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# COMPETITION IN PERIODIC MEDIA: I – EXISTENCE OF PULSATING FRONTS

LÉO GIRARDIN<sup>1</sup>

ABSTRACT. This paper is concerned with the existence of pulsating front solutions in space-periodic media for a bistable two-species competition–diffusion Lotka–Volterra system. Considering highly competitive systems, a simple ‘high frequency or small amplitudes’ algebraic sufficient condition for the existence of pulsating fronts is stated. This condition is in fact sufficient to guarantee that all periodic coexistence states vanish and become unstable as the competition becomes large enough.

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## INTRODUCTION

This is the first part of a sequel to our previous article with Grégoire Nadin [20]. In this prequel, we studied the sign of the speed of bistable traveling wave solutions of the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1(1 - u_1) - k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R} \\ \partial_t u_2 - d \partial_{xx} u_2 = r u_2(1 - u_2) - \alpha k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R}. \end{cases}$$

We proved that, as  $k \rightarrow +\infty$ , the speed of the traveling wave connecting  $(1, 0)$  to  $(0, 1)$  converges to a limit which has exactly the sign of  $\alpha^2 r - d$ . In particular,

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if  $\alpha = r = 1$  and if  $k$  is large enough, the more motile species is the invader: this is what we called the “unity is not strength” result.

In view of this result, it would seem natural to try to generalize it in heterogeneous spaces, that is to systems with non-constant coefficients. Is the more motile species still the invading one?

The first obstacle toward this generalization is that of the existence of traveling fronts – or of some suitable generalization of these – for such a problem. Indeed, while past work had already established the existence of competitive bistable traveling waves in the case of homogeneous spaces (recall for instance Gardner [19] and Kan-On [24]), to the best of our knowledge, there is at this time no such pre-established result in the case of fully heterogeneous spaces (see the recent review of Guo and Wu [22]).

One of the main difficulties regarding this existence problem is of course the combination of unboundedness and heterogeneity. This yields additional difficulties (for instance, there are multiple non-equivalent definitions of the principal eigenvalue [4] and convenient integration-wise boundary conditions are lacking). Therefore, it is likely easier to first treat a simple case. With this in mind, we focus in this article on a simple, yet relevant application-wise heterogeneity: the periodic one. We hope to pave the way for a possible future generalization.

Periodic spaces are likely the type of unbounded heterogeneous spaces we know best how to handle mathematically and thus a literature about scalar equations in periodic spaces has been developed during the past few years. Concerning scalar reaction–diffusion in periodic spaces and with “KPP”-type non-linearities, important results have been established recently by Berestycki and his collaborators [1, 2, 3] (see also Nadin [26, 27] in space-time periodic media). We will rely a lot on these scalar results. Regarding bistable non-linearities, we refer to the work of Ding, Hamel and Zhao [14] and Zlatos [31].

For the sake of simplicity, we will assume that diffusion and interspecific competition rates are constant. We expect our main ideas to be generalizable to systems with periodic diffusion and interspecific competition rates, but we also expect a lot of technical details to get messy and there might very well be some major issues. As a counterpart to this loss in generality, we will be able to treat a much larger class of growth–saturation terms since the explicit form of these will not be prescribed a priori. We will only require some reasonable “KPP non-linearities” assumptions.

Since our final goal is to study the limits of these pulsating fronts as the competition becomes infinite, we will only consider systems in which competition is the main underlying mechanism, that is for large values of the interspecific competition rate. A first consequence of this approach is that our system will always be bistable. A second consequence is that segregation phenomena will be involved quite frequently. Competition-induced segregation in homogeneous spaces have been a main center of interest of Dancer, Terracini and others since the nineties ([5, 6, 7, 8, 9, 10, 11, 12] among others). They basically confirmed the intuitive idea that competitors tend to live in different ecological niches.

To investigate the existence of bistable pulsating fronts connecting two extinction states, we have at our disposal recent abstract results about monotone semiflows stated by Weinberger [29] (monostable case) and Fang and Zhao [17] (bistable case). Even though both articles were mostly concerned by scalar equations, they were careful enough to include monotone systems, such as two-species competitive ones,

in their framework. Notice that Yu and Zhao [30] used a similar framework to prove, in the weak competition case, the existence of monostable pulsating fronts connecting two extinction states despite the presence of an intermediate coextinction state (Weinberger’s framework requires no intermediate stationary state) (see also Fang–Yu–Zhao [16] for a similar work in space-time periodic media).

The core idea of Fang and Zhao’s theorem is as follows: provided a bistable monotone problem, if all intermediate stationary states are unstable and if they are invaded by the stable states, then bistable traveling waves do exist. While these hypotheses might be easily verified for some problems (say, scalar or space-homogeneous), in the case exposed here, real issues arise from the segregation phenomenon. Indeed, stable intermediate segregated periodic coexistence states might a priori exist. Therefore it is natural to wonder whether periodicity might induce some simple, yet relevant, sufficient condition to enforce the non-existence of segregated periodic coexistence states. We will indeed state one such condition and will show that this condition is moreover sufficient to guarantee that all remaining periodic stationary states are unstable and invaded by the stable ones.

The following pages will be organized as follows: in the first section, the core hypotheses and framework will be precisely formulated and the main results stated. The second section will be dedicated to the proof of the existence of pulsating front solutions; in particular, we will perform a quite thorough study of the stability of periodic coexistence states.

The study of the limit as  $k \rightarrow +\infty$  of these pulsating fronts will be the object of the second part [21].

## 1. PRELIMINARIES AND MAIN RESULTS

Let  $d, k, \alpha, L > 0$ ,  $C = (0, L) \subset \mathbb{R}$  and  $(f_1, f_2) : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$   $L$ -periodic with respect to its second variable. For any  $u : \mathbb{R}^2 \rightarrow [0, +\infty)$  and  $i \in \{1, 2\}$ , we refer to  $(t, x) \mapsto f_i(u(t, x), x)$  as  $f_i[u]$ . Our interest lies in the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1[u_1] - k u_1 u_2 \\ \partial_t u_2 = d \partial_{xx} u_2 + u_2 f_2[u_2] - \alpha k u_1 u_2 \end{cases} \quad (\mathcal{P}_k)$$

### 1.1. Preliminaries.

#### 1.1.1. Redaction conventions.

- Mirroring the definition of  $f_1[u]$  and  $f_2[u]$ , for any function of two real variables  $f$  and any real-valued function  $u$  of two real variables,  $f[u]$  will refer to  $(t, x) \mapsto f(u(t, x), x)$ . For any real-valued function  $u$  of one real variable,  $f[u]$  will refer to  $x \mapsto f(u(x), x)$ . For any function  $f$  of one real variable and any real-valued function  $u$  of one or two real variables,  $f[u]$  will simply refer to  $f \circ u$ .
- For the sake of brevity, although we could index everything  $((\mathcal{P}), u_1, u_2, \dots)$  on  $k$  and  $d$ , the dependencies on  $k$  or  $d$  will mostly be implicit and will only be made explicit when it definitely facilitates the reading.
- Since we consider the limit of this system when  $k \rightarrow +\infty$ , many (but finitely many) results will only be true when “ $k$  is large enough”. Hence, we define by induction the positive number  $k^*$ , whose value is initially 1 and is updated each time a statement is only true when “ $k$  is large enough” in the

following way: if the statement is true for any  $k \geq k^*$ , the value of  $k^*$  is unchanged; if, conversely, there exists  $K > k^*$  such that the statement is true for any  $k \geq K$  but false for any  $k \in [k^*, K)$ , the value of  $k^*$  becomes that of  $K$ . In the text, we will indifferently write “for  $k$  large enough” or “provided  $k^*$  is large enough”. Moreover, when  $k$  indexes appear, they a priori indicate that we are considering families indexed on (equivalently, functions defined on)  $[k^*, +\infty)$ , but for the sake of brevity, when sequential arguments imply extractions of sequences and subsequences indexed themselves on increasing elements of  $[k^*, +\infty)^{\mathbb{N}}$ , we will not explicitly define these sequences of indexes and will simply stick with the indexes  $k$ , reindexing along the course of the proof the considered objects. In such a situation, the statement “as  $k \rightarrow +\infty$ ” should be understood unambiguously.

- Periodicity will always implicitly mean  $L$ -periodicity (unless explicitly stated otherwise). For any functional space  $X$  on  $\mathbb{R}$ ,  $X_{per}$  denotes the subset of  $L$ -periodic elements of  $X$ .
- We will use the classical partial order on the space of functions from any  $\Omega \subset \mathbb{R}^N$  to  $\mathbb{R}$ :  $g \leq h$  if and only if, for any  $x \in \Omega$ ,  $g(x) \leq h(x)$ , and  $g < h$  if and only if  $g \leq h$  and  $g \neq h$ . We recall that when  $g < h$ , there might still exist  $x \in \Omega$  such that  $g(x) = h(x)$ . If, for any  $x \in \Omega$ ,  $g(x) < h(x)$ , we use the notation  $g \ll h$ . In particular, if  $g \geq 0$ , we say that  $g$  is non-negative, if  $g > 0$ , we say that  $g$  is non-negative non-zero, and if  $g \gg 0$ , we say that  $g$  is positive (and we define similarly non-positive, non-positive non-zero and negative functions). Eventually, if  $g_1 \leq h \leq g_2$ , we write  $h \in [g_1, g_2]$ , if  $g_1 < h < g_2$ , we write  $h \in (g_1, g_2)$ , and if  $g_1 \ll h \ll g_2$ , we write  $h \in \langle g_1, g_2 \rangle$ .
- We will also use the partial order on the space of vector functions  $\Omega \rightarrow \mathbb{R}^{N'}$  naturally derived from the preceding partial order. It will involve similar notations.
- The periodic principal eigenvalue of a second order elliptic operator  $\mathcal{L}$  with periodic coefficients will be generically referred to as  $\lambda_{1,per}(-\mathcal{L})$ . Recall (from Berestycki–Hamel–Roques [2] for instance) that the periodic principal eigenvalue of  $\mathcal{L}$  is the unique real number  $\lambda$  such that there exists a periodic function  $\varphi \gg 0$  satisfying:

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi \text{ in } \mathbb{R} \\ \|\varphi\|_{L^\infty(C)} = 1 \end{cases}$$

The Dirichlet principal eigenvalue of an elliptic operator  $\mathcal{L}$  in a sufficiently smooth domain  $\Omega$  will be referred to as  $\lambda_{1,Dir}(-\mathcal{L}, \Omega)$ . Since our framework is spatially one-dimensional, such elliptic operators will involve first and second derivatives with respect to the spatial variable  $x$ .

1.1.2. *Hypotheses on the reaction.* For any  $i \in \{1, 2\}$ , we have in mind functions  $f_i$  such that the reaction term  $uf_i[u]$  is of logistic type (also known as KPP type). At least, we want to cover the largest possible class of  $(u, x) \mapsto \mu(x) - \nu(x)u$ . This is made precise by the following assumptions.

- ( $\mathcal{H}_1$ )  $f_i$  is  $\mathcal{C}^1$  with respect to its first variable up to 0 and Hölder-continuous with respect to its second variable with a Hölder exponent larger than or equal to  $\frac{1}{2}$ .
- ( $\mathcal{H}_2$ ) There exists a constant  $m_i > 0$  such that  $f_i[0] \geq m_i$ .

( $\mathcal{H}_3$ )  $f_i$  is decreasing with respect to its first variable and there exists  $a_i > 0$  such that, if  $u > a_i$ , then for any  $x \in \mathbb{R}$   $f_i(u, x) < 0$ .

*Remark.* If  $f_i$  is in the class of all  $(u, x) \mapsto \mu(x) - \nu(x)u$ , then  $\mu, \nu \in C_{per}^{0,1/2}(\mathbb{R})$ ,  $\mu \gg 0$ ,  $\nu \gg 0$ . More generally, from ( $\mathcal{H}_1$ ), ( $\mathcal{H}_2$ ) and the periodicity of  $f_i[0]$ , it follows immediately that there exists a constant  $M_i > m_i$  such that  $f_i[0] \leq M_i$ . Without loss of generality, we assume that  $m_i$  and  $M_i$  are optimal, that is  $m_i = \min_{\overline{C}} f_i[0]$  and  $M_i = \max_{\overline{C}} f_i[0]$ .

We refer to  $\max(M_1, M_2)$  (resp.  $\min(m_1, m_2)$ ) as  $M$  (resp.  $m$ ).

Furthermore, we need a coupled hypothesis on the pair  $(f_1, f_2)$ .

( $\mathcal{H}_{freq}$ ) The constants  $d, M_1$  and  $M_2$  satisfy  $L < \pi \left( \frac{1}{\sqrt{M_1}} + \sqrt{\frac{d}{M_2}} \right)$ .

*Remark.* Even if this might not be clear right now, this is the key hypothesis. ( $\mathcal{H}_{freq}$ ) means that, given a fixed amplitude, we consider high frequencies, or equivalently, given a fixed frequency, we consider low amplitudes. This sufficient condition for existence might be a bit relaxed but the best condition we can give is very verbose and only slightly better. See the proof of Proposition 2.4, which is where ( $\mathcal{H}_{freq}$ ) plays its role.

**1.2. Two main results and a conjecture.** Using known results about scalar equations and periodic principal eigenvalues [2], the following lemma is quite straightforward (as will show Subsection 1.3.3).

**Lemma 1.1.** *Assume that  $f_1$  and  $f_2$  satisfy ( $\mathcal{H}_1$ ), ( $\mathcal{H}_2$ ) and ( $\mathcal{H}_3$ ).*

*The set of all periodic stationary states of the problem ( $\mathcal{P}$ ) contains  $(0, 0)$ , which is unstable, and a pair  $\{(\tilde{u}_1, 0), (0, \tilde{u}_2)\}$  with  $(\tilde{u}_1, \tilde{u}_2) \in \mathcal{C}_{per}^2(\mathbb{R}, (0, +\infty)^2)$ .*

As usual in the literature concerning competitive systems, hereafter, the stationary states with exactly one null component are referred to as *extinction states* whereas the stationary states with no null component are referred to as *coexistence states*. The extinction states of ( $\mathcal{P}$ ) are periodic and some of its coexistence states may be periodic as well.

Our contribution to the study of the stationary states is the following theorem.

**Theorem 1.2.** *Assume that  $f_1$  and  $f_2$  satisfy ( $\mathcal{H}_1$ ), ( $\mathcal{H}_2$ ) and ( $\mathcal{H}_3$ ) and that  $(f_1, f_2)$  satisfies ( $\mathcal{H}_{freq}$ ).*

*Then there exists  $k^* > 0$  such that, for any  $k > k^*$ , each extinction state is locally asymptotically stable and any periodic coexistence state is unstable.*

*Furthermore, let  $(u_{1,k}, u_{2,k})_{k > k^*}$  be a family of  $\mathcal{C}_{per}^2(\mathbb{R}, \mathbb{R}^2)$  such that, for any  $k > k^*$ ,  $(u_{1,k}, u_{2,k})$  is an unstable periodic stationary state of ( $\mathcal{P}_k$ ). Then  $(u_{1,k}, u_{2,k})$  converges in  $\mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^2)$  to  $(0, 0)$  as  $k \rightarrow +\infty$ .*

*Remark.* We stress that we did not investigate the existence nor the countability of the subset of periodic coexistence states. We stress as well that we did not investigate at all aperiodic coexistence states. We believe that a sharper description of the set of stationary states of ( $\mathcal{P}$ ) could follow from bifurcation arguments (see Hutson–Lou–Mischaikow [23] or Furter–López-Gómez [18]). Since it was not our point at all (instability of periodic coexistence states was only a required step toward existence of pulsating fronts), we chose to leave this subject as an open question.

Thanks to the previous theorem, it is then possible to prove the following existence theorem.

**Theorem 1.3.** *Assume that  $f_1$  and  $f_2$  satisfy  $(\mathcal{H}_1)$ ,  $(\mathcal{H}_2)$  and  $(\mathcal{H}_3)$  and that  $(f_1, f_2)$  satisfies  $(\mathcal{H}_{freq})$ .*

*Then there exists  $k^* > 0$  such that, for any  $k > k^*$ , the problem  $(\mathcal{P})$  admits a bistable pulsating front solution connecting the two extinction states.*

To end this subsection, let us present an important conjecture about the existence problem and about the sharpness of  $(\mathcal{H}_{freq})$ . We did not address this question but hopefully others will.

**Conjecture.** *Neither  $(\mathcal{H}_{freq})$  nor the nonexistence of a stable periodic coexistence state are necessary conditions for the existence of a bistable pulsating front solution connecting the two extinction states.*

*Furthermore, there exists a non-empty set of parameters  $(L, d, \alpha, k, f_1, f_2)$  such that no such pulsating front exists.*

We point out that, according to the present work, any of the following two conditions enforces that either  $(\mathcal{H}_{freq})$  is not satisfied or  $k \leq k^*$ :

- the existence of a stable periodic coexistence state;
- the nonexistence of a bistable pulsating front solution.

Moreover, our work will show that, if  $k > k^*$ , any stable periodic coexistence state has the “close to segregation” form (which will be rigorously defined later on; roughly speaking, “close to segregation” periodic coexistence states converge as  $k \rightarrow +\infty$  to a non-trivial periodic coexistence state satisfying  $u_1 u_2 = 0$ ). This important property might be the starting point of a future work on the preceding conjecture.

### 1.3. A few more preliminaries.

1.3.1. *Compact embeddings of Hölder spaces.* We recall a well-known result of functional analysis.

**Proposition 1.4.** *Let  $(a, a') \in (0, +\infty)^2$  and  $n, n', \beta, \beta'$  such that  $(a, a') = (n + \beta, n' + \beta')$ ,  $n$  and  $n'$  are non-negative integers and  $\beta$  and  $\beta'$  are in  $(0, 1]$ .*

*If  $a \leq a'$ , then the canonical embedding  $i : \mathcal{C}^{n', \beta'}(C) \hookrightarrow \mathcal{C}^{n, \beta}(C)$  is continuous and compact.*

It will be clear later on that this problem naturally involves uniform bounds in  $\mathcal{C}^{0, 1/2}$  and in  $\mathcal{C}^{2, 1/2}$ . Therefore, we fix once and for all  $\beta \in (0, \frac{1}{2})$  and we will use systematically the compact embeddings  $\mathcal{C}^{n, 1/2} \hookrightarrow \mathcal{C}^{n, \beta}$ , meaning that uniform bounds in  $\mathcal{C}^{n, 1/2}$  yield relative compactness in  $\mathcal{C}^{n, \beta}$ .

1.3.2. *Existence and uniqueness for the evolution system.*

**Proposition 1.5.** *Let  $k > 0$ . Equipped with an initial non-negative condition  $(u_{1,0}, u_{2,0}) \in \mathcal{C}^{0, 1/2}(\mathbb{R}, \mathbb{R}^2)$ , the problem  $(\mathcal{P})$  is well-posed: there exists a unique non-negative entire solution  $(u_1, u_2) \in \mathcal{C}^{1, 1/4}([0, +\infty), \mathcal{C}^{2, 1/2}(\mathbb{R}, \mathbb{R}^2))$ .*

*Furthermore, if  $(u_{1,0}, u_{2,0}) > 0$ , then  $(u_1, u_2) \gg 0$ , and if  $(u_{1,0}, u_{2,0}) \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^2)$ , then  $(u_1, u_2) \in \mathcal{C}^1([0, +\infty), \mathcal{C}_{per}^2(\mathbb{R}, \mathbb{R}^2))$ .*

*Remark.* We do not give a fully detailed proof of this statement. Ideas similar to those given in Berestycki–Hamel–Roques [2, Remark 2.7] suffice. The existence of solutions for the truncated system in  $(-n, n)$  with Dirichlet boundary conditions can be proved with Pao’s super- and sub-solutions theorem for competitive systems [28].

### 1.3.3. Extinction states.

**Lemma 1.6.** *The periodic principal eigenvalues of  $-\frac{d^2}{dx^2} - f_1[0]$  and  $-d\frac{d^2}{dx^2} - f_2[0]$  are negative.*

*Proof.* This follows from  $(\mathcal{H}_2)$  and the monotonicity of the periodic principal eigenvalue with respect to the zeroth order term of the elliptic operator. Indeed, for instance:

$$\lambda_{1,per} \left( -\frac{d^2}{dx^2} - f_1[0] \right) \leq \lambda_{1,per} \left( -\frac{d^2}{dx^2} - m_1 \right) = -m_1 < 0.$$

□

From this lemma and hypotheses  $(\mathcal{H}_1)$  and  $(\mathcal{H}_3)$ , a fundamental result from Berestycki–Hamel–Roques [2] can be applied.

**Theorem 1.7.** *For any  $\delta > 0$  and any  $i \in \{1, 2\}$ , the equation:*

$$-\delta z'' = z f_i[z]$$

*admits a unique positive solution in  $\mathcal{C}_{per}^2(\mathbb{R})$ .*

Hereafter,  $\tilde{u}_1$  and  $\tilde{u}_2$  are the respective unique positive periodic solutions of:

$$-z'' = z f_1[z],$$

$$-dz'' = z f_2[z].$$

$(\tilde{u}_1, 0)$  and  $(0, \tilde{u}_2)$  are indeed the extinction states of any  $(\mathcal{P}_k)$ .

1.3.4. *Monotone evolution system.* One of the most important specificities of two-species competitive systems is that, up to a slight transformation, they are monotone systems. It is the key behind the results of Fang–Zhao [17] and Weinberger [29]. Let us recall this transformation.

**Lemma 1.8.** *Let  $J : z \mapsto \tilde{u}_2 - z$ , for any  $z \in \mathcal{C}_{per}^2(\mathbb{R})$  or  $z \in \mathcal{C}^1([0, +\infty), \mathcal{C}_{per}^2(\mathbb{R}))$  (with a slight abuse of notation). Let  $k > k^*$  and let  $(u_1, u_2)$  be a solution of  $(\mathcal{P})$  and  $v_2 = J(u_2)$ .*

*Then  $(u_1, v_2)$  satisfies the following cooperative problem with periodicity conditions:*

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1 f_1[u_1] + k u_1 (-\tilde{u}_2 + v_2) \\ \partial_t v_2 - d \partial_{xx} v_2 = \tilde{u}_2 f_2[\tilde{u}_2] - (\tilde{u}_2 - v_2) f_2[\tilde{u}_2 - v_2] + \alpha k u_1 (\tilde{u}_2 - v_2). \end{cases} \quad (\mathcal{M}_k)$$

**Corollary 1.9.** *Any solution  $(u_1, u_2)$  of  $(\mathcal{P})$  with initial condition  $(0, 0) < (u_{1,0}, u_{2,0}) < (\tilde{u}_1, \tilde{u}_2)$  satisfies  $(0, 0) \ll (u_1, u_2) \ll (\tilde{u}_1, \tilde{u}_2)$ .*



1.3.5. *Segregated reaction terms.* As  $k \rightarrow +\infty$ , the following functions will naturally appear:

$$\begin{aligned}\eta : (z, x) &\mapsto f_1\left(\frac{z}{\alpha}, x\right) z^+ - \frac{1}{d} f_2\left(-\frac{z}{d}, x\right) z^-, \\ \gamma : (z, x) &\mapsto f_1(0, x) z^+ - \frac{1}{d} f_2(0, x) z^-, \end{aligned}$$

where  $z^+ = \max(z, 0)$  and  $z^- = -\min(z, 0)$  so that  $z = z^+ - z^-$ .

1.3.6. *Derivatives of the reaction terms.* We will denote  $g_i$  the partial derivative of  $(u, x) \mapsto u f_i(u, x)$  with respect to  $u$ :

$$g_i : (u, x) \mapsto f_i(u, x) + u \partial_1 f_i(u, x) \text{ for all } i \in \{1, 2\}.$$

## 2. EXISTENCE OF PULSATING FRONTS

2.1. **Aim: Fang–Zhao’s theorem.** We recall that, for any  $k > k^*$  and any  $t > 0$ , the Poincaré’s map  $Q_t$  associated with  $(\mathcal{M})$  is defined as the operator:

$$Q_t : \mathcal{C}(\mathbb{R}, \mathbb{R}^2) \cap [(0, 0), (\tilde{u}_1, \tilde{u}_2)] \rightarrow \mathcal{C}(\mathbb{R}, \mathbb{R}^2) \cap [(0, 0), (\tilde{u}_1, \tilde{u}_2)]$$

which associates with some initial condition  $(u_{1,0}, v_{2,0})$  the solution  $(u_1, v_2)$  of  $(\mathcal{M})$  evaluated at time  $t > 0$ .

From Fang and Zhao [17], we know that  $(\mathcal{M})$  admits a pulsating front solution connecting  $(\tilde{u}_1, \tilde{u}_2)$  to  $(0, 0)$  if:

- (1)  $(0, 0)$  and  $(\tilde{u}_1, \tilde{u}_2) \gg (0, 0)$  are locally asymptotically stable periodic stationary states of  $(\mathcal{M})$  and all intermediate periodic stationary states of  $(\mathcal{M})$  are unstable;
- (2) for any intermediate periodic stationary state  $(u_1, v_2)$ , the sum of the spreading speeds associated with front-like initial data connecting respectively  $(\tilde{u}_1, \tilde{u}_2)$  to  $(u_1, v_2)$  and  $(u_1, v_2)$  to  $(0, 0)$  is positive (notice that these sub-problems are of monostable type);
- (3) and if, for any  $t > 0$ ,  $Q_t$  satisfies the following hypotheses:
  - (a)  $Q_t$  is spatially periodic;
  - (b)  $Q_t$  is continuous with respect to the topology of the locally uniform convergence;
  - (c)  $Q_t$  is strongly monotone, in the sense that if  $(u_1, v_2) > (u^1, v^2)$ , then:
 
$$Q_t((u_1, v_2)) \gg Q_t((u^1, v^2));$$
  - (d)  $Q_t$  is compact with respect to the topology of the locally uniform convergence;

It is quite standard to check that the last four hypotheses are indeed satisfied. The verification of the first two, on the contrary, is the object of the remaining of this paper.

### 2.2. Stability of all extinction states.

**Proposition 2.1.** *Provided  $k^*$  is large enough,  $(\tilde{u}_1, 0)$  and  $(0, \tilde{u}_2)$  are locally asymptotically stable.*

*Remark.* For the case  $k = 1$ , the proof of the local asymptotic stability of the extinction states was done by Dockery and his coauthors [15] with the help of Mora’s theorem [25]. It works here too with a very slight adaptation; we give the proof for the sake of completeness.

*Proof.* Thanks to Mora’s theorem [25], we know that  $(\tilde{u}_1, 0)$  is asymptotically stable if the periodic principal eigenvalue of the elliptic part of the monotone problem  $(\mathcal{M})$  linearized at  $(\tilde{u}_1, \tilde{u}_2) = (u, J(0))$  is positive. Therefore we consider the differential operator  $\mathcal{A}_{(\tilde{u}_1, 0)} : \mathcal{C}_{per}^2(\mathbb{R}) \rightarrow \mathcal{C}_{per}(\mathbb{R})$  defined as:

$$\mathcal{A}_{(\tilde{u}_1, 0)} = \begin{pmatrix} \frac{d^2}{dx^2} + g_1[\tilde{u}_1] & k\tilde{u}_1 \\ 0 & d\frac{d^2}{dx^2} + f_2[0] - \alpha k\tilde{u}_1 \end{pmatrix}$$

From the special “triangular” form of  $\mathcal{A}_{(\tilde{u}_1, 0)}$ , it is clear that:

$$\min(\text{sp}(-\mathcal{A}_{(\tilde{u}_1, 0)})) = \min\left(\lambda_{1,per}\left(-\frac{d^2}{dx^2} - g_1[\tilde{u}_1]\right), \lambda_{1,per}\left(-d\frac{d^2}{dx^2} - (f_2[0] - \alpha k\tilde{u}_1)\right)\right).$$

By monotonicity of the periodic principal eigenvalue and  $(\mathcal{H}_3)$ , we obtain:

$$\lambda_{1,per}\left(-\frac{d^2}{dx^2} - g_1[\tilde{u}_1]\right) > \lambda_{1,per}\left(-\frac{d^2}{dx^2} - f_1[\tilde{u}_1]\right).$$

For any  $k$  large enough,  $f_2[0] - \alpha k\tilde{u}_1 < f_2[\tilde{u}_2]$  holds, so that:

$$\lambda_{1,per}\left(-d\frac{d^2}{dx^2} - (f_2[0] - \alpha k\tilde{u}_1)\right) > \lambda_{1,per}\left(-d\frac{d^2}{dx^2} - f_2[\tilde{u}_2]\right).$$

Moreover, from the equation solved by  $\tilde{u}_1$ ,  $\tilde{u}_1$  is actually an eigenfunction for the following eigenvalue:

$$\lambda_{1,per}\left(-\frac{d^2}{dx^2} - f_1[\tilde{u}_1]\right) = 0.$$

Similarly,

$$\lambda_{1,per}\left(-d\frac{d^2}{dx^2} - f_2[\tilde{u}_2]\right) = 0.$$

Thus:

$$\lambda_{1,per}(-\mathcal{A}_{(\tilde{u}_1, 0)}) > 0.$$

The same proof holds for  $(0, \tilde{u}_2)$ . □

**2.3. Instability of all periodic coexistence states.** In this subsection, we prove that  $(\mathcal{M})$  admits no stable periodic stationary states in  $\langle(0, 0), (\tilde{u}_1, \tilde{u}_2)\rangle$ .

For any  $k > k^*$ , let:

$$S_k \subset \mathcal{C}_{per}^2(\mathbb{R}, \mathbb{R}^2)$$

be the set of periodic solutions of the following problem:

$$\begin{cases} -u_1'' = u_1 f_1[u_1] - k u_1 u_2 \\ -d u_2'' = u_2 f_2[u_2] - \alpha k u_1 u_2 \\ u_1 \in \langle 0, \tilde{u}_1 \rangle \\ u_2 \in \langle 0, \tilde{u}_2 \rangle. \end{cases}$$

Any  $(u_1, u_2) \in S$  is a periodic coexistence state.

2.3.1. *Basic properties of periodic coexistence states.*

**Lemma 2.2.** *Let  $k > k^*$ . Any  $(u_1, u_2) \in S$  satisfies:*

$$\begin{cases} k \min u_2 \leq \max f_1 [\max u_1] \\ \alpha k \min u_1 \leq \max f_2 [\max u_2] \\ \min f_1 [\min u_1] \leq k \max u_2 \\ \min f_2 [\min u_2] \leq \alpha k \max u_1, \end{cases}$$

*each extrema being implicitly over  $\bar{C}$ .*

*Proof.* We only prove the first inequality, the three others being proved similarly.

Let  $\bar{x} \in \bar{C}$  such that  $u_1(\bar{x}) = \max u_1$ . Since  $u_1 \in \mathcal{C}^2(\mathbb{R})$ ,  $u_1''(\bar{x}) \leq 0$ , that is:

$$\max u_1 f_1 [\max u_1] \geq \max u_1 k u_2(\bar{x}).$$

Since  $u_1 > 0$ , we can divide by  $\max u_1$ . The claimed result easily follows.  $\square$

*Remark.* This lemma will be used together with  $m > 0$  to prove that  $ku_1$  and  $ku_2$  stay non-zero as  $k \rightarrow +\infty$ . Thus, for the forthcoming study, it is not sufficient to merely assume that  $\lambda_{1,per} \left( -\frac{d^2}{dx^2} - f_1[0] \right)$  and  $\lambda_{1,per} \left( -d\frac{d^2}{dx^2} - f_2[0] \right)$  are negative (as was done for instance by Dockery and his collaborators [15]).

**Proposition 2.3.** *As  $k \rightarrow +\infty$ , the family  $(S_k)_{k>k^*}$  is relatively compact in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R}, \mathbb{R}^2)$ .  $(0, 0)$  is one of its limit points. Any other limit point  $(u_{1,seg}, u_{2,seg}) \in \mathcal{C}_{per}^{0,\beta}(\mathbb{R}, \mathbb{R}^2)$  is called a periodic segregated state and is such that  $\alpha u_{1,seg} - du_{2,seg}$  is a non-zero sign-changing solution in  $\mathcal{C}_{per}^{2,\beta}(\mathbb{R})$  of the following elliptic equation:*

$$-z'' = \eta[z].$$

*Proof.* Let  $k > k^*$ .

Multiplying by  $u_{1,k}$  the first equation of the stationary system and integrating over  $C$  yields easily:

$$\begin{aligned} \|u'_{1,k}\|_{L^2(C)} &\leq M_1 \|u_{1,k}\|_{L^2(C)} \\ &\leq M_1 \|\tilde{u}_1\|_{L^2(C)}, \end{aligned}$$

whence, for all  $(x, y) \in C^2$ :

$$|u_{1,k}(x) - u_{1,k}(y)| \leq M_1 \|\tilde{u}_1\|_{L^2(C)} |x - y|^{1/2}.$$

Moreover,  $\|u_{1,k}\|_{L^\infty(C)} \leq \|\tilde{u}_1\|_{L^\infty(C)}$ , and therefore  $(u_{1,k})_{k>k^*}$  is uniformly bounded in  $\mathcal{C}^{0,1/2}(C)$  and relatively compact in  $\mathcal{C}^{0,\beta}(C)$ . The same proof holds for  $(u_2)_{k>k^*}$ .

Let  $(u_{1,\infty}, u_{2,\infty}) \in \mathcal{C}_{per}^{0,\beta}(\mathbb{R}, \mathbb{R}^2)$  be a limit point of  $(S_k)_{k>k^*}$ . There exists a sequence of periodic coexistence states  $((u_{1,k}, u_{2,k}))_{k>k^*}$  whose limit in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R}, \mathbb{R}^2)$  is  $(u_{1,\infty}, u_{2,\infty})$ . By elliptic regularity and thanks to the following equation:

$$-\alpha u''_{1,k} + dv''_{2,k} = \alpha u_{1,k} f_1[u_{1,k}] - u_{2,k} f_2[u_{2,k}],$$

which holds for any  $k > k^*$  and is obtained by linear combination of the equations of the stationary system,  $(\alpha u_{1,k} - du_{2,k})$  converge in  $\mathcal{C}_{per}^{2,\beta}(\mathbb{R})$  to  $v = \alpha u_{1,\infty} - du_{2,\infty} \in \mathcal{C}_{per}^{2,\beta}(\mathbb{R})$ .

Multiplying by a test function  $\varphi \in \mathcal{D}(\mathbb{R})$  the equation defining  $u_{1,k}$ , integrating and dividing by  $k$ , we obtain easily that  $(u_{1,k} u_{2,k})$  converges as  $k \rightarrow +\infty$  in  $\mathcal{D}'(\mathbb{R})$

to 0. Hence  $u_{1,\infty}u_{2,\infty} = 0$  and then  $\alpha u_{1,\infty} = v^+$  and  $du_{2,\infty} = v^-$ . In particular,  $v$  satisfies as claimed:

$$-v'' = \eta[v]$$

Let:

$$\begin{aligned} C_1 &= \{x \in C \mid v(x) > 0\}, \\ C_2 &= \{x \in C \mid v(x) < 0\}, \\ \Gamma &= \{x \in C \mid v(x) = 0\}, \end{aligned}$$

so that:

$$C \subset C_1 \cup C_2 \cup \Gamma \subset \overline{C}.$$

Exactly four cases are a priori possible:

- (1)  $C_1 = C$ : then by continuity  $v = \alpha u_{1,\infty}$  in  $\overline{C}$  whereas  $u_{2,\infty} = 0$  in  $\overline{C}$ , hence  $u_{1,\infty} \in \mathcal{C}_{per}^{2,\beta}(\mathbb{R})$  is a non-negative non-zero solution of

$$-u_{1,\infty}'' = u_{1,\infty} f_1[u_{1,\infty}]$$

in  $\mathbb{R}$ , and eventually by the elliptic strong minimum principle  $u_{1,\infty} \gg 0$ , meaning that  $u_{1,\infty} = \tilde{u}_1$ , and  $C_2 = \Gamma = \emptyset$ ;

- (2)  $C_2 = C$ : then similarly  $C_1 = \Gamma = \emptyset$ ,  $u_{1,\infty} = 0$  and  $u_{2,\infty} = \tilde{u}_2$ ;  
 (3)  $C_1 \neq \emptyset$  and  $C_2 \neq \emptyset$ .  
 (4)  $C_1 = \emptyset$  and  $C_2 = \emptyset$ :  $\Gamma = C$ ,  $u_{1,\infty}$  and  $u_{2,\infty}$  are uniformly 0;

It is easily seen that Lemma 2.2 excludes the cases 1 (use the second inequality) and 2 (use the first inequality).  $\square$

**Proposition 2.4.** *The following set equalities hold:*

$$\begin{aligned} \{z \in \mathcal{C}_{per}^2(\mathbb{R}) \mid -z'' = \gamma[z]\} &= \{0\}, \\ \{z \in \mathcal{C}_{per}^2(\mathbb{R}) \mid -z'' = \eta[z]\} &= \{-d\tilde{u}_2, 0, \alpha\tilde{u}_1\}. \end{aligned}$$

*Proof.* In the  $\gamma$  case, solutions of constant sign are excluded by:

$$\begin{aligned} \lambda_{1,per} \left( -\frac{d^2}{dx^2} - f_1[0] \right) &< 0, \\ \lambda_{1,per} \left( -d\frac{d^2}{dx^2} - f_2[0] \right) &< 0. \end{aligned}$$

In the  $\eta$  case, solutions of constant sign are unique (see Berestycki–Hamel–Roques [2]) and are exactly  $\alpha\tilde{u}_1$  and  $-d\tilde{u}_2$ . It only remains to prove that non-zero sign-changing solutions are excluded, and up to a shift of  $C$  it suffices to prove that non-zero sign-changing solutions which are equal to 0 at 0 and  $L$  are excluded.

For any  $x \in \mathbb{R}$ , any  $f \in \mathcal{C}_{per}^0(\mathbb{R}, [m, M])$  and any  $\delta \in \{1, d\}$ , let  $R(x, f, \delta) > 0$  such that:

$$\lambda_{1,Dir} \left( -\delta\frac{d^2}{dx^2} - f, B(x, R(x, f, \delta)) \right) = 0.$$

Since the following function:

$$R \mapsto \lambda_{1,Dir} \left( -\delta\frac{d^2}{dx^2} - f, B(x, R) \right)$$

is continuous, decreasing and has positive and negative values (its limits as  $R \rightarrow 0$  or  $R \rightarrow +\infty$  are respectively  $+\infty$  and  $\lambda_{1,per} \left( -\delta\frac{d^2}{dx^2} - f \right) < 0$ , as proved in [2]),  $R(x, f, \delta)$  is uniquely defined. Since  $\lambda_{1,Dir} \left( -\delta\frac{d^2}{dx^2} - f, B(x, R) \right)$  is non-increasing

with respect to  $f$  and decreasing with respect to  $R$ , it is easy to check that  $f \mapsto R(x, f, \delta)$  is non-increasing.

Remark that  $R(x, f, \delta)$  and  $\lambda_{1,Dir} \left( -\delta \frac{d^2}{dx^2} - f, B(x, R(x, f, \delta)) \right)$  do not depend on  $x$  if  $f$  does not depend on  $x$ . Remark that, in such a case,  $R(0, f, \delta)$  can be easily determined analytically and is equal to  $\frac{\pi}{2} \sqrt{\frac{\delta}{f}}$ .

With these notations,  $(\mathcal{H}_{freq})$  means:

$$L < 2(R(0, M_1, 1) + R(0, M_2, d)).$$

Let  $z$  be a solution of  $-z'' = \gamma[z]$  or a solution of  $-z'' = \eta[z]$ . Let:

$$C_+ = z^{-1}((0, +\infty)) \cap C,$$

$$C_- = z^{-1}((-\infty, 0)) \cap C.$$

Assume by contradiction that both are non-empty. Let  $n$  be the number of zeros of  $z$  in  $C$ . Then:

- in virtue of the Hopf lemma, of:

$$\min \left( \min_{x \in \overline{C}} R(x, f_1[0], 1), \min_{x \in \overline{C}} R(x, f_2[0], d) \right) > 0$$

and of the continuity of  $z$ ,  $n$  is finite and odd, say  $n = 2p + 1$  with  $p$  a non-negative integer, and  $C_+$  and  $C_-$  both have precisely  $p + 1$  connected components, each of them being a one-dimensional ball (that is an interval); let  $(x_i^+)_{1 \leq i \leq p+1}$  (resp.  $(x_i^-)_{1 \leq i \leq p+1}$ ) be the ordered centers of the connected components of  $C_+$  (resp.  $C_-$ );

- in the  $\gamma$  case:

$$\begin{aligned} |C_+| &= 2 \sum_{i=1}^{p+1} R(x_i^+, f_1[0], 1) \\ &\geq 2 \sum_{i=1}^{p+1} R(x_i^+, M_1, 1) \\ &\geq 2(p+1) R(0, M_1, 1) \\ &\geq 2R(0, M_1, 1), \end{aligned}$$

and similarly:

$$\begin{aligned} |C_-| &= 2 \sum_{i=1}^{p+1} R(x_i^-, f_2[0], d) \\ &\geq 2R(0, M_2, d), \end{aligned}$$

whence we get the contradiction;

- in the  $\eta$  case:

$$\begin{aligned} |C_+| &= 2 \sum_{i=1}^{p+1} R \left( x_i^+, f_1 \left[ \frac{z}{\alpha} \right], 1 \right) \\ &\geq 2 \sum_{i=1}^{p+1} R(x_i^+, f_1[0], 1), \end{aligned}$$

$$\begin{aligned} |C_-| &= 2 \sum_{i=1}^{p+1} R \left( x_i^-, f_2 \left[ -\frac{z}{d} \right], d \right) \\ &\geq 2 \sum_{i=1}^{p+1} R \left( x_i^-, f_2 [0], d \right) \end{aligned}$$

yield a similar contradiction.  $\square$

**Corollary 2.5.** *Any family  $(u_{1,k}, u_{2,k})_{k > k^*}$  of periodic coexistence states converges in  $C_{per}^{0,\beta}(\mathbb{R}, \mathbb{R}^2)$  as  $k \rightarrow +\infty$  to  $(0, 0)$ .*

*Remark.* This result has a very natural interpretation from an ecological point of view: if the wavelength of the distribution of resources is small enough, or if the resources are rare enough even in the most favorable areas, the species are not able to settle periodically in a favorable habitat smaller than the wavelength. Either one of them is strong enough to overcome unfavorable areas while eliminating the competitor and then it settles in the whole habitat, either both go extinct. Basically, at a given average intrinsic growth rate, the more fragmented the habitat is, the higher the chances of extinction are.

**Lemma 2.6.** *There exists  $R_1 \in (0, +\infty)$  and  $R_2 \in (R_1, +\infty)$  such that, provided  $k^*$  is large enough, for any  $k > k^*$  and any  $(u_{1,k}, u_{2,k}) \in S_k$ :*

$$R_1 \leq \frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha \|u_{1,k}\|_{L^\infty(C)}} \leq R_2.$$

*Remark.* Proof inspired by Dancer–Du [8, Lemma 2.1].

*Proof.* By contradiction, assume that there exists a sequence of periodic coexistence states  $((u_{1,k}, u_{2,k}))_{k > k^*}$  such that  $\left( \frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha \|u_{1,k}\|_{L^\infty(C)}} \right)_{k > k^*}$  is neither bounded from above nor from below by a positive constant. By symmetry, we can assume without loss of generality that it is not bounded from below by a positive constant. Up to extraction,  $\frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha \|u_{1,k}\|_{L^\infty(C)}} \rightarrow 0$  as  $k \rightarrow +\infty$ .

Suppose first that  $(\alpha k \|u_{1,k}\|_{L^\infty(C)})_{k > k^*}$  is bounded. Necessarily,  $k \|u_{2,k}\|_{L^\infty(C)} \rightarrow 0$  as  $k \rightarrow +\infty$ .

For any non-negative  $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ , the following problem:

$$-z'' = z f_1 [z] - z f$$

with periodicity conditions has a unique positive periodic solution  $z_f$  if and only if:

$$\lambda_{1,per} \left( -\frac{d^2}{dx^2} - (f_1 - f) \right) < 0$$

(see Berestycki–Hamel–Roques [2]). Moreover,  $z_f$  depends continuously on  $f$  as a map from  $\mathcal{C}_{per}(C)$  into itself (see Berestycki–Rossi [4]). Hence  $u_{1,k} = z_{k u_{2,k}} \rightarrow z_0$  as  $k \rightarrow +\infty$ , where  $z_0$  solves:

$$-z_0'' = z_0 f_1 [z_0]$$

with periodicity conditions (that is  $u[0] = \tilde{u}_1$ ). Since  $k \|\tilde{u}_1\|_{L^\infty(C)} \rightarrow +\infty$ , we get a contradiction.

Hence  $(\alpha k \|u_{1,k}\|_{L^\infty(C)})_{k > k^*}$  is unbounded. Up to extraction, we can assume that  $k \|u_{1,k}\|_{L^\infty(C)} \rightarrow +\infty$ .

For any  $k > k^*$ , let  $\hat{u}_{1,k} = \frac{u_{1,k}}{\|u_{1,k}\|_{L^\infty(C)}}$ ,  $\hat{u}_{2,k} = \frac{u_{2,k}}{\|u_{2,k}\|_{L^\infty(C)}}$ . Clearly,  $(\hat{u}_{1,k}, \hat{u}_{2,k})$  satisfies:

$$\begin{cases} -\hat{u}_{1,k}'' = \hat{u}_{1,k} f_1 [\|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k}] - k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \\ -d\hat{u}_{2,k}'' = \hat{u}_{2,k} f_2 [\|u_{2,k}\|_{L^\infty(C)} \hat{u}_{2,k}] - \alpha k \|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k}. \end{cases}$$

From there, it follows with the same estimates as in the proof of Proposition 2.3 that  $\hat{u}_{1,k}$  and  $\hat{u}_{2,k}$  converge up to extraction in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R})$ . Let  $\hat{u}_{1,\infty}$  and  $\hat{u}_{2,\infty}$  be their limits; for any  $i \in \{1, 2\}$   $\|\hat{u}_{i,\infty}\|_{L^\infty(C)} = 1$ , hence  $u_{i,\infty} \neq 0$ .

Then, we consider the system above in  $\mathcal{D}'(C)$ . Let  $\varphi \in \mathcal{D}(C)$  and use it as a test function. On the second line, we see that, since:

$$\int (d\hat{u}_{2,k}'' + \hat{u}_{2,k} f_2 [\|u_{2,k}\|_{L^\infty(C)} \hat{u}_{2,k}]) \varphi$$

is  $k$ -uniformly bounded, the same is true of:

$$\int \alpha k \|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi.$$

Thus:

$$\int k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi = \frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha \|u_{1,k}\|_{L^\infty(C)}} \int (\alpha k \|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi) \rightarrow 0$$

Therefore, considering the first line, we see that, by dominated convergence, the limit satisfies in the distributional sense:

$$-\hat{u}_{1,\infty}'' = \hat{u}_{1,\infty} f_1 [\|u_{1,\infty}\|_{L^\infty(C)} \hat{u}_{1,\infty}].$$

Since  $\hat{u}_{1,\infty}$  is in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R})$ , it is actually a solution in  $\mathcal{C}_{per}^{2,\beta}(\mathbb{R})$  by classical elliptic regularity. In virtue of the elliptic strong minimum principle,  $\hat{u}_{1,\infty} \gg 0$ . But it is also true, using the same arguments as before, that  $\hat{u}_{1,\infty} \hat{u}_{2,\infty} = 0$ , hence  $\hat{u}_{2,\infty} = 0$ , which is indeed a contradiction.  $\square$

**Lemma 2.7.** *Let  $((u_{1,k}, u_{2,k}))_{k > k^*}$  be a sequence of periodic coexistence states. Then  $((ku_{1,k}, ku_{2,k}))_{k > k^*}$  is  $k$ -uniformly bounded in  $L^\infty(C)$ .*

*Proof.* From Lemma 2.6, it suffices to assume that there exists a sequence  $((u_1, u_2))_{k > k^*}$  such that  $k\|u_{1,k}\|_{L^\infty(C)} \rightarrow +\infty$  as  $k \rightarrow +\infty$  and to get a contradiction.

With the same notations as in the proof of Lemma 2.6, up to extraction we can assume that  $\hat{u}_{1,k} \rightarrow \hat{u}_{1,\infty}$  and  $\hat{u}_{2,k} \rightarrow \hat{u}_{2,\infty}$  in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R})$ . We have for any  $i \in \{1, 2\}$   $\|\hat{u}_{i,\infty}\|_{L^\infty(C)} = 1$ , hence  $u_{i,\infty} \neq 0$ . Considering the limit of the equation satisfied by  $\hat{u}_{2,k}$  in  $\mathcal{D}'(C)$  shows that  $\hat{u}_{1,\infty} \hat{u}_{2,\infty} = 0$ . Thanks to Lemma 2.6, up to extraction, we can assume that there exists  $l > 0$  such that  $\frac{\alpha \|u_{1,k}\|_{L^\infty(C)}}{\|u_{2,k}\|_{L^\infty(C)}} \rightarrow l$ . Moreover, considering the equation satisfied by  $\hat{u}_{1,k}$  in  $\mathcal{D}'(C)$  shows that, for any  $\varphi \in \mathcal{D}(C)$ :

$$\int k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi$$

is  $k$ -uniformly bounded.

Multiplying the equation defining  $\hat{u}_{1,k}$  by  $l$  and subtracting from it the equation defining  $\hat{u}_{2,k}$  yields:

$$\begin{aligned} -l\hat{u}_{1,k}'' + d\hat{u}_{2,k}'' &= l\hat{u}_{1,k} f_1 [\|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k}] - \hat{u}_{2,k} f_2 [\|u_{2,k}\|_{L^\infty(C)} \hat{u}_{2,k}] \\ &\quad + \left( \frac{\alpha \|u_{1,k}\|_{L^\infty(C)}}{\|u_{2,k}\|_{L^\infty(C)}} - l \right) k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k}. \end{aligned}$$

Considering it in  $\mathcal{D}'(C)$ , passing to the limit (with, in virtue of Corollary 2.5,  $\|u_{i,k}\|_{L^\infty(C)} \rightarrow 0$ ) and defining  $v = l\hat{u}_{1,\infty} - d\hat{u}_{2,\infty}$ , it becomes:

$$-v'' = \gamma[v].$$

By classical elliptic regularity,  $v$  is actually a solution in  $\mathcal{C}_{per}^{2,\beta}(\mathbb{R})$ . Then Proposition 2.4 implies  $l\hat{u}_{1,\infty} = d\hat{u}_{2,\infty}$ , but together with  $\hat{u}_{1,\infty}\hat{u}_{2,\infty} = 0$  and the fact that the pair  $(u_{1,\infty}, u_{2,\infty})$  is non-zero, this is a contradiction.  $\square$

**Lemma 2.8.** *Provided  $k^*$  is large enough, the following lower bound holds:*

$$\inf_{k > k^*} \inf_{(u_1, u_2) \in S_k} \min \left\{ \min_{\overline{C}}(ku_1), \min_{\overline{C}}(ku_2) \right\} > 0$$

*Proof.* Let  $((u_{1,k}, u_{2,k}))_{k > k^*}$ . For any  $i \in \{1, 2\}$  and any  $k > k^*$ , let  $U_{i,k} = ku_{i,k}$ .  $(U_{1,k}, U_{2,k})$  satisfies the following system:

$$\begin{cases} -U_{1,k}'' = U_{1,k}f_1\left[\frac{U_{1,k}}{k}\right] - U_{1,k}U_{2,k} \\ -dU_{2,k}'' = U_{2,k}f_2\left[\frac{U_{2,k}}{k}\right] - \alpha U_{1,k}U_{2,k}. \end{cases}$$

Since  $U_{1,k}$  and  $U_{2,k}$  are  $k$ -uniformly bounded in  $L^\infty(C)$  in virtue of Lemma 2.7, we can prove with the same arguments as before that, for any  $i \in \{1, 2\}$  and up to extraction,  $U_{i,k}$  converges in  $\mathcal{C}_{per}^{0,\beta}(\mathbb{R})$  to some  $U_{i,\infty} \geq 0$ , and by Lemma 2.2 (third and fourth inequalities),  $U_{i,\infty} \neq 0$ . The limits satisfy the remarkable following system:

$$\begin{cases} -U_{1,\infty}'' = U_{1,\infty}f_1[0] - U_{1,\infty}U_{2,\infty} \\ -dU_{2,\infty}'' = U_{2,\infty}f_2[0] - \alpha U_{1,\infty}U_{2,\infty}. \end{cases}$$

At first this system is to be understood in the distributional sense, but once more thanks to classical elliptic regularity  $U_{1,\infty}$  and  $U_{2,\infty}$  are actually in  $\mathcal{C}_{per}^{2,\beta}(\mathbb{R})$ . Thanks to the elliptic strong minimum principle, for any  $i \in \{1, 2\}$ ,  $U_{i,\infty} \gg 0$ .

In  $C$ ,  $-\frac{U_{1,\infty}''}{U_{1,\infty}} = f_1[0] - U_{2,\infty} \leq M_1$ . Integration over  $C$  yields:

$$\int_C f_1[0] = - \int_C \left| \frac{U_{1,\infty}'}{U_{1,\infty}} \right|^2 + \int_C U_{2,\infty} \leq \int_C U_{2,\infty}.$$

Similarly,

$$\int_C f_2[0] \leq \int_C U_{1,\infty}.$$

Then  $(\mathcal{H}_2)$  shows that  $(U_{1,\infty}, U_{2,\infty})$  is at positive distance of the origin in  $L^1(C)$ , and then in  $L^\infty(C)$  by classical embeddings. Harnack's inequality yields eventually that  $\min\left(\min_{\overline{C}}(U_{1,\infty}), \min_{\overline{C}}(U_{2,\infty})\right)$  is bounded from below by a real number  $\epsilon > 0$ . By uniform convergence and provided  $k^*$  is large enough, the infimum of the sequence  $\left(\min\left\{\min_{\overline{C}}(ku_{1,k}), \min_{\overline{C}}(ku_{2,k})\right\}\right)_{k > k^*}$  is greater than, say,  $\frac{3\epsilon}{4}$ . This  $\epsilon$  depends on  $m, C$ , but neither on the limit point  $(U_{1,\infty}, U_{2,\infty})$  nor on the choice of a convergent subsequence of  $((u_1, u_2))_{k > k^*}$ , whence the bound holds for any convergent subsequence of  $((u_1, u_2))_{k > k^*}$ . Furthermore, the bound does not depend on the choice of the sequence  $((u_1, u_2))_{k > k^*}$  itself, whence it holds for any convergent subsequence of any sequence.

The conclusion on the whole set is a standard compactness argument.  $\square$



2.3.2. *Instability of periodic coexistence states close to  $(0, 0)$ .*

**Lemma 2.9.** *Provided  $k^*$  is large enough, for any  $(u_1, u_2) \in S$ , the differential operator  $\mathcal{A}_{(u_1, u_2)} : \mathcal{C}_{per}^2(\mathbb{R}) \rightarrow \mathcal{C}_{per}(\mathbb{R})$  defined as:*

$$\mathcal{A}_{(u_1, u_2)} = \begin{pmatrix} \frac{d^2}{dx^2} + g_1[u_1] - ku_2 & ku_1 \\ \alpha ku_2 & d \frac{d^2}{dx^2} + g_2[u_2] - \alpha ku_1 \end{pmatrix}$$

*is strongly positive.*

*Proof.* It is well-known that  $\mathcal{A}_{(u_1, u_2)}$  is strongly positive (i.e. satisfies the strong minimum principle) if there exists a pair of positive functions whose image by  $-\mathcal{A}_{(u_1, u_2)}$  is itself non-negative (see for instance Figueiredo–Mitidieri [13]). From  $(\mathcal{H}_1)$ , if  $k$  is large enough, there exists a constant  $R > 0$  which depends only on  $x \mapsto \partial_1 f_1(0, x)$  and  $x \mapsto \partial_1 f_2(0, x)$  such that:

$$\begin{cases} \partial_1 f_1[u_1] \in [-R, 0] \\ \partial_1 f_2[u_2] \in [-R, 0]. \end{cases}$$

From here, it is easy to check that, up to extraction and using the notations of the proof of Lemma 2.8,

$$-\mathcal{A}_{(u_1, k, u_2, k)} \begin{pmatrix} U_{1, \infty} \\ U_{2, \infty} \end{pmatrix} \rightarrow \begin{pmatrix} U_{1, \infty} U_{2, \infty} \\ \alpha U_{1, \infty} U_{2, \infty} \end{pmatrix}$$

uniformly in  $C$  as  $k \rightarrow +\infty$ .

This limit being positive, thanks to standard compactness arguments, we get indeed the claimed statement.  $\square$

**Proposition 2.10.** *For any  $k > k^*$ , any  $(u_1, u_2) \in S$  is unstable.*

*Proof.* Thanks to Mora’s theorem [25], we know that  $(u_1, u_2)$  is unstable if the principal eigenvalue of the elliptic part of the monotone problem  $(\mathcal{M})$  linearized at  $(u_1, J(u_2))$  is negative. It is easy to verify that the linearized operator is in fact:

$$\mathcal{A}_{(u_1, u_2)} = \begin{pmatrix} \frac{d^2}{dx^2} + g_1[u_1] - ku_2 & ku_1 \\ \alpha ku_2 & d \frac{d^2}{dx^2} + g_2[u_2] - \alpha ku_1 \end{pmatrix}$$

$\mathcal{A}_{(u_1, u_2)}$  being strongly positive (see Lemma 2.9), it is injective and, up to a restriction of its codomain, it is invertible. Krein–Rutman’s theorem and a well-known routine involving the compact canonical embedding  $\mathcal{C}^{2, \beta}(C) \hookrightarrow \mathcal{C}_{loc}^{0, \beta}(C)$  prove the existence of the periodic principal eigenvalue  $\lambda_{1, per}(-\mathcal{A}_{(u_1, u_2)})$ .

Now, we have to prove that  $\lambda_{1, per}(-\mathcal{A}_{(u_1, u_2)}) < 0$ . Recall the following characterization from Krein–Rutman’s theorem:

$$\lambda_{1, per}(-\mathcal{A}_{(u_1, u_2)}) = \inf \left\{ \lambda \in \mathbb{R} \mid \exists \varphi \in \mathcal{C}_{per}^2(\mathbb{R}, (0, +\infty)^2) \quad (-\mathcal{A}_{(u_1, u_2)} - \lambda) \varphi \leq 0 \text{ in } \mathbb{R} \right\}.$$

Therefore, we only need to find some  $\lambda < 0$  and some  $\varphi \in \mathcal{C}_{per}^2(\mathbb{R}, (0, +\infty)^2)$  satisfying:

$$(-\mathcal{A}_{(u_1, u_2)} - \lambda) \varphi \leq 0.$$

Using  $(\mathcal{H}_1)$ , it is easy to check that there exists a constant  $R > 0$  which depends only on  $x \mapsto \partial_1 f_1(0, x)$  and  $x \mapsto \partial_1 f_2(0, x)$  such that:

$$\begin{aligned}
 (-\mathcal{A}_{(u_1, u_2)}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} -u_1^2 \partial_1 f_1 [u_1] - k u_1 u_2 \\ -u_2^2 \partial_1 f_2 [u_2] - \alpha k u_1 u_2 \end{pmatrix} \\
 &\leq \begin{pmatrix} (R u_1 - k u_2) u_1 \\ (R u_2 - \alpha k u_1) u_2 \end{pmatrix} \\
 &\leq -\min \left\{ \min_{\bar{C}} (k u_2 - R u_1), \min_{\bar{C}} (\alpha k u_1 - R u_2) \right\} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
 \end{aligned}$$

In virtue of Lemma 2.8, provided  $k^*$  is large enough, for any  $K > k^*$  and any  $(u_{1,K}, u_{2,K}) \in S_K$ :

$$\min \left\{ \min_{\bar{C}} (K u_{2,K} - R u_{1,K}), \min_{\bar{C}} (\alpha K u_{1,K} - R u_{2,K}) \right\} > 0.$$

Consequently it holds for  $k$  and  $(u_1, u_2)$ .

Now, if we define  $\lambda$  as  $-\min \left\{ \min_{\bar{C}} (k u_2 - R u_1), \min_{\bar{C}} (\alpha k u_1 - R u_2) \right\}$  and  $\varphi$  as  $(u_1, u_2)$ , it is obvious that  $(-\mathcal{A}_{(u_1, u_2)} - \lambda) \varphi \leq 0$ . Therefore,  $(u_1, u_2)$  is unstable.  $\square$

**2.4. Counter-propagation.** In this subsection, we prove the so-called counter-propagation hypothesis. Let us recall from Fang–Zhao [17] that, since every intermediate periodic stationary state is unstable (Proposition 2.10), their set is totally unordered.

**Proposition 2.11.** *Let  $k > k^*$  and  $(u_1, u_2) \in S$ .*

*Let  $c_+^*((u_1, \tilde{u}_2 - u_2), (\tilde{u}_1, \tilde{u}_2)) \in \mathbb{R}$  and  $c_-^*((u_1, \tilde{u}_2 - u_2), (0, 0)) \in \mathbb{R}$  be the spreading speeds associated with front-like initial data connecting respectively  $(\tilde{u}_1, \tilde{u}_2)$  to  $(u_1, \tilde{u}_2 - u_2)$  and  $(u_1, \tilde{u}_2 - u_2)$  to  $(0, 0)$ .*

*Then:*

$$c_+^*((u_1, \tilde{u}_2 - u_2), (\tilde{u}_1, \tilde{u}_2)) + c_-^*((u_1, \tilde{u}_2 - u_2), (0, 0)) > 0.$$

*Remark.* At least formally, since  $(u_1, u_2)$  vanishes as  $k \rightarrow +\infty$ , we have:

$$\begin{aligned}
 c_+^*((u_1, \tilde{u}_2 - u_2), (\tilde{u}_1, \tilde{u}_2)) &\rightarrow c_+^*((0, \tilde{u}_2), (\tilde{u}_1, \tilde{u}_2)), \\
 c_-^*((u_1, \tilde{u}_2 - u_2), (0, 0)) &\rightarrow c_-^*((0, \tilde{u}_2), (0, 0)).
 \end{aligned}$$

It is easily seen that the first limit is in fact the spreading speed of the scalar KPP pulsating front connecting  $\tilde{u}_1$  to 0 for the equation  $\partial_t u_1 - \partial_{xx} u_1 = u_1 f_1 [u_1]$  whereas the second one is in fact the spreading speed of the scalar KPP pulsating front connecting  $\tilde{u}_2$  to 0 for the equation  $\partial_t u_2 - d \partial_{xx} u_2 = u_2 f_2 [u_2]$ . These limiting speeds are both positive. Hence, heuristically, we expect that both  $c_+^*((u_1, \tilde{u}_2 - u_2), (\tilde{u}_1, \tilde{u}_2))$  and  $c_-^*((u_1, \tilde{u}_2 - u_2), (0, 0))$  are positive whenever  $k$  is large enough, and this is indeed what we will prove.

*Proof.* Let  $k > k^*$ ,  $(u_1, u_2) \in S$ ,  $\mathcal{A}_{(u_1, u_2)}$  be the associated linear elliptic operator defined as in Lemma 2.9,  $t > 0$ ,  $Q_t$  be the semiflow associated with  $(\mathcal{M})$  and  $Q_t^{u, lin}$  be the linear semiflow associated with  $\partial_t - \mathcal{A}_{(u_1, u_2)}$ . We intend to use Weinberger's theory [29, Theorem 2.4] in order to establish that:

$$c_+^*((u_1, \tilde{u}_2 - u_2), (\tilde{u}_1, \tilde{u}_2)) \geq \inf_{\mu > 0} \frac{-\lambda_{1, per}(-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1, u_2)})}{\mu}.$$

(The exponential relation between the periodic principal eigenvalue of the elliptic operator  $\mathcal{A}_{(u_1, u_2)}$  and that of the semiflow  $Q_t^{u, lin}$  is classical and not detailed here.)

On one hand, to apply [29, Theorem 2.4], we have to find  $\delta \in (0, 1)$  and  $\eta_+ > 0$  such that, for all  $(v_1, v_2) \in [(0, 0), (\eta_+, \eta_+)]$ :

$$Q_t [(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)] - (u_1, \tilde{u}_2 - u_2) \geq (1 - \delta) Q_t^{u, lin} [(v_1, v_2)],$$

that is such that:

$$\delta Q_t^{u, lin} [(v_1, v_2)] \geq Q_t^{u, lin} [(v_1, v_2)] + (u_1, \tilde{u}_2 - u_2) - Q_t [(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)].$$

On the other hand, by definition of  $Q_t^{u, lin}$ , for all  $\varepsilon > 0$ , we have the existence of  $\eta_\varepsilon > 0$  such that, if  $(v_1, v_2) \in [(0, 0), (\eta_\varepsilon, \eta_\varepsilon)]$ :

$$\left| Q_t^{u, lin} [(v_1, v_2)] + (u_1, \tilde{u}_2 - u_2) - Q_t [(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)] \right| \leq \varepsilon \max \left( \frac{\max v_1}{C}, \frac{\max v_2}{C} \right).$$

Hence it would be sufficient to show, for all  $(v_1, v_2) \in [(0, 0), (\eta_\varepsilon, \eta_\varepsilon)]$ , the following inequality:

$$\varepsilon \max \left( \frac{\max v_1}{C}, \frac{\max v_2}{C} \right) \leq \delta \min \left( \min Q_t^{u, lin} [(v_1, v_2)]_1, \min Q_t^{u, lin} [(v_1, v_2)]_2 \right),$$

which is a straightforward consequence of the positivity of  $\mathcal{A}_{(u_1, u_2)}$  and of the instability of  $(u_1, u_2)$  (fixing for instance  $\delta = \frac{1}{2}$  and then choosing  $\varepsilon$  small enough). Finally we define  $\eta_+ = \eta_\varepsilon$ .

Applying the same sketch of proof and being careful with the signs, we prove the existence of  $\eta_- > 0$  such that, for all  $(v_1, v_2) \in [(0, 0), (\eta_-, \eta_-)]$ :

$$-Q_t [-(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)] + (u_1, \tilde{u}_2 - u_2) \geq \frac{1}{2} Q_t^{u, lin} [(v_1, v_2)],$$

whence a second inequality is established:

$$c_-^* ((u_1, \tilde{u}_2 - u_2), (0, 0)) \geq \inf_{\mu > 0} \frac{-\lambda_{1, per} (-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1, u_2)})}{\mu}.$$

It is worthy to point out that both spreading speeds are estimated from below by the same quantity.

To conclude, we just have to notice the following inequality, true for all  $\mu > 0$ :

$$\lambda_{1, per} (-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1, u_2)}) \leq -\mu^2 \min(1, d) + \lambda_{1, per} (-\mathcal{A}_{(u_1, u_2)}) < 0.$$

In particular, from:

$$\frac{-\lambda_{1, per} (-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1, u_2)})}{\mu} \geq \inf_{\mu > 0} \left( \mu \min(1, d) - \frac{\lambda_{1, per} (-\mathcal{A}_{(u_1, u_2)})}{\mu} \right),$$

we deduce the following estimate:

$$\inf_{\mu > 0} \frac{-\lambda_{1, per} (-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1, u_2)})}{\mu} \geq 2\sqrt{\min(1, d) |\lambda_{1, per} (-\mathcal{A}_{(u_1, u_2)})|} > 0.$$

□

**2.5. Existence of pulsating fronts connecting both extinction states.** We are now able to state rigorously the existence of pulsating fronts thanks to Fang–Zhao [17].

**Theorem 2.12.** *For any  $k > k^*$ , there exists  $c \in \mathbb{R}$  and  $(\varphi_1, \varphi_2) \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^2)$  such that the following properties hold.*

- (1)  $\varphi_1$  and  $\varphi_2$  are respectively non-increasing and non-decreasing with respect to their first variable, generically noted  $\xi$ .
- (2)  $\varphi_1$  and  $\varphi_2$  are periodic with respect to their second variable, generically noted  $x$ .
- (3) As  $\xi \rightarrow +\infty$ ,

$$\max_{x \in [0, L]} |(\varphi_1, \varphi_2)(-\xi, x) - (\tilde{u}_1, 0)(x)| + \max_{x \in [0, L]} |(\varphi_1, \varphi_2)(\xi, x) - (0, \tilde{u}_2)(x)| \rightarrow 0.$$

- (4)  $(u_1, u_2) : (t, x) \mapsto (\varphi_1, \varphi_2)(x - ct, x)$  is a classical solution of  $(\mathcal{P})$ .

*Remark.* For any  $\xi_0 \in \mathbb{R}$ ,  $(\xi, x) \mapsto (\varphi_1, \varphi_2)(\xi + \xi_0, x)$  is a pulsating front solution of  $(\mathcal{P})$  as well.

Regarding the regularity of  $(\varphi_1, \varphi_2)$ , we recall that, even if Fang–Zhao [17] (as well as Weinberger [29]) worked in the framework of continuous functions, by classical parabolic regularity, a continuous solution of  $(\mathcal{P})$  is in  $\mathcal{C}_{loc}^1(\mathbb{R}, \mathcal{C}_{loc}^2(\mathbb{R}, \mathbb{R}^2))$ . Hence  $(\varphi_1, \varphi_2)$  is a fortiori in  $\mathcal{C}_{loc}^1(\mathbb{R}^2, \mathbb{R}^2)$ . This can be improved provided  $f_1$  and  $f_2$  are  $\mathcal{C}^1$  with respect to  $x$ . Indeed, differentiating  $(\mathcal{P})$  with respect to  $t$  and  $x$  shows similarly that  $\partial_t(u_1, u_2) \in \mathcal{C}_{loc}^1(\mathbb{R}, \mathcal{C}_{loc}^2(\mathbb{R}, \mathbb{R}^2))$  and  $\partial_x(u_1, u_2) \in \mathcal{C}_{loc}^1(\mathbb{R}, \mathcal{C}_{loc}^2(\mathbb{R}, \mathbb{R}^2))$ . In such a case,  $(\varphi_1, \varphi_2)$  is at least in  $\mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ .

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