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► **To cite this version:**

Jordan Emme. Spectral measure at zero for self-similar tilings. *Moscow Mathematical Journal*, 2017, 17 (1), pp.35 - 49. hal-01328019v1

HAL Id: hal-01328019

<https://hal.science/hal-01328019v1>

Submitted on 7 Jun 2016 (v1), last revised 23 Oct 2017 (v2)

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Spectral measure at zero for self-similar tilings

Jordan Emme *

June 7, 2016

Abstract

The goal of this paper is to study the action of the group of translations over self-similar tilings in the euclidian space \mathbb{R}^d . It investigates the behaviour near zero for spectral measures for such dynamical systems. Namely the paper gives a Hölder asymptotic expansion near zero for these spectral measures. It is a generalization to higher dimension of a result by Bufetov and Solomyak who studied self similar-suspension flows for substitutions in [4]. The study of such asymptotics mostly involves the understanding of the deviations of some ergodic averages.

1 Introduction

Spectral analysis plays an important role in dynamical systems and ergodic theory. Some properties of the spectrum of the Koopman operator or of a group of operators acting on a space translate into dynamical properties (in our case the group \mathbb{R}^d acts on the space of tilings and this defines a group of operators acting on the functions of the tiling space). In such a light, spectral measure hold some amount of information concerning dynamical systems. Spectral analysis of some dynamical systems via the spectral measures were done in [7] for instance.

We are interested in this paper in a particular family of dynamical systems which is thoroughly studied: those who arise from substitutions or self similar tilings. Substitutions are essentially combinatorial objects and have no geometry a priori. However it is possible to interpret infinite words as tilings of the real line (or real semiline depending in which context we are working) and thus have some (limited) geometry. Self-similar tilings are objects that arise when looking for a naturel generalization of this in higher dimension.

Bufetov and Solomyak have studied the deviation of ergodic averages for the action of \mathbb{R}^d by translation over self-similar tilings in [3]. Such results generalize famous works by Bufetov for $d = 1$ in [1] or [2].

The precise study of the deviation of ergodic averages allows the understanding of the modulus of continuity for suspension flows over substitution dynamical systems in [4]. One of the result is that there is a Hölder asymptotic expansion on balls centered at zero for the spectral measure of self-similar suspension flows (which we could see as self similar tilings of the real line). The Hölder exponent is explicitly computed.

The aim of this paper is to give a generalization of this result for spectral measures of self-similar tilings for $d \geq 1$. We prove that there is always a Hölder asymptotic expansion regardless of the choice of d and give an explicit computation of the Hölder exponent.

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Section 2 is dedicated to defining self-similar tilings and their associated dynamical systems along with some basic properties.

Section 3 exposes some of the results that were obtained in [3, 4] which are crucial to the proof of our main theorem. We also define the principal object of the paper in this section: spectral measures.

In Section 4 we state the main result and give its proof based on the study of the asymptotics of the ergodic integrals and the finitely-additive measures from [3].

1.1 Result

In this article we study a particular class of tilings of the euclidian space \mathbb{R}^d obtained with the following process. We fix a finite amount of tiles called prototiles. We endow the set of tilings of \mathbb{R}^d by the prototiles with a topology detailed in Section 2. We choose a real constant λ and consider the images of these tiles by the homothety of factor λ . If we can tile these images with translates of the prototiles, then the way to do it defines a substitution rule ζ . If this rule ζ satisfies to certain conditions, then iterating it over a prototile spans bigger and bigger patches (i.e. union of tiles) which eventually tiles the whole space. Taking the closure of the orbit of this tiling via the action of \mathbb{R}^d by translation defines a space \mathfrak{X}_ζ . Some other conditions on ζ ensure that the dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d)$ is uniquely ergodic and we denote its unique ergodic probability measure μ . Then for any function real valued function $f \in \mathfrak{X}_\zeta$ to \mathbb{R} in $L^1(\mathfrak{X}_\zeta, \mu)$, one can associate a spectral measure σ_f . The following theorem states that this measure is a Radon measure on balls centered at zero after renormalization. This can be seen as a Hölder regularity property. The definitions of the technical conditions of this theorem such as cylindrical functions or the quantity $m_{\Phi_2^-}(f)$ are given respectively in Definitions 3.4 and 3.5.

Theorem 1.1. *Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a set of prototiles in \mathbb{R}^d and ζ a primitive non periodic tile-substitution over those prototiles defining a self-similar tiling with finite local complexity. Let λ be the real expansion constant of ζ and S be its incidence matrix with eigenvalues*

$$\theta_1 > \theta_2 > |\theta_3| \geq \dots \geq |\theta_m|$$

satisfying $\theta_2 > \theta_1^{\frac{d-1}{d}}$. Let f be a cylindrical function with zero mean $\int f d\mu = 0$ and $m_{\Phi_2^-}(f) \neq 0$. Let σ_f be its associated spectral measure on \mathbb{R}^d . Then there exists a non-trivial positive σ -finite Radon measure η on \mathbb{R}_+ such that:

$$\lim_{N \rightarrow \infty} \frac{\sigma_f(B(0, a\lambda^{-N}))}{\lambda^{-N(2d-2\alpha)}} = \eta([0, a]), \quad \text{for all } a > 0 \text{ such that } \eta(\{a\}) = 0,$$

where

$$\alpha = \frac{d \log(\theta_2)}{\log(\theta_1)} \in (d-1, d).$$

This theorem is a natural generalization of [4, Theorem 6.1] which gives the same behaviour for $d=1$.

Remark that in this theorem, we have:

$$\theta_1 = \lambda^d.$$

1.2 Outline of the paper

Section 2 is devoted to defining the framework. Namely, what a self similar tiling is. We recall some famous results. Namely that a primitive tile-substitution ζ with finite local complexity defines a uniquely ergodic dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d)$ where \mathfrak{X}_ζ is the set of tilings of the euclidian space \mathbb{R}^d obtained with the substitution rules. We also remind how to subdivide a certain tiling using the implicit substitution rules of ζ .

With these subdivisions in mind, we recall in Section 3 the construction of finitely additive measures. For a thorough and detailed construction of such objects, we refer the reader to [1]. These are capital for the understanding of ergodic deviations. Namely, we state the main theorem of [3] which gives a precise behaviour of Birkhoff sums depending on a decomposition of \mathbb{R}^d in stable subspaces for the transpose of the incidence matrix of the tile-substitution ζ .

Section 4 is devoted to proving Theorem 1.1. We start by stating the spectral theorem which allows to define the spectral measures. The proof of the main theorem is then divided in 5 steps. In the first step we integrate a test function (which approximates the characteristic function of an interval) with respect to the spectral measure. The spectral isomorphism allows to express our integral in terms of ergodic sums on balls of radius R . The second step uses Theorem 3.6 on ergodic deviations in order to estimate the rest of our ergodic sum as R goes to infinity. Step 3 is just a change of variables in the main term, using the “self-similarity” properties of the finitely additive measures. We can then compute the limit as R goes to infinity. Step 4 is devoted to the regularity of this limit and how to extend it so that it becomes a bilinear form, right continuous on continuous compactly supported functions. Step 5 introduces a technical lemma which allows to make the link between our limit and distribution and then with Radon measures.

Acknowledgements

I would like to thank ALEXANDER BUFETOV and BORIS SOLOMYAK for asking me the question this paper is about and for their very helpful advice and the enlightening discussions we had about this problem.

2 Self-similar tilings

In all that follows, we are working in the euclidian space \mathbb{R}^d . We are interested in tiling the euclidian space \mathbb{R}^d (which comes with its origin), with tiles which are compact, and the interior of their closure. Let us give a more formal definition:

Definition 2.1. Fix a set of types $\{1, \dots, m\} \subset \mathbb{N}$. A tile T in \mathbb{R}^d is a pair (A, i) where $A = \text{supp}(T)$ (the support of T) is a closed compact subset of \mathbb{R}^d such that $A = \overline{A}$ and $i \in \{1, \dots, m\}$ is the type of the tile denoted by $l(T)$.

Fix an alphabet $\mathcal{A} = \{T_1, \dots, T_m\}$ where $T_i = (A_i, i)$. The tiles T_i are called prototiles.

Definition 2.2. A patch P is a finite union of tiles of disjoint interiors. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches whose tiles are translates of prototiles in \mathcal{A} .

Definition 2.3. Define the translate of a tile T by a vector $x \in \mathbb{R}^d$:

$$T + x := (\text{supp}(T) + x, l(T)).$$

Define the translate of a patch P in the same manner.

After having fixed a finite set of prototiles we are interested in tilings of the euclidian space only with translates of these prototiles. Namely, a tiling \mathcal{T} of \mathbb{R}^d by tiles in $\mathcal{A} = \{T_1, \dots, T_m\}$ is given by a covering of \mathbb{R}^d in the following way:

$$\mathbb{R}^d = \bigcup_{v \in V} \text{supp}(T_{i(v)}) + v$$

and $(\text{supp}(T_{i(v)}) + v) \cap (\text{supp}(T_{i(w)}) + w) = \emptyset$ whenever $v \neq w$.

Here V denotes a countable set of vectors in \mathbb{R}^d and i is an application from V to $\{1, \dots, m\}$.

In other words, a tiling of the space with tiles in \mathcal{A} is a covering of this space with translates of the prototiles in \mathcal{A} .

Let us now define what a tile substitution is.

Definition 2.4. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an expansive linear map. A map $\zeta : \mathcal{A} \rightarrow \mathcal{P}_{\mathcal{A}}$, where $\mathcal{P}_{\mathcal{A}}$ denotes the set of patches with tiles in \mathcal{A} , is called a tile-substitution with expansion ϕ if $\text{supp}(\zeta(T_i)) = \phi(\text{supp}(T_i))$.

Definition 2.5. For any tile-substitution ζ , define a substitution matrix S such that $S_{i,j}$ is the number of tiles T_i appearing in the patch $\zeta(T_j)$.

We say that ζ is primitive if the matrix S is primitive.

Definition 2.6. A tiling \mathcal{T} of \mathbb{R}^d is self-affine if $\zeta(\mathcal{T}) = \mathcal{T}$ for a primitive tile-substitution ζ . If the expansion map ϕ is an homothety (i.e $\exists \lambda > 1, \phi(x) = \lambda x$), then \mathcal{T} is self-similar. The number λ is called the real expansion constant.

We extend the definition of ζ to $\mathcal{P}_{\mathcal{A}}$ and to the set of tilings of \mathbb{R}^d by tiles in \mathcal{A} (as it is usually done for substitutions).

In order to extend the definition to the set of patches $\mathcal{P}_{\mathcal{A}}$, we first define the application ζ with real expansion constant λ on the set of tiles which are translates of prototiles in \mathcal{A} . Let T be a translate of prototile T_i for a certain i by a vector $x \in \mathbb{R}^d$, i.e. $T = T_i + x$. Then we define $\zeta(T) = \zeta(T_i) + \lambda x$. Now in order to define ζ on the set of patches we do the following:


Let P be a patch in $\mathcal{P}_{\mathcal{A}}$. There exists a finite set of vectors $\{v_1, \dots, v_n\}$ in \mathbb{R}^d and a finite set of integers $\{i_1, \dots, i_n\}$ in $\llbracket 1, m \rrbracket$ such that:


$$P = \bigcup_{j=1}^m T_{i_j} + v_j.$$

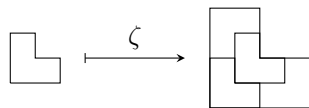
Then, define the patch $\zeta(P)$ in the following manner:

$$\zeta(P) = \bigcup_{j=1}^m \zeta(T_{i_j}) + \phi(v_j).$$

Remark 2.7. If θ_1 denotes the Perron-Frobenius eigenvalue of S , we have $\theta_1 = \lambda^d$ [8].

Example 2.8. We give an example of such a tiling. The chair tiling has a set of prototiles given by  and its rotations by $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$.

We give the tile substitution rule only for the tile , the other mappings being the same up to a rotation.



The chair tiling has expansion constant $\lambda = 2$, hence $\phi : z \mapsto 2z$, and substitution matrix

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Remark that the Perron Frobenius eigenvalue is indeed $\theta = 4 = \lambda^2$. A famous class of such tilings is found in [6].

Definition 2.9. Given a tile-substitution ζ , we define the tiling space \mathfrak{X}_ζ :

A tiling \mathcal{T} of \mathbb{R}^d is in \mathfrak{X}_ζ if every finite patch of \mathcal{T} is a subpatch of $\zeta^k(\mathcal{T}_i)$ for some $k \in \mathbb{N}$ and some $i \in \llbracket 1, m \rrbracket$.

We endow this set with a topology defined by the following metric:

Two tilings in \mathfrak{X}_ζ are ϵ close (with $\epsilon > 0$) if they agree on a ball of radius $1/\epsilon$ around the origin, up to a translation by ϵ or less. For more details the reader can refer to [8].

Definition 2.10. Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be an alphabet. Let ζ be a tile substitution with expansion map ϕ . A tile T is said to be a tile of order $k \in \mathbb{Z}$ if there exist $i \in \llbracket 1, m \rrbracket$ and $y \in \mathbb{R}^d$ such that $\text{supp}(T) - y = \phi^k(\text{supp}(T_i))$

Definition 2.11. We extend the definition of the tile-substitution ζ to tiles of order $k \in \mathbb{Z}$ and note it $\phi^k \zeta$:

$$\begin{aligned} \phi^k \mathcal{A} &:= \{(\phi^k \text{supp}(T_i), i)\}_{i \in \llbracket 1, m \rrbracket}, \\ \forall i \in \llbracket 1, m \rrbracket, (\phi^k \zeta)((\phi^k(\text{supp}(T_i)), i)) &:= \phi^k(\text{supp}(\zeta(T_i))). \end{aligned}$$

This allows to define the subdivision map:

$$\Upsilon_k : \mathfrak{X}_{\phi^k \zeta} \mapsto \mathfrak{X}_{\phi^{k-1} \zeta}.$$

This divides every tile of order k into subtiles according to the substitution rule. Note that is well defined if and only if the substitution is non periodic from the following:

Theorem 2.12 ([9]). *The map $\zeta : \mathfrak{X}_\zeta \rightarrow \mathfrak{X}_\zeta$ is injective if and only if ζ is a non periodic substitution*

With this division, we define inductively, for any tiling $\mathcal{T} \in \mathfrak{X}_\zeta$ and for any integer k , the subdivided tiling $\mathcal{T}^{(k)}$ in the following way:

$$\begin{aligned} \mathcal{T}^{(0)} &:= \mathcal{T}, \\ \mathcal{T}^{(k)} &= \begin{cases} \Upsilon_k^{-1} \dots \Upsilon_1^{-1}(\mathcal{T}), & \text{if } k > 0, \\ \Upsilon_{k+1} \dots \Upsilon_0(\mathcal{T}), & \text{if } k < 0, \end{cases} \quad \mathcal{T}^{(k)} \in \mathfrak{X}_{\phi^k \zeta}. \end{aligned}$$

The space of tilings $i \in \llbracket 1, m \rrbracket$ by tiles of order k is essential for the introduction of finitely additive measures in the next section.

Let ζ be a primitive tile substitution in \mathbb{R}^d over the finite set of tiles $\mathcal{A} = \{T_1, \dots, T_m\}$ and S its incidence matrix. We denote by $\theta_1 > |\theta_2| \geq \dots \geq |\theta_m|$ the eigenvalues of the matrix S^t

(θ_1 being real, greater than one and of multiplicity one by Perron-Frobenius Theorem). Remark that \mathbb{R}^d acts on \mathfrak{X}_ζ by translation. For any vector $x \in \mathbb{R}^d$ let h_x be the application from \mathfrak{X}_ζ that sends a tiling \mathcal{T} to its translate by vector $-x$ i.e. $\mathcal{T} - x$. This defines a continuous action of \mathbb{R}^d on \mathfrak{X}_ζ by translation.

This defines a dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d)$. Remark also that iterating ζ over a certain tile can converge towards a tiling of the space \mathbb{R}^d . This tiling is of course in the tiling space \mathfrak{X}_ζ .

Let us now remind that in the special case of dimension one, that is to say substitutions, we generally limit our study to the case where a substitution is primitive. This is also the case for tile substitutions, we study in this paper only the primitive case. But we also add the property of finite local complexity:

Definition 2.13. A tiling \mathcal{T} has finite local complexity (FLC) if for any positive real number r , there is only a finite number of different patches of diameter r in \mathcal{T} up to translations.

A tile substitution ζ is FLC if every tiling in the tiling space \mathfrak{X}_ζ has finite local complexity.

Theorem 2.14. [8] *If ζ is a primitive substitution with finite local complexity then $(\mathfrak{X}_\zeta, \mathbb{R}^d)$ is uniquely ergodic. We note μ the unique ergodic probability measure of the dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d)$.*

This theorem has for immediate consequence the following lemma:

Lemma 2.15. [8] *If ζ is a primitive substitution with finite local complexity, then μ is invariant under the substitution action i.e. $\zeta_*\mu := \mu \circ \zeta^{-1} = \mu$.*

Proof. It is an easy check that the measure $\zeta_*\mu$ is invariant under the action of \mathbb{R}^d since ζ acts on vectors as an homothety of ratio λ and μ is invariant under the action of \mathbb{R}^d by translation. Hence, by unique ergodicity, $\zeta_*\mu = \mu$. \square

3 Finitely-additive measures and Ergodic deviation

We now define finitely additive measures as done in [3].

Definition 3.1. The rapidly expanding subspace E^{++} is the linear span of Jordan cells of S^t whose eigenvalues θ satisfy:

$$|\theta| > \theta_1^{\frac{d-1}{d}} = \lambda^{d-1}.$$

In a similar fashion, denote by E^+ the linear span of Jordan cells of S^t whose eigenvalues are greater than 1.

Equivalently, denote \tilde{E}^{++} the rapidly expanding subspace of S .

We define, for any vector $v \in E^+$ and any tiling $\mathcal{T} \in \mathfrak{X}_\zeta$ on tiles of order $k \in \mathbb{Z}$, a finitely-additive $\Phi_{v, \mathcal{T}}^+$ measure as follows:

$$\Phi_{v, \mathcal{T}}^+(\text{supp}(T)) = ((S^t)^k v)_j, \text{ if } \exists k \in \mathbb{Z}, y \in \mathbb{R}^d T = \phi^k(T_j) - y \in \mathcal{T}^{(k)}$$

The definition of such finitely additive measures can be extended to Lipschitz domains. This is done in [3].

The following lemma gives a “self-similarity” property for finitely additive measures after their extension to Lipschitz domains.

Lemma 3.2. [3, Lemma 3.2] *For any $v \in E^{++}$, there exists finitely-additive measures $\Phi_{v, \mathcal{T}}^+$ defined on the ring of sets generated by Lipschitz domain in \mathbb{R}^d . Moreover, if $S^t v = \theta v$, they satisfy, for any Lipschitz domain Ω , the following:*

$$\Phi_{v, \zeta(\mathcal{T})}^+(\phi(\Omega)) = \Phi_{\theta v, \mathcal{T}}^+(\Omega) = \theta \Phi_{v, \mathcal{T}}^+(\Omega).$$

Lemma 3.3. [3, Lemma 3.3] Suppose that $v \in E^{++}$ belongs to the S^t invariant subspace corresponding to a Jordan block of size $s \geq 1$ with eigenvalue θ and that $\|v\| = 1$. Then for a Lipschitz domain Ω , there exists a positive constant C_1 such that for any positive real number R and $\Omega_R = R\Omega$:

$$\left| \Phi_{v, \mathcal{T}}^+(\Omega_R) \right| \leq C_1 (\log R)^{s-1} R^\alpha, \quad \text{where } \alpha = \frac{d \log |\theta|}{\log \theta_1}.$$

Before stating the main theorem on the estimates of deviation in ergodic sums, let us remark that it only gives estimates for cylindrical functions. Let us define cylindrical functions:

Definition 3.4. A function $f : \mathfrak{X}_\zeta \rightarrow \mathbb{R}$ in $L^1(\mathfrak{X}_\zeta, \mu)$ is cylindrical if it depends only on the tile containing the origin. More precisely, there is a family of functions $(\Psi_i)_{i \in \llbracket 1, m \rrbracket}$ where $\Psi_i : \text{supp}(T_i) \rightarrow \mathbb{R}$ and $\Psi_i \in L^1(\text{supp}(T_i), \text{Leb}_d)$, Leb_d being the Lebesgue measure on \mathbb{R}^d , such that:

$$f(\mathcal{T}) = \Psi_i(x) \text{ if } 0 \in \text{supp}(T_i) - x, \quad T_i - x \in \mathcal{T}.$$

Now, define for any cylindrical function f :

Definition 3.5. For any cylindrical function f , for any vector $v = (v_1, \dots, v_m) \in \mathbb{C}^m$,

$$m_{\Phi_v^-}(f) = \sum_{i=1}^m v_i \int_{\text{supp}(T_i)} \Psi_i(x) dx.$$

Now denote by l the dimension of E^{++} (and \tilde{E}^{++}) and choose bases $(u_i)_{i \leq l}$ and $(\tilde{u}_i)_{i \leq l}$ of E^{++} and \tilde{E}^{++} . Let us denote $\Phi_{u_i, \mathcal{T}}^+ = \Phi_{u_i, \mathcal{T}}^+$ and $m_{\Phi_{\tilde{u}_i}^-} = m_{\Phi_{\tilde{u}_i}^-}$.

We can state the theorem describing the deviation of ergodic averages for the dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d, \mu)$ on Lipschitz domains:

Theorem 3.6. [3, Theorem 4.3] Let ζ be a primitive substitution with finite local complexity such that the dynamical system $(\mathfrak{X}_\zeta, \mathbb{R}^d)$ is non periodic and let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Then there exists a positive constant C depending on ζ and Ω such that, for any cylindrical function f , any tiling $\mathcal{T} \in \mathfrak{X}_\zeta$ and any $R > 0$,

$$\left| \int_{\Omega_R} f \circ h_x(\mathcal{T}) dx - \text{Leb}_d(\Omega_R) \int_{\mathfrak{X}_\zeta} f d\mu - \sum_{n=2}^l \Phi_{u_n, \mathcal{T}}^+(\Omega_R) \cdot m_{\Phi_{\tilde{u}_n}^-}(f) \right| \leq CR^{d-1} (\log R)^s \int_{\mathfrak{X}_\zeta} |f| d\mu$$

where s is the maximal size of the Jordan block corresponding to eigenvalues satisfying $|\theta_1| = \theta_1^{\frac{d-1}{d}}$. If there are no such eigenvalues, then $s = 0$.

4 Spectral measures at zero

Recall that for a measure m on a space \mathbb{R}^d , the Fourier transform \hat{m} of m is a function from \mathbb{R}^d onto itself defined by:

$$\forall x \in \mathbb{R}^d, \hat{m}(x) = \int_{\omega \in \mathbb{R}^d} e^{2i\pi \langle \omega, x \rangle} dm(\omega).$$

We start by stating the spectral theorem:

Theorem 4.1. Let $f \in L^2(\mathfrak{X}_\zeta, \mu)$. There exists a positive measure σ_f on \mathbb{R}^d called the spectral measure defined by $\widehat{\sigma}_f(x) := \langle f \circ h_x, f \rangle$ for all $x \in \mathbb{R}^d$ such that the function mapping the function on \mathfrak{X}_ζ , $(\mathcal{T} \mapsto f \circ h_x(\mathcal{T}))$ to the function $(\omega \mapsto e^{2i\pi \langle \omega, x \rangle})$ extends to an isometry J between a closed subspace of $L^2(\mathfrak{X}_\zeta, \mu)$ and $L^2(\mathbb{R}^d, \sigma_f)$.

We refer the reader to [5] for details on this theorem, but in particular, it states that:

$$\forall x \in \mathbb{R}^d, \int_{\omega \in \mathbb{R}^d} e^{2i\pi\langle \omega, x \rangle} d\sigma_f(\omega) = \langle f \circ h_x, f \rangle,$$

where

$$\langle f \circ h_x, f \rangle = \int_{\mathcal{T} \in \mathfrak{X}_\zeta} f(\mathcal{T}) f \circ h_x(\mathcal{T}) d\mu(\mathcal{T}).$$

This theorem introduces the spectral measures on which Theorem 1.1 gives results. In order to prove Theorem 1.1 we also need the set:

$$\mathcal{S}_F(\mathbb{R}_+) = \{f \in C^\infty(\mathbb{R}_+, \mathbb{R}) \mid \forall \alpha, \beta \in \mathbb{N}, \sup_{x \in \mathbb{R}_+} |x^\alpha f^{(\beta)}(x)| < \infty \text{ and } \forall k > 0, f^{(k)}(0) = 0\}$$

and the continuous function $\text{Rad} : \mathcal{S}_F(\mathbb{R}_+) \rightarrow \mathcal{S}(\mathbb{R}^d)$ (where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions on \mathbb{R}^d) defined by:

$$\forall \psi \in \mathcal{S}_F(\mathbb{R}_+), \forall x \in \mathbb{R}^d, \text{Rad}(\psi)(x) = \psi(\|x\|).$$

4.1 Proof of Theorem 1.1

4.1.1 Step 1: using the spectral isomorphism

Recall that Theorem 4.1 states that:

$$\forall x \in \mathbb{R}^d, \int_{\omega \in \mathbb{R}^d} e^{2i\pi\langle \omega, x \rangle} d\sigma_f(\omega) = \langle f \circ h_x, f \rangle.$$

Moreover, the spectral isomorphism J satisfies $J^{-1}(\omega \mapsto e^{2i\pi\langle \omega, x \rangle}) = \mathcal{T} \mapsto f \circ h_x(\mathcal{T})$ for all x in \mathbb{R}^d .

Let us analyse the behaviour of σ_f near zero. In order to do this, we fix a function $\psi \in \mathcal{S}_F(\mathbb{R}_+)$ and study the integral $\int |\text{Rad}(\psi)(R\omega)|^2 d\sigma_f(\omega)$ for all $R > 0$.

Applying the Inverse Fourier Transform, which preserves $\mathcal{S}(\mathbb{R}^d)$ and the radial property of $\text{Rad}(\psi)$, we obtain:

$$\text{Rad}(\psi)(R\omega) = R^{-d} \int_{x \in \mathbb{R}^d} e^{2i\pi\langle \omega, x \rangle} \widehat{\text{Rad}(\psi)}(x/R) dx.$$

By the spectral isomorphism, this function of ω maps, by j^{-1} , to:

$$R^{-d} \int_{x \in \mathbb{R}^d} f \circ h_x(\mathcal{T}) \cdot \widehat{\text{Rad}(\psi)}(x/R) dx$$

which is a function of \mathcal{T} in $L^2(\mathfrak{X}_\zeta, \mu)$. Since the spectral isomorphism preserves the L^2 norm, we obtain:

$$\int_{\omega \in \mathbb{R}^d} |\text{Rad}(\psi)(R\omega)|^2 d\sigma_f(\omega) = R^{-2d} \int_{\mathcal{T} \in \mathfrak{X}_\zeta} \left| \int_{x \in \mathbb{R}^d} f \circ h_x(\mathcal{T}) \cdot \widehat{\text{Rad}(\psi)}(x/R) dx \right|^2 d\mu(\mathcal{T}).$$

Now let us change coordinates in the right-hand side member and use spherical coordinates. Let u be the mapping from spherical coordinates to euclidian coordinates on \mathbb{R}^d . Since Fourier

transform preserves radial functions, $\widehat{\text{Rad}}(\psi)(\rho, 0, \dots, 0) = \widehat{\text{Rad}}(\psi)(u(\rho, \phi_1, \dots, \phi_{d-1}))$. For simplicity, in all that follows, let us write $\widehat{\text{Rad}}(\psi)(\rho)$ instead of $\widehat{\text{Rad}}(\psi)(\rho, 0, \dots, 0)$. Then the integral

$$\int_{x \in \mathbb{R}^d} f \circ h_x(\mathcal{T}) \cdot \widehat{\text{Rad}}(\psi)(x/R) dx$$

can be written:

$$\int_{\rho=0}^{+\infty} \int_{[0, \pi]^{d-1}} \int_{\phi_{d-1}=0}^{2\pi} f \circ h_{u(\rho, \phi_1, \dots, \phi_{d-1})}(\mathcal{T}) \cdot \widehat{\text{Rad}}(\psi)(\rho/R) \cdot \rho^{d-1} \prod_{j=1}^{d-2} (\sin^{d-1-j}(\phi_j)) d\phi_{d-1} \dots d\phi_1 d\rho$$

or:

$$\int_{\rho=0}^{+\infty} \widehat{\text{Rad}}(\psi)(\rho/R) \int_{[0, \pi]^{d-1}} \int_{\phi_{d-1}=0}^{2\pi} f \circ h_{u(\rho, \phi_1, \dots, \phi_{d-1})}(\mathcal{T}) \rho^{d-1} \prod_{j=1}^{d-2} (\sin^{d-1-j}(\phi_j)) d\phi_{d-1} \dots d\phi_1 d\rho$$

Now, an integration by parts with regards to the variable ρ yields, differentiating $\widehat{\text{Rad}}(\psi)(\rho/R)$ and integrating $\int_{[0, \pi]^{d-1}} \int_{\phi_{d-1}=0}^{2\pi} f \circ h_{u(\rho, \phi_1, \dots, \phi_{d-1})}(\mathcal{T}) \rho^{d-1} \prod_{j=1}^{d-2} (\sin^{d-1-j}(\phi_j)) d\phi_{d-1} \dots d\phi_1$ yields:

$$\int_{x \in \mathbb{R}^d} f \circ h_x(\mathcal{T}) \cdot \widehat{\text{Rad}}(\psi)(x/R) dx = 2i\pi R^{-1} \int_{\rho=0}^{+\infty} S(f, \mathcal{T}, \rho) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho/R) d\rho, \quad (1)$$

where

$$S(f, \mathcal{T}, \rho) = \int_{x \in B(0, \rho)} f \circ h_x(\mathcal{T}) dx.$$

4.1.2 Step 2: estimating the rest

From Theorem 3.6 and Lemma 3.3, and since $\int f d\mu = 0$, we have:

$$S(f, \mathcal{T}, \rho) = \Phi_{2, \mathcal{T}}^+(f)(\rho) \cdot m_{\Phi_2^-}(f) + \mathcal{R}(\rho),$$

where

$$|\mathcal{R}(\rho)| \leq C_1 \max(1, \rho^{\alpha-\varepsilon})$$

with $\alpha = \frac{d \log(\theta_2)}{\log(\theta_1)}$ and some $\varepsilon > 0$ small enough.

Since $\mathcal{F}(x_1 \text{Rad}(\psi)(x))$ is in $\mathcal{S}(\mathbb{R}^d)$, we have:

$$|\mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho)| \leq C_{\psi, \alpha} \min(1, \rho^{-\alpha-1}).$$

Finally,

$$2\pi R^{-1} \left| \int_{\rho=0}^{+\infty} \mathcal{R}(\rho) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho) d\rho \right| \leq C_1 C_{\psi, \alpha} 2\pi R^{-1} \left(1 + \int_1^R \rho^{\alpha-\varepsilon} d\rho + \int_R^{+\infty} \rho^{\alpha-\varepsilon} (\rho/R)^{-\alpha-1} d\rho \right)$$

hence

$$2\pi R^{-1} \left| \int_{\rho=0}^{+\infty} \mathcal{R}(\rho) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho) d\rho \right| = O(R^{\alpha-\varepsilon}) \quad (2)$$

since $\alpha \in (d-1, d)$.

4.1.3 Step 3: change of variables in the main term

Now that the error term in $\int_{\rho=0}^{+\infty} S(f, \mathcal{T}, \rho) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho/R) d\rho$ is estimated, let us study the main term in Equation (1). To that end, we will assume that

$$R = \lambda^N = \theta_1^{\frac{N}{d}}, N \geq 1.$$

Let $\rho = \lambda^N r = Rr$, and let us use Lemma 3.2 to do the following renormalization:

$$\Phi_{2, \mathcal{T}}^+(\rho) = \Phi_{2, \mathcal{T}}^+(\lambda^N r) = \theta_2^N \Phi_{2, \zeta^{-N}(\mathcal{T})}^+(r) = R^\alpha \Phi_{2, \zeta^{-N}(\mathcal{T})}^+(r).$$

After this change of variables, we have:

$$\begin{aligned} 2i\pi R^{-1} \int_{\rho=0}^{+\infty} (\Phi_{2, \mathcal{T}}^+(\rho) \cdot m_{\Phi_2^-}(f) + \mathcal{R}(\rho)) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(\rho/R) d\rho \\ = \\ 2i\pi R^\alpha m_{\Phi_2^-}(f) \int_{r=0}^{+\infty} \Phi_{2, \zeta^{-N}(\mathcal{T})}^+(r) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(r) dr. \end{aligned}$$

The estimation of the rest in Equation 2 and this last equality can be used to get that

$$\begin{aligned} \int_{\omega \in \mathbb{R}^d} |\text{Rad}(\psi)(R\omega)|^2 d\sigma_f(\omega) \\ = \\ 4\pi^2 R^{2\alpha-2d} (m_{\Phi_2^-}(f))^2 \int_{\mathcal{T} \in \mathfrak{X}_\zeta} \left| \int_{r=0}^{+\infty} \Phi_{2, \mathcal{T}}^+(r) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(r) dr \right|^2 d\mu(\mathcal{T}) + O(R^{2\alpha-2d-2\varepsilon}), \end{aligned}$$

as $R \rightarrow +\infty$, since μ is ζ invariant by Lemma 2.15. Finally, with $R = \lambda^N$

$$\lim_{N \rightarrow +\infty} \frac{\int_{\omega \in \mathbb{R}^d} |\text{Rad}(\psi)(\lambda^N \omega)|^2 d\sigma_f(\omega)}{\lambda^{N(2\alpha-2d)}} = (2\pi m_{\Phi_2^-}(f))^2 \int_{\mathcal{T} \in \mathfrak{X}_\zeta} \left| \int_{r=0}^{+\infty} \Phi_{2, \mathcal{T}}^+(r) \mathcal{F}(x_1 \text{Rad}(\psi)(x))(r) dr \right|^2 d\mu(\mathcal{T}).$$

We can remark that the right term is non zero by hypothesis on $m_{\Phi_2^-}(f)$ and because this is true for all choice of ψ .

4.1.4 Step 4: regularity of the limit

This limit is a quadratic form on $\mathcal{S}_F(\mathbb{R}_+)$, let us write the associated bilinear form on $\mathcal{S}_F(\mathbb{R}_+) \times \mathcal{S}_F(\mathbb{R}_+)$:

$$\begin{aligned} Q(\psi_1, \psi_2) &:= \lim_{N \rightarrow +\infty} \frac{\int_{\omega \in \mathbb{R}^d} \text{Rad}(\psi_1)(\lambda^N \omega) \text{Rad}(\psi_2)(\lambda^N \omega) d\sigma_f(\omega)}{\lambda^{N(2\alpha-2d)}} = \\ &(2\pi m_{\Phi_2^-}(f))^2 \int_{\mathcal{T} \in \mathfrak{X}_\zeta} \left(\int_{r=0}^{+\infty} \Phi_{2, \mathcal{T}}^+(r) \mathcal{F}(x_1 \text{Rad}(\psi_1)(x))(r) dr \int_{r=0}^{+\infty} \Phi_{2, \mathcal{T}}^+(r) \mathcal{F}(x_1 \text{Rad}(\psi_2)(x))(r) dr \right) d\mu(\mathcal{T}). \end{aligned}$$

Let us remark that this bilinear form depends solely on the product $\psi_1 \psi_2$, is not identically zero, continuous and nonnegative on nonnegative functions.

Let us also prove the following:

Lemma 4.2. *Every function in $C_0^\infty(\mathbb{R}_+, \mathbb{R})$ is the uniform limit of a sequence of function in $\mathcal{S}_F(\mathbb{R}_+)$.*

Proof. To see this, it suffices to take a function f in $C_0^\infty(\mathbb{R})$ and a sequence of functions $(f_n)_{n \in \mathbb{N}^*} \in C(\mathbb{R})$ such that:

- $\forall n \in \mathbb{N}^*, \forall x \in [-\frac{1}{n}, \frac{1}{n}], f_n(x) = f(0)$.
- $(f_n)_{n \in \mathbb{N}^*}$ converges uniformly towards f .
- $\forall n \in \mathbb{N}^*, f_n$ is compactly supported.

Then we take a regularizing sequence ρ_m and remark that for all $n \in \mathbb{N}^*$ and $m > n$, $\rho_m * f_n$ is in $C_0^\infty(\mathbb{R})$ and is flat in 0. Thus there exists a strictly increasing sequence $(m_n)_{n \in \mathbb{N}^*} \in \mathbb{N}^*$ such that the sequence $(\rho_{m_n} * f_n)_{n \in \mathbb{N}^*}$, which consists of C_0^∞ functions flat in 0, uniformly converges to f , thus proving the lemma. \square

Let us also remark that:

$$\forall \psi_1, \psi_2 \in \mathcal{S}_F(\mathbb{R}_+), \quad Q(\psi_1^2, \psi_2) < Q(\psi_1, \psi_1) \|\psi_2\|_\infty.$$

As a consequence, for all ψ_1 in $\mathcal{S}_F(\mathbb{R}_+)$, $\psi_2 \mapsto Q(\psi_1^2, \psi_2)$ can be continuously extended to all ψ_2 in $C_0^\infty(\mathbb{R}_+, \mathbb{R})$ so that it is a distribution nonnegative on nonnegative functions.

4.1.5 Step 5: a technical lemma

Let us prove the following lemma.

Lemma 4.3. *There exists a σ -finite positive Radon measure η on \mathbb{R}_+ such that for any $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}_+, \mathbb{R})$ we have:*

$$Q(\psi_1, \psi_2) = \int_{\mathbb{R}_+} \psi_1 \psi_2 d\eta.$$

Proof. In order to prove this lemma, we need to use a classical result from the theory of distribution stating that if a distribution D supported on \mathbb{R}_+ is nonnegative on nonnegative functions in $C_0^\infty(\mathbb{R}_+, \mathbb{R})$ then there exists a σ -finite positive Radon measure η supported on \mathbb{R}_+ such that

$$D(\varphi) = \int_{\mathbb{R}_+} \varphi d\eta.$$

First we observe that for any nonnegative $\psi_1 \in \mathcal{S}_F(\mathbb{R}_+)$, the functional $\varphi \mapsto Q(\psi_1^2, \varphi)$ is a distribution on \mathbb{R}_+ that is nonnegative on $\varphi \geq 0$. Thus there exists a measure η_{ψ_1} such that:

$$\forall \varphi \in C_0^\infty(\mathbb{R}_+, \mathbb{R}), \quad Q(\psi_1^2, \varphi) = \int_{\mathbb{R}_+} \varphi d\eta_{\psi_1}.$$

Remark that $Q((\psi_1 \psi_2)^2, \varphi) = Q(\psi_1^2, \psi_2^2 \varphi)$, hence

$$d\eta_{\psi_1 \psi_2} = \psi_2^2 d\eta_{\psi_1}, \quad \text{for } \psi_1, \psi_2 \in \mathcal{S}_F(\mathbb{R}_+).$$

Thus, taking any $\psi > 0$, $d\eta = \frac{1}{\psi^2} d\eta_\psi$ does not depend on the choice of ψ .

Now let $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}_+, \mathbb{R})$:

$$Q(\psi_1, \psi_2) := Q(\psi^2, \frac{\psi_1 \psi_2}{\psi^2}) = \int_{\mathbb{R}_+} \psi_1 \psi_2 d\eta,$$

which proves the lemma. □

Let us now conclude that the following formula:

$$\lim_{N \rightarrow +\infty} \frac{\int_{\omega \in \mathbb{R}^d} \text{Rad}(\psi_1)(\lambda^N \omega) \text{Rad}(\psi_2)(\lambda^N \omega) d\sigma_f(\omega)}{\lambda^{N(2\alpha-2d)}} = \int_{\mathbb{R}_+} \psi_1 \psi_2 d\eta$$

also holds for characteristic functions of intervals $[0, a]$ where a is a point of continuity of the measure η .

Indeed, let us choose $a \in \mathbb{R}^+$ such that $\eta(\{a\}) = 0$, let $\psi = \psi_1 = \psi_2 = \chi_{[0, a]}$ and choose sequences of functions ψ_n^+, ψ_n^- in $C_0^\infty(\mathbb{R}_+, \mathbb{R})$, flat in 0, approximating ψ respectively from above and below, converging pointwise to ψ and uniformly on the complementary of $(a - \delta, a + \delta)$ for any $\delta > 0$.

By continuity of η over a which is not an atom, we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \eta((a - \delta, a + \delta)) < \varepsilon.$$

As a consequence,

$$\lim_{N \rightarrow +\infty} \int_{\mathbb{R}_+} (\psi_n^+ \psi_n^+ - \psi_n^- \psi_n^-) d\eta = 0,$$

and so the formula is still true for characteristic function $\chi_{[0, a]}$ whenever $\eta(\{a\}) = 0$ which proves Theorem 1.1.

References

- [1] A. Bufetov. Finitely-additive measures on the asymptotic foliations of a Markov compactum. *Mosc. Math. J.*, 14(2):205–224, 426, 2014.
- [2] A. Bufetov. Limit theorems for translation flows. *Ann. of Math. (2)*, 179(2):431–499, 2014.
- [3] A. Bufetov and B. Solomyak. Limit theorems for self-similar tilings. *Comm. Math. Physics*, 319(3):761–789, 2013.
- [4] A. Bufetov and B. Solomyak. On the modulus of continuity for spectral measures in substitution dynamics. *Adv. Math.*, 260:84–129, 2014.
- [5] A. Katok and J-P. Thouvenot. Spectral properties and combinatorial constructions in ergodic theory. In *Handbook of dynamical systems. Vol. 1B*, pages 649–743. Elsevier B. V., Amsterdam, 2006.
- [6] R. Penrose. Pentaplexity: a class of nonperiodic tilings of the plane. *Math. Intelligencer*, 2(1):32–37, 1979/80.
- [7] M. Queffélec. Étude spectrale de substitutions. *C. R. Acad. Sci. Paris Sér. I Math.*, 297(6):317–320, 1983.
- [8] E. Robinson, Jr. Symbolic dynamics and tilings of \mathbb{R}^d . In *Symbolic dynamics and its applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 81–119. Amer. Math. Soc., Providence, RI, 2004.
- [9] B. Solomyak. Nonperiodicity implies unique composition for self-similar translationally finite tilings. *Disc. Comp. Geom*, 20(2):265–279, 1998.