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Abstract

We wish to use graded structures [KrVu87], [Vu01] on differential operators and quasimodular forms on classical groups and show that these structures provide a tool to construct p-adic measures and p-adic L-functions on the corresponding non-archimedean weight spaces.

An approach to constructions of automorphic L-functions on unitary groups and their p-adic avatars is presented. For an algebraic group G over a number field K these L functions are certain Euler products $L(s, \pi, r, \chi)$. In particular, our constructions cover the L-functions in [Shi00] via the doubling method of Piatetski-Shapiro and Rallis.

A p-adic analogue of $L(s, \pi, r, \chi)$ is a p-adic analytic function $L_p(s, \pi, r, \chi)$ of p-adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers defined through the normalized critical values $L^*(s, \pi, r, \chi)$ of the corresponding complex analytic L-function. We present a method using arithmetic nearly-holomorphic forms and general quasi-modular forms, related to algebraic automorphic forms. It gives a technique of constructing p-adic zeta-functions via quasi-modular forms and their Fourier coefficients.

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Introduction

Let p be a prime number. Our purpose is to indicate a link of Krasner graded structures [KrVu87], [Kr80], [Vu01] and constructions of p-adic L-functions via distributions and quasimodular forms on classical groups.

Krasner's graded structures are flexible and well adapted to various applications, e.g. the rings and modules of differential operators on classical groups and non-archimedean weight spaces.

Arithmetical modular forms

belong traditionally to the world of arithmetic, but also to the worlds of geometry, algebra and analysis. On the other hand, it is very useful, to attach zeta-functions (or L-functions) to mathematical objects of various nature as certain generating functions (or as certain Euler products).

Examples of modular forms, quasimodular forms, zeta functions, and L-functions

Eisenstein series $E_k = 1 + \dfrac{2}{\zeta(1-k)}\sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n$ ($\in \mathcal{M}$, modular forms for even $k \geq 4, q = e^{2\pi i z}$), and $E_2 \in \mathcal{QM}$ *is a quasimodular form.* The ring of quasimodular forms, closed under differential operator $D = q\dfrac{d}{dq} = \dfrac{1}{2\pi i}\dfrac{d}{dz}$, used in arithmetic, $\zeta(s)$ is the Riemann zeta function, $\zeta(-1) = -\dfrac{1}{12}$, $E_2 = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n$ which is also a p-adic modular form (due to J.-P.Serre,[Se73], p.211)

Elliptic curves $E : y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, A.Wiles's modular forms $f_E = \sum_{n=1}^{\infty} a_n q^n$ with $a_p = p - Card E(\mathbb{F}_p)$ $(p \not| 4a^3 + 27b^2)$, and the L-function $L(E, s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Zeta-functions or L-functions attached to various mathematical objects as certain Euler products.

- L-functions link such objects to each other (a general form of functoriality);
- Special L-values answer fundamental questions about these objects in the form of a number (complex or p-adic).

Computing these numbers use integration theory of Dirichlet-Hecke characters along p-adic and complex valued measures.

This approach originates in the Dirichlet class number formula using the L-values in order to compute class numbers of algebraic number fields through Dirichlet's L-series $L(s, \chi)$.

The Millenium BSD Conjecture gives the rank of an elliptic curve E as order of $L(E, s)$ at s=1 (i.e. the residue of its logarithmic derivative, see [MaPa], Ch.6).

1 Graded groups in the sense of Krasner

Definition 1.1. *Let G be a multiplicative group with the neutral element 1. A graduation of G is a mapping $\gamma : \Delta \to Sg(G), \delta \in \Delta$ of a set Δ "the set of grades of γ ") to the set $Sg(G)$ of subgroups of G such that G is the direct decomposition*

$$G = \bigoplus_{\delta\in\Delta} G_\delta, g = (g_\delta)_{\delta\in\Delta}$$

The group G endowed with such graduation is called graded group, and g_δ are Krasner grade components of g with grade δ.

Remarks

- The set $H = \bigcup_{\delta \in \Delta} G_\delta$ is called *the homogeneous part* of G for γ.
- $x \in H$ are called *homogeneous elements* of G for γ
- the elements $\delta \in \Delta$ are called *grades* of γ
- $\delta \in \Delta$ are called *significant* (resp. empty) according as
$$G_\delta \neq \{1\}, (resp. G_\delta = \{1\})$$
and $\Delta^* = \{\delta \in \Delta; G_\delta \neq \{1\}\}$ *the significative part* of Δ. The graduation γ is called *strict* if $\Delta = \Delta^*$.

The choice of Δ is very flexible: we give examples of p-adic characters.

1.1 Graded rings and modules

Moreover, M. Krasner introduced useful notions of graded rings and modules.

Let $(A; x+y, xy)$ be a ring (not necessarily associative) and let $\gamma : \Delta \to Sg(A; x+y)$ be a graduation of its addivive group. The graduation γ is called a graduation of a ring $(A; x+y, xy) = A$ if, for all $\xi, \eta \in \Delta$ there exists a $\zeta \in \Delta$ such that $A_\xi A_\eta \subset A_\zeta$.

1.2 Examples for p-adic groups $\mathcal{X}$, and group rings

use the Tate field $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, the completion of $\bar{\mathbb{Q}}_p$, which is a fundamental object in p-adic analysis, and thanks to Krasner we know that $\mathbb{C}_p$ is algebraically closed, see [Am75], Théorème 2.7.1 ("Lemme de Krasner"), [Kr74]. This famous result allows to develop analytic functions and analytic spaces over $\mathbb{C}_p$ ([Kr74] , Tate, Berkovich...). and we embed $incl_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

1) Algebraically, a p-adic measure μ on $\mathcal{X}$ is an element of the completed group ring $A[[\mathcal{X}]]$, A any p-adic subring of $\mathbb{C}_p$.

2) The p-adic L-function of μ is given by the evaluation $\mathcal{L}_\mu(y) := y(\mu)$ on the group $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$ of $\mathbb{C}_p^*$-valued characters of $\mathcal{X}$. *The values $\mathcal{L}_\mu(y_j)$ on algebraic characters $y_j \in \mathcal{Y}^{alg}$ determine $\mathcal{L}_\mu$ iff they satisfy Kummer-type congruences.* 3) Our setting: a p-adic torus $T = \mathcal{X}$ of a unitary group G attached to a CM field K over $\mathbb{Q}$, a quadratic extension of a totally real field F, and an n-dimensional hermitian K-vector space V. Elements of $\mathcal{Y}^{alg}$ are identified with some algebraic characters of the torus T of the unitary group.

2 An extension problem.

From a subset $J = \mathcal{Y}^{alg}$ of classical weights in $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$, via the A-module $\mathcal{QM}$ of quasimodular forms, extend a mapping to the group ring $A[\mathcal{Y}]$:

$$\mathcal{L} : \mathcal{Y}^{alg} \longrightarrow \underset{\circlearrowleft\, \mathcal{H}_A}{\overset{\curvearrowright \mathcal{D}_A}{\mathcal{QM}}} \longrightarrow \mathbb{C}_p,\ y_j \overset{\ell}{\mapsto} \mathcal{L}(y_j), y_j \in \mathcal{Y}^{alg}$$

($\mathcal{H}_A$ a Hecke algebra, $\mathcal{D}_A$ a ring of differential operators over A), or even to all continuous functions $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$, or just to local-analytic functions $\mathcal{C}^{loc-an}(\mathcal{X}, \mathbb{C}_p)$ such that the values $\mathcal{L}(y_j)$ on $y_j \in \mathcal{Y}^{alg}$ are given algebraic L-values under the embedding $incl_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

Advantages of the A-module $\mathcal{QM}$:

1) nice Fourier expansion (q-expansions);

2) action of D and of the ring of differential operators $\mathcal{D}_A = A[D]$

3) action of the Hecke algebra $\mathcal{H}_A$;

4) projection $\pi_\alpha : \mathcal{QM}^\alpha \to \mathcal{QM}^\alpha$ to finite rank component ("generalized eigenvectors of Atkin's U-operator") for any non-zero Hecke eigenvalue α of level p ; ℓ goes through $\mathcal{QM}^\alpha$ if $U^*(\ell) = \alpha^*\ell$.

Solution (extension of $\mathcal{L}$ to $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$) is given by the abstract Kummer-type congruences:

$$\forall x \in \mathcal{X}, \ \sum_j \beta_j y_j(x) \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \ell(y_j) \equiv 0 \mod p^N (\beta_j \in A).$$

These congruences imply the *p-adic analytic continuation of the Riemann zeta function*:

2.1 Example: Mazur's p-adic integral and interpolation

For any natural number $c > 1$ not divisible by p, there exists a p-adic measure μ_c on $\mathcal{X} = \mathbb{Z}_p^*$, such that the special values

$$\zeta(1-k)(1-p^{k-1}) = \frac{\int_{\mathbb{Z}_p^*} x^{k-1} d\mu_c(x)}{1-c^k} \in \mathbb{Q}, (k \geq 2 \text{ even })$$

produce the *Kubota-Leopoldt p-adic zeta-function* $\zeta_p : \mathcal{Y}_p \to \mathbb{C}_p$ on the space $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ as the *p-adic Mellin transform*

$$\zeta_p(y) = \frac{\int_{\mathbb{Z}_p^*} y(x) d\mu_c(x)}{1 - cy(c)} = \frac{\mathcal{L}_{\mu_c}(y)}{1 - cy(c)},$$

with a single simple pole at $y = y_p^{-1} \in \mathcal{Y}_p$, where $y_p(x) = x$ the inclusion character $\mathbb{Z}_p^* \hookrightarrow \mathbb{C}_p^*$ and $y(x) = \chi(x)x^{k-1}$ is a typical arithmetical character ($y = y_p^{-1}$ becomes $k = 0$, $s = 1 - k = 1$).

Explicitly: Mazur's measure is given by $\mu_c(a + p^v\mathbb{Z}_p) =$
$\frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v})$, $B_1(x) = x - \frac{1}{2}$,
see [LangMF], Ch.XIII, ℓ is given by the *constant term of Eisenstein series.*

2.2 Various exampes

The choice of Δ is quite flexible: we use the following examples:
1) any direct decomposition $G = \bigoplus_{\delta \in \Delta} G_\delta$ gives a graduation where Δ is the set of homogeneous components and $\gamma = \mathrm{Id} : \Delta \to \Delta$,
2) the regular representation of a commutative group algebra $A = K[G]$ with respect to characters of G, and of a subgroup $H \subset G$ gives a graduation of A
3) in particular, a Hodge structure of a complex vector space V is a certain spectral decomposition for the real algebraic group $\mathbf{S} = \mathbb{C}^*$

Applications of the graduation $G = \bigoplus_{\delta \in \Delta} G_\delta$ are similar to the spectral theory, and the set Δ extends the notion of the spectrum. The homogeneous components g_δ in $g = (g_\delta)_{\delta \in \Delta}$, are similar to the Fourier coefficients, they allow to extract essential information about $g \in G$. The same applies to *graded rings and modules* : $A = \bigoplus_{\delta\in\Delta} A_\delta$, $M = \bigoplus_{\delta\in\Delta} M_\delta$, the homogeneous Krasner grade components a_δ, m_δ simplify the p-adic constructions componentwise.

2.3 Krasner grade components for proving Kummer-type congruences for L and zeta-values

For more general L-function $L(f, s)$, of an automorphic form f one can prove certain Kummer-type congruences component-by-component using various Krasner grade components, with respect to *weights, Hecke-Dirichlet characters, eigenvalues of Hecke operators acting on spaces automorphic forms (including Atkin-type U_p-operators), and the classical Fourier coefficients of quasimodular forms.*

It turns out that certain critical L-values $L(f, s)$, expressed through Petersson-type product $\langle f, g_s\rangle$, reduces to $\langle \pi_\alpha(f), g_s\rangle$, where g_s is an explicit arithmetical automorphic form, $\alpha \neq 0$ is a eigenvalue attached to f, and $\pi_\alpha(f)$ is the component given by the α-characteristic projection, known to be in a fixed finite dimensional space (known for Siegel modular case, and extends to the unitary case).

3 Graded structures and p-adic measures.

For a compact and totally disconnected topological space X, a p-adic ring R, let $C(X, R)$ be the R-module of continuous maps from X to R (with respect of the p-adic topology on R). For a M a p-adically complete R-module, an *M-valued measure on* X is an element of the R-module

$$Meas(X, M) = Hom_{\mathbb{Z}_p}(C(X, \mathbb{Z}_p), M) \xrightarrow{\sim} Hom_R(C(X, R), M).$$

3.1 An idea of construction of measures

uses certain *families* $\{\phi_j\}_{j\in J}$ *of test functions*, components of grade $j \in J$ in the group ring $R[Y] \subset C(X, R)$. In this case $J \subset \Delta = Y$. A given family $m_j = \mu(\phi_j) \in M$ extends to a measure μ on the R-linear span $R\langle\{\phi_j\}_{j\in J}\rangle \subset C(X, R)$ provided that certain Kummer-type congruences are satisfied for $\mu(\phi_j) \in M$.

Suppose X is a profinite abelian group. Then $Meas(X, R)$ is identified with the completed group ring $R[[X]]$ so that if a measure μ is identified with $f \in R[[X]]$, then for any continuous character $\chi : X \to R_1^*$ with R_1 a p-adic R-algebra, $\mu(\chi) = \chi(f)$ (see Section 6.1. Measures: generalities of [EHLS]).

3.2 Example: p-adic characters of $\mathbb{Z}_p^*$

This group $G = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ is used for constructions of p-adic zeta functions via the p-adic integration theory and the decomposition $Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*) = Hom_{cont}(\Gamma, \mathbb{C}_p^*) \times Hom_{cont}(\mathbb{Z}_p^{*\,tors}, \mathbb{C}_p^*)$, where $\Gamma = (1 + p\mathbb{Z}_p)^* = \overline{\langle 1 + p\rangle}$ for $p \neq 2$, when $Hom_{cont}(\Gamma, \mathbb{C}_p^*)$ which is a p-adic analytic Lie group.

A graduation and an analytic structure on G are given by the above choice of the subgroup of tame characters $G_1 = Hom_{cont}(\mathbb{Z}_p^{*tors}, \mathbb{C}_p^*)$ (a finite subgroup), and the subgroup of wild characters $G_0 = Hom_{cont}(\Gamma, \mathbb{C}_p^*) = U = \{1 + t \in \mathbb{C}_p \ |t|_p < 1\}$ (a p-adic disc).

The Mellin transform of a p-adic distribution μ on $\mathbb{Z}_p^*$ gives an analytic function on the group of p-adic characters

$$y \mapsto \mathcal{L}_\mu(y) = \int_{\mathbb{Z}_p^*} y(x) d\mu(y), \ y \in \mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*).$$

According to *Iwasawa's theorem, there is one-to-one correspondence* between p-adic distributions μ on $\mathbb{Z}_p^*$ and bounded analytic functions on $\mathcal{Y}$.

3.3 Methods of construction of p-adic L-functions

- p-adic interpolation (starting from Kubota-Leopoldt in 1964);
- abstract Kummer congruences using a family of test elements (e.g. binomial coefficients as functions with values in $\mathbb{Z}_p$, or certain characters with values in $\bar{\mathbb{Q}}_p^*$, such as elements of Y^{class}
- the use of p-adic Cauchy integral (Shnirelman integral), e.g. a theorem of Amice-Fresnel, see [Ko80], p.120. Let $f(z) = \sum_n a_n z^n \in \mathbb{C}_p[[z]]$ have the property that the coefficients a_n can be p-adically interpolated i.e., there exists a continuous function $\phi : \mathbb{Z}_p \to \mathbb{C}_p$ such that $\phi(n) = a_n$. Then f (whose disc of convergence must be the open unit disc $D_0(1^-)$ is the restriction to $D_0(1^-)$ of a Krasner analytic function $\tilde{f}$ on the complement of $D_1(1^-)$, see also [Kr74].
- The Mellin transform of a p-adic measure μ on $\mathcal{X}$ gives an analytic function on the p-adic weight space $\mathcal{Y}$, group of p-adic characters of $\mathcal{X}$:

$$y \mapsto \mathcal{L}_\mu(y) = \int_{\mathcal{X}} y(x) d\mu(y), \ y \in \mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*).$$

4 Automorphic L-functions and their p-adic analogues

Our main objects in this talk are automorphic L-functions and their p-adic analogues.

For an algebraic group G over a number field K these L functions are defined as certain Euler products. More precisely, we apply our constructions for the L-functions studied in Shimura's book [Shi00].

EXAMPLE 4.2 ($G = \mathrm{GL}(2), K = \mathbb{Q}, L_f(s) = \sum_{n \geq 1} a_n n^{-s}, s \in \mathbb{C}.$). *Here $f(z) = \sum_{n\geq 1} a_n q^n$ is a modular form on the upper-half plane*

$$H = \{z \in \mathbb{C}, \mathrm{Im}\,(z) > 0\} = \mathrm{SL}(2)/SO(2), q = e^{2\pi i z}.$$

An Euler product has the form

$$L_f(s) = \prod_{p \ primes} (1 - a_p p^{-s} + \psi_f(p) p^{k-1-2s})^{-1}$$

where k is the weight and ψ_f the Dirichlet character of f. It is defined iff the automorphic representation π_f attached to f is irreducible, and π_f is generated by the lift $\tilde{f}$ of f to the group $G(\mathbb{A})$.

4.1 A p-adic analogue of $L_f(s)$ (Manin-Mazur)

It is a p-adic analytic function $L_{f,p}(s,\chi)$ of p-adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers

$$L_f^*(s,\chi)/\omega^{\pm} \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p \text{ (the Tate field)}$$

for $1 \le s \le k-1$, $\omega^{\pm}$ are periods of f where the complex analytic L function of f is defined for all $s \in \mathbb{C}$ so that in the absolutely convergent case $\mathrm{Re}(s) > (k+1)/2$,

$$L_f^*(s,\chi) = (2\pi)^{-s}\Gamma(s)\sum_{n\ge 1}\chi(n)a_n n^{-s}$$

which extends to holomorphic function with a functional equation. According to Manin and Shimura, this number is algebraic if the period $\omega^{\pm}$ is chosen according to the parity $\chi(-1)(-1)^{-s} = \pm 1$.

4.2 Constructions of p-adic analogues of complex L-functions

An irreducible automorphic representation π of adelic group $G(\mathbb{A}_K)$:

$$L(s,\pi,r,\chi) = \prod_{\mathfrak{p}_v\ primes\ in\ K}\prod_{j=1}^{m}(1-\beta_{j,\mathfrak{p}_v}N\mathfrak{p}_v^{-s})^{-1},\ \text{where } \pi = \pi(f).$$

is an Euler product giving an L-function, where $v \in \Sigma_K$ (places in K), $\mathfrak{p} = \mathfrak{p}_v$, $\alpha_{i,\mathfrak{p}}$ the Satake parameters of $\pi = \bigotimes_v \pi_v$,

$$\prod_{j=1}^{m}(1-\beta_{j,\mathfrak{p}}X) = \det(1_m - r(\mathrm{diag}(\alpha_{i,\mathfrak{p}})X)),$$

$h_v = \mathrm{diag}(\alpha_{i,\mathfrak{p}}) \in {}^LG(\mathbb{C})$ (the Langlands group), $r : {}^LG(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ a finite dimensional representation, and $\chi : \mathbb{A}_K^*/K^* \to \mathbb{C}^*$ is a character of finite order.

Constructions extend to general automorphic representations on Shimura varieties via the following tools:

- Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms)
- Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or throught the
- CM-values (special points on Shimura varieties),

4.3 Accessible cases: symplectic and unitary groups

- $G = \mathrm{GL}_1$ over $\mathbb{Q}$ (Kubota-Leopoldt-Mazur) for the Dirichlet L-function $L(s, \chi)$.
- $G = \mathrm{GL}_1$ over a totally real field F (Deligne-Ribet, using algebraicity result by Klingen).
- $G = \mathrm{GL}_1$ over a CM-field K, i.e. a totally imaginary extension of a totally real field F (N.Katz, Manin-Vishik).
- the Siegel modular case $G = \mathrm{GSp}_n$ (the Siegel modular case, $F = \mathbb{Q}$).
- General symplectic and unitary groups over a CM-field K Certain Euler products in Chapter 5 of [Shi00], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_{n+m}(K) | \alpha\varphi\alpha^* = \nu(\alpha)\varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$, or $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, $n = m$, see also Ch.Skinner and E.Urban [MC] and Shimura G., [Shi00].

4.4 Nearly holomorphic modular forms for GL(2)

in N.M.Katz [Ka76] used real-analytic and quasimodular forms coming from derivatives of holomorphic forms, and p-adic modular forms.

A relation real-analytic $\leftrightarrow$ p-adic modular forms comes from the notion of p-adic modular forms invented by J.-P.Serre [Se73] as p-adic limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, using the ring M_p of p-adic modular forms contains $\mathcal{M} = \oplus_{k\geq 0}\mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains E_2 as an element with the q-expansion $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$. On the other hand, the function of $z = x + iy$,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (that is, its coefficients are polynomials of S over $\mathbb{Q}$). Then $\mathcal{N} = \mathbb{Z}[\tilde{E}_2, E_4, E_6]$ is the ring of such forms, $\tilde{E}_2|_{S=0} = E_2$ and E_2 is a p-adic modular form.

Elements of the ring $\mathcal{QM} = \mathcal{N}|_{S=0} = \mathbb{Z}[E_2, E_4, E_6]$ are called quasimodular forms hence such forms are all p-adic modular forms. These phenomena are quite general and used in computations and proofs. S.Boecherer extended these results to the Siegel modular case.

5 Automorphic L-functions attached to symplectic and unitary groups

Let us briefly describe the L-functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [Shi00], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_{n+m}(K) | \alpha\varphi\alpha^* = \nu(\alpha)\varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$, or $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, $n = m$, see also Ch.Skinner and E.Urban [MC] and Shimura G., [Shi00].

5.1 The groups and automorphic forms in [Shi00]

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. Take $\eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and define

$$\begin{array}{ll} G = \mathrm{Sp}(n, F) & \text{(Case Sp)} \\ G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha\eta_n\alpha^* = \eta_n\} & \text{(Case UT = unitary tube)} \\ G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha T \alpha^* = T\} & \text{(Case UB = unitary ball)} \end{array}$$

according to three cases. Assume $F = \mathbb{Q}$ for a while. The group of the real points G_∞ acts on the associated domain

$$\mathcal{H} = \begin{cases} \{z \in M(n, n, \mathbb{C}) \mid {}^t z = z, \mathrm{Im}\,(z) > 0\} & \text{(Case Sp)} \\ \{z \in M(n, n, \mathbb{C}) \mid i(z^* - z) > 0\} & \text{(Case UT)} \\ \{z \in M(p, q, \mathbb{C}) \mid 1_q - z^* z > 0\} & \text{(Case UB).} \end{cases}$$

$(p, q), p + q = n$, being the signature of iT . Here $z^* = {}^t\bar{z}$ and $>$ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g, z)$, $g \in G_\infty$, $z \in \mathcal{H}$ taking values in $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$.

5.2 Shimura's arithmeticity [Shi00], p-adic zeta functions and nearly-holomorphic forms on classical groups

Automorphic L-functions and their p-adic analogues can be obtained for quite general automorphic representations on Shimura varieties by constructing p-adic distributions out of algebraic numbers attached to automorphic forms. This means that these numbers satisfy certain Kummer-type congruences established in different ways: via

- Normalized Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or through the
- CM-values (special points on Shimura varieties), see The Iwasawa Main Conjecture for GL(2) by C. Skinner and E. Urban, [MC], Shimura G., Arithmeticity in the theory of automorphic forms [Shi00].

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated.

In order to prove the congruences needed for the p-adic constructions, we use a simplification due to nearly-holomorphic and general quasi-modular forms, related to algebraic automorphic forms. In this paper, a new method of constructing p-adic zeta-functios is presented using general quasi-modular forms and their Fourier coefficients.

5.3 Algebraicity and congruences of the critical values of the zeta functions

of automorphic forms on unitary and symplectic groups, we follow the review by H.Yoshida [YS] of Shimura's book [Shi00].

Shimura's mathematics developed by stages:

(A) Complex multiplication of abelian varieties =>

(B) The theory of canonical models = Shimura varieties =>

(C) Critical values of zeta functions and periods of automorphic forms.

(B) includes (A) as 0-dimensional special case of canonical models. The relation of (B) and (C) is more involved, but (B) provides a solid foundation of the notion of the arithmetic automorphic forms. Also unitary Shimura varieties have recently attracted much interest (in particular by Ch. Skinner and E. Urban), see [MC], in relation with the proof of the The Iwasawa Main Conjecture for GL(2).

5.4 Integral representations and critical values of the zeta functions

In Cases Sp and UT, Eisenstein series $E(z,s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behavior and those values of $\sigma \in 2^{-1}\mathbb{Z}$ at which $E(z,\sigma)$ is nearly holomorphic and arithmetic are studied in [Shi00]. This is achieved by proving a relation giving passage from s to $s-1$ for $E(z,s)$, involving a differential operator, then examining Fourier coefficients of Eisenstein series using the theory of confluent hypergeometric functions on tube domains.

For a Hecke eigenform f on $G_{\mathbb{A}}$ and an algebraic Hecke character χ on the idele group of K (in Case Sp, $K = F$), the zeta function $\mathcal{Z}(s,f,\chi)$*, an Euler product extended over prime ideals of* F*, the degree of the Euler factor is* $2n+1$ *in Case Sp,* $4n$ *in Case UT, and* $2n$ *in Case UB*, except for finitely many prime ideals, see Theorem 19.8 on Euler products of Chapter 5, [Shi00].

This zeta function is almost the same as the so called standard L-function attached to f twisted by χ but it turns out to be more general in the unitary case, see also [EHLS].

Main results on critical values of the L-functions studied in Shimura's book [Shi00] is stated in Theorem 28.5, 28.8 (Cases Sp, UT), and in Theorem 29.5 in Case UB.

5.5 Shimura's Theorem: algebraicity of critical values in Cases Sp and UT, p.234 of [Shi00]

Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume, in the notations of Chapter 7 of [Shi00] onr the weights k_v, μ_v, ℓ_v, that

Case Sp	$2n+1-k_v+\mu_v \le 2\sigma_0 \le k_v-\mu_v$, where $\mu_v=0$ if $[k_v]-l_v \in 2\mathbb{Z}$ and $\mu_v=1$ if $[k_v]-l_v \notin 2\mathbb{Z}$; $\sigma_0-k_v+\mu_v$ for every $v\in\mathbf{a}$ if $\sigma_0>n$ and $\sigma_0-1-k_v+\mu_v \in 2\mathbb{Z}$ for every $v\in\mathbf{a}$ if $\sigma_0\le n$.
Case UT	$4n-(2k_{v\rho}+\ell_v) \le 2\sigma_0 \le m_v-\lvert k_v-k_{v\rho}-\ell_v\rvert$ and $2\sigma_0-\ell_v \in 2\mathbb{Z}$ for every $v\in\mathbf{a}$.

Further exclude the following cases

(A)	Case Sp	$\sigma_0=n+1, F=\mathbb{Q}$ and $\chi^2=1$;
(B)	Case Sp	$\sigma_0=n+(3/2), F=\mathbb{Q}; \chi^2=1$ and $[k]-\ell\in 2\mathbb{Z}$
(C)	Case Sp	$\sigma_0=0, \mathfrak{c}=\mathfrak{g}$ and $\chi=1$;
(D)	Case Sp	$0<\sigma_0\le n, \mathfrak{c}=\mathfrak{g}, \chi^2=1$ and the conductor of χ is $\mathfrak{g}$;
(E)	Case UT	$2\sigma_0=2n+1, F=\mathbb{Q}, \chi_1=\theta,$ and $k_v-k_{v\rho}=\ell_v$;
(F)	Case UT	$0\le 2\sigma_0<2n, \mathfrak{c}=\mathfrak{g}, \chi_1=\theta^{2\sigma_0}$ and the conductor of χ is $\mathfrak{r}$

Then

$$\mathcal{Z}(\sigma_0,\mathbf{f},\chi)/\langle\mathbf{f},\mathbf{f}\rangle \in \pi^{n|m|+d\varepsilon}\bar{\mathbb{Q}},$$

where $d=[F:\mathbb{Q}]$, $|m|=\sum_{v\in\mathbf{a}} m_v$, and

$$\varepsilon=\begin{cases}(n+1)\sigma_0-n^2-n, & \text{Case Sp}, k\in\mathbb{Z}^{\mathbf{a}},\text{ and } \sigma_0>n_0),\\ n\sigma_0-n^2, & \text{Case Sp}, k\notin\mathbb{Z}^{\mathbf{a}}, \text{or} \sigma_0\le n_0),\\ 2n\sigma_0-2n^2+n & \text{Case UT}\end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon}\in\mathbb{Z}$ in all cases; if $k\notin\mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0+k_v\in\mathbb{Z}$, so that $n|m|+d\varepsilon\in\mathbb{Z}$.

5.6 A p-adic analogue of Shimura's Theorem (Cases Sp and UT)

represent algebraic parts of critical values as values of certain p-adic analytic zeta function, Mellin transform $y\mapsto\mathcal{L}_\mu(y)$ of a p-adic distributions μ on $\mathcal{X}=T$, an analytic function on the group $\mathcal{Y}$ as above.

The construction of μ uses congruences of Kummer type between the Fourier coefficients of modular forms. Suppose that we are given some L-function $\mathcal{Z}^*(s,\mathbf{f},\chi)$ attached to a quasimodular form $\mathbf{f}$ and assume that for infinitely many "critical pairs" $(s_j;\chi_j)$ one has an integral representation $\mathcal{Z}^*(s_j,\mathbf{f},\chi_j)=\langle\mathbf{f},h_j\rangle$ with all $h_j=\sum_{\mathcal{T}} b_{j,\mathcal{T}} q^{\mathcal{T}}\in\mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing $\mathbf{f}$ and defined over $\bar{\mathbb{Q}}$. We want to prove the following Kummer-type congruences:

$$\forall x\in\mathbb{Z}_p^*,\ \sum_j \beta_j\chi_j x^{k_j}\equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j\frac{\mathcal{Z}^*(s,\mathbf{f},\chi)}{\langle\mathbf{f},\mathbf{f}\rangle}\equiv 0 \mod p^N$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$$

Computing the Petersson products of a given quasimodular form $\mathbf{f}(Z) = \sum_{\mathcal{T}} a_{\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another quasimodular form $h(Z) = \sum_{\mathcal{T}} b_{\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ uses a linear form

$$\ell_{\mathbf{f}} : h \mapsto \frac{\langle \mathbf{f}, h\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle} \text{ defined over a subfield } k \subset \bar{\mathbb{Q}}.$$

5.7 Using algebraic and p-adic modular forms

There are several methods to compute various L-values starting from the constant term of the Eisenstein series in [Se73],

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \frac{\Gamma(k)}{(2\pi i)^k} \sum_{(c,d)}{}'(cz+d)^{-k} \quad (\text{for} k \geq 4),$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [Shi00]:

the Rankin-Selberg method,

the doubling method (pull-back method).

A known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_n^k(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ a Dirichlet character.

Theorem (the case of even genus n ([Pa91], [CourPa]), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \leq s \leq k-n$ through the Petersson product $\langle f, \theta_{T_0} \delta^r E\rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0.

5.8 Using p-adic doubling method

Theorem (p-adic interpolation of $D(s, f, \chi)$)

(1) (the case of odd genus (Boecherer-Schmidt, [BS00])

Assume that n is arbitrary genus, and a prime p ordinary then there exists a p-adic interpolation of $D(s, f, \chi)$

(2) *(Anh-Tuan Do (non-ordinary case, PhD Thesis, 2014)), via the doubling method)*

Assume that n is arbitrary genus, and p an arbitrary prime not dividing level of f then there exists a p-adic interpolation of $D(s, f, \chi)$.

Proof uses the following Boecherer-Garrett-Shimura identity (a pull-back formula, related to the Basic Identity of Piatetski-Shapiro and Rallis, [GPSR]), which allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l+2s, \chi) D(l+2s-n, f, \chi) f = \langle f(w), E^{2n}_{l,\nu,\chi,s}(\mathrm{diag}[z, w])_w,$$

Here $k = l+\nu, \nu \geq 0, \Lambda(l+2s, \chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, $E^{2n}_{l,\nu,\chi,s}$ a higher twist of a Siegel-Eisenstein series on $(z, w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [BS00]).

5.9 Injecting nearly-holomorphic forms into p-adic modular forms

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], J. reine angew. Math., [Ich13] allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into p-adic modular forms. Via the Fourier expansions, the image of this injection is represented by certain quasi-modular holomorphic forms like $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, with algebraic Fourier expansions. This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L-values, which we would like to develop here. In fact, the realization of nearly holomorphic forms as p-adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [U14], Chapter 10.

Urban only treats the elliptic modular case in that paper, but I believe he and Skinner are working on applications of a more general theory. This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms.

5.10 Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$ -valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$ where T runs through the set B_n of all half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)), P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S.

A geometric construction of such arithmetical forms uses the algebraic theory af moduli spaces of abelian varieties. Following [Ha81], consider the columns $Z_1, Z_2, \dots Z_n$ of $Z \in \mathbb{H}_n$ and the $\mathbb{Z}$-lattice L_Z in $\mathbb{C}^n$ generated by $\{E_1, E_2, \dots E_n, Z_1, Z_2, \dots Z_n\}$, where $E_1, E_2, \dots E_n$ are the columns of the identity matrix E. The torus $A_Z = \mathbb{C}^n/L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \to \mathbb{H}_n$ whose fiber over the point $Z \in \mathbb{H}_n$ is A_Z . Let us consider the quotient space $\mathbb{H}_n/\Gamma(N)$ of the Siegel upper half space $\mathbb{H}_n$ of degree n by the integral symplectic group

$$\Gamma(N) = \left\{\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \middle| \begin{array}{c} A_\gamma \equiv D_\gamma \equiv 1_n \\ B_\gamma \equiv C_\gamma \equiv 0_n \end{array}\right\},$$

giving the moduli space classifying principally polarized abelian schemes of relative dimension n with a symplectic level N structure.

5.11 Applications to constructions of p-adic L-functions

We present here a survey of some methods of construction of p-adic L-functions. Two important ideas that are not as well known as they should be are developed biefly in this section.

There exist two kinds of L-functions

- Complex L-functions (Euler products) on $\mathbb{C} = \mathrm{Hom}(\mathbb{R}_+^*; \mathbb{C}^*)$.
- p-adic L-functions on the $\mathbb{C}_p$-analytic group $\mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ (Mellin transforms $\mathcal{L}_\mu$ of p-adic measures μ on $\mathbb{Z}_p^*$).

Both are used in order to obtain a number (L-value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$$

and we may compare the complex and p-adic L-values at many points.

How to define and to compute p-adic L-functions?

The Mellin transform of a p-adic distribution μ on $\mathbb{Z}_p^*$ gives an analytic function on the group of p-adic characters

$$y \mapsto \mathcal{L}_\mu(y) = \int_{\mathbb{Z}_p^*} y(x)d\mu(x),\ \ y \in \mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L-values. Here we present a new method to construct p-adic L-functions via quasimodular forms.

How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L-function $L_f^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" $(s_j;\chi_j)$ one has an integral representation $L_f^*(s,\chi) = \langle f, h_j\rangle$ with all $h_j = \sum_{\mathcal{T}} b_{j,\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$. We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*,\ \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \frac{L_f^*(s_j,\chi)_j}{\langle f, f\rangle} \equiv 0 \mod p^N$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$$

Computing the Petersson products of a given modular form $f(Z) = \sum_{\mathcal{T}} a_{\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_{\mathcal{T}} b_{\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ uses a linear form

$$\ell_f : h \mapsto \frac{\langle f, h\rangle}{\langle f, f\rangle}$$

defined over a subfield $k \subset \bar{\mathbb{Q}}$.

5.12 Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients: $\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i} (i = 1, \ldots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$.

Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j}\beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.

Using *Krasner graded structures* in general, the abstract Kummer congruences are checked for a family of test elements (e.g. certain p-adic Dirichlet characters with values in $\bar{\mathbb{Q}}_p^*$, viewed as homogeneous elements of grade $j \in J \subset \mathcal{Y}$, with above $J = \mathcal{Y}^{alg}$.

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