

computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_m(K) \mid \alpha \varphi^t \alpha^\rho = \nu(\alpha) \varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ or $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$, see also Ch.Skinner and E.Urban [MC] and Shimura G., [Shi00].

2.1 The groups and automorphic forms studied by Shimura in [Shi00]

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. Take $\eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and define

$$\begin{aligned} G &= \mathrm{Sp}(n, F) && \text{(Case Sp)} \\ G &= \{\alpha \in \mathrm{GL}_{2n}(K) \mid \alpha \eta_n \alpha^* = \eta_n\} && \text{(Case UT = unitary tube)} \\ G &= \{\alpha \in \mathrm{GL}_{2n}(K) \mid \alpha T \alpha^* = T\} && \text{(Case UB = unitary ball)} \end{aligned}$$

according to three cases. Assume $F = \mathbb{Q}$ for a while. The group of the real points G_∞ acts on the associated domain

$$\mathcal{H} = \begin{cases} \{z \in M(n, n, \mathbb{C}) \mid {}^t z = z, \mathrm{Im}(z) > 0\} & \text{(Case Sp)} \\ \{z \in M(n, n, \mathbb{C}) \mid i(z^* - z) > 0\} & \text{(Case UT)} \\ \{z \in M(p, q, \mathbb{C}) \mid 1_q - z^* z > 0\} & \text{(Case UB)}, \end{cases}$$

with $(p, q), p + q = n$, being the signature of iT . Here $z^* = {}^t \bar{z}$ and $>$ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g, z)$, $g \in G_\infty$, $z \in \mathcal{H}$ taking values in $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$.

2.2 Shimura's arithmeticity in the theory of automorphic forms [Shi00], p -adic zeta functions and nearly-holomorphic forms on classical groups

Automorphic L-functions via general quasi-modular forms. Automorphic L -functions and their p -adic avatars can be obtained for quite general automorphic representations on Shimura varieties by constructing p -adic distributions out of algebraic numbers attached to automorphic forms. These

numbers satisfy certain Kummer-type congruences established in different ways: via

- Normalized Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or through the
- CM-values (special points on Shimura varieties), see [MC] by C. Skinner and E. Urban, and [Shi00] by Shimura.

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated.

In order to prove the congruences needed for the p -adic constructions, we use a simplification due to nearly-holomorphic and general quasi-modular forms, related to algebraic automorphic forms. In this paper, a new method of constructing p -adic zeta-functions is presented using general quasi-modular forms and their Fourier coefficients.

In order to describe both algebraicity and congruences of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups, we follow the review by H.Yoshida [YS] of Shimura's book "Arithmeticity in the theory of automorphic forms" [Shi00], which describes Shimura's mathematics developed by stages:

- (A) Complex multiplication of abelian varieties =>
- (B) The theory of canonical models = Shimura varieties =>
- (C) Critical values of zeta functions and periods of automorphic forms.

(B) includes (A) as 0-dimensional special case of canonical models. The relation of (B) and (C) is more involved, but (B) provides a solid foundation of the notion of the arithmetic automorphic forms. Also unitary Shimura varieties have recently attracted much interest (in particular by Ch. Skinner and E. Urban), see [MC], in relation with the proof of the The Iwasawa Main Conjecture for $GL(2)$.

2.3 Integral representations and critical values of the zeta functions

Automorphic forms are assumed scalar valued in this part. For Cases Sp and UT, Eisenstein series $E(z, s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behaviour and those values of $\in 2^{-1}\mathbb{Z}$ at which $E(z, \sigma)$ is nearly holomorphic and arithmetic, are studied in [Shi00]. This is achieved by proving a relation giving passage from s to $s - 1$ for $E(z, s)$, involving a differential operator, then examining Fourier

coefficients of Eisenstein series using the theory of confluent hypergeometric functions on tube domains.

For a Hecke eigenform f on $G_{\mathbb{A}}$ and an algebraic Hecke character χ on the idele group of K (in Case Sp, $K = F$), the zeta function $\mathcal{Z}(s, f, \chi)$ is defined. Viewing it as an Euler product extended over prime ideals of F , the degree of the Euler factor is $2n + 1$ in Case Sp, $4n$ in Case UT, and $2n$ in Case UB, except for finitely many prime ideals, see Chapter 5 of [Shi00].

This zeta function is almost the same as the so called standard L -function attached to f twisted by χ but it turns out to be more general in the unitary case, see also [EHLS].

Main results on critical values of the L -functions studied in Shimura's book [Shi00] is stated in Theorem 28.5, 28.8 (Cases Sp, UT), and in Theorem 29.5 in Case UB.

Theorem 2.1 (algebraicity of critical values in Cases Sp and UT)

Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell} |x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume the following conditions (in the notations of Chapter 7 of [Shi00] for the weight k_v, μ_v, ℓ_v)

- Case Sp $2n + 1 - k_v - \mu_v - \leq k_v - \mu_v$, where $\mu_v = 0$ if $[k_v] - l_v \in 2\mathbb{Z}$ and $\mu_v = 1$ if $[k_v] - l_v \notin 2\mathbb{Z}$; $\sigma_0 - k_v + \mu_v$ for every $v \in \mathbf{a}$ if $\sigma_0 > n$ and $\sigma_0 - 1 - k_v + \mu_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$ if $\sigma_0 \leq n$.
- Case UT $4n - (2k_{v\rho} + \ell_v) \leq 2\sigma_0 \leq m_v - [k_v - k_{v\rho} - \ell_v]$ and $2\sigma_0 - \ell_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$.

Further exclude the following cases

- (A) Case Sp $\sigma_0 = n + 1, F = \mathbb{Q}$ and $\chi^2 = 1$;
- (B) Case Sp $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$
- (C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$;
- (D) Case Sp $0 < \sigma_0 \leq n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is \mathfrak{g} ;
- (E) Case UT $2\sigma_0 = 2n + 1, F = \mathbb{Q}, \chi_1 = \theta$, and $k_v - k_{v\rho} = \ell_v$;
- (F) Case UT $0 \leq 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is \mathfrak{r}

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) / \langle \mathbf{f}, \mathbf{f} \rangle \in \bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp, } k \in \mathbb{Z}^{\mathbf{a}}, \text{ and } \sigma_0 > n_0, \\ n\sigma_0 - n^2, & \text{Case Sp, } k \notin \mathbb{Z}^{\mathbf{a}}, \text{ or } \sigma_0 \leq n_0, \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

We establish a p -adic analogue of Theorem 28.8 (in Cases Sp and UT) representing algebraic parts of critical values as values of certain p -adic analytic zeta functions.

3 Constructing p -adic zeta-functions via quasi-modular forms

We present here a new method of constructing p -adic zeta-functions based on the use of general quasi-modular forms on classical groups.

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated. We present a method of simplification using nearly-holomorphic and general quasi-modular forms, related to algebraic automorphic forms. It gives a new method of constructing p -adic zeta-functions using general quasi-modular forms and their Fourier coefficients. The symmetric space

$$\mathcal{H} = G(\mathbb{R}) / ((\text{maximal-compact subgroup})\mathcal{K} \times \text{Center})$$

parametrizes certain families of abelian varieties A_z ($z \in \mathcal{H}$) so that $F \subset \text{End}(A_z) \otimes \mathbb{Q}$. The CM-points z correspond to a maximal multiplication ring $\text{End}(A_z)$.

For the group $GL(2)$, N.Katz used arithmetical elements (real-analytic and p -adic)

instead of holomorphic forms in these representation spaces. These elements correspond also to quasi-modular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators. A relation real-analytic \leftrightarrow p -adic modular forms comes from the notion of p -adic modular forms invented by J.-P.Serre [Se73] as p -adic limits of q -expansions of modular forms with rational coefficients for $\Gamma = \text{SL}_2(\mathbb{Z})$. The present method of constructing p -adic automorphic L -functions uses general quasi-modular forms, and their link to algebraic p -adic modular forms.

Real-analytic and p -adic modular forms

In Serre's case for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the ring M_p of p -adic modular forms contains $\mathcal{M} = \bigoplus_{k \geq 0} \mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains as element with the q -expansion $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$. On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of S over \mathbb{Q}). Let \mathcal{N} be the ring of such forms. Then $\tilde{E}_2|_{S=0} = E_2$ and it was proved by J.-P. Serre that E_2 is a p -adic modular form. Elements of the ring $QM = \mathcal{N}|_{S=0} = 0$ will be called general quasi-modular forms. These phenomena are quite general and can be used in computations and proofs. In June 2014 in a talk in Grenoble, S.Boecherer extended these results to the Siegel modular case.

Using algebraic and p -adic modular forms

There are several methods to compute various L -values starting from the constant term of the Eisenstein series in [Se73],

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n = \frac{\Gamma(k)}{(2\pi i)^k} \sum_{(c,d)} '(cz+d)^{-k} \quad (\text{for } k \geq 4),$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [Shi00]:

- the Rankin-Selberg method,
- the doubling method (pull-back method).

A known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_n^k(\Gamma)$ of genus n (with local factors of degree $2n + 1$) and χ a Dirichlet character.

Theorem ([CourPa], the case of even genus n , via the Rankin-Selberg method) gives a p -adic interpolation of the normalized critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1 + n - k \leq s \leq k - n$ through the Petersson product $\langle f, \theta_{T_0} \delta^r E \rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0 .

Theorem 3.1 (p -adic interpolation of $D(s, f, \chi)$) (1) (the case of odd genus (Boecherer-Schmidt, [BS00]))

Assume that n is arbitrary genus, and a prime p ordinary then there exists a p -adic interpolation of $D(s, f, \chi)$

(2) (Anh-Tuan Do (non-ordinary case, PhD Thesis of March 2014)), via the doubling method)

Assume that n is arbitrary genus, and p an arbitrary prime not dividing level of f then there exists a p -adic interpolation of $D(s, f, \chi)$.

Proof uses the following Boecherer-Garrett-Shimura identity (a pull-back formula) which allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l + 2s, \chi)D(l + 2s - n, f, \chi)f = \langle f(w), E_{l, \nu, \chi, s}^{2n}(\text{diag}[z, w])_w \rangle,$$

Here $k = l + \nu$, $\nu \geq 0$, $\Lambda(l + 2s, \chi)$ is a product of special values of Dirichlet L -functions and Γ -functions, $E_{l, \nu, \chi, s}^{2n}$ a higher twist of a Siegel-Eisenstein series on $(z, w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [BS00]).

A p -adic construction uses congruences for the L -values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms. In the present approach of computing the Petersson products and L -values, an injection of algebraic nearly holomorphic modular forms into p -adic modular forms is used.

3.1 Injecting nearly-holomorphic forms into p -adic modular forms

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], [Ich13], allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into p -adic modular forms. Via the Fourier expansions, the image of this injection is represented by certain quasi-modular holomorphic forms like $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$, with algebraic Fourier expansions. This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L -values, which we would like to develop here. In fact, the realization of nearly holomorphic forms as p -adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [U14], Chapter 10.

Urban only treats the elliptic modular case in that paper, but I believe he and Skinner are working on applications of a more general theory. This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p -adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p -adic. Parts of these results are also valid for vector-valued modular forms.

3.2 Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of \mathbb{C} are certain \mathbb{C}^d -valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$ where T runs through the set B_n of all half-integral semi-positive matrices, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\text{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S .

3.3 Review of the algebraic theory

Consider the columns Z_1, Z_2, \dots, Z_n of $Z \in \mathbb{H}_n$ and the \mathbb{Z} -lattice L_Z in \mathbb{C}^n generated by $\{E_1, E_2, \dots, E_n, Z_1, Z_2, \dots, Z_n\}$, where E_1, E_2, \dots, E_n are the columns of the identity matrix E . The torus $A_Z = \mathbb{C}^n/L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \rightarrow \mathbb{H}_n$ whose fiber over the point $Z \in \mathbb{H}_n$ is A_Z . Let us consider the quotient space $\mathbb{H}_n/\Gamma(N)$ of the Siegel upper half space \mathbb{H}_n of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \left(\begin{array}{cc|c} A_\gamma & B_\gamma & A_\gamma \equiv D_\gamma \equiv 1_n \\ C_\gamma & D_\gamma & B_\gamma \equiv C_\gamma \equiv 0_n \end{array} \right) \right\}$$

If $N > 3$, $\Gamma(N)$ acts without fixed points on $\mathcal{A} = \mathcal{A}_n$ and the quotient is a smooth algebraic family $\mathcal{A}_{n,N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$ defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N -th root of 1. For positive integers n and N , $\mathcal{H}_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension n with a symplectic level N structure.

3.4 Applications to constructions of p -adic L -functions

We present here a survey of some methods of construction of p -adic L -functions. Two important ideas that are not as well known as they should be are developed briefly in this section.

There exist two kinds of L -functions

- Complex L -functions (Euler products) on $\mathbb{C} = \text{Hom}(\mathbb{R}_+^*; \mathbb{C}^*)$.
- p -adic L -functions on the \mathbb{C}_p -analytic group $\text{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ (Mellin transforms \mathcal{L}_μ of p -adic measures μ on \mathbb{Z}_p^*).

Both are used in order to obtain a number (L -value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\mathbb{Q}}_p$$

and we may compare the complex and p -adic L -values at many points.

How to define and to compute p -adic L -functions? The Mellin transform of a p -adic distribution μ on \mathbb{Z}_p^* gives an analytic function on the group of p -adic characters

$$x \mapsto \mathcal{L}_\mu(x) = \int_{\mathbb{Z}_p^*} x(y) d\mu(y), \quad x \in X_{\mathbb{Z}_p^*} = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$$

A general idea is to construct p -adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L -values. Here we present a new method to construct p -adic L -functions via quasi-modular forms:

3.5 How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L -function $L_f^*(s, \chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" $(s_j; \chi_j)$ one has an integral representation $\boxed{L_f^*(s, \chi) = \langle f, h_j \rangle}$ with all $h_j = \sum_T b_{j,T} q^T \in \mathcal{M}$ in a certain finite-dimensional space \mathcal{M} containing f and defined over $\bar{\mathbb{Q}}$. We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \quad \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \pmod{p^N} \implies \sum_j \beta_j \frac{L_f^*(s, \chi)}{\langle f, f \rangle} \equiv 0 \pmod{p^N}$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ k_j = s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$$

Using the above expression for $\ell_f(h_j) = \sum_i \gamma_i j b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,T_i} \equiv 0 \pmod{p^N}.$$

The last congruence is done by elementary check on Fourier coefficients b_{j,T_i} .

Using Krasner graded structures in general, the abstract Kummer congruences proved for a family of test elements (e.g. certain p -adic Dirichlet characters with values in $\bar{\mathbb{Q}}_p^*$, viewed as homogeneous elements of grade j).

3.6 From holomorphic to nearly holomorphic and p -adic modular forms

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional \mathbb{Q} -vector space $\mathcal{N} \subset \mathcal{N}_\rho(\overline{\mathbb{Q}})$ (of nearly holomorphic modular forms).

Usually, h_j can be expressed through the functions $\delta_{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$. Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\overline{\mathbb{Q}}$ -vector space $\mathcal{N} = \mathcal{N}(\mathbb{Q})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\mathbb{Q}) := \bigcup_{m \geq 0} \mathcal{N}_{k,r}(\mathbb{Q})(Np^m, Q)$$

(of the depth r): to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms. This step controls Petersson products using conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h \rangle = \alpha^{-m} \langle f^0, h|U(p)^m \rangle = \langle f^0, \pi^\alpha(h) \rangle.$$

From holomorphic to nearly holomorphic and p -adic modular forms : Ichikawa's mapping

2) Let us apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \rightarrow \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p -adic Siegel modular forms. Notice also that the realization of nearly holomorphic forms as p -adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [U14], Chapter 10.

Let us assume algebraically,

$$h_j = \sum_T b_{j,T}(S)q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computations become much easier, as the action of δ^{k_j} becomes simply a k_j -power of the Ramanujan Θ -operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T(0) q^T$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasi-modular forms $\kappa(h_j(\chi_j))$ which can be explicitly evaluated using the Θ -operator.

3.7 Computing with Siegel modular forms over a ring A

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p -adic). Consider modular forms over a ring $A = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[[T]], \dots$ as certain formal Fourier expansions over A . Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S -degree of the polynomial Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$h_j = \sum_T b_{j,T}(S)q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j -power of the Ramanujan Θ -operator

$$\Theta : \sum_T a_T(S)q^T \in \mathcal{N}(A),$$

over R , and denote by $\mathcal{N}_{\rho,r}(\Gamma, A)$ the A -module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional F -vector space over the fraction field $F = \text{Frac}(A)$.

3.8 Types of modular forms

- \mathcal{M}_ρ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \text{GL}_n \rightarrow \text{GL}_d$)
- $\mathcal{QM} = \mathcal{N}|_{S=0}$ (quasi-modular vector-valued forms attached to ρ)
- \mathcal{N}_ρ (holomorphic vector-valued Siegel modular forms, algebraic p -adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $\mathcal{QM}_{\rho,r} = \kappa(\mathcal{N}_{\rho,r})$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0)q^T$, with the notation $\mathcal{R}_{n,\infty} = \mathbb{C}[[q_{11}, \dots, q_{nn}]][[q_{ij}, q_{ij}^{-1}]_{ij=1,\dots,n}]$

- : $\mathcal{M}_{\rho,r}^{\flat}(R, \Gamma)) = F_c(t_p(\mathcal{N}_{\rho,r}(R, \Gamma))) \subset \mathcal{R}_{n,p}^d$ where

$$\mathcal{R}_{n,p} = \mathbb{C}_p[[q_{11}, \dots, q_{nn}]] [q_{ij}, q_{ij}^{-1}]_{ij=1, \dots, n}$$

Let us fix the level Γ , the depth r , and a subring R of $\bar{\mathbb{Q}}$, then all the R -modules $\mathcal{M}_{\rho,r}(R, \Gamma)$, $\mathcal{N}_{\rho,r}(R, \Gamma)$, $Q\mathcal{M}_{\rho,r}(R, \Gamma)$, $\mathcal{M}_{\rho,r}^{\flat}(R, \Gamma)$ are then locally free of finite rank.

In interesting cases, there is an inclusion

$$Q\mathcal{M}_{\rho,r}(R, \Gamma) \hookrightarrow \mathcal{M}_{\rho,r}^{\flat}(R, \Gamma).$$

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p -adic modular form, see [Se73], p.211.

4 Applications to unitary groups

Although we treat here only the Siegel modular case here, the results can be extended to the general Sp - and unitary cases (UT in Shimura's terminology).

4.1 Towards a construction of p -adic L -functions for unitary group.

In a recent paper, [EHLS] by Ellen Eischen, Michael Harris, Jianshu Li, Christopher Skinner (Submitted on 4 Feb 2016) completed key steps towards a construction of p -adic L -functions for unitary group. More precisely, in 2006, the last three authors proposed an approach to constructing p -adic L -functions for unitary groups. Building on more recent results, including the first author's construction of Eisenstein measures, this paper provides the calculations of local ζ -integrals occurring in the Euler product (including at p) and also continues to develop the substantial geometric framework that will be necessary for concluding the construction of p -adic L -functions in Part III. On the other hand, in [KiNa16] by Toshiyuki Kikuta, Shoyu Nagaoka, some important congruence relations with respect to some Hermitian modular forms are given. We wish to interpret them as a passage from the p -adic modular forms to algebraic quasi-modular forms, simplifying constructions of p -adic L -functions mentioned above.

4.2 The use of the Krasner graded structures on weight spaces

General graded structures [KrVu87], [Kr80], [Vu01] can be used on the rings and modules of differential operators and quasimodular forms on classical groups, but also on the corresponding on weight spaces.

Our purpose is to use them in order to construct p -adic measures and p -adic L -functions on the corresponding non-archimedean weight spaces.

In the definition of M. Krasner, without the commutativity hypothesis, the structure of a graded group is determined by the underlying structure of the abstract group, and only the homogeneous part is equipped with a partial composition, induced by that of the group.

Moreover, M. Krasner introduces useful notions of graded ring and module, and their homogeneous parts called *annéïde et moduloïde* where the suffixes *oïde*, *eïde* mean that the composition structure is partial in principle. In our situation the underlying abstract group is a certain p -adic analytic group Y of characters with values in \mathbb{C}_p^* which contains a subset Y^{class} of algebraic "motivic" points. Here $\bar{\mathbb{Q}}_p$ be an algebraic closure of \mathbb{Q}_p , $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the Tate field, the completion of $\bar{\mathbb{Q}}_p$, which is a fundamental object in p -adic analysis, and thanks to Krasner we know that \mathbb{C}_p is algebraically closed [Am75], Théorème 2.7.1 (Lemme de Krasner), [Kr74]. This famous result allows to develop theories of analytic functions and analytic spaces over \mathbb{C}_p ([Kr74], Tate, Berkovich...). We fix an embedding $incl_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, so that elements of Y^{class} are identified with some characters with values in $\bar{\mathbb{Q}}^*$.

More detailed study of the homogeneous part leads to the pargraduate case, see [KrVu87], [Vu01].

4.3 Methods of construction of p -adic L -functions.

We briefly recall basic methods of construction of p -adic L -functions. These methods work in the framework of graded structures on a certain group of p -adic characters, and eventually, on certain non-commutative Iwasawa algebras:

1) p -adic interpolation (starting from Kubota-Leopoldt in 1964); see also [Am75], following Ch. Pisot, "...les séries d'interpolation sont un instrument de choix dans l'étude des fonctions analytiques, il est destiné à faciliter aux chercheurs l'accès à l'analyse p -adique".

2) abstract Kummer congruences using a family of test elements (e.g. binomial coefficients as functions with values in \mathbb{Z}_p , or certain characters with values in $\bar{\mathbb{Q}}_p^*$, such as elements of Y^{class}

3) the use of a p -adic Cauchy integral (the Shnirelman integral), see [Ko80], p.129.

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