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Regularity of Stochastic Kinetic Equations

Ennio Fedrizzi ^{*}, Franco Flandoli[†], Enrico Priola [‡], Julien Vovelle [§]

Abstract

We consider regularity properties of stochastic kinetic equations with multiplicative noise and drift term which belongs to a space of mixed regularity (L^p -regularity in the velocity-variable and Sobolev regularity in the space-variable). We prove that, in contrast with the deterministic case, the SPDE admits a unique weakly differentiable solution which preserves a certain degree of Sobolev regularity of the initial condition without developing discontinuities. To prove the result we also study the related degenerate Kolmogorov equation in Bessel-Sobolev spaces and construct a suitable stochastic flow.

1 Introduction

We consider the linear Stochastic Partial Differential Equation (SPDE) of kinetic transport type

$$d_t f + (v \cdot D_x f + F \cdot D_v f) dt + D_v f \circ dW_t = 0, \quad f|_{t=0} = f_0 \quad (1)$$

and the associated stochastic characteristics described by the stochastic differential equation (SDE)

$$\begin{cases} dX_t &= V_t dt, & dV_t &= F(X_t, V_t) dt + dW_t \\ X(0) &= x_0, & V(0) &= v_0. \end{cases} \quad (2)$$

Here $t \in [0, T]$, $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $f_0 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $x_0, v_0 \in \mathbb{R}^d$ and $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$; the operation $D_v f \circ dW_t =$

^{*}I2M, Aix-Marseille Univ, CNRS, UMR 7373, Centrale Marseille, 13453 Marseille, France. Email address: ennio.fedrizzi@centrale-marseille.fr.

[†]Dipartimento di Matematica, Largo Bruno Pontecorvo 5, Università di Pisa, 56127 Pisa, Italy. Email address: flandoli@dma.unipi.it.

[‡]Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy. Email address: enrico.priola@unito.it

[§]Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France Email address: vovelle@math.univ-lyon1.fr

$\sum_{\alpha=1}^d \partial_{v_\alpha} f \circ dW_t^\alpha$ will be understood in the Stratonovich sense, in order to preserve (a priori only formally) the relation $df(t, X_t, V_t) = 0$, when (X_t, V_t) is a solution of the SDE; we use Stratonovich not only for this mathematical convenience, but also because, in the spirit of the so called Wong-Zakai principle, the Stratonovich sense is the natural one from the physical view-point as a limit of correlated noise with small time-correlation. The physical meaning of the SPDE (1) is the transport of a scalar quantity described by the function $f(t, x, v)$ (or the evolution of a density $f(t, x, v)$, when $\operatorname{div}_v F = 0$, so that $FD_v f = \operatorname{div}_v(Ff)$), under the action of a fluid - or particle - motion described by the SDE (2), where we have two force components: a “mean” (large scale) component $F(x, v)$, plus a fast fluctuating perturbation given by $\frac{dW_t}{dt}$. Under suitable assumptions and more technical work one can consider more elaborate and flexible noise terms, space dependent, of the form $\sum_{k=1}^\infty \sigma_k(x) dW_t^k$ (see [CF], [DFV14] for examples of assumptions on a noise with this structure and [FGV01] for physical motivations), but for the purpose of this paper it is sufficient to consider the simplest noise $dW_t = \sum_{k=1}^d e_k dW_t^k$, $\{e_k\}_{k=1, \dots, d}$ being an orthonormal base of \mathbb{R}^d .

Our aim is to show that noise has a regularizing effect on both the SDE (2) and the SPDE (1), in the sense that it provides results of existence, uniqueness and regularity under assumptions on F which are forbidden in the deterministic case. Results of this nature have been proved recently for other equations of transport type, see for instance [FGP10], [FF13b], [F11], [BFGM], but here, for the first time, we deal with the case of “degenerate” noise, because dW_t acts only on a component of the system. It is well known that the kinetic structure has good “propagation” properties from the v to the x component; however, for the purpose of regularization by noise one needs precise results which are investigated here for the first time and are technically quite non trivial. Let us describe more precisely the result proved here. First, we investigate the SDE (2) under the assumption (see below for more details) that F is in the mixed regularity space $L^p(\mathbb{R}_v^d; W^{s,p}(\mathbb{R}_x^d; \mathbb{R}^d))$ for some $s \in (\frac{2}{3}, 1)$ and $p > 6d$; this means that we require

$$\int_{\mathbb{R}^d} \|F(\cdot, v)\|_{W^{s,p}}^p dv < \infty,$$

where $W^{s,p} = W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$ is a fractional Sobolev space (cf. Hypothesis 1 and the comments after this assumption; see also Sections 3.1 and 3.2 for more details). Thus our drift is only L^p in the “good” v -variable in which the noise acts and has Sobolev regularity in the other x -variable. This is particularly clear in the special case of

$$F(x, v) = \varphi(v)G(x), \quad (3)$$

where $G \in W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$, $\varphi \in L^p(\mathbb{R}^d)$ and $p > 6d$ with $s \in (\frac{2}{3}, 1)$. Just to mention in the case of full-noise action, the best known assumption to get pathwise uniqueness (cf. [KR05]) is that F must belong to $L^p(\mathbb{R}^N; \mathbb{R}^N)$, $p > N$ (in our case $N = 2d$).

According to a general scheme (see [Ver80], [KR05], [FGP10], [FF11], [F11], [Pr12], [FF13a], [FF13b], [BFGM], [CdR], [WZ15], [WZ]) to study regularity properties of the stochastic characteristics one first needs to establish precise regularity results for solutions to associated Kolmogorov equations. In our case such equations are degenerate elliptic equations of the type

$$\lambda \psi(x, v) - \frac{1}{2} \Delta_v \psi(x, v) - v \cdot D_x \psi(x, v) - F(x, v) \cdot D_v \psi(x, v) = g(x, v), \quad (4)$$

where $\lambda > 0$ is given (see Section 3.3). We prove an optimal regularity result for (4) in Bessel-Sobolev spaces (see Theorem 10). Such result requires basic L^p -estimates proved in [Bo02] and [BCLP10] and non-standard interpolation techniques for functions from \mathbb{R}^d with values in Bessel-Sobolev spaces (see in particular the proofs of Theorem 7 and Lemma 9).

The results of Section 3 are exploited in Section 4 to prove existence of strong solutions to (2) and pathwise uniqueness. Moreover, we can also construct a continuous stochastic flow, injective and surjective, hence a flow of homeomorphisms. These maps are locally γ -Hölder continuous for every $\gamma \in (0, 1)$. We cannot say that they are diffeomorphisms; however, we can show that for any t and \mathbb{P} -a.s. the random variable $Z_t = (X_t, V_t)$ admits a distributional derivative with respect to $z_0 = (x_0, v_0)$. Moreover, for any t and $p > 1$, the weak derivative $D_z Z_t \in L^p_{loc}(\Omega \times \mathbb{R}^{2d})$ (i.e., $D_z Z_t \in L^p(\Omega \times K)$, for any compact set $K \subset \mathbb{R}^{2d}$; see Theorem 33). These results are a generalization to the kinetic (hence degenerate noise) case of theorems in [FF13b].

Well-posedness for kinetic SDEs (2) with non-Lipschitz drift has been recently investigated: strong existence and uniqueness have been recently proved in [CdR] and [WZ15]. Moreover, a stochastic flow of diffeomorphisms has been obtained in [WZ] even with a multiplicative noise. In [WZ15] and [WZ] the drift is assumed to be β -Hölder continuous in the x -variable with $\beta > \frac{2}{3}$ and Dini continuous in the v -variable. The results here are more general even concerning the regularity in the x -variable (see also Section 2.1). We stress that well-posedness is not true without noise, as the counter-examples given by Propositions 2 and 3 show.

Based on our results on the stochastic flow, we prove in Section 5 that if the initial condition f_0 is sufficiently smooth, the SPDE (1) admits a weakly differentiable solution and provide a representation formula (see Theorem 38). Moreover, the solution of equation (1) in the spatial variable is of class $W^{1,r}_{loc}(\mathbb{R}^{2d})$, for every $r \geq 1$, \mathbb{P} -a.s., at every time $t \in [0, T]$. Such regularity result is not true without noise: Proposition 3 gives an example where solutions develop discontinuities from smooth initial conditions and with drift in the class considered here. Moreover, assuming in addition that $\operatorname{div}_v F \in L^\infty(\mathbb{R}^{2d})$ we prove uniqueness of weakly differentiable solutions (see Theorem 39).

The results presented here may also serve as a preliminary for the investigation of properties of interest in the theory of kinetic equations, where again we see a regularization by noise. In a forthcoming paper we shall investigate the mixing property

$$\|f_t\|_{L^\infty_x(L^1_v)} \leq C(t) \|f_0\|_{L^1_x(L^\infty_v)},$$

with $C(t)$ diverging as $t \rightarrow 0$, to see if it holds when the noise is present in comparison to the deterministic case (cf. [GS02] and [H-K10]). Again the theory of stochastic flows, absent without noise under our assumptions, is a basic ingredient for this analysis.

The paper is constructed as follows. We begin by introducing in the next section some necessary notation and presenting some examples that motivate our study. In Section 3 we state some well-posedness results for an associated degenerate elliptic equation (see Theorem 10, which contains the main result of this section). These results will be used in Section 4 to solve the stochastic equation of characteristics associated to (1). This is a degenerate stochastic equation, but we can prove existence and uniqueness of strong solutions (see Theorem 21), generating a weakly differentiable

flow of homeomorphisms (see Theorems 32 and 33). Using all these tools, we can finally show in Section 5 that the stochastic kinetic equation (1) is well-posed in the class of weakly differentiable solutions.

2 Notation and Examples

We will either use a dot or $\langle \cdot, \cdot \rangle$ to denote the scalar product in \mathbb{R}^d and $|\cdot|$ for the Euclidian norm. Other norms will be denoted by $\|\cdot\|$, and for the sup norm we shall use both $\|\cdot\|_\infty$ and $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. $C_b(\mathbb{R}^d)$ denotes the Banach space of all real continuous and bounded functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ endowed with the sup norm; $C_b^1(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the subspace of all functions which are differentiable on \mathbb{R}^d with bounded and continuous partial derivatives on \mathbb{R}^d ; for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, $C^\alpha(\mathbb{R}^d) \subset C^0$ is the space of α -Hölder continuous functions on \mathbb{R}^d ; $C_c^\infty(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the space of all infinitely differentiable functions with compact support. C, c, K will denote different constants, and we use subscripts to indicate the parameters on which they depend.

Throughout the paper, we shall use the notation z to denote the point $(x, v) \in \mathbb{R}^{2d}$. Thus, for a scalar function $g(z) : \mathbb{R}^{2d} \rightarrow \mathbb{R}$, $D_z g$ will denote the vector in \mathbb{R}^{2d} of derivatives with respect to all variables $z = (x, v)$, $D_x g \in \mathbb{R}^d$ denotes the vector of derivatives taken only with respect to the first d variables and similarly for $D_v g(z)$. We will have to work with spaces of functions of different regularity in the x and v variables: we will then use subscripts to distinguish the space and velocity variables, as in Hypothesis 1.

Let us state the regularity assumptions we impose on the force field F .

Hypothesis 1 *The function $F : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ is a Borel function such that*

$$\int_{\mathbb{R}^d} \|F(\cdot, v)\|_{H_p^s}^p dv < \infty \quad (5)$$

where $s \in (2/3, 1)$ and $p > 6d$. We write that $F \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d; \mathbb{R}^d))$.

The Bessel space $H_p^s = H_p^s(\mathbb{R}^d; \mathbb{R}^d)$ is defined by the Fourier transform (see Section 3). According to Remark 5, condition (5) can also be rewritten using the related fractional Sobolev spaces $W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$ instead of $H_p^s(\mathbb{R}^d; \mathbb{R}^d)$. In the sequel we will also write $H_p^s(\mathbb{R}^d)$ instead of $H_p^s(\mathbb{R}^d; \mathbb{R}^d)$ when no confusion may arise.

2.1 Examples

Without noise, when F is only in the space $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$ for some $s > \frac{2}{3}$ and $p > 6d$, the equation for the characteristics

$$\begin{aligned} x' &= v, & v' &= F(x, v) \\ x(0) &= x_0, & v(0) &= v_0 \end{aligned} \quad (6)$$

and the associated kinetic transport equation

$$D_t f + v \cdot D_x f + F \cdot D_v f = 0, \quad f|_{t=0} = f_0 \quad (7)$$

may have various types of pathologies. We shall mention here some of them in the very simple case of $d = 1$,

$$F(x, v) = \pm \theta(x, v) \operatorname{sign}(x) |x|^\alpha \quad (8)$$

for some $\alpha \in \left(\frac{1}{2}, 1\right)$, $\theta \in C_c^\infty(\mathbb{R}^2)$.

First, note that this function belongs to $L^p(\mathbb{R}_v; H_p^s(\mathbb{R}_x))$ for for some $s > \frac{2}{3}$ and $p > 6$ (to check this fact one can use the Sobolev embedding theorem: $H_q^1(\mathbb{R}) \subset H_p^s(\mathbb{R})$ if $\frac{1}{p} = \frac{1}{q} - 1 + s$).

Thus F satisfies our Hypothesis 1. On the other hand when $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, the function $\operatorname{sign}(x) |x|^\alpha$ is not in $C_{loc}^\gamma(\mathbb{R})$ for any $\gamma > 2/3$ and the results of [WZ15], [WZ] do not apply.

Let us come to the description of the pathologies of characteristics and kinetic equation when $F(x, v) = \pm \theta(x, v) |x|^\alpha$.

Proposition 2 *In $d = 1$, if $\theta \in C_c^\infty(\mathbb{R}^2)$, $\theta = 1$ on $B(0, R)$ for some $R > 0$, $F(x, v) = \theta(x, v) \operatorname{sign}(x) |x|^\alpha$, then system (6) with initial condition $(x_0, 0)$ has infinitely many solutions. In particular, for small time (depending on R and α), $(x_t, v_t) = (x_0 + At^\beta, A\beta t^{\beta-1})$, with (β, A) satisfying (9) below, and also $A = 0$, are solutions.*

Proof. Let us check that $(x_t, v_t) = (At^\beta, A\beta t^{\beta-1})$ with the specified values of (β, A) and a small range of t , are solutions. We have $x'_t = v_t$,

$$\begin{aligned} v'_t - F(x, v_t) &= A\beta(\beta - 1)t^{\beta-2} - \operatorname{sign}(x_t) |x_t|^\alpha \\ &= A\beta(\beta - 1)t^{\beta-2} - \operatorname{sign}(A) |A|^\alpha t^{\alpha\beta} = 0 \end{aligned}$$

for $\alpha\beta = \beta - 2$ and $A\beta(\beta - 1) = \operatorname{sign}(A) |A|^\alpha$, namely

$$\beta = \frac{2}{1 - \alpha}, \quad A = \pm \left(\frac{1}{\beta(\beta - 1)} \right)^{\frac{1}{1-\alpha}} = \pm \left(\frac{(1 - \alpha)^2}{2(1 + \alpha)} \right)^{\frac{1}{1-\alpha}}. \quad (9)$$

■

With a little greater effort one can show, in this specific example, that every solution (x_t, v_t) from the initial condition $(0, 0)$ has, for small time, the form $(x_t, v_t) = (A(t - t_0)^\beta, A\beta(t - t_0)^{\beta-1}) \mathbf{1}_{t \geq t_0}$ for some $t_0 \geq 0$, or it is $(x_t, v_t) = (0, 0)$ ((β, A) always given by (9)) and that existence and uniqueness holds from any other initial condition, even from points of the form $(0, v_0)$, $v_0 \neq 0$, around which F is not Lipschitz continuous. Given $T > 0$ and $R > 0$ large enough, there is thus, at every time $t \in [0, T]$, a set $\Lambda_t \subset \mathbb{R}^2$ of points “reached from $(0, 0)$ ”, which is the set

$$\Lambda_t = \left\{ \left(A(t - t_0)^\beta, A\beta(t - t_0)^{\beta-1} \right) \in \mathbb{R}^2 : t_0 \in [0, t] \right\}.$$

Using this family of sets one can construct examples of non uniqueness for the transport equation (7), because a solution $f(t, x, v)$ is not uniquely determined on Λ_t . However, these examples are not striking since the region of non-uniqueness, $\cup_{t \geq 0} \Lambda_t$, is thin and

one could say that uniqueness is restored by a modification of f on a set of measure zero. But, with some additional effort, it is also possible to construct an example with $F(x, v) = \pm\theta(x, v)|x|^\alpha$. In this case, for some negative m (depending on R and α), one can construct infinitely many solutions (x_t, v_t) starting from any point in a segment $(x_0, 0)$, $x_0 \in [m, 0)$. Indeed, $(x_t, v_t) = (x_0, 0)$ is a solution, but there are also solutions leaving $(x_0, 0)$ which will have $v_t > 0$, at least for some small time interval. Then one obtains that the solution $f(t, x, v)$ is not uniquely determined on a set of positive Lebesgue measure.

More relevant, for a simple class of drift as the one above, is the phenomenon of loss of regularity. Preliminary, notice that, when F is Lipschitz continuous, system (6) generates a Lipschitz continuous flow and, using it, one can show that, for every Lipschitz continuous $f_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$, the transport equation (7) has a unique solution in the class of continuous functions $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ that are Lipschitz continuous in (x, v) , uniformly in t . The next proposition identifies an example with non-Lipschitz F where this persistence of regularity is lost. More precisely, even starting from a smooth initial condition, unless it has special symmetry properties, there is a solution with a point of discontinuity. This pathology is removed by noise, since we will show that with sufficiently good initial condition, the unique solution $f(t, z)$ is of class $W_{loc}^{1,r}(\mathbb{R}^2)$ for every $r \geq 1$ and $t \in [0, T]$ a.s., hence in particular continuous. However, in the stochastic case, we do not know whether the solution is Lipschitz under our assumptions, whereas presumably it is under the stronger Hölder assumptions on F of [WZ].

Proposition 3 *In $d = 1$, if $\theta \in C_c^\infty(\mathbb{R}^2)$, $\theta = 1$ on $B(0, R)$ for some $R > 0$, $F(x, v) = \theta(x, v) \text{sign}(x)|x|^\alpha$, then system (6) has a unique local solution on any domain not containing the origin, for every initial condition. For every $t_0 > 0$ (small enough with respect to R), the two initial conditions $(At_0^\beta, -A\beta t_0^{\beta-1})$ with (β, A) given by (9) produce the solution*

$$(x_t, v_t) = \left(A(t_0 - t)^\beta, -A\beta(t_0 - t)^{\beta-1} \right)$$

for $t \in [0, t_0]$, and $(x_{t_0}, v_{t_0}) = (0, 0)$. As a consequence, the transport equation (7) with any smooth f_0 such that $f_0(At_0^\beta, -A\beta t_0^{\beta-1}) \neq f_0(-At_0^\beta, A\beta t_0^{\beta-1})$ for some $t_0 > 0$, has a solution with a discontinuity at time t_0 at position $(x, v) = (0, 0)$.

Proof. The proof is elementary but a full proof is lengthy. We limit ourselves to a few simple facts, without proving that system (6) is forward well posed (locally in time) and the transport equation (7) is also well posed in the set of weak solutions. We only stress that the claim $(x_{t_0}, v_{t_0}) = (0, 0)$ when the initial condition is $(At_0^\beta, -A\beta t_0^{\beta-1})$ can be checked by direct computation (as in the previous proposition) and the discontinuity of the solution f of (7) is a consequence of the transport property, namely the fact that whenever f is regular we have

$$f(t, x_t, v_t) = f_0(x_0, v_0) \tag{10}$$

where (x_t, v_t) is the unique solution with initial condition (x_0, v_0) . Hence we have this identity for points close (but not equal) to the coalescing ones mentioned above, where

the forward flow is regular and a smooth initial condition f_0 gives rise to a smooth solution; but then, from identity (10) in nearby points, the limit

$$\lim_{(x,v) \rightarrow (0,0)} f(t_0, x, v)$$

does not exist if t_0 is as above and $f_0 \left(At_0^\beta, -A\beta t_0^{\beta-1} \right) \neq f_0 \left(-At_0^\beta, A\beta t_0^{\beta-1} \right)$. ■

3 Well-posedness for degenerate Kolmogorov equations in Bessel-Sobolev spaces

3.1 Preliminaries on functions spaces and interpolation theory

Here we collect basic facts on Bessel and Besov spaces (see [BL76], [Tri78] and [Ste70] for more details). In the sequel if X and Y are real Banach spaces then $Y \subset X$ means that Y is continuously embedded in X .

The Bessel (potential) spaces are defined as follows (cf. [BL76] page 139 and [Ste70] page 135). For the sake of simplicity we only consider $p \in [2, \infty)$ and $s \in \mathbb{R}_+$.

First one considers the Bessel potential J^s ,

$$J^s f = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2} \mathcal{F}f]$$

where \mathcal{F} denotes the Fourier transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, $d \geq 1$. Then we introduce

$$H_p^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : J^s f \in L^p(\mathbb{R}^d)\}$$

(clearly $H_p^0(\mathbb{R}^d) = L^p(\mathbb{R}^d)$). This is a Banach space endowed with the norm $\|f\|_{H_p^s} = \|J^s f\|_p$, where $\|\cdot\|_p$ is the usual norm of $L^p(\mathbb{R}^d)$ (we identify functions with coincide a.e.). It can be proved that

$$H_p^s(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : \mathcal{F}^{-1}[|\cdot|^s \mathcal{F}f] \in L^p(\mathbb{R}^d)\} \quad (11)$$

and an equivalent norm in $H_p^s(\mathbb{R}^d)$ is

$$\|f\|_{s,p} = \|f\|_p + \|\mathcal{F}^{-1}[|\cdot|^s \mathcal{F}f]\|_p \simeq \|\mathcal{F}^{-1}[(1 + |\cdot|^s) \mathcal{F}f]\|_p.$$

To show this characterization one can use that

$$(1 + 4\pi^2|x|^2)^{s/2} = (1 + (2\pi|x|^s)) [\mathcal{F}\phi(x) + 1], \quad x \in \mathbb{R}^d,$$

for some $\phi \in L^1(\mathbb{R}^d)$ (see page 134 in [Ste70]), and basic properties of convolution and Fourier transform. We note that

$$H_p^k(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d) \quad (12)$$

if $k \geq 0$ is an integer with equivalence of norms (here $W^{k,p}(\mathbb{R}^d)$ is the usual Sobolev space; $W^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$); see Theorem 6.2.3 in [BL76]. However if s is not an integer we only have (see Theorem 6.4.4 in [BL76] or [Ste70] page 155)

$$H_p^s(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d) \quad (13)$$

where $W^{s,p}(\mathbb{R}^d)$ is a fractional Sobolev space (see below). We have (cf. Theorem 6.2.3 in [BL76])

$$H_p^{s_2}(\mathbb{R}^d) \subset H_p^{s_1}(\mathbb{R}^d)$$

if $s_2 > s_1$ and, moreover, $C_c^\infty(\mathbb{R}^d)$ is dense in any $H_p^s(\mathbb{R}^d)$.

One can compare Bessel spaces with Besov spaces $B_{p,q}^s(\mathbb{R}^d)$ (see, for instance, Theorem 6.2.5 in [BL76]). Let $p, q \geq 2$, $s \in (0, 2)$, to simplify notation.

If $s \in (0, 1)$ then $B_{p,q}^s(\mathbb{R}^d)$ consists of functions $f \in L^p(\mathbb{R}^d)$ such that

$$[f]_{B_{p,q}^s} = \left(\int_{\mathbb{R}^d} \frac{dh}{|h|^{d+sq}} \left(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p dx \right)^{q/p} \right)^{1/q} < \infty.$$

Thus we have

$$B_{p,p}^s(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d) \quad (14)$$

with equivalence of norms. However if $s = 1$, $B_{p,q}^1(\mathbb{R}^d)$ consists of all functions $f \in L^p(\mathbb{R}^d)$ such that

$$[f]_{B_{p,q}^1} = \left(\int_{\mathbb{R}^d} \frac{dh}{|h|^{d+q}} \left(\int_{\mathbb{R}^d} |f(x+2h) - 2f(x+h) + f(x)|^p dx \right)^{q/p} \right)^{1/q} < \infty.$$

Thus we only have $B_{p,p}^1(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$. Note that $B_{p,q}^s(\mathbb{R}^d)$ is a Banach space endowed with the norm: $\|\cdot\|_p + [\cdot]_{B_{p,q}^s}$. Similarly, if $s \in (1, 2)$, then $B_{p,q}^s(\mathbb{R}^d)$ consists of functions $f \in W^{1,p}(\mathbb{R}^d)$ such that

$$[f]_{B_{p,q}^s} = \sum_{i=1}^d \left(\int_{\mathbb{R}^d} \frac{dh}{|h|^{d+sq}} \left(\int_{\mathbb{R}^d} |\partial_{x_i} f(x+h) - \partial_{x_i} f(x)|^p dx \right)^{q/p} \right)^{1/q} < \infty. \quad (15)$$

Moreover, $C_c^\infty(\mathbb{R}^d)$ is dense in any $B_{p,q}^s(\mathbb{R}^d)$ and

$$B_{p,q}^{s_2}(\mathbb{R}^d) \subset B_{p,q}^{s_1}(\mathbb{R}^d), \quad 0 < s_1 < s_2 < 2, \quad p \geq 2. \quad (16)$$

We also have the following result (cf. Theorem 6.4.4 in [BL76])

$$B_{p,2}^s(\mathbb{R}^d) \subset H_p^s(\mathbb{R}^d) \subset B_{p,p}^s(\mathbb{R}^d), \quad (17)$$

$s \in (0, 2)$, $p \geq 2$. Next we state a result for which we have not found a precise reference in the literature. This is useful to give an equivalent formulation to Hypothesis 1 (cf. Remark 5). The proof is given in Appendix.

Proposition 4 *Let $p > 2$, s, s' such that $0 < s < s' < 1$. We have*

$$W^{s',p}(\mathbb{R}^d) \subset B_{p,2}^s(\mathbb{R}^d) \subset H_p^s(\mathbb{R}^d).$$

It is important to notice that Besov spaces are real interpolation spaces (for the definition of interpolation spaces $(X, Y)_{\theta,q}$ with X and Y real Banach spaces and $Y \subset X$ see Chapter 1 in [Lu95] or [BL76]). As a particular case of Theorem 6.2.4 in [BL76] we have for $0 \leq s_0 < s_1 \leq 2$, $\theta \in (0, 1)$, $p \geq 2$,

$$(H_p^{s_0}(\mathbb{R}^d), H_p^{s_1}(\mathbb{R}^d))_{\theta,p} = B_{p,p}^s(\mathbb{R}^d) \quad (18)$$

with $s = (1 - \theta)s_0 + \theta s_1$. Moreover, it holds (see Theorem 6.4.5 in [BL76]):

$$(B_{p,p}^{s_0}(\mathbb{R}^d), B_{p,p}^{s_1}(\mathbb{R}^d))_{\theta,p} = B_{p,p}^s(\mathbb{R}^d) \quad (19)$$

with $0 < s_0 < s_1 < 2$, $s = (1 - \theta)s_0 + \theta s_1$, $\theta \in (0, 1)$.

3.2 Interpolation of functions with values in Banach spaces

We follow Section VII in [LP64] and [Cw74]. Let A_0 be a real Banach space. We will consider the Banach space $L^p(\mathbb{R}^d; A_0)$, $1 \leq p < \infty$, $d \geq 1$. As usual this consists of all strongly measurable functions f from \mathbb{R}^d into A_0 such that the real valued function $\|f(x)\|_{A_0}$ belongs to $L^p(\mathbb{R}^d)$. We have

$$\|f\|_{L^p(\mathbb{R}^d; A_0)} = \left(\int_{\mathbb{R}^d} \|f(x)\|_{A_0} dx \right)^{1/p}, \quad f \in L^p(\mathbb{R}^d; A_0).$$

If A_1 is another real Banach spaces with $A_1 \subset A_0$ we can define the Banach space

$$L^p(\mathbb{R}^d; (A_0, A_1)_{\theta, q}),$$

by using the interpolation space $(A_0, A_1)_{\theta, q}$, $q \in (1, \infty)$, $p \geq 1$ and $\theta \in (0, 1)$. One can prove that

$$(L^p(\mathbb{R}^d; A_0), L^p(\mathbb{R}^d; A_1))_{\theta, q} = L^p(\mathbb{R}^d; (A_0, A_1)_{\theta, q}). \quad (20)$$

with equivalence of norms (see [LP64] and [Cw74]). In the sequel we will often use, for $s \geq 0$, $p \geq 2$,

$$L^p(\mathbb{R}^d; H_p^s(\mathbb{R}^d)). \quad (21)$$

We will often identify this space with the Banach space $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$ of all measurable functions $f(x, v)$, $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f(\cdot, v) \in H_p^s(\mathbb{R}^d)$, for a.e. $v \in \mathbb{R}^d$, and, moreover (see (11))

$$\int_{\mathbb{R}^d} \|f(\cdot, v)\|_{H_p^s}^p dv = \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |\mathcal{F}_x^{-1}[(1 + |\cdot|^s)\mathcal{F}_x f(\cdot, v)](x)|^p dx < \infty \quad (22)$$

(here \mathcal{F}_x denotes the partial Fourier transform in the x -variable; as usual we identify functions which coincide a.e.). As a norm we consider

$$\|f\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} = \left(\int_{\mathbb{R}^d} \|f(\cdot, v)\|_{H_p^s}^p dv \right)^{1/p}. \quad (23)$$

Also $L^p(\mathbb{R}_v^d; L^p(\mathbb{R}_x^d))$ can be identified with $L^p(\mathbb{R}^{2d})$. Similarly, we can define $L^p(\mathbb{R}_v^d; B_{p,p}^s(\mathbb{R}_x^d))$. Using (17) we have

$$L^p(\mathbb{R}^d; H_p^s(\mathbb{R}^d)) \subset L^p(\mathbb{R}^d; B_{p,p}^s(\mathbb{R}^d)), \quad (24)$$

$p \geq 2$, $0 < s < 2$. Finally using (20) and (19) we get for $0 < s_0 < s_1 < 2$, $\theta \in (0, 1)$, $p \geq 2$,

$$(L^p(\mathbb{R}^d; (B_{p,p}^{s_0}(\mathbb{R}^d)), L^p(\mathbb{R}^d; (B_{p,p}^{s_1}(\mathbb{R}^d))))_{\theta, p} = L^p(\mathbb{R}^d; B_{p,p}^s(\mathbb{R}^d)), \quad (25)$$

with $s = (1 - \theta)s_0 + \theta s_1$.

In the sequel when no confusion may arise, we will simply write $L^p(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d; \mathbb{R}^k)$, $k \geq 1$, $p \in [1, \infty)$. Thus a function $U : \mathbb{R}^d \rightarrow \mathbb{R}^k$ belongs to $L^p(\mathbb{R}^d)$ if all its components $U_i \in L^p(\mathbb{R}^d)$, $i = 1, \dots, k$. Moreover, $\|U\|_{L^p} = \left(\sum_{i=1}^k \|U_i\|_{L^p}^p \right)^{1/p}$. This convention about vector-valued functions will be used for other function spaces as well.

Remark 5 Proposition 4 and formula (13) show that Hypothesis 1 is equivalent to the following one: $F : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ is a Borel function such that

$$\int_{\mathbb{R}^d} \|F(\cdot, v)\|_{W^{s,p}}^p dv < \infty, \quad (26)$$

where $s \in (2/3, 1)$ and $p > 6d$.

3.3 Regularity results in Bessel-Sobolev spaces

Here $\mathbb{R}^N = \mathbb{R}^{2d}$ and $z = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$. Let also $p \in (1, \infty)$, $s \in (0, 1)$ and $\lambda > 0$. This section is devoted to the study of the equation

$$\begin{aligned} \lambda\psi(z) - \frac{1}{2}\Delta_v\psi(z) - v \cdot D_x\psi(z) - F(z) \cdot D_v\psi(z) &= g(z) \\ &= \lambda\psi(z) - \frac{1}{2}\text{Tr}(QD^2\psi(z)) - \langle Az, D\psi(z) \rangle - \langle B(z), D\psi(z) \rangle \end{aligned}$$

where $A = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$ are $(2d \times 2d)$ -matrices, $B = \begin{pmatrix} 0 \\ F \end{pmatrix} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$. We shall start by considering the simpler equation with $B = 0$, i.e.,

$$\lambda\psi(z) - \frac{1}{2}\Delta_v\psi(z) - v \cdot D_x\psi(z) = \lambda\psi(z) - \mathcal{L}\psi(z) = g(z), \quad z \in \mathbb{R}^{2d}. \quad (27)$$

Recall that $D_v\psi$ and $D_x\psi$ denote respectively the gradient of ψ in the v -variables and in the x -variables; moreover, $D_v^2\psi$ indicates the Hessian matrix of ψ with respect to the v -variables (we have $\Delta_v\psi = \text{Tr}(D_v^2\psi)$).

Definition 6 The space $X_{p,s}$ consists of all functions $f \in W^{1,p}(\mathbb{R}^{2d})$ such that D_v^2f and $v \cdot D_xf$ belong to $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$. Recall that

$$\|D_v^2f\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p = \int_{\mathbb{R}^d} \sum_{i,j=1}^d \|\partial_{v_i v_j}^2 f(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv.$$

It turns out that $X_{p,s}$ is a Banach space endowed with the norm:

$$\|f\|_{X_{p,s}} = \|f\|_{W^{1,p}(\mathbb{R}^{2d})} + \|D_v^2f\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} + \|v \cdot D_xf\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}. \quad (28)$$

If $f \in X_{p,s}$ then $(\lambda f - \mathcal{L}f) \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$ (see (27)). With a slight abuse of notation, we will still write $f \in X_{p,s}$ for vector valued functions $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$, meaning that all components $f_i : \mathbb{R}^{2d} \rightarrow \mathbb{R}$, $i = 1 \dots 2d$ belong to $X_{p,s}$.

The following theorem improves results in [Bo02] and [BCLP10]. In particular it shows that there exists the weak derivative $D_x\psi \in L^p(\mathbb{R}^{2d})$ so that (27) admits a strong solution ψ which solves equation (27) in distributional sense.

Theorem 7 *Let $\lambda > 0$, $p \geq 2$, $s \in (1/3, 1)$ and $g \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$. There exists a unique solution $\psi = \psi_\lambda \in X_{p,s}$ to equation (27). Moreover, we have*

$$\lambda\|\psi\|_{L^p(\mathbb{R}^{2d})} + \sqrt{\lambda}\|D_v\psi\|_{L^p(\mathbb{R}^{2d})} + \|D_v^2\psi\|_{L^p(\mathbb{R}^{2d})} + \|v \cdot D_x\psi\|_{L^p(\mathbb{R}^{2d})} \leq C\|g\|_{L^p(\mathbb{R}^{2d})} \quad (29)$$

with $C = C(d, p) > 0$ and

$$\|D_x \psi\|_{L^p(\mathbb{R}^{2d})} \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad (30)$$

with $C(\lambda) = C(\lambda, s, p, d) > 0$ and $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. In addition there exists $c = c(s, p, d) > 0$ such that

$$\begin{aligned} & \lambda \|\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} + \sqrt{\lambda} \|D_v \psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} + \|D_v^2 \psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \\ & + \|v \cdot D_x \psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \leq c \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \end{aligned} \quad (31)$$

Proof. *Uniqueness.* Let $\psi \in X_{p,s}$ be a solution. Multiplying both sides of equation (27) by $|\psi|^{p-2}\psi$ and integrating by parts we obtain

$$\begin{aligned} \lambda \|\psi\|_{L^p(\mathbb{R}^{2d})}^p + \frac{(p-1)}{2} \sum_{k=1}^d \int_{\mathbb{R}^{2d}} |\psi|^{p-2} |\partial_{v_k} \psi|^2 dz + \int_{\mathbb{R}^{2d}} (v \cdot D_x \psi) |\psi|^{p-2} \psi dz \\ = \int_{\mathbb{R}^{2d}} g |\psi|^{p-2} \psi dz \end{aligned}$$

(this identity can be rigorously proved by approximating ψ by smooth functions). Note that there exists the weak derivative

$$D_x(|\psi|^p) = p|\psi|^{p-2}\psi D_x \psi \in L^1(\mathbb{R}^{2d})$$

and so

$$\int_{\mathbb{R}^{2d}} (v \cdot D_x \psi) |\psi|^{p-2} \psi dz = \frac{1}{p} \int_{\mathbb{R}^{2d}} v \cdot D_x(|\psi|^p) dz = 0.$$

It follows easily that

$$\|\psi\|_{L^p(\mathbb{R}^{2d})} \leq \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}^{2d})} \quad (32)$$

which implies uniqueness of solutions for the linear equation (27).

Existence. I Step. We prove existence of solutions and estimates (29) and (30).

Let us first introduce the Ornstein-Uhlenbeck semigroup

$$\begin{aligned} P_t g(z) = P_t g(x, v) &= \int_{\mathbb{R}^{2d}} g(e^{tA} z + y) N(0, Q_t) dy \\ &= \int_{\mathbb{R}^{2d}} g(x + tv + y_1, v + y_2) N(0, Q_t) dy, \quad g \in C_c^\infty(\mathbb{R}^{2d}), \quad t \geq 0, \end{aligned} \quad (33)$$

where $N(0, Q_t)$ is the Gaussian measure with mean 0 and covariance matrix

$$Q_t = \int_0^t e^{sA} Q e^{sA^*} ds = \int_0^t e^{sA} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{\mathbb{R}^d} \end{pmatrix} e^{sA^*} ds = \begin{pmatrix} \frac{1}{3} t^3 \mathbb{I}_{\mathbb{R}^d} & \frac{1}{2} t^2 \mathbb{I}_{\mathbb{R}^d} \\ \frac{1}{2} t^2 \mathbb{I}_{\mathbb{R}^d} & t \mathbb{I}_{\mathbb{R}^d} \end{pmatrix} \quad (34)$$

(A^* denotes the adjoint matrix). By the Young inequality (cf. the proof of Lemma 13 in [Pr15]) we know that $P_t g$ is well-defined also for any $g \in L^p(\mathbb{R}^{2d})$, z a.e.; moreover $P_t : L^p(\mathbb{R}^{2d}) \rightarrow L^p(\mathbb{R}^{2d})$, for any $t \geq 0$, and

$$\|P_t g\|_{L^p(\mathbb{R}^{2d})} \leq \|g\|_{L^p(\mathbb{R}^{2d})}, \quad g \in L^p(\mathbb{R}^{2d}), \quad t \geq 0. \quad (35)$$

Let us consider, for any $\lambda > 0$, $z \in \mathbb{R}^d$, $g \in C_c^\infty(\mathbb{R}^{2d})$,

$$\psi(z) = G_\lambda g(z) = \int_0^{+\infty} e^{-\lambda t} P_t g(z) dt. \quad (36)$$

Using the Jensen inequality, the Fubini theorem and (35) it is easy to prove that $G_\lambda g$ is well defined for $g \in L^p(\mathbb{R}^{2d})$, z a.e., and belongs to $L^p(\mathbb{R}^{2d})$. Moreover, for any $p \geq 1$,

$$G_\lambda : L^p(\mathbb{R}^{2d}) \rightarrow L^p(\mathbb{R}^{2d}), \quad \|G_\lambda g\|_p \leq \frac{\|g\|_p}{\lambda}, \quad \lambda > 0, \quad g \in L^p(\mathbb{R}^{2d}). \quad (37)$$

Note that $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d)) \subset L^p(\mathbb{R}_v^d; W^{s,p}(\mathbb{R}_x^d))$ (see (13)). Let us consider a sequence $(g_n) \in C_c^\infty(\mathbb{R}^{2d})$ such that

$$g_n \rightarrow g \text{ in } L^p(\mathbb{R}_v^d; W^{s,p}(\mathbb{R}_x^d)).$$

Arguing as in [Pr15, Lemma 13] one can show that there exist classical solutions ψ_n to (27) with g replaced by g_n . Moreover, $\psi_n = G_\lambda g_n$. By [Pr15, Theorem 11], which is based on results in [BCLP10], we have that

$$\|D_v^2 \psi_n\|_{L^p(\mathbb{R}^{2d})} \leq C \|g_n\|_{L^p(\mathbb{R}^{2d})}, \quad (38)$$

$\lambda > 0$, $n \geq 1$, $C = C(p, d)$. Using also (37) we deduce easily that (ψ_n) and $(D_v^2 \psi_n)$ are both Cauchy sequences in $L^p(\mathbb{R}^{2d})$. Let us denote by $\psi \in L^p(\mathbb{R}^{2d})$ the limit function; it holds that $\psi = G_\lambda g$ and $D_v^2 \psi \in L^p(\mathbb{R}^{2d})$.

Passing to the limit in (27) when ψ and g are replaced by ψ_n and g_n we obtain that ψ solves (27) in a weak sense ($v \cdot D_x \psi$ is intended in distributional sense). By (38) as $n \rightarrow \infty$ we also get

$$\begin{aligned} \|D_v^2 \psi\|_{L^p(\mathbb{R}^{2d})} &\leq C \|g\|_{L^p(\mathbb{R}^{2d})}, \quad \|\psi\|_{L^p(\mathbb{R}^{2d})} \leq \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}^{2d})} \\ \text{and } \|v \cdot D_x \psi\|_{L^p(\mathbb{R}^{2d})} &\leq C \|g\|_{L^p(\mathbb{R}^{2d})}. \end{aligned} \quad (39)$$

To prove (29) it remains to show the estimate for $D_v \psi$. This follows from

$$\|D_v \psi\|_{L^p(\mathbb{R}^{2d})}^p = \int_{\mathbb{R}^d} \|D_v \psi(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx \leq (\|\psi\|_{L^p(\mathbb{R}^{2d})})^{p/2} (\|D_v^2 \psi\|_{L^p(\mathbb{R}^{2d})})^{p/2}. \quad (40)$$

To prove that $\psi \in W^{1,p}(\mathbb{R}^{2d})$ it is enough to check that

$$\psi \in L^p(\mathbb{R}_v^d; W^{1,p}(\mathbb{R}_x^d)). \quad (41)$$

Thus we have to prove that $\psi(\cdot, v) \in W^{1,p}(\mathbb{R}^d)$ for a.e. v and

$$\int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |D_x \psi(x, v)|^p dx < \infty.$$

To this purpose we will use a result in [Bo02] and interpolation theory. We consider $\eta \in C_c^\infty(\mathbb{R})$ such that $\text{Supp}(\eta) \subset [-1, 1]$ and $\int_{-1}^1 \eta(t) dt > 0$.

Setting $f(t, z) = \eta(t)\psi(z)$, where ψ solves (27) we have that $f \in L^p(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$. In order to apply Corollary 2.2 in [Bo02] we note that, for $z = (x, v) \in \mathbb{R}^{2d}$, $t \in \mathbb{R}$,

$$\partial_t f(t, z) + v \cdot D_x f(t, z) = \eta'(t)\psi(z) - \eta(t)g(z) + \lambda\eta(t)\psi(z) - \frac{1}{2}\eta(t)\Delta_v \psi(z).$$

Since $D_v^2 \psi \in L^p(\mathbb{R}^{2d})$ we deduce that $\partial_t f + v \cdot D_x f$ and $D_v^2 f$ both belong to $L^p(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$.

By Corollary 2.2 in [Bo02] and (39) we get easily that $\psi(\cdot, v) \in H_p^{2/3}(\mathbb{R}^d)$, for $v \in \mathbb{R}^d$ a.e., and

$$\int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |\mathcal{F}_x^{-1}[(1 + |\cdot|^{2/3})\mathcal{F}_x \psi(\cdot, v)](x)|^p dx \leq \left(\frac{\lambda+1}{\lambda}\right)^{2p/5} c \|g\|_{L^p(\mathbb{R}^{2d})}^p,$$

$\lambda > 0$, with $c = c(p, d)$, i.e.,

$$\psi = G_\lambda g \in L^p(\mathbb{R}_v^d; H_p^{2/3}(\mathbb{R}_x^d)) \quad \text{and} \quad \|G_\lambda g\|_{L^p(\mathbb{R}_v^d; H_p^{2/3}(\mathbb{R}_x^d))} \leq \left(\frac{\lambda+1}{\lambda}\right)^{2/5} c_1 \|g\|_{L^p(\mathbb{R}^{2d})}. \quad (42)$$

By (20) and (18) with $s_0 = 0$ and $s_1 = 2/3$ we can interpolate between (42) and the estimate $\|G_\lambda g\|_{L^p(\mathbb{R}^d; L^p(\mathbb{R}^d))} \leq \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}^{2d})}$ (see Proposition 1.2.6 in [Lu95]) and get, for $\epsilon \in (0, 2/3)$,

$$\|G_\lambda g\|_{L^p(\mathbb{R}_v^d; W^{2/3-\epsilon, p}(\mathbb{R}_x^d))} \leq c_\epsilon(\lambda) \|g\|_{L^p(\mathbb{R}^{2d})}. \quad (43)$$

with $c_\epsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Suppose now that $g \in L^p(\mathbb{R}_v^d; W^{1, p}(\mathbb{R}_x^d))$ and fix $k = 1, \dots, d$. By approximating g with regular functions, it is not difficult to prove that there exists the weak derivative $\partial_{x_k} \psi \in L^p(\mathbb{R}^{2d})$, and

$$\partial_{x_k} \psi(z) = \partial_{x_k} G_\lambda g(z) = \int_0^{+\infty} e^{-\lambda t} P_t(\partial_{x_k} g)(z) dt. \quad (44)$$

Arguing as in (43) we obtain that $\partial_{x_k} \psi \in L^p(\mathbb{R}_v^d; W^{2/3-\epsilon, p}(\mathbb{R}_x^d))$ and

$$\|\partial_{x_k} \psi\|_{L^p(\mathbb{R}_v^d; W^{2/3-\epsilon, p}(\mathbb{R}_x^d))} \leq c_\epsilon(\lambda) \|\partial_{x_k} g\|_{L^p(\mathbb{R}^{2d})}, \quad k = 1, \dots, d,$$

so that

$$\|G_\lambda g\|_{L^p(\mathbb{R}_v^d; B_{p,p}^{1+2/3-\epsilon}(\mathbb{R}_x^d))} \leq c_\epsilon(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; W^{1, p}(\mathbb{R}_x^d))}. \quad (45)$$

Taking into account (18), (19) and (20) we can interpolate between (43) and (45) (see also (21) and (22)) and get

$$\begin{aligned} G_\lambda : L^p(\mathbb{R}_v^d; W^{s, p}(\mathbb{R}_x^d)) &= (L^p(\mathbb{R}^d; H_p^0(\mathbb{R}^d)), L^p(\mathbb{R}^d; W^{1, p}(\mathbb{R}^d)))_{s, p} \\ &\longrightarrow (L^p(\mathbb{R}^d; W^{2/3-\epsilon, p}(\mathbb{R}^d)), L^p(\mathbb{R}^d; B_{p,p}^{5/3-\epsilon}(\mathbb{R}^d)))_{s, p} = L^p(\mathbb{R}_v^d; B_{p,p}^{s+2/3-\epsilon}(\mathbb{R}_x^d)). \end{aligned} \quad (46)$$

Since $L^p(\mathbb{R}_v^d; B_{p,p}^{s+2/3-\epsilon}(\mathbb{R}_x^d)) \subset L^p(\mathbb{R}_v^d; W^{1, p}(\mathbb{R}_x^d))$ with ϵ small enough (recall that $s \in (1/3, 1)$) we finally obtain that

$$G_\lambda : L^p(\mathbb{R}_v^d; W^{s, p}(\mathbb{R}_x^d)) \rightarrow L^p(\mathbb{R}_v^d; W^{1, p}(\mathbb{R}_x^d)) \quad (47)$$

is linear and continuous. Moreover, we have with $\psi = G_\lambda g$

$$\int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |\partial_{x_k} \psi(x, v)|^p dx \leq C'(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; W^{s,p}(\mathbb{R}_x^d))}^p \leq C''(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p, \quad (48)$$

$k = 1, \dots, d$, where $C'(\lambda)$ and $C''(\lambda)$ tend to 0 as $\lambda \rightarrow \infty$ (recall the estimates (43) and (45)). This proves (30) and (41).

II Step. We prove the last assertion (31).

The main problem is to show that

$$\begin{aligned} \int_{\mathbb{R}^d} \|D_v^2 \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv &= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |\mathcal{F}_x^{-1}[(1 + |\cdot|^s) \mathcal{F}_x D_v^2 \psi(\cdot, v)](x)|^p dx \\ &= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |D_v^2 (\mathcal{F}_x^{-1}[(1 + |\cdot|^s) \mathcal{F}_x \psi(\cdot, v)])(x)|^p dx \leq C \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p \end{aligned}$$

(\mathcal{F}_x denotes the Fourier transform in the x -variable) with $\psi = G_\lambda g$. We introduce

$$h_s(x, v) = \mathcal{F}_x^{-1}[(1 + |\cdot|^s) \mathcal{F}_x g(\cdot, v)](x),$$

$x, v \in \mathbb{R}^d$. We know that $h_s \in L^p(\mathbb{R}^{2d})$ by our hypothesis on g . A straightforward computation based on the Fubini theorem shows that

$$\mathcal{F}_x^{-1}[(1 + |\cdot|^s) \mathcal{F}_x \psi(\cdot, v)](x) = G_\lambda h_s(x, v).$$

By using (39) (with g replaced by h_s and ψ by $G_\lambda h_s$) we easily obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \|D_v^2 \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv &= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |D_v^2 G_\lambda h_s(x, v)|^p dx \\ &\leq C \|h_s\|_{L^p(\mathbb{R}^{2d})}^p = C \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p, \end{aligned} \quad (49)$$

where $C = C(d, p)$. Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \|D_v \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv &= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dx \\ &\leq \frac{C}{(\lambda)^{p/2}} \|h_s\|_{L^p(\mathbb{R}^{2d})}^p = \frac{C}{(\lambda)^{p/2}} \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p \end{aligned} \quad (50)$$

and $\|\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} = \|G_\lambda h_s\|_{L^p(\mathbb{R}^{2d})} \leq \frac{1}{\lambda} \|G_\lambda h_s\|_{L^p(\mathbb{R}^{2d})} = \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}$. The proof is complete. ■

Lemma 8 *Assume as in Theorem 7 that $g \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$, $s \in (1/3, 1)$. Moreover, suppose that $p > d$. Then the solution $\psi = G_\lambda g$ to (27) verifies also*

$$\sup_{v \in \mathbb{R}^d} \|D_v \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)} \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad \lambda > 0, \quad (51)$$

where $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. Using the notation introduced in the previous proof, we have for any $v \in \mathbb{R}^d$, a.e.,

$$\|D_v \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dx.$$

By (49), (50) and the Fubini theorem we know that

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dv + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |D_v^2 G_\lambda h_s(x, v)|^p dv < \infty.$$

It follows that, for $x \in \mathbb{R}^d$ a.e.,

$$\|D_v G_\lambda h_s(x, \cdot)\|_{W^{1,p}(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dv + \int_{\mathbb{R}^d} |D_v^2 G_\lambda h_s(x, v)|^p dv < \infty.$$

In order to prove (51) with $C(\lambda) \rightarrow 0$, we consider $r \in (0, 1)$ such that $rp > d$. Let us fix $x \in \mathbb{R}^d$, a.e.; by the previous estimate the mapping $v \mapsto D_v G_\lambda h_s(x, v)$ belongs to $W^{r,p}(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$.

We can apply the Sobolev embedding theorem (see page 203 in [Tri78]) and get that $v \mapsto D_v G_\lambda h_s(x, v)$ in particular is bounded and continuous on \mathbb{R}^d . Moreover,

$$\sup_{v \in \mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p \leq c \|D_v G_\lambda h_s(x, \cdot)\|_{W^{r,p}(\mathbb{R}^d)}^p, \quad (52)$$

where $c = c(p, d, r)$. Integrating with respect to x we get

$$\int_{\mathbb{R}^d} \left[\sup_{v \in \mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p \right] dx \leq c \int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{W^{r,p}(\mathbb{R}^d)}^p dx.$$

By (18) we know that $(L^p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{r,p} = W^{r,p}(\mathbb{R}^d)$. Applying Corollary 1.2.7 in [Lu95] we obtain that, for any $f \in W^{1,p}(\mathbb{R}^d)$,

$$\|f\|_{W^{r,p}(\mathbb{R}^d)} \leq c(r, p) (\|f\|_{L^p(\mathbb{R}^d)})^{1-r} \cdot (\|f\|_{W^{1,p}(\mathbb{R}^d)})^r.$$

It follows that

$$\begin{aligned} & \sup_{v \in \mathbb{R}^d} \|D_v \psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p = \sup_{v \in \mathbb{R}^d} \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dx \\ & \leq \int_{\mathbb{R}^d} \left[\sup_{v \in \mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p \right] dx \leq c \int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{W^{r,p}(\mathbb{R}^d)}^p dx \\ & \leq c' \int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{L^p(\mathbb{R}^d)}^{p(1-r)} \cdot \|D_v G_\lambda h_s(x, \cdot)\|_{W^{1,p}(\mathbb{R}^d)}^{pr} dx \\ & \leq c' \left(\int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx \right)^{1-r} \cdot \left(\int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{W^{1,p}(\mathbb{R}^d)}^p dx \right)^r. \end{aligned}$$

Now we easily obtain (51) using (49) and (50), since

$$\int_{\mathbb{R}^d} \|D_v G_\lambda h_s(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx = \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} |D_v G_\lambda h_s(x, v)|^p dx \leq C(\lambda) \|g\|_{L^p(\mathbb{R}^d; H_p^s(\mathbb{R}^d))}$$

with $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. ■

We complete the study of the regularity of solutions to equation (27) with the next result in which we strengthen the assumptions of Lemma 8. Note that the next assumption on p holds when $p > 6d$ as in Hypothesis 1.

Lemma 9 *Let $\lambda > 0$, $s \in (2/3, 1)$ and $g \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$. In addition assume that $p(s - \frac{1}{3}) > 2d$, then the following statements hold.*

(i) *The solution $\psi = G_\lambda g$ (see (37)) is bounded and Lipschitz continuous on \mathbb{R}^{2d} . Moreover there exists the classical derivative $D_x \psi$ which is continuous and bounded on \mathbb{R}^{2d} and, for $\lambda > 0$,*

$$\|\psi\|_\infty + \|D_x \psi\|_\infty \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad \text{with } C(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (53)$$

(ii) *$D_v \psi \in W^{1,p}(\mathbb{R}^{2d})$ (so in particular there exist weak partial derivatives $\partial_{x_i} \partial_{v_j} \psi \in L^p(\mathbb{R}^{2d})$, $i, j = 1, \dots, d$) and*

$$\|D_v \psi\|_{W^{1,p}(\mathbb{R}^{2d})} \leq c(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad \lambda > 0, \quad \text{with } c = c(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (54)$$

Proof. (i) The boundedness of ψ follows easily from estimates (29) and (30) using the Sobolev embedding since in our case $p > 2d$. Let us concentrate on proving the Lipschitz continuity.

First we recall a Fubini type theorem for fractional Sobolev spaces (see [Str68]):

$$W^{\gamma,p}(\mathbb{R}^{2d}) = \left\{ f \in L^p(\mathbb{R}^{2d}) : \int_{\mathbb{R}^d} \|f(x, \cdot)\|_{W^{\gamma,p}(\mathbb{R}^d)}^p dx + \int_{\mathbb{R}^d} \|f(\cdot, v)\|_{W^{\gamma,p}(\mathbb{R}^d)}^p dv < \infty \right\}, \quad (55)$$

$\gamma \in (0, 1]$ (with equivalence of the respective norms). Let $\eta \in (0, s + 2/3 - 1)$ be such that

$$\eta p > 2d. \quad (56)$$

We will prove that $D_x \psi \in W^{\eta,p}(\mathbb{R}^{2d})$ so that by the Sobolev embedding $W^{\eta,p}(\mathbb{R}^{2d}) \subset C_b^{\eta-2d/p}(\mathbb{R}^{2d})$ (see page 203 in [Tri78]) we get the assertion. According to (55) we check that

$$\int_{\mathbb{R}^d} \|D_x \psi(\cdot, v)\|_{W^{\eta,p}(\mathbb{R}^d)}^p dv \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p, \quad (57)$$

and

$$\int_{\mathbb{R}^d} \|D_x \psi(x, \cdot)\|_{W^{\eta,p}(\mathbb{R}^d)}^p dx \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p. \quad (58)$$

with $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Estimate (57) follows by (46) which gives $\psi \in L^p(\mathbb{R}_v^d; B_{p,p}^{\eta+1}(\mathbb{R}_x^d))$ with $\eta = s - \epsilon - 1/3$.

Let us concentrate on (58). We still use the interpolation theory results of Section 3.2 but here in addition to (22) we also need to identify $L^p(\mathbb{R}^d; H_p^s(\mathbb{R}^d))$ with the Banach space $L^p(\mathbb{R}_x^d; H_p^s(\mathbb{R}_v^d))$ of all measurable functions $f(x, v)$, $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f(x, \cdot) \in H_p^s(\mathbb{R}^d)$, for $x \in \mathbb{R}^d$ a.e., $0 \leq s \leq 2$, and, moreover $\int_{\mathbb{R}^d} \|f(x, \cdot)\|_{H_p^s}^p dx < \infty$. As a norm one considers

$$\|f\|_{L^p(\mathbb{R}_x^d; H_p^s(\mathbb{R}_v^d))} = \left(\int_{\mathbb{R}^d} \|f(x, \cdot)\|_{H_p^s(\mathbb{R}^d)}^p dx \right)^{1/p}. \quad (59)$$

Similarly, we identify $L^p(\mathbb{R}^d; B_{p,p}^s(\mathbb{R}^d))$ with the Banach space $L^p(\mathbb{R}_x^d; B_{p,p}^s(\mathbb{R}_v^d))$ of all measurable functions $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f(x, \cdot) \in B_{p,p}^s(\mathbb{R}^d)$, for x a.e., and $\int_{\mathbb{R}^d} \|f(x, \cdot)\|_{B_{p,p}^s(\mathbb{R}^d)}^p dx < \infty$. By (29) and (30) in Theorem 7 and using (44) we find with $\psi = G_\lambda g$

$$\begin{aligned} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |D_x \psi(x, v)|^p dv &= \int_{\mathbb{R}^d} \|D_x \psi(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p, \\ \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |D_v^2(D_x \psi)(x, v)|^p dv &= \int_{\mathbb{R}^d} \|D_v^2(D_x \psi)(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx \leq c \|g\|_{L^p(\mathbb{R}_v^d; H_p^1(\mathbb{R}_x^d))}^p, \end{aligned} \quad (60)$$

with $c = c(d, p) > 0$. Thus we can consider the following linear maps ($s' \in (1/3, 1)$ will be fixed below)

$$\begin{aligned} D_x G_\lambda &: L^p(\mathbb{R}_v^d; H_p^{s'}(\mathbb{R}_x^d)) \rightarrow L^p(\mathbb{R}^{2d}) = L^p(\mathbb{R}_x^d; L^p(\mathbb{R}_v^d)), \\ D_x G_\lambda &: L^p(\mathbb{R}_v^d; H_p^1(\mathbb{R}_x^d)) \rightarrow L^p(\mathbb{R}_x^d; H_p^2(\mathbb{R}_v^d)). \end{aligned} \quad (61)$$

Interpolating, choosing $s' \in (1/3, 1)$ such that

$$s' < 2s - 1,$$

we get (see (18) and (20) with $\theta = \frac{s-s'}{1-s'} > 1/2$)

$$\begin{aligned} D_x G_\lambda &: L^p(\mathbb{R}_v^d; W^{s,p}(\mathbb{R}_x^d)) = \left(L^p(\mathbb{R}_v^d; H_p^{s'}(\mathbb{R}_x^d)), L^p(\mathbb{R}_v^d; H_p^1(\mathbb{R}_x^d)) \right)_{\theta,p} \\ &\longrightarrow \left(L^p(\mathbb{R}_x^d; L^p(\mathbb{R}_v^d)), L^p(\mathbb{R}_x^d; H_p^2(\mathbb{R}_v^d)) \right)_{\theta,p} = L^p(\mathbb{R}_x^d; B_{p,p}^{2\theta}(\mathbb{R}_v^d)) \end{aligned} \quad (62)$$

and by the estimates in (60) we find

$$\int_{\mathbb{R}^d} \|D_x(G_\lambda g)(x, \cdot)\|_{B_{p,p}^{2\theta}(\mathbb{R}^d)}^p dx \leq C'(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p$$

(recall that $H_p^s(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d)$). Since $\eta < 2/3$ we have $B_{p,p}^{2\theta}(\mathbb{R}^d) \subset W^{\eta,p}(\mathbb{R}^d)$ (cf. (16)) and we finally get (58).

(ii) We fix $j = 1, \dots, d$ and prove the assertion with $D_v \psi$ replaced by $\partial_{v_j} \psi$.

By Theorem 7 we already know that there exists $D_v \partial_{v_j} \psi \in L^p(\mathbb{R}^{2d})$. Therefore to show the assertion it is enough to check that there exists the weak derivative

$$D_x(\partial_{v_j} \psi) = \partial_{v_j}(D_x \psi) \in L^p(\mathbb{R}^{2d}). \quad (63)$$

We use again (62) with the same θ . Since $2\theta > 1$ we know in particular that $D_x G_\lambda g \in L^p(\mathbb{R}_x^d; W^{1,p}(\mathbb{R}_v^d))$. Thus we have that there exists the weak derivative $\partial_{v_j} D_x \psi(x, \cdot)$, for x a.e., and

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |\partial_{v_j} D_x \psi(x, v)|^p dv = \int_{\mathbb{R}^d} \|\partial_{v_j} D_x \psi(x, \cdot)\|_{L^p(\mathbb{R}^d)}^p dx \leq C(\lambda) \|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p. \quad (64)$$

This finishes the proof. ■

Now we study the complete equation

$$\lambda\psi(z) - \frac{1}{2}\Delta_v\psi(z) - v \cdot D_x\psi(z) - F(z) \cdot D_v\psi(z) = g(z), \quad z = (x, v) \in \mathbb{R}^{2d}, \quad (65)$$

assuming that $F \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$ (cf. (22) and (23)). From the previous results we obtain (see also Definition 6)

Theorem 10 *Let $s \in (2/3, 1)$ and p be such that $p(s - \frac{1}{3}) > 2d$. Assume that*

$$g, F \in L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d)).$$

Then there exists $\lambda_0 = \lambda_0(s, p, d, \|F\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}) > 0$ such that for any $\lambda > \lambda_0$ there exists a unique solution $\psi = \psi_\lambda \in X_{p,s}$ to (65) and moreover

$$\begin{aligned} \lambda\|\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} + \sqrt{\lambda}\|D_v\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} + \|D_v^2\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \\ + \|v \cdot D_x\psi\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \leq C\|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \end{aligned} \quad (66)$$

with $C = C(s, p, d, \|F\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}) > 0$. We also have

$$\sup_{v \in \mathbb{R}^d} \|D_v\psi(\cdot, v)\|_{H_p^s(\mathbb{R}^d)} \leq C(\lambda)\|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad \text{with } C(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (67)$$

Moreover, $\psi \in C_b^1(\mathbb{R}^{2d})$, i.e., ψ is bounded on \mathbb{R}^{2d} and there exist the classical derivatives $D_x\psi$ and $D_v\psi$ which are bounded and continuous on \mathbb{R}^{2d} ; we also have with $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$

$$\|\psi\|_\infty + \|D_x\psi\|_{L^p(\mathbb{R}^{2d})} + \|D_x\psi\|_\infty + \|D_v\psi\|_\infty \leq C(\lambda)\|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}. \quad (68)$$

Finally, $D_v\psi \in W^{1,p}(\mathbb{R}^{2d})$ and

$$\|D_v\psi\|_{W^{1,p}(\mathbb{R}^{2d})} \leq c(\lambda)\|g\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad c = c(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (69)$$

Proof. First note that, since $p > 2d$, the boundedness of ψ follows by the Sobolev embedding (recall also (28)). Similarly the second estimate in (68) follows from (69).

We consider the Banach space $Y = L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$ and use an argument similar to the one used in the proof of Proposition 5 in [DFPR13]. Introduce the operator $T_\lambda : Y \rightarrow Y$,

$$T_\lambda f := F \cdot D_v(G_\lambda f), \quad f \in Y,$$

where G_λ is defined in (36). It is not difficult to check that $T_\lambda f \in Y$ for $f \in Y$. Indeed by Lemma 8 we get

$$\begin{aligned} \int_{\mathbb{R}^d} \|T_\lambda f(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv &\leq \sup_{v \in \mathbb{R}^d} \|D_v(G_\lambda f)(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \|F(\cdot, v)\|_{H_p^s(\mathbb{R}^d)}^p dv \\ &\leq C(\lambda)\|f\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p \|F\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}^p. \end{aligned}$$

It is clear that T_λ is linear and bounded. Moreover we find easily that there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ we have that the operator norm of T_λ is less than $1/2$.

Let us fix $\lambda > \lambda_0$. Since T_λ is a strict contraction, there exists a unique solution $f \in Y$ to

$$f - T_\lambda f = g. \quad (70)$$

We write $f = (\mathbb{I} - T_\lambda)^{-1}g \in Y$.

Uniqueness. Let ψ_1 and ψ_2 be solutions in $X_{p,s}$. Set $w = \psi_1 - \psi_2$. We know that

$$\lambda w(z) - \frac{1}{2}\Delta_v w(z) - v \cdot D_x w(z) - F(z) \cdot D_v w(z) = 0.$$

We have $\lambda w - \frac{1}{2}\Delta_v w - v \cdot D_x w = f \in Y$. By uniqueness (see Theorem 7) we get that $w = G_\lambda f$. Hence, for z a.e.,

$$0 = f(z) - F(z) \cdot D_v w(z) = f(z) - F(z) \cdot D_v G_\lambda f(z).$$

Since T_λ is a strict contraction we obtain that $f = 0$ and so $\psi_1 = \psi_2$.

Existence. It is not difficult to prove that

$$\psi = \psi_\lambda = G_\lambda(\mathbb{I} - T_\lambda)^{-1}g, \quad (71)$$

is the unique solution to (65).

Regularity of ψ and estimates. All the assertions follow easily from (71) since $(\mathbb{I} - T_\lambda)^{-1}g \in Y$ and we can apply Theorem 7, Lemmas 8 and 9. ■

In the Appendix we will also present a result on the stability of the PDE (65), see Lemma 40 .

4 Regularity of the characteristics

We will prove existence of a stochastic flow for the SDE (2) assuming Hypothesis 1.

We can rewrite our SDE as follows. Set $Z_t = (X_t, V_t) \in \mathbb{R}^{2d}$, $z_0 = (x_0, v_0)$ and introduce the functions $b(x, v) = A \cdot z + B(z) : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$, where

$$A = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad RR^* = Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad B = RF = \begin{pmatrix} 0 \\ F \end{pmatrix} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}. \quad (72)$$

With this new notation, (2) can be rewritten as

$$\begin{cases} dZ_t = b(Z_t) dt + R \cdot dW_t \\ Z_0 = z_0 \end{cases} \quad (73)$$

or

$$\begin{cases} dZ_t = (A \cdot Z_t + B(Z_t))dt + R \cdot dW_t \\ Z_0 = z_0 \end{cases} . \quad (74)$$

We have

$$\begin{aligned} X_t &= x_0 + \int_0^t V_s ds = x_0 + tv_0 + \int_0^t (t-s)F(X_s, V_s) ds + \int_0^t W_s ds, \\ V_t &= v_0 + \int_0^t F(X_s, V_s) ds + W_t. \end{aligned} \quad (75)$$

4.1 Strong well posedness

To prove strong well posedness for (73) we will also use solutions U with values in \mathbb{R}^{2d} of

$$\begin{aligned} \lambda U(z) - \frac{1}{2} \text{Tr}(QD^2U(z)) - \langle Az, DU(z) \rangle - \langle B(z), DU(z) \rangle &= B(z), \\ \text{i.e., } \lambda U(z) - \mathcal{L}U(z) &= B(z) \end{aligned} \quad (76)$$

(defined componentwise at least for λ large enough). Note that $U = \begin{pmatrix} 0 \\ \tilde{u} \end{pmatrix}$ where

$$\lambda \tilde{u}(z) - \mathcal{L}\tilde{u}(z) = F(z)$$

is again defined componentwise ($\tilde{u} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$).

Remark 11 *In the following, according to (72), we will say that the singular diffusion Z_t (the noise acts only on the last d coordinates $\{e_{d+1}, \dots, e_{2d}\}$) or the associated Kolmogorov operator*

$$\mathcal{L}f(z) = \frac{1}{2} \Delta_v f(z) + \langle b(z), Df(z) \rangle,$$

$b(z) = Az + B(z)$, are hypoelliptic to refer to the fact that the vectors

$$\{e_{d+1}, \dots, e_{2d}, Ae_{d+1}, \dots, Ae_{2d}\}$$

generate \mathbb{R}^{2d} . Equivalently using Q given in (72) and the adjoint matrix A^* we have that the symmetric matrix $Q_t = \int_0^t e^{sA} Q e^{sA^*} ds$ is positive definite for any $t > 0$ (cf. (34)). Note that

$$\det(Q_t) = ct^{4d}, \quad t > 0.$$

We collect here some preliminary results, which we will later need. Recall the OU process

$$\begin{cases} dL_t = AL_t dt + R dW_t \\ L_0 = z \in \mathbb{R}^{2d} \end{cases}, \quad \text{i.e., } L_t = L_t^z = e^{tA}z + \int_0^t e^{(t-s)A} R dW_s. \quad (77)$$

Using the fact that L_t is hypoelliptic, for any $t > 0$, one gets that the law of L_t is equivalent to the Lebesgue measure in \mathbb{R}^{2d} (see for example the proof of the next lemma). We also have the following result.

Lemma 12 *Let (L_t^z) be the OU process solution of (77). Let $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for $q > 2d$. Then there exists a constant C depending on q, d and T such that*

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\int_0^T f(L_s^z) ds \right] \leq C \|f\|_{L^q(\mathbb{R}^{2d})}. \quad (78)$$

Proof. We need to compute

$$\mathbb{E} \left[\int_0^T f(L_s^z) ds \right] = \int_0^T P_s f(z) ds,$$

where P_t is the Ornstein-Uhlenbeck semigroup introduced in (33). By changing variable and using the Hölder inequality we find, for $t \in [0, T]$, $z \in \mathbb{R}^{2d}$,

$$\begin{aligned} |P_t f(z)| &= \left| c_d \int_{\mathbb{R}^{2d}} f(e^{tA}z + \sqrt{Q_t}y) e^{-\frac{|y|^2}{2}} dy \right| \leq c_q \left(\int_{\mathbb{R}^{2d}} |f(e^{tA}z + \sqrt{Q_t}y)|^q dy \right)^{1/q} \\ &= \frac{c_q}{(\det(Q_t))^{1/2q}} \left(\int_{\mathbb{R}^{2d}} |f(e^{tA}z + w)|^q dw \right)^{1/q} = \frac{c_q}{(\det(Q_t))^{1/2q}} \|f\|_{L^q(\mathbb{R}^{2d})}. \end{aligned}$$

with c_q independent of z . We now have to study when

$$\int_0^t \frac{1}{(\det(Q_s))^{1/2q}} ds < \infty. \quad (79)$$

By a direct computation for $s \rightarrow 0^+$

$$(\det(Q_s))^{1/2q} \sim c(s^{4d})^{1/2q},$$

hence the result follows for $q > 2d$. ■

We state now the classical Khas'minskii lemma for an OU process. The original version of this lemma ([Kh59], or [Sz98, Section 1, Lemma 2.1]) is stated for a Wiener process, but the proof only relies on the Markov property of the process, so that its extension to this setting requires no modification.

Lemma 13 (Khas'minskii 1959) *Let (L_t^z) be our $2d$ -dimensional OU process starting from z at time 0 and $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ be a positive Borel function. Then, for any $T > 0$ such that*

$$\alpha = \sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\int_0^T f(L_t^z) dt \right] < 1, \quad (80)$$

we also have

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\exp \left(\int_0^T f(L_t^z) dt \right) \right] < \frac{1}{1 - \alpha}. \quad (81)$$

We now introduce a generalization of the previous Khas'minskii lemma which we will use to prove the Novikov condition, allowing us to apply Girsanov's theorem.

Proposition 14 *Let (L_t) be the OU process solution of (77). Let $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for $q > 2d$. Then, there exists a constant K_f depending on d, q, T and continuously depending on $\|f\|_{L^q(\mathbb{R}^{2d})}$ such that*

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\exp \left(\int_0^T |f(L_s^z)| ds \right) \right] = K_f < \infty. \quad (82)$$

Proof. From Lemma 12, for any $a > 1$ s.t. $q/a > 2d$ we get

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\int_0^T |f|^a(L_s^z) ds \right] \leq C \|f\|_{L^q(\mathbb{R}^{2d})}^a.$$

Setting $\varepsilon = (C\|f\|_{L^q}^a)^{-1} \wedge 1$, we apply Young's inequality: $|f(z)| \leq \frac{\varepsilon}{a}|f(z)|^a + C_\varepsilon \frac{a-1}{a}$ and Khas'minskii's Lemma 13 replacing f with $\frac{\varepsilon}{a}|f|^a$ to get

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\exp \left(\int_0^T |f(L_s^z)| ds \right) \right] \leq \sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\exp \left(\int_0^T \frac{\varepsilon}{a} |f(L_s^z)|^a ds \right) \right] e^{Tc_{\varepsilon,a}} \leq \frac{1}{1-\alpha} e^{CT} < \infty.$$

■

The next result can be proved by using the Girsanov theorem (cf. [IW89] and [LS01]).

Theorem 15 *Suppose that in (73) we have $F \in L^p(\mathbb{R}^{2d}; \mathbb{R}^d)$ with $p > 4d$. Then the following statements hold.*

(i) *Equation (73) is well posed in the weak sense.*

(ii) *For any $z \in \mathbb{R}^{2d}$, $T > 0$ the law in the space of continuous functions $C([0, T]; \mathbb{R}^{2d})$ of the solution $Z = (Z_t) = (Z_t^z)$ to the equation (73) is equivalent to the law of the OU process $L = (L_t) = (L_t^z)$.*

(iii) *For any $t > 0$, $z \in \mathbb{R}^{2d}$, the law of Z_t is equivalent to the Lebesgue measure in \mathbb{R}^{2d} .*

Proof. (i) Existence. We argue similarly to the proof of Theorem IV.4.2 in [IW89]. Let $T > 0$. Starting from an Ornstein-Uhlenbeck process (cf. (77))

$$L_t = L_t^z = z + \int_0^t AL_s ds + RW_t, \quad t \geq 0$$

defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which it is defined an \mathbb{R}^d -valued Wiener process $(W_t) = W$, we can define the process

$$H_t := W_t - \int_0^t F(L_r) dr, \quad t \in [0, T]. \quad (83)$$

Since $p > 4d$, Proposition 14 with $f = F^2$ provides the Novikov condition ensuring that the process

$$\Phi_t = \exp \left(\int_0^t \langle F(L_s), dW_s \rangle - \frac{1}{2} \int_0^t |F(L_s)|^2 ds \right), \quad t \in [0, T],$$

is an \mathcal{F}_t -martingale. Then, by the Girsanov theorem $(H_t)_{t \in [0, T]}$ is a d -dimensional Wiener process on $(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s \leq T}, \mathbb{Q})$, where \mathbb{Q} is the probability measure on (Ω, \mathcal{F}_T) having density $\Phi = \Phi_T$ with respect to \mathbb{P} . We have that on the new probability space

$$L_t = L_t^z = z + \int_0^t AL_s ds + \int_0^t RF(L_s) ds + RH_t, \quad t \in [0, T]$$

(cf. (72)). Hence $L = (L_t)$ is a solution to (73) on $(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s \leq T}, \mathbb{Q})$.

Uniqueness. To prove weak uniqueness we use some results from [LS01]. First note that the process

$$V_t = v_0 + \int_0^t F(X_s, V_s) ds + W_t \quad (84)$$

(cf. (75)) is a process of diffusion type according to Definition 7 in page 118 of [LS01, Section 4.2]. Indeed, since $X_t = x_0 + \int_0^t V_s ds$ we have

$$V_t = v_0 + \int_0^t F\left(x_0 + \int_0^s V_r dr, V_s\right) ds + W_t$$

and the process $(b_s(V))_{s \in [0, T]} = (F(x_0 + \int_0^s V_r dr, V_s))_{s \in [0, T]}$ is (\mathcal{F}_t^V) -adapted (here \mathcal{F}_t^V is the σ -algebra generated by $\{V_s, s \in [0, t]\}$).

We can apply to $V = (V_t)$ Theorem 7.5 on page 257 of [LS01] (see also paragraph 7.2.7 in [LS01]): since $\int_0^T |b_s(V)|^2 ds < \infty$, \mathbb{P} -a.s., we obtain that

$$\mu_V \sim \mu_W \text{ on } \mathcal{B}(C([0, T]; \mathbb{R}^d)),$$

i.e. the laws of $V = (V_t)_{t \in [0, T]}$ and $W = (W_t)_{t \in [0, T]}$ are equivalent. Moreover, by [LS01, Theorem 7.7], the Radon-Nykodim derivative $\frac{\mu_V}{\mu_W}(x)$, $x \in C([0, T]; \mathbb{R}^d)$, verifies

$$\frac{\mu_V}{\mu_W}(W) = \exp\left(\int_0^T \langle b_s(W), dW_s \rangle - \frac{1}{2} \int_0^T |b_s(W)|^2 ds\right).$$

It follows that, for any Borel set $B \in \mathcal{B}(C([0, T]; \mathbb{R}^d))$,

$$\mathbb{E}[1_B(V)] = \mathbb{E}^{\mu_W}\left[1_B \frac{\mu_V}{\mu_W}\right] = \mathbb{E}\left[1_B(W) \exp\left(\int_0^T \langle b_s(W), dW_s \rangle - \frac{1}{2} \int_0^T |b_s(W)|^2 ds\right)\right];$$

this shows easily that uniqueness in law holds.

Clearly (iii) follows from (ii). Let us prove (ii).

(ii) The processes $L = (L_t)$ and $Z = (Z_t)$, $t \in [0, T]$, satisfy the same equation (73) in $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}, (H_t))$ and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, (W_t))$ respectively. Therefore, by weak uniqueness, the laws of L and Z on $C([0, T]; \mathbb{R}^{2d})$ are the same (under the probability measures \mathbb{Q} and \mathbb{P} respectively). Hence, for any Borel set $J \subset C([0, T]; \mathbb{R}^{2d})$, we have

$$\mathbb{E}[1_J(Z)] = \mathbb{E}[1_J(L) \Phi].$$

Since $W_t = (\langle L_t, e_{d+1} \rangle, \dots, \langle L_t, e_{2d} \rangle)$ we see that each W_s is measurable with respect to the σ -algebra generated by the random variable L_s , $s \leq T$. By considering L as a random variable with values in $C([0, T]; \mathbb{R}^{2d})$, we obtain that

$$\Phi = \exp[G(L)]$$

for some measurable function $G : M = C([0, T]; \mathbb{R}^{2d}) \rightarrow \mathbb{R}$. Using the laws μ_Z of Z and μ_L of L we find

$$\int_M 1_J(\omega) \mu_Z(d\omega) = \mathbb{E}[1_J(Z)] = \mathbb{E}[1_J(L) \exp[G(L)]] = \int_M 1_J(\omega) \exp[G(\omega)] \mu_L(d\omega).$$

Finally note that $|G(\omega)| < \infty$, for any $\omega \in M$ μ_L -a.s. (indeed $\int_M |G(\omega)| \mu_L(d\omega) = \mathbb{E}[|G(L)|] < \infty$). It follows that $\exp[G(\omega)] > 0$, for any $\omega \in M$ μ_L -a.s., and this shows that μ_L is equivalent to μ_Z . ■

We can now prove that the result of Lemma 12 holds also replacing the OU process L_t with Z_t .

Lemma 16 *Let Z_t^z be a solution of (73). Let $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for some $q > 2d$. Then there exists a constant C depending on q, d and T such that*

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\int_0^T f(Z_s^z) ds \right] \leq C \|f\|_{L^q(\mathbb{R}^{2d})} \quad (85)$$

and a constant K_f depending on q, d, T and continuously depending on $\|f\|_{L^q(\mathbb{R}^{2d})}$ for which

$$\sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\exp \left(\int_0^T f(Z_s^z) ds \right) \right] \leq K_f. \quad (86)$$

Proof. Recall that $F \in L^p(\mathbb{R}^{2d})$ for $p > 4d$. As seen in the previous proof, the laws of L_t and Z_t are the same under the \mathbb{Q} and \mathbb{P} respectively. Then, applying Hölder's inequality with $1/a + 1/a' = 1$ we have

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^T f(Z_s) ds \right] = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T f(L_s) ds \right] \leq \mathbb{E}^{\mathbb{P}} \left[\int_0^T f^a(L_s) ds \right]^{1/a} \mathbb{E}^{\mathbb{P}} [\Phi^{a'}]^{1/a'}.$$

Taking $a > 1$ small enough so that $q/a > 2d$, we can apply Lemma 12 to $|f|^a$ and control the first expectation on the right hand side with a constant times $\|f\|_{L^q(\mathbb{R}^{2d})}$. Then we write

$$\Phi^{a'} = \exp \left(\int_0^T \langle a' F(L_s), dW_s \rangle - \frac{1}{2} \int_0^T |a' F(L_s)|^2 ds + \frac{(a')^2 - a'}{2} \int_0^T |F(L_s)|^2 ds \right),$$

which has finite expectation due to Proposition 14. Both these estimates are uniform in z , so that (85) follows. Similarly, we have

$$\mathbb{E}^{\mathbb{P}} \left[\exp \left(\int_0^T f(Z_s) ds \right) \right] \leq \mathbb{E}^{\mathbb{P}} \left[\exp \left(2 \int_0^T f(L_s) ds \right) \right]^{1/2} \mathbb{E}^{\mathbb{P}} [\Phi^2]^{1/2}.$$

Both terms on the right hand side are finite due to Proposition 14: this proves (86). ■

Recall that we are always assuming Hypothesis 1.

Lemma 17 *Any process (Z_t) which is solution of the SDE (73) has finite moments of any order, uniformly in $t \in [0, T]$: for any $q \geq 2$*

$$\mathbb{E}[|Z_t^z|^q] \leq C_{z,q,d,T} < \infty. \quad (87)$$

Proof. Recall that, setting $Z_t^z = Z_t$,

$$Z_t = z + \int_0^t F(Z_s) ds + \int_0^t AZ_s ds + \int_0^t R dW_s.$$

It follows from (86) that for any $q \geq 1$, $\mathbb{E}[|\int_0^T F(Z_t) dt|^q] \leq C$. Using this bound, the Burkholder inequality for stochastic integrals and the Grönwall lemma we obtain the assertion. ■

In the proof of strong uniqueness of solutions of the SDE (73) we will have to deal with a new SDE with a Lipschitz drift coefficient, but a diffusion which only

has derivatives in L^p . However, following an idea of Veretennikov [Ver80], we can deal with increments of the diffusion coefficient on different solutions by means of the process A_t defined in (88). The following lemma generalizes Veretennikov's result to our degenerate kinetic setting and even provides bounds on the exponential of the process A_t . It will be a key element to prove continuity of the flow associated to (73) and will also be used in subsection 4.3 to study weak derivatives of the flow.

Lemma 18 *Let Z_t, Y_t be two solutions of (73) starting from $z, y \in \mathbb{R}^{2d}$ respectively, $U : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$, $U \in X_{p,s} \cap C_b^1$ (see Definition 6), and set*

$$A_t = \int_0^t 1_{\{Z_s \neq Y_s\}} \frac{\| [DU(Z_s) - DU(Y_s)]R \|^2_{HS}}{|Z_s - Y_s|^2} ds, \quad (88)$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Then, A_t is a well-defined, real valued, continuous, adapted, increasing process such that $\mathbb{E}[A_T] < \infty$, for every $t \in [0, T]$

$$\int_0^t \left\| [DU(Z_s^z) - DU(Y_s^y)]R \right\|_{HS}^2 ds = \int_0^t |Z_s^z - Y_s^y|^2 dA_s \quad (89)$$

and for any $k \in \mathbb{R}$, uniformly with respect to the initial conditions z, y :

$$\sup_{z, y \in \mathbb{R}^{2d}} \mathbb{E}[e^{kA_T}] < \infty. \quad (90)$$

Proof. Recall that $B = \begin{pmatrix} 0 \\ F \end{pmatrix}$ and $\| [DU(Z_s) - DU(Y_s)]R \|^2_{HS} = |D_v \tilde{u}(Z_s) - D_v \tilde{u}(Y_s)|^2$.

We have

$$\begin{aligned} |D_v \tilde{u}(Z_s) - D_v \tilde{u}(Y_s)| &= \left| \sum_{i=1}^{2d} (Z_s - Y_s)^i \int_0^1 D_i D_v \tilde{u}(rZ_s + (1-r)Y_s) dr \right| \\ &\leq |Z_s - Y_s| \int_0^1 |DD_v \tilde{u}(rZ_s + (1-r)Y_s)| dr. \end{aligned}$$

Set $Z_t^r = rZ_t + (1-r)Y_t$ (the process $(Z_t^r)_{t \geq 0}$ depends on $r \in [0, 1]$). We will first prove that

$$\mathbb{E} \left[\int_0^1 dr \int_0^t |DD_v \tilde{u}(Z_s^r)|^2 ds \right] < \infty, \quad t > 0. \quad (91)$$

By setting $F_s^r = [rF(Z_s) + (1-r)F(Y_s)]$ and $z^r = rz + (1-r)y$, we obtain, for any $r \in [0, 1]$,

$$Z_t^r = z^r + \int_0^t \begin{pmatrix} 0 \\ F_s^r \end{pmatrix} ds + \begin{pmatrix} 0 \\ W_t \end{pmatrix} + \int_0^t AZ_s^r ds.$$

Since $\int_0^T |F_s^r|^2 ds \leq C \int_0^T |F(Z_s)|^2 + |F(Y_s)|^2 ds$, using Hölder's inequality and Lemma 16 we get for all $k \in \mathbb{R}$

$$\sup_{z, y} \mathbb{E} \left[\exp \left(k \int_0^T |F_s^r|^2 ds \right) \right] \leq C_k < \infty, \quad (92)$$

where the constant C_k depends on k, p, T and $\|F\|_{L^p(\mathbb{R}^{2d})}$, but is uniform in z, y and r .

We can use again the Girsanov theorem (cf. the proof of Theorem 15). The process

$$\tilde{W}_t := W_t + \int_0^t F_s^r dr, \quad t \in [0, T]$$

is a d -dimensional Wiener process on $(\Omega, (\mathcal{F}_s)_{s \leq T}, \mathcal{F}_T, \mathbb{Q})$, where \mathbb{Q} is the probability measure on (Ω, \mathcal{F}_T) having the density ρ_r with respect to \mathbb{P} ,

$$\rho_r = \exp\left(\int_0^T -\langle F_s^r, dW_s \rangle - \frac{1}{2} \int_0^T |F_s^r|^2 ds\right).$$

Recalling the Ornstein-Uhlenbeck process L_t (starting at z^r), i.e.,

$$L_t = e^{tA} z^r + W_A(t), \quad \text{where } W_A(t) = \int_0^t e^{(t-s)A} R dW_s, \quad (93)$$

we have:

$$Z_t^r = L_t + \int_0^t e^{(t-s)A} R F_s^r ds.$$

Hence

$$Z_t^r = e^{tA} z^r + \int_0^t e^{(t-s)A} d\tilde{W}_s$$

is an OU process on $(\Omega, (\mathcal{F}_s)_{s \leq T}, \mathcal{F}_T, \rho_r \mathbb{P})$.

We now find, by the Hölder inequality, for some $a > 1$ such that $1/a + 1/a' = 1$,

$$\begin{aligned} \mathbb{E}\left[\rho_r^{-1/a} \rho_r^{1/a} \int_0^t |DD_v \tilde{u}(Z_s^r)|^2 ds\right] &\leq C_T \left(\mathbb{E}\left[\rho_r \int_0^t |DD_v \tilde{u}(Z_s^r)|^{2a} ds\right]\right)^{1/a} (\mathbb{E}[\rho_r^{-a'/a}])^{1/a'} \\ &\leq C_T \left(\mathbb{E}\left[\rho_r \int_0^t |DD_v \tilde{u}(Z_s^r)|^{2a} ds\right]\right)^{1/a}, \end{aligned} \quad (94)$$

for any $t \in [0, T]$. Observe that the bound on the moments of ρ_r is uniform in the initial conditions $z, y \in \mathbb{R}^{2d}$ due to (92). Setting $f(z) = |DD_v \tilde{u}(z)|^{2a}$ and using the Girsanov Theorem, assertion (91) follows from Lemma 12 if we fix $a > 1$ such that $q = p/2a > 2d$.

Therefore, the process A_t is well defined and $\mathbb{E}[A_t] < \infty$ for all $t \in [0, T]$. (89) and the other properties of A_t follow.

To prove the exponential integrability of the process A_t we proceed in a way similar to [FF13a, Lemma 4.5]. Using the convexity of the exponential function we get

$$\mathbb{E}\left[e^{kA_T}\right] \leq \mathbb{E}\left[\exp\left(k \int_0^T \int_0^1 |DD_v \tilde{u}(Z_s^r)|^2 dr ds\right)\right] \leq \int_0^1 \mathbb{E}\left[\exp\left(k \int_0^T |DD_v \tilde{u}(Z_s^r)|^2 ds\right)\right] dr$$

and we can continue as above (superscripts denote the probability measure used to take expectations)

$$\begin{aligned} \sup_{z,y} \mathbb{E}^{\mathbb{P}} \left[e^{kA_T} \right] &\leq \sup_{z,y} \int_0^1 \mathbb{E}^{\mathbb{P}} \left[\rho_r^{-1/a} \rho_r^{1/a} \exp \left(k \int_0^T |DD_v \tilde{u}(Z_s^r)|^2 ds \right) \right] dr \\ &\leq C_T \sup_{z,y} \int_0^1 \mathbb{E}^{\mathbb{Q}} \left[\exp \left(ak \int_0^T |DD_v \tilde{u}(Z_s^r)|^2 ds \right) \right]^{1/a} dr \\ &\leq C_T \sup_{z,y} \int_0^1 \mathbb{E}^{\mathbb{P}} \left[\exp \left(ak \int_0^T |DD_v \tilde{u}(L_s)|^2 ds \right) \right]^{1/a} dr. \end{aligned}$$

The last integral is finite due to Lemma 14 because $p/2 > 2d$. The proof is complete. \blacksquare

Proposition 19 (Itô formula) *If $\varphi : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ belongs to $X_{p,s} \cap C_b^1$ and Z_t is a solution of (73), for any $0 \leq s \leq t \leq T$ the following Itô formula holds:*

$$\varphi(Z_t) = \varphi(Z_s) + \int_s^t [b(Z_r) \cdot D\varphi(Z_r) + \frac{1}{2} \Delta_v \varphi(Z_r)] dr + \int_s^t D_v \varphi(Z_r) dW_r. \quad (95)$$

Proof. Note that we can use (iii) in Theorem 15 to give a meaning to the critical term $\int_s^t \Delta_v \varphi(Z_r) dr$. The result then follows approximating φ with regular functions and using Lemma 16.

Let $\varphi_\varepsilon \in C_c^\infty \rightarrow \varphi$ in $X_{p,s}$. φ_ε satisfy the assumptions of the classical Itô formula, which provides an analogue of (95) for $\varphi_\varepsilon(Z_t)$. For any fixed t , the random variables $\varphi_\varepsilon(Z_t) \rightarrow \varphi(Z_t)$ \mathbb{P} -almost surely. Using that $D\varphi$ is bounded and almost surely $F(Z_r)$ and AZ_r are in $L^1(0, T)$ (this follows by Lemma 16 and Lemma 17 respectively), the dominated convergence theorem gives the convergence of the first term in the Lebesgue integral. For the second term we use Lemma 16 with $f = \Delta_v \varphi_\varepsilon - \Delta_v \varphi$ (recall that $p > 6d$):

$$\mathbb{E} \left[\int_s^t \Delta_v \varphi_\varepsilon(Z_r) - \Delta_v \varphi(Z_r) dr \right] \leq C \|\Delta_v \varphi_\varepsilon - \Delta_v \varphi\|_{L^p(\mathbb{R}^{2d})} \rightarrow 0.$$

In the same way, one can show that $\mathbb{E} \left[\int_s^t |D_v \varphi_\varepsilon(Z_r) - D_v \varphi(Z_r)|^2 dr \right]$ converges to zero, which implies the convergence of the stochastic integral by the Itô isometry. \blacksquare

Remark 20 *Using the boundedness of φ , it is easy to generalize the above Itô formula (95) to $\varphi^\alpha(Z_t)$ for any $\alpha \geq 2$.*

We can finally prove the well-posedness in the strong sense of the degenerate SDE (73). A different proof of this result in a Hölder setting is contained in [CdR], but no explicit control on the dependence on the initial data is given there, so that a flow cannot be constructed. See also the more recent results of [WZ15]. We here present a different, and in some sense more constructive, proof. This approach, based on ideas introduced in [FGP10], [KR05], [FF13a], will even allow us to obtain some regularity results on certain derivatives of the solution. We will use Theorem 10 from Section 3.3, which provides the regularity $X_{p,s} \cap C_b^1(\mathbb{R}^{2d})$ of solutions of (76).

Theorem 21 *Equation (73) is well posed in the strong sense.*

Proof. Since we have weak well posedness by (i) of Theorem 15, the Yamada-Watanabe principle provides strong existence as soon as strong uniqueness holds. Therefore, we only need to prove strong uniqueness. This can be done by using an appropriate change of variables which transforms equation (73) into an equation with more regular coefficients. This method was first introduced in [FGP10], where it is used to prove strong uniqueness for a non degenerate SDE with a Hölder drift coefficient.

Here, the SDE is degenerate and we only need to regularize the second component of the drift coefficient, $F(\cdot)$, which is not Lipschitz continuous. We therefore introduce the auxiliary PDE (76) with λ large enough such that

$$\|U_\lambda\|_{L^\infty(\mathbb{R}^{2d})} + \|DU_\lambda\|_{L^\infty(\mathbb{R}^{2d})} < 1/2 \quad (96)$$

holds (see (68)). In the following we will always use this value of λ and to ease notation we shall drop the subscript for the solution U_λ of (76), writing $U_\lambda = U$.

Let Z_t be one solution to (73) starting from $z \in \mathbb{R}^{2d}$. Since

$$Z_t = z + \int_0^t B(Z_s) ds + \int_0^t AZ_s ds + RW_t,$$

and $U \in X_{p,s} \cap C_b^1$ (see Theorem 10), by the Itô formula of Proposition 19 we have

$$\begin{aligned} U(Z_t) &= U(z) + \int_0^t DU(Z_s)R dW_s + \int_0^t \mathcal{L}U(Z_s) ds \\ &= U(z) + \int_0^t DU(Z_s)R dW_s + \lambda \int_0^t U(Z_s) ds - \int_0^t B(Z_s) ds. \end{aligned}$$

Using the SDE to rewrite the last term we find

$$U(Z_t) = U(z) + \int_0^t DU(Z_s)R dW_s + \lambda \int_0^t U(Z_s) ds - Z_t + z + \int_0^t AZ_s ds + RW_t$$

and so

$$Z_t = U(z) - U(Z_t) + \int_0^t DU(Z_s)R dW_s + \lambda \int_0^t U(Z_s) ds + z + \int_0^t AZ_s ds + RW_t. \quad (97)$$

Let now Y_t be another solution starting from $y \in \mathbb{R}^{2d}$ and let

$$\gamma(x) = x + U(x), \quad x \in \mathbb{R}^{2d}. \quad (98)$$

We have $\gamma(z) - \gamma(y) = z - y + U(z) - U(y)$, and so $|z - y| \leq |U(z) - U(y)| + |\gamma(z) - \gamma(y)|$. Since we have chosen λ such that $\|DU\|_{L^\infty(\mathbb{R}^{2d})} < 1/2$, there exist finite constants $C, c > 0$ such that

$$c|\gamma(z) - \gamma(y)| \leq |z - y| \leq C|\gamma(z) - \gamma(y)|, \quad \forall z, y \in \mathbb{R}^{2d}. \quad (99)$$

We find

$$d\gamma(Z_t) = (\lambda U(Z_t) + AZ_t) dt + (DU(Z_t) + \mathbb{I})R \cdot dW_t \quad (100)$$

and

$$\begin{aligned} \gamma(Z_t) - \gamma(Y_t) &= z - y + U(z) - U(y) + \int_0^t [DU(Z_s) - DU(Y_s)]R \cdot dW_s \\ &\quad + \lambda \int_0^t [U(Z_s) - U(Y_s)]ds + \int_0^t A(Z_s - Y_s) ds. \end{aligned} \quad (101)$$

For $a \geq 2$, let us apply Itô formula to $|\gamma(Z_t) - \gamma(Y_t)|^a = (\sum_{i=1}^{2d} [\gamma(Z_t) - \gamma(Y_t)]_i^2)^{a/2}$:

$$\begin{aligned} d[|\gamma(Z_t) - \gamma(Y_t)|^a] &= a|\gamma(Z_t) - \gamma(Y_t)|^{a-2}(\gamma(Z_t) - \gamma(Y_t)) \cdot d(\gamma(Z_t) - \gamma(Y_t)) \\ &\quad + \frac{a}{2}|\gamma(Z_t) - \gamma(Y_t)|^{a-4} \sum_{i,j=1}^{2d} \sum_{k=1}^d \left\{ (a-2)(\gamma(Z_t) - \gamma(Y_t))_i (\gamma(Z_t) - \gamma(Y_t))_j \right. \\ &\quad \quad \left. + \delta_{i,j} |\gamma(Z_t) - \gamma(Y_t)|^2 \right\} \\ &\quad \times \left[(DU(Z_t) - DU(Y_t))R \right]_{k,i} \left[(DU(Z_t) - DU(Y_t))R \right]_{k,j} dt \\ &\leq a|\gamma(Z_t) - \gamma(Y_t)|^{a-2} \left\{ (\gamma(Z_t) - \gamma(Y_t)) \cdot [DU(Z_t) - DU(Y_t)]R \cdot dW_t \right. \\ &\quad + (\gamma(Z_t) - \gamma(Y_t)) \cdot \left(\lambda[U(Z_t) - U(Y_t)] + A(Z_t - Y_t) \right) dt \\ &\quad \left. + C_{a,d} \left\| [DU(Z_t) - DU(Y_t)]R \right\|_{HS}^2 dt \right\}. \end{aligned}$$

Note that Z_t has finite moments of all orders, and U is bounded, so that also the process $\gamma(Z_t)$ has finite moments of all orders. Using also that DU is a bounded function, we deduce that the stochastic integral is a martingale M_t :

$$M_t = \int_0^t a|\gamma(Z_s) - \gamma(Y_s)|^{a-2}(\gamma(Z_s) - \gamma(Y_s)) \cdot [DU(Z_s) - DU(Y_s)]R \cdot dW_s$$

As in [KR05] and [FF11] we now consider the following process

$$B_t = \int_0^t 1_{\{Z_s \neq Y_s\}} \frac{\left\| [DU(Z_s) - DU(Y_s)]R \right\|_{HS}^2}{|\gamma(Z_s) - \gamma(Y_s)|^2} ds \leq C^2 A_t, \quad (102)$$

where we have used the equivalence (99) between $|Z_t - Y_t|$ and $|\gamma(Z_t) - \gamma(Y_t)|$ and A_t is the process defined by (88) and studied in Lemma 18. Just as the process A_t , also B_t has finite moments, and even its exponential has finite moments. With these notations at hand we can rewrite

$$\begin{aligned} d[|\gamma(Z_t) - \gamma(Y_t)|^a] &\leq a|\gamma(Z_t) - \gamma(Y_t)|^{a-2}(\gamma(Z_t) - \gamma(Y_t)) \cdot \left(\lambda[U(Z_t) - U(Y_t)] + A(Z_t - Y_t) \right) dt \\ &\quad + dM_t + C_{a,d}|\gamma(Z_t) - \gamma(Y_t)|^a dB_t. \end{aligned}$$

Again by Itô formula we have

$$\begin{aligned} d\left(e^{-C_{a,d}B_t}|\gamma(Z_t) - \gamma(Y_t)|^a\right) &= -C_{a,d}e^{-C_{a,d}B_t}|\gamma(Z_t) - \gamma(Y_t)|^a dB_t \\ &+ e^{-C_{a,d}B_t}\left\{a|\gamma(Z_t) - \gamma(Y_t)|^{a-2}(\gamma(Z_t) - \gamma(Y_t)) \cdot \left(\lambda[U(Z_t) - U(Y_t)] + A(Z_t - Y_t)\right) dt \right. \\ &\quad \left. + dM_t + C_{a,d}|\gamma(Z_t) - \gamma(Y_t)|^a dB_t\right\}. \end{aligned} \quad (103)$$

The term $e^{-C_{a,d}B_t}dM_t$ is still the differential of a zero-mean martingale. Integrating and taking the expected value we find

$$\begin{aligned} \mathbb{E}\left[e^{-C_{a,d}B_t}|\gamma(Z_t) - \gamma(Y_t)|^a\right] &= |\gamma(z) - \gamma(y)|^a + \mathbb{E}\left[\int_0^t e^{-C_{a,d}B_s} a|\gamma(Z_s) - \gamma(Y_s)|^{a-2} \right. \\ &\quad \left. \times (\gamma(Z_s) - \gamma(Y_s)) \cdot \left(\lambda[U(Z_s) - U(Y_s)] + A(Z_s - Y_s)\right) ds\right]. \end{aligned}$$

Using again the equivalence (99) between $|Z_t - Y_t|$ and $|\gamma(Z_t) - \gamma(Y_t)|$ and the fact that U is Lipschitz continuous, this finally provides the following estimate:

$$\mathbb{E}\left[e^{-C_{a,d}B_t}|Z_t - Y_t|^a\right] \leq C\left\{|z - y|^a + \int_0^t \mathbb{E}\left[e^{-C_{a,d}B_s}|Z_s - Y_s|^a\right] ds\right\}.$$

By Grönwall's inequality, there exists a finite constant C' such that

$$\mathbb{E}\left[e^{-C_{a,d}B_t}|Z_t - Y_t|^a\right] \leq C'|z - y|^a. \quad (104)$$

Using that B_t is increasing and a.s. $B_T < \infty$, taking $z = y$ we get for any fixed $t \in [0, T]$ that $\mathbb{P}(Z_t \neq Y_t) = 0$. Strong uniqueness follows by the continuity of trajectories. This completes the proof. ■

Corollary 22 *Using the finite moments of the exponential of the process B_t , we can also prove that for any $a \geq 2$,*

$$\mathbb{E}\left[|Z_t - Y_t|^a\right] \leq C|z - y|^a. \quad (105)$$

Proof. Using Hölder's inequality and for an appropriate constant c , we have

$$\begin{aligned} \mathbb{E}\left[|Z_t - Y_t|^a\right] &= \mathbb{E}\left[e^{cB_t}e^{-cB_t}|Z_t - Y_t|^a\right] \\ &\leq C\left(\mathbb{E}\left[e^{-2cB_t}|Z_t - Y_t|^{2a}\right]\right)^{1/2} \leq C|z - y|^a. \end{aligned}$$

■

4.2 Stochastic Flow

Many of the proofs of results contained in this section follow closely the proofs of [Ku84, Chapter II.2] or [Ku90, Chapter 4.5]. To avoid reporting lengthy computations from those references, we will often content ourselves with describing how to adapt the classical proofs to our setting.

We stress that the main ingredient is the quantitative control on the continuous dependence on the initial data of solutions of the SDE (73), which was already obtained in Corollary 22.

We will repeatedly use the transformation γ introduced in (98) and the Itô formula (100), which we rewrite as

$$d\gamma(Z_t^z) = \tilde{b}(Z_t) dt + \tilde{\sigma}(Z_t) \cdot dW_t,$$

where $\tilde{b}(z) = \lambda U(z) + Az$ is Lipschitz continuous and $\tilde{\sigma}(z) = (DU(z) + \mathbb{I}_{2d})R$ is bounded.

4.2.1 Continuity

Lemma 23 *Let a be any real number. Then there is a positive constant C_a independent of $t \in [0, T]$ and $z \in \mathbb{R}^{2d}$ such that*

$$\mathbb{E}\left[\left(1 + |Z_t^z|^2\right)^a\right] \leq C_{a,d}(1 + |z|^2)^a. \quad (106)$$

Proof. Using the boundedness of the solution U of the PDE (76) (see (96)) one can show the equivalence

$$c(1 + |z|^2) \leq 1 + |\gamma(z)|^2 \leq C(1 + |z|^2).$$

Set $\gamma_t = \gamma(Z_t^z)$. Then, it is enough to prove that $\mathbb{E}\left[(1 + |\gamma_t|^2)^a\right] \leq C_{a,d}(1 + |\gamma(z)|^2)^a$. Set $f(z) := (1 + |z|^2)$. The idea is to apply Itô formula to $g(\gamma_t)$, where $g(z) = f^a(z)$. Since

$$\frac{\partial g}{\partial z_i}(z) = 2af^{a-1}(z)z_i, \quad \frac{\partial^2 g}{\partial z_i \partial z_j}(z) = 4a(a-1)f^{a-2}(z)z_i z_j + 2af^{a-1}(z)\delta_{i,j},$$

we see that

$$\begin{aligned} g(\gamma_t) - g(\gamma(z)) &= 2a \int_0^t f^{a-1}(\gamma_s) \gamma_s \cdot \tilde{\sigma}(\gamma_s) \cdot dW_s + 2a \int_0^t f^{a-1}(\gamma_s) \gamma_s \cdot \tilde{b}(\gamma_s) ds \\ &\quad + \sum_{i,j,k=1}^{2d} \int_0^t 2af^{a-2}(\gamma_s) \left[(a-1)\gamma_s^i \gamma_s^j + \delta_{i,j} f(\gamma_s) \right] \tilde{\sigma}^{k,i}(\gamma_s) \tilde{\sigma}^{k,j}(\gamma_s) ds. \end{aligned} \quad (107)$$

Here we have used the relation $d\langle \gamma_t, \gamma_t \rangle = \tilde{\sigma}(\gamma_t) \tilde{\sigma}^t(\gamma_t) dt$. Since γ_t has finite moments, the first term on the right hand side of (107) is a martingale with zero mean. Note that $f(z) \geq 1$, so that $f^{a-1} \leq f^a$ and $|z| \leq f^{1/2}(z)$. Moreover, since $\tilde{\sigma}$ is bounded and \tilde{b} is Lipschitz continuous, $|\tilde{b}(z)| \leq C(1 + |z|) \leq C f^{1/2}(z)$. Using all this, we can see that the second and third term on the right hand side of (107) are dominated by a constant times $\int_0^t g(\gamma_s) ds$. Therefore, taking expectations in (107) we have

$$\mathbb{E}[g(\gamma_t)] - g(\gamma_0) \leq C_{a,d} \int_0^t \mathbb{E}[g(\gamma_s)] ds,$$

and the result follows by Grönwall's lemma. ■

Proposition 24 *Let Z_t^z be the unique strong solution to the SDE (73) given by Theorem 21 and starting from the point $z \in \mathbb{R}^{2d}$. For any $a > 2$, $s, t \in [0, T]$ and $z, y \in \mathbb{R}^{2d}$ we have*

$$\mathbb{E}\left[|Z_t^z - Z_s^y|^a\right] \leq C_{a,d,\lambda,T} \left\{ |z - y|^a + (1 + |z|^a + |y|^a) |t - s|^{a/2} \right\}. \quad (108)$$

Proof. Assume $t > s$. It suffice to show that

$$\mathbb{E}\left[|Z_s^z - Z_s^y|^a\right] \leq C |z - y|^a \quad (109)$$

$$\mathbb{E}\left[|Z_t^z - Z_s^z|^a\right] \leq C(1 + |z|^a) |t - s|^{a/2}. \quad (110)$$

The first inequality was obtained in Corollary 22. To prove the second inequality we use the equivalence (99) between Z_t and $\gamma(Z_t)$. We use the Itô formula (100) for $\gamma(Z_t)$ and $\gamma(Z_s)$: we can control the differences of the first and last term using the fact that U and DU are bounded, together with Burkholder's inequality

$$\begin{aligned} \mathbb{E}\left[|Z_t^z - Z_s^z|^a\right] &\leq C \mathbb{E}\left[|\gamma(Z_t^z) - \gamma(Z_s^z)|^a\right] \\ &\leq C_{a,d} \left\{ \left| \int_s^t \|\lambda U\|_\infty^2 dr \right|^{a/2} + \mathbb{E}\left[\left| \int_s^t AZ_r^z dr \right|^a \right] + \|DU\|_\infty \mathbb{E}\left[|R(W_t - W_s)|^a \right] \right\} \end{aligned}$$

and for the linear part we use Hölder's inequality and Lemma 23:

$$\mathbb{E}\left[\left| \int_s^t AZ_r^z dr \right|^a \right] \leq (t - s)^{a/2} \mathbb{E}\left[\int_s^t |AZ_r^z|^a dr \right] \leq C(t - s)^{a/2} \int_s^t (1 + |z|^2)^{a/2} dr.$$

■

Applying Kolmogorov's regularity theorem (see [Ku84, Theorem I.10.3]), we immediately obtain the following

Theorem 25 *The family of random variables (Z_t^z) , $t \in [0, T]$, $z \in \mathbb{R}^d$, admits a modification which is locally α -Hölder continuous in z for any $\alpha < 1$ and β -Hölder continuous in t for any $\beta < 1/2$.*

From now on, we shall always use the continuous modification of Z provided by this theorem.

4.2.2 Injectivity and Surjectivity

The proofs of the injectivity and surjectivity are inspired by [Ku84] and are similar to the ones given in Section 5 of [FF13a]. Thus proofs of the main results in this section are given in Appendix.

To obtain the injectivity of the flow, we review the computations of Proposition 24: we now want to allow the exponent a to be negative. The proofs of the following two lemmas are given in Appendix.

Lemma 26 *Let a be any real number and $\varepsilon > 0$. Then there is a positive constant $C_{a,d}$ (independent of ε) such that for any $t \in [0, T]$ and $z, y \in \mathbb{R}^{2d}$*

$$\mathbb{E}\left[\left(\varepsilon + |Z_t^z - Z_t^y|^2 \right)^a \right] \leq C_{a,d} (\varepsilon + |z - y|^2)^a. \quad (111)$$

Corollary 27 *Let ε tend to zero in Lemma 26. Then, by monotone convergence, we have:*

$$\mathbb{E}\left[|Z_t^z - Z_t^y|^a\right] \leq C_{a,d}|z - y|^a. \quad (112)$$

Considering the case $a < 0$ we get that $z \neq y \Rightarrow Z_t^z \neq Z_t^y$ a.s. for any $t \in [0, T]$.

Kunita in [Ku84] calls the previous property “weak injectivity”. This intermediate result allows to obtain the following lemma.

Lemma 28 *For $t \in [0, T]$ set $\eta_t(z, y) := |Z_t^z - Z_t^y|^{-1}$. Then for any $a > 2$ there exists a constant $C = C_{a,d,\lambda,T}$ such that for any $\delta > 0$*

$$\begin{aligned} \mathbb{E}\left[|\eta_t(z, y) - \eta_{t'}(z', y')|^a\right] &\leq C \delta^{-2a} \left[|z - z'|^a + |y - y'|^a\right. \\ &\quad \left. + (1 + |z|^a + |z'|^a + |y|^a + |y'|^a)|t - t'|^{a/2}\right] \end{aligned}$$

holds for any $t, t' \in [0, T]$ and $|z - z'| \geq \delta, |y - y'| \geq \delta$.

Theorem 29 *The map $Z_t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is one to one, for any $t \in [0, T]$ almost surely.*

Proof. Take $a/2 > 2(d+1)$ in Lemma 28. Kolmogorov’s theorem states that $\eta_t(z, y)$ is continuous in (t, z, y) in the domain $\{(t, z, y) | t \in [0, T], |z - y| \geq \delta\}$. Since δ is arbitrary, it is also continuous in the domain $\mathcal{D} := \{(t, z, y) | t \in [0, T], z \neq y\}$. Note that \mathcal{D} has at most two connected components, both intersecting the hyperplane $\{t = 0\}$. Then, since η_0 is finite, η_t must be finite on all of \mathcal{D} . Therefore, if $z \neq y$, $Z_t^z \neq Z_t^y$, and the theorem is proved. ■

Surjectivity will follow from the next lemma which is similar to [Ku84, Lemma II.4.2]. Theorem 31 below can be proved using an homotopy argument, as in [Ku84, pag. 226]. Both proofs will be given in Appendix.

Lemma 30 *Let $\widehat{\mathbb{R}}^{2d}$ be the one point compactification (Alexandrov compactification) of \mathbb{R}^{2d} . For $z \in \mathbb{R}^{2d} \setminus \{0\}$ set $\widehat{z} := z/|z|^2$, $\widehat{z} := \infty$ for $z = 0$ and define for every $t \in [0, T]$*

$$\eta_t(\widehat{z}) := \begin{cases} \frac{1}{1 + |Z_t^z|} & \text{if } z \in \mathbb{R}^{2d}, \\ 0 & \text{if } \widehat{z} = 0. \end{cases}$$

Then, for any positive a there exists a constant $C = C_{a,d,\lambda,T}$ such that

$$\mathbb{E}\left[|\eta_t(\widehat{z}) - \eta_{t'}(\widehat{y})|^a\right] \leq C\left(|\widehat{z} - \widehat{y}|^a + |t - t'|^{a/2}\right), \quad \widehat{y}, \widehat{z} \in \widehat{\mathbb{R}}^{2d}, t, t' \in [0, T].$$

Theorem 31 *The map $Z_t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is onto for any $t \in [0, T]$ almost surely.*

4.2.3 The flow

We resume the results we have obtained so far for the flow associated to the SDE (73) in the following theorem.

Theorem 32 *The unique strong solution $Z_t = (X_t, V_t)$ of the SDE (2) defines a stochastic flow of Hölder continuous homeomorphisms ϕ_t .*

Proof. The map $Z_t(\omega)$ is Hölder continuous, see Theorem 25, it is one to one by Theorem 29 and it is onto by Theorem 31. Hence the inverse map $(Z_t(\omega))^{-1}$ is well defined, one to one and onto. We claim that it is also continuous. Indeed, since the map $\widehat{Z}_t(\omega)$ is one to one and continuous from the compact space $\widehat{\mathbb{R}}^{2d}$ into itself, it is a closed map. Hence the inverse map $(\widehat{Z}_t(\omega))^{-1}$ is continuous, and so is its restriction to \mathbb{R}^{2d} . ■

4.3 Regularity of the derivatives

Although F is not even weakly differentiable, from the reformulation (97) of equation (73) it is reasonable to expect differentiability of the flow, since the derivatives DX_t , DV_t with respect to the initial conditions (x, v) formally solve suitable SDEs with well-defined, integrable coefficients. We have the following result.

Theorem 33 *Let $\phi_t(z)$ be the flow associated to (73) provided by Theorem 32. Then, for any $t \in [0, T]$, \mathbb{P} -a.s., the random variable $\phi_t(z)$ admits a weak distributional derivative with respect to z ; moreover $D_z \phi_t \in L_{loc}^p(\Omega \times \mathbb{R}^{2d})$, for any $p \geq 1$.*

Proof. *Step 1.* (Bounds on difference quotients) It is sufficient to prove the existence and regularity of $D_{z_i} \phi_t$ for some fixed $i \in \{1, \dots, 2d\}$. We omit to write i and set $e = e_i$.

Introduce for every $h > 0$ the stochastic processes

$$\theta_t^h(z) = \frac{\phi_t(z + he) - \phi_t(z)}{h}, \quad \xi_t^h(z) = \frac{\gamma(\phi_t(z + he)) - \gamma(\phi_t(z))}{h}, \quad (113)$$

where $\gamma(z) = z + U(z)$ as in (98). It is clear that they have finite moments of all orders because ϕ and $\gamma(\phi)$ do. The two processes are also equivalent in the sense that there exist constants C_1, C_2 such that

$$C_1 |\xi_t^h(z)| \leq |\theta_t^h(z)| \leq C_2 |\xi_t^h(z)|. \quad (114)$$

This follows from (99). To fix the ideas, consider the case $i > d$. We have

$$\begin{aligned} \xi_t^h &= e + \frac{1}{h} \left[U(z + he) - U(z) \right] \\ &+ \int_0^t \frac{\lambda}{h} \left[U(\phi_s(z + he)) - U(\phi_s(z)) \right] + A\theta_t^h(z) \, ds \\ &+ \frac{1}{h} \int_0^t \left[D_v U(\phi_s(z + he)) - D_v U(\phi_s(z)) \right] R \cdot dW_s. \end{aligned} \quad (115)$$

Proceeding as in the proof of Theorem 21 above we have

$$\begin{aligned}
d|\xi_t^h|^p &\leq p|\xi_t^h|^{p-2}\xi_t^h \cdot \left(\frac{\lambda}{h}[U(\phi_t(z+he)) - U(\phi_t(z))] + A\theta_t^h(z)\right) dt \\
&\quad + \frac{p}{h}|\xi_t^h|^{p-2}\xi_t^h \cdot \left[D_v U(\phi_t(z+he)) - D_v U(\phi_t(z))\right] R \cdot dW_t \\
&\quad + \frac{C_{p,d}}{h^2}|\xi_t^h|^{p-2} \left\|D_v U(\phi_t(z+he)) - D_v U(\phi_t(z))R\right\|_{HS}^2 dt \\
&= p|\xi_t^h|^{p-2}\xi_t^h \cdot \left(\frac{\lambda}{h}[U(\phi_t(z+he)) - U(\phi_t(z))] + A\theta_t^h(z)\right) dt \\
&\quad + dM_t^h + C_{p,d}|\xi_t^h|^{p-2}|\theta_t^h|^2 dA_t,
\end{aligned}$$

where the process A_t is defined as in (88), but with $Z = \phi(z+he)$ and $Y = \phi(z)$, and for every $h > 0$, dM_t^h is the differential of a martingale because DU is bounded and ξ_t^h has finite moments. Setting $C_p = (C_2)^2 C_{p,d}$ we get

$$\begin{aligned}
d\left(e^{-C_p A_t} |\xi_t^h|^p\right) &= -C_p e^{-C_p A_t} |\xi_t^h|^p dA_t + e^{-C_p A_t} d(|\xi_t^h|^p) \\
&\leq e^{-C_p A_t} p|\xi_t^h|^{p-2}\xi_t^h \cdot \left(\frac{\lambda}{h}[U(\phi_t(z+he)) - U(\phi_t(z))] + A\theta_t^h(z)\right) dt + e^{-C_p A_t} dM_t^h.
\end{aligned} \tag{116}$$

After integrating and taking expectations we find

$$\begin{aligned}
\mathbb{E}\left[e^{-C_p A_t} |\xi_t^h|^p\right] &\leq \left|e + \frac{1}{h}[U(z+he) - U(z)]\right|^p \\
&\quad + \int_0^t \mathbb{E}\left[e^{-C_p A_s} p|\xi_s^h|^{p-2}\xi_s^h \cdot \left(\frac{\lambda}{h}[U(\phi_s(z+he)) - U(\phi_s(z))] + A\theta_s^h(z)\right)\right] ds \\
&\leq C(1 + \|DU\|_{L^\infty(\mathbb{R}^{2d})}^p) + \int_0^t C(\lambda\|DU\|_{L^\infty(\mathbb{R}^{2d})} + |A|)\mathbb{E}\left[e^{-C_p A_s} p|\xi_s^h|^p\right] ds.
\end{aligned}$$

A similar estimate holds for the case $i \leq d$. We now apply Grönwall's inequality and proceeding as in the proof of Corollary 22 we finally get that

$$\mathbb{E}\left[|\theta_t^h|^p\right] \leq C\mathbb{E}\left[|\xi_t^h|^p\right] \leq C_{p,d,T,\lambda} < \infty. \tag{117}$$

Step 2. (Derivative of the Flow) Remark that, due to the boundedness of DU , the bound (117) is uniform in h and z , and we get

$$\sup_{z \in \mathbb{R}^{2d}} \sup_{h \in (0,1]} \mathbb{E}\left[|\theta_t^h|^p\right] \leq C_{p,d,T,\lambda} < \infty. \tag{118}$$

We can then apply [BF13, Corollary 3.5] and obtain the existence of the weak derivative for the flow $D\phi_t \in L_{loc}^p(\Omega \times \mathbb{R}^{2d})$. ■

Remark 34 *Since the bound (117) is also uniform in time, applying [BF13, Theorem 3.6] one would also get the existence of the weak derivative as a process $D\phi_t$ belonging to $L_{loc}^p([0, T] \times \mathbb{R}^{2d})$ with probability one, and the weak convergence $\theta_t^h \rightharpoonup D\phi_t$ in $L_{loc}^p(\Omega \times [0, T] \times \mathbb{R}^{2d})$.*

5 Stochastic kinetic equation

We present here results on the stochastic kinetic equation (1). The first result concerns existence of solutions with a certain Sobolev regularity (see Theorem 38). The second one is about uniqueness of solutions (see Theorem 39).

We will use the results of the previous sections together with results similar to the ones given in [FF13b] to approximate the flow associated to the equation of characteristics. We report them in the Appendix for the sake of completeness. To prove that some degree of Sobolev regularity of the initial condition is preserved one has to deal with weakly differentiable solutions, according to the definition introduced in [FF13b] for solutions of the stochastic transport equation.

Recall that, as observed in Section 2, by point 2 of the next definition and Sobolev embedding, weakly differentiable solutions of the stochastic kinetic equation are a.s. continuous in the space variable, for every $t \in [0, T]$; this is in contrast with the deterministic kinetic equation, where solutions can be discontinuous (see Proposition 3). In the sequel, given a Banach space E we denote by $C^0([0, T]; E)$ the Banach space of all continuous functions from $[0, T]$ into E endowed with the supremum norm.

Definition 35 *Assume that F satisfies Hypothesis 1. We say that f is a weakly differentiable solution of the stochastic kinetic equation (1) if*

1. $f : \Omega \times [0, T] \times \mathbb{R}^{2d} \rightarrow \mathbb{R}$ is measurable, $\int_{\mathbb{R}^{2d}} f(t, z) \varphi(z) dz$ (well defined by property 2 below) is progressively measurable for each $\varphi \in C_c^\infty(\mathbb{R}^{2d})$;
2. $\mathbb{P}\left(f(t, \cdot) \in \cap_{r \geq 1} W_{loc}^{1,r}(\mathbb{R}^{2d})\right) = 1$ for every $t \in [0, T]$ and both f and Df are in $\cap_{r \geq 1} C^0([0, T]; L^r(\Omega \times \mathbb{R}^{2d}))$;
3. setting $b(z) = A \cdot z + B(z)$, $b : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$, see (72), for every $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and $t \in [0, T]$, with probability one, one has

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} f(t, z) \varphi(z) dz + \int_0^t \int_{\mathbb{R}^{2d}} b(z) \cdot Df(s, z) \varphi(z) dz ds \\ &= \int_{\mathbb{R}^{2d}} f_0(z) \varphi(z) dz + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^{2d}} f(s, z) \partial_{v_i} \varphi(z) dz \right) dW_s^i \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} f(s, z) \Delta_v \varphi(z) dz ds. \end{aligned}$$

Remark 36 *The process $s \mapsto Y_s^i := \int_{\mathbb{R}^{2d}} f(s, z) \partial_{v_i} \varphi(z) dz$ is progressively measurable by property 1 and $\int_0^T |Y_s^i|^2 ds < \infty$ \mathbb{P} -a.s. by property 2, hence the Itô integral is well defined.*

Remark 37 *The term $\int_0^t \int_{\mathbb{R}^{2d}} b(z) \cdot Df(s, z) \varphi(z) dz ds$ is well defined with probability one because of the integrability properties of b (assumptions) and Df (property 2).*

In the next result the inverse of ϕ_t will be denoted by ϕ_0^t .

Theorem 38 *If F satisfies Hypothesis 1 and $f_0 \in \cap_{r \geq 1} W^{1,r}(\mathbb{R}^{2d})$, then $f(t, z) := f_0(\phi_0^t(z))$ is a weakly differentiable solution of the stochastic kinetic equation (1).*

Proof. The proof follows the one of [FF13b, Theorem 10]. We divide it into several steps.

Step 1 (preparation). The random field $(\omega, t, z) \mapsto f_0(\phi_0^t(z)(\omega))$ is jointly measurable and $(\omega, t) \mapsto \int_{\mathbb{R}^{2d}} f_0(\phi_0^t(z)(\omega)) \varphi(z) dz$ is progressively measurable for each $\varphi \in C_c^\infty(\mathbb{R}^{2d})$. Hence part 1 of Definition 35 is true. To prove part 2 and 3 we approximate $f(t, z)$ by smooth fields $f_n(t, z)$.

Let $f_{0,n}$ be a sequence of smooth functions which converges to f_0 in $W^{1,r}(\mathbb{R}^{2d})$, for any $r \geq 1$, and so uniformly on \mathbb{R}^{2d} by the Sobolev embedding. This can be done for instance by using standard convolution with mollifiers. Moreover suppose that F_n are smooth approximations converging to F in $L^p(\mathbb{R}^{2d})$ (p is given in Hypothesis 1), let $\phi_{t,n}$ be the regular stochastic flow generated by the SDE (74) where B is replaced by $B_n = RF_n$ and let $\phi_{0,n}^t$ be the inverse flow. Then $f_n(t, z) := f_{0,n}(\phi_{0,n}^t(z))$ is a smooth solution of

$$df_n = -(v \cdot D_x f_n + F_n \cdot D_v f_n) dt - D_v f_n \circ dW_t$$

and thus for every $\varphi \in C_c^\infty(\mathbb{R}^{2d})$, $t \in [0, T]$ and bounded r.v. Y , it satisfies

$$\begin{aligned} & \mathbb{E} \left[Y \int_{\mathbb{R}^{2d}} f_n(t, z) \varphi(z) dz \right] + \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(z) \cdot Df_n(s, z) \varphi(z) dz ds \right] \\ &= \mathbb{E} \left[Y \int_{\mathbb{R}^{2d}} f_{0,n}(z) \varphi(z) dz \right] + \sum_{i=1}^d \mathbb{E} \left[Y \int_0^t \left(\int_{\mathbb{R}^{2d}} f_n(s, z) \partial_{v_i} \varphi(z) dz \right) dW_s^i \right] \\ &+ \frac{1}{2} \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} f_n(s, z) \Delta_v \varphi(z) dz ds \right]. \end{aligned} \quad (119)$$

We shall pass to the limit in each one of these terms. We are forced to use this very weak convergence due to the term

$$\mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(z) \cdot Df_n(s, z) \varphi(z) dz ds \right], \quad (120)$$

where we may only use weak convergence of Df_n .

Step 2 (convergence of f_n to f). We claim that, uniformly in n and for every $r \geq 1$,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} \mathbb{E} \left[|f_n(t, z)|^r \right] dz \leq C_r, \quad (121)$$

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} \mathbb{E} \left[|Df_n(t, z)|^r \right] dz \leq C_r. \quad (122)$$

Let us show how to prove the second bound; the first one can be obtained in the same way. The key estimate is the bound (135) on the derivative of the flow, which is proved in Appendix. We use the representation formula for f_n and the Hölder inequality to obtain

$$\left(\int_{\mathbb{R}^{2d}} \mathbb{E} \left[|Df_n(t, z)|^r \right] dz \right)^2 \leq \sup_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[|D\phi_{0,n}^t(z)|^{2r} \right] \int_{\mathbb{R}^{2d}} \mathbb{E} \left[|Df_{0,n}(\phi_{0,n}^t(z))|^{2r} \right] dz.$$

The first term on the right-hand side can be uniformly bounded using Lemma 42. Also the last integral can be bounded uniformly: changing variables (all functions are regular) we get

$$\int_{\mathbb{R}^{2d}} \mathbb{E} \left[|Df_{0,n}(\phi_{0,n}^t(z))|^{2r} \right] dz = \int_{\mathbb{R}^{2d}} |Df_{0,n}(y)|^{2r} \mathbb{E} [J_{\phi_{t,n}}(y)] dy,$$

where $J_{\phi_{t,n}}(y)$ is the Jacobian determinant of $\phi_{t,n}(y)$. Then we conclude using again the Hölder inequality, (135) and the boundedness of $(f_{0,n})$ in $W^{1,r}(\mathbb{R}^{2d})$ (for every $r \geq 1$). Remark that all the bounds obtained are uniform in n and t .

We can now consider the convergence of f_n to f . Let us first prove that, given $t \in [0, T]$ and $\varphi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\mathbb{P} - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{2d}} f_n(t, z) \varphi(z) dz = \int_{\mathbb{R}^{2d}} f(t, z) \varphi(z) dz \quad (123)$$

(convergence in probability). Using the representation formulas $f_n = f_{0,n}(\phi_{0,n}^t)$, $f = f_0(\phi_0^t)$ and Sobolev embedding $W^{1,4d} \hookrightarrow C^{1/2}$ we have $(\text{Supp}(\varphi) \subset B_R$ where B_R is the ball of radius $R > 0$ and center 0)

$$\begin{aligned} \left| \int_{\mathbb{R}^{2d}} (f_n(t, z) - f(t, z)) \varphi(z) dz \right| &\leq \|f_{0,n} - f_0\|_{L^\infty(\mathbb{R}^{2d})} \|\varphi\|_{L^1(\mathbb{R}^{2d})} \\ &\quad + C \|\varphi\|_{L^\infty(\mathbb{R}^{2d})} \int_{B_R} |\phi_{0,n}^t(z) - \phi_0^t(z)|^{1/2} dz. \end{aligned}$$

The first term converges to zero by the uniform convergence of $f_{0,n}$ to f_0 . From Lemma 41 we get

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{B_R} |\phi_{0,n}^t(z) - \phi_0^t(z)| dz \right] = 0,$$

and the convergence in probability (123) follows. This allows to pass to the limit in the first and in the last term of equation (119) using the uniform bound (121) and the Vitali convergence theorem. Similarly, we can show that, given $\varphi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\mathbb{P} - \lim_{n \rightarrow \infty} \int_0^T \left| \int_{\mathbb{R}^{2d}} (f_n(t, z) - f(t, z)) \varphi(z) dz \right|^2 dt = 0 \quad (124)$$

and allows to pass to the limit in the stochastic integral term of (119). Hence, one can easily show convergence of all terms in (119) except for the one in (120) which will be treated in Step 4.

Step 3 (a bound for f). Let us prove property 2 of Definition 35. The key estimate is property (122) obtained in the previous step.

Recall we have already obtained the convergence (123) and the uniform bound (122) on Df_n . We can then apply [FF13b, Lemma 16] which gives $\mathbb{P}(f(t, \cdot) \in W_{loc}^{1,r}(\mathbb{R}^{2d})) = 1$ for any $r \geq 1$ and $t \in [0, T]$, and

$$\mathbb{E} \left[\int_{B_R} |Df(t, z)|^r dz \right] \leq \limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_{B_R} |Df_n(t, z)|^r dz \right] \leq C_r,$$

for every $R > 0$ and $t \in [0, T]$. Hence, by monotone convergence we have

$$\sup_{t \in [0, T]} \mathbb{E} \left[\int_{\mathbb{R}^{2d}} |Df(t, z)|^r dz \right] \leq C_r. \quad (125)$$

A similar bound can be proved for f itself using (121), the convergence in probability (123) and the Vitali convergence theorem.

Step 4 (passage to the limit). Finally, we prove that we can pass to the limit in equation (119) and deduce that f satisfies property 3 of Definition 35. It remains to consider the term $\mathbb{E}[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(s, z) \cdot Df_n(s, z) \varphi(z) dz ds]$. Since $F_n \rightarrow F$ in $L^p(\mathbb{R}^{2d})$, it is sufficient to use a suitable weak convergence of Df_n to Df . Precisely, for $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(z) \cdot Df_n(s, z) \varphi(z) dz ds \right] \\ & \quad - \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} b(z) \cdot Df(s, z) \varphi(z) dz ds \right] = I_n^{(1)}(t) + I_n^{(2)}(t); \\ I_n^{(1)}(t) &= \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} (F_n(z) - F(z)) \cdot D_v f_n(s, z) \varphi(z) dz ds \right]; \\ I_n^{(2)}(t) &= \mathbb{E} \left[Y \int_0^t \int_{\mathbb{R}^{2d}} \varphi(z) b(z) \cdot (Df_n(s, z) - Df(s, z)) dz ds \right]. \end{aligned}$$

We have to prove that both $I_n^{(1)}(t)$ and $I_n^{(2)}(t)$ converge to zero as $n \rightarrow \infty$. By the Hölder inequality, for all $t \in [0, T]$

$$I_n^{(1)}(t) \leq C \|F_n - F\|_{L^p(\mathbb{R}^{2d})} \sup_{t \in [0, T]} \mathbb{E} \left[\|Df_n(t, \cdot)\|_{L^{p'}(\mathbb{R}^{2d})} \right]$$

where $1/p + 1/p' = 1$ and $C = C_{Y, T, \varphi}$. Thus, from (122), $I_n^{(1)}(t)$ converges to zero as $n \rightarrow \infty$. Let us treat $I_n^{(2)}(t)$. Using the integrability properties shown above we can change the order of integration. The function

$$h_n(s) := \mathbb{E} \left[\int_{\mathbb{R}^{2d}} Y \varphi(z) b(z) \cdot (Df_n(s, z) - Df(s, z)) dz \right], \quad s \in [0, T],$$

converges to zero as $n \rightarrow \infty$ for almost every s and satisfies the assumptions of the Vitali convergence theorem (we shall prove these two claims in Step 5 below). Hence $I_n^{(2)}(t)$ converges to zero.

Now we may pass to the limit in equation (119) and from the arbitrariness of Y we obtain property 3 of Definition 35.

Step 5 (auxiliary facts). We have to prove the two properties of $h_n(s)$ claimed in Step 4. For every $s \in [0, T]$ [FF13b, Lemma 16] gives

$$\mathbb{E} \left[\int_{\mathbb{R}^{2d}} \partial_{z_i} f(s, z) \varphi(z) Y dz \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\mathbb{R}^{2d}} \partial_{z_i} f_n(s, z) \varphi(z) Y dz \right], \quad (126)$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and bounded r.v. Y . Since the space $C_c^\infty(\mathbb{R}^{2d})$ is dense in $L^p(\mathbb{R}^{2d})$, we may extend the convergence property (126) to all $\varphi \in L^p(\mathbb{R}^{2d})$ by means of the bounds (122) and (125), which proves the first claim.

Moreover, for every $\varepsilon > 0$ there is a constant $C_{Y,\varepsilon}$ such that $(\text{Supp}(\varphi) \subset B_R)$

$$\sup_{n \geq 1} \int_0^T h_n^{1+\varepsilon}(s) ds \leq C_{Y,\varepsilon} \|b\varphi\|_{L^p}^{1+\varepsilon} \left\{ \left(\mathbb{E} \int_0^T \int_{B_R} |Df_n(s, z)|^r dz ds \right)^{\frac{1+\varepsilon}{r}} + \left(\mathbb{E} \int_0^T \int_{B_R} |Df(s, z)|^r dz ds \right)^{\frac{1+\varepsilon}{r}} \right\}$$

for a suitable r depending on ε (we have used Hölder inequality; cf. page 1344 in [FF13b]). The bounds (122) and (125) imply that $\int_0^T h_n^{1+\varepsilon}(s) ds$ is uniformly bounded, and the Vitali theorem can be applied. The proof is complete. ■

We now present the uniqueness result for weakly differentiable solutions. The proof seems to be of independent interest.

Theorem 39 *If F satisfies Hypothesis 1 and, moreover, $\text{div}_v F \in L^\infty(\mathbb{R}^{2d})$ ($\text{div}_v F$ is understood in distributional sense) weakly differentiable solutions are unique.*

Proof. By linearity of the equation we just have to show that the only solution starting from $f_0 = 0$ is the trivial one.

Step 1 (f^2 is a solution). We prove that for any solution f , the function f^2 is still a weak solution of the stochastic kinetic equation. Take test functions of the form $\varphi_\zeta^n(z) = \rho_n(\zeta - z)$, where $(\rho_n)_n$ is a family of standard mollifiers (ρ_n has support in $B_{1/n}$). Let $\zeta = (\xi, \nu) \in \mathbb{R}^{2d}$, $f_n(t, \zeta) = (f(t, \cdot) \star \rho_n)(\zeta)$. By definition of solution we get that, \mathbb{P} -a.s.,

$$f_n(t, \zeta) + \int_0^t b(\zeta) \cdot Df_n(s, \zeta) ds + \int_0^t D_v f_n(t, \zeta) \circ dW_s = \int_0^t R_n(s, \zeta) ds,$$

$$R_n(s, \zeta) = \int_{\mathbb{R}^{2d}} (b(\zeta) - b(z)) \cdot D_z f(s, z) \rho_n(\zeta - z) dz.$$

The functions f_n are smooth in the space variable. For any fixed $\zeta \in \mathbb{R}^{2d}$, by the Itô formula we get

$$df_n^2 = 2f_n df_n$$

$$= -2f_n b \cdot Df_n dt - 2f_n D_v f_n \circ dW_t + 2f_n R_n dt.$$

Now we multiply by $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and integrate over \mathbb{R}^{2d} . Using the Itô integral we pass to the limit as $n \rightarrow \infty$ and find, \mathbb{P} -a.s.,

$$\int_{\mathbb{R}^{2d}} f_n^2(t, \zeta) \varphi(\zeta) d\zeta - \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} f_n^2(s, \zeta) \Delta_v \varphi(\zeta) d\zeta ds + \int_0^t \int_{\mathbb{R}^{2d}} \varphi(\zeta) b(\zeta) \cdot Df_n^2(s, \zeta) d\zeta ds$$

$$- \int_0^t \int_{\mathbb{R}^{2d}} f_n^2(s, \zeta) D_v \varphi(\zeta) d\zeta \cdot dW_s = 2 \int_0^t \int_{\mathbb{R}^{2d}} f_n(s, \zeta) R_n(s, \zeta) \varphi(\zeta) d\zeta ds. \quad (127)$$

Recall that

$$b(z) = A \cdot z + \begin{pmatrix} 0 \\ F(z) \end{pmatrix} \in \mathbb{R}^{2d}.$$

Let us fix $t \in [0, T]$. By definition of weakly differentiable solution it is not difficult to pass to the limit in probability as $n \rightarrow \infty$ in all the terms in the left hand side of (127). Indeed, we can use that, for every $t \in [0, T]$, $r \geq 1$, $f_n(t, \cdot) \rightarrow f(t, \cdot)$ in $W_{loc}^{1,r}(\mathbb{R}^{2d})$, \mathbb{P} -a.s., together with the bounds

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} \mathbb{E} \left[|f_n(t, z)|^r \right] dz \leq C_r, \quad \sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} \mathbb{E} \left[|Df_n(t, z)|^r \right] dz \leq C_r, \quad (128)$$

and the Vitali theorem. For instance, if $\text{Supp}(\varphi) \subset B_R$ we have

$$\begin{aligned} J_n(t) &= \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} |f_n^2(s, \zeta) - f^2(s, \zeta)| |\Delta_v \varphi(\zeta)| d\zeta ds \\ &\leq C_\varphi \int_0^T \mathbb{E} \int_{B_R} |f_n^2(s, \zeta) - f^2(s, \zeta)| d\zeta ds, \end{aligned}$$

and, for any $s \in [0, T]$, \mathbb{P} -a.e ω , $k_n(\omega, s) = \int_{B_R} |f_n^2(\omega, s, \zeta) - f^2(\omega, s, \zeta)| d\zeta \rightarrow 0$ as $n \rightarrow \infty$. From (128) we deduce easily that

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^T k_n^2(s) ds \right] < \infty$$

and so by the Vitali theorem we get $\int_0^T \mathbb{E} \left[\int_{B_R} |f_n^2(s, \zeta) - f^2(s, \zeta)| d\zeta \right] ds \rightarrow 0$, as $n \rightarrow \infty$, which implies $\lim_{n \rightarrow \infty} J_n(t) = 0$. In order to show that

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^{2d}} |f_n(s, \zeta) R_n(s, \zeta) \varphi(\zeta)| d\zeta ds \right] \leq C_\varphi \mathbb{E} \left[\int_0^t \int_{B_R} |f_n(s, \zeta) R_n(s, \zeta)| d\zeta ds \right] \rightarrow 0$$

as $n \rightarrow \infty$, it is enough to prove that for fixed ω , \mathbb{P} -a.s., and $s \in [0, T]$ we have

$$\int_{B_R} |f_n(s, \zeta) R_n(s, \zeta)| d\zeta \rightarrow 0, \quad (129)$$

as $n \rightarrow \infty$. Indeed once (129) is proved, using the bounds (128) and the Hölder inequality we get

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^t \int_{B_R} |f_n(s, \zeta) R_n(s, \zeta)| d\zeta \right]^2 ds \leq C_R \sup_{n \geq 1} \mathbb{E} \left[\int_0^t \int_{B_R} |f_n(s, \zeta) R_n(s, \zeta)|^2 d\zeta ds \right] < \infty.$$

Thus we can apply the Vitali theorem and deduce the assertion. Let us check (129).

By Sobolev regularity of weakly differentiable solutions we know that

$$\sup_{n \geq 1} \sup_{\zeta \in B_R} |f_n(s, \zeta)| = M < \infty.$$

Hence it is enough to prove that $\int_{B_R} |R_n(s, \zeta)| d\zeta \rightarrow 0$. Recall that

$$R_n(s, \zeta) = b(\zeta) \cdot Df_n(s, \zeta) - [(b \cdot Df(s, \cdot)) * \rho_n](s, \zeta).$$

Using the fact that $b \in L^p_{loc}(\mathbb{R}^{2d})$, with p given in Hypothesis 1, the Hölder inequality and basic properties of convolutions we have

$$\begin{aligned} \int_{B_R} |b(\zeta) \cdot Df_n(s, \zeta) - b(\zeta) \cdot Df(s, \zeta)| d\zeta &\rightarrow 0, \\ \int_{B_R} \left| [(b \cdot Df(s, \cdot)) * \rho_n](s, \zeta) - b(\zeta) \cdot Df(s, \zeta) \right| d\zeta &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. This shows that (129) holds. We have proved that also f^2 is a weakly differentiable solution of the stochastic kinetic equation.

Step 2 (f is identically zero). Due to the integrability properties of f , the stochastic integral in Itô's form is a martingale; it follows that the function $g(t, z) = \mathbb{E}[f^2(t, z)]$ belongs to $C^0([0, T]; W^{1,r}(\mathbb{R}^{2d}))$ for any $r \geq 1$ and satisfies, for any $\varphi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\int_{\mathbb{R}^{2d}} g(t, z) \varphi(z) dz + \int_0^t \int_{\mathbb{R}^{2d}} b(z) \cdot Dg(s, z) \varphi(z) dz ds = \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} g(s, z) \Delta_v \varphi(z) dz ds.$$

We have, for any $s \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} b(z) \cdot Dg(s, z) \varphi(z) dz &= \int_{\mathbb{R}^{2d}} v \cdot D_x g(s, z) \varphi(z) dz + \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s, z) \varphi(z) dz \\ &= - \int_{\mathbb{R}^{2d}} v \cdot D_x \varphi(z) g(s, z) dz + \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s, z) \varphi(z) dz. \end{aligned}$$

Now we fix $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta = 1$ on the ball B_1 of center 0 and radius 1. By considering the test functions:

$$\varphi_{nm}(x, v) = \eta(x/n) \eta(v/m), \quad (x, v) = z \in \mathbb{R}^{2d},$$

$n, m \geq 1$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} g(t, z) \eta(x/n) \eta(v/m) dz - \frac{1}{n} \int_0^t \int_{\mathbb{R}^{2d}} \eta(v/m) v \cdot D\eta(x/n) g(s, z) dz ds \\ &+ \int_0^t \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s, z) \eta(x/n) \eta(v/m) dz ds = \frac{1}{2m^2} \int_0^t \int_{\mathbb{R}^{2d}} \eta(x/n) g(s, z) \Delta \eta(v/m) dz ds. \end{aligned}$$

Now we fix $m \geq 1$ and pass to the limit as $n \rightarrow \infty$ by the Lebesgue theorem. We infer

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} g(t, z) \eta(v/m) dz + \int_0^t \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s, z) \eta(v/m) dz ds \\ &= \frac{1}{2m^2} \int_0^t \int_{\mathbb{R}^{2d}} g(s, z) \Delta \eta(v/m) dz ds. \end{aligned}$$

Passing to the limit as $m \rightarrow \infty$ we arrive at

$$\int_{\mathbb{R}^{2d}} g(t, z) dz = - \int_0^t \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s, z) dz ds.$$

Since in particular $g(t, z) \in C^0([0, T]; W^{1,r}(\mathbb{R}^{2d}))$, with $r = \frac{p}{p-1}$, we obtain

$$\int_{\mathbb{R}^{2d}} g(t, z) dz = \int_0^t \int_{\mathbb{R}^{2d}} \operatorname{div}_v F(z) g(s, z) dz ds \leq \|\operatorname{div}_v F(z)\|_\infty \int_0^t \int_{\mathbb{R}^{2d}} g(s, z) dz ds.$$

Applying the Grönwall lemma we get that g is identically zero and this proves uniqueness for the kinetic equation. ■

6 Appendix

Proof of Proposition 4. By (17) we only have to prove the first inclusion. Let $f \in W^{s',p}(\mathbb{R}^{2d})$. Recall that

$$[f]_{B_{p,2}^s}^2 = \int_{\mathbb{R}^{2d}} \frac{dh}{|h|^{2d+2s}} \left(\int_{\mathbb{R}^{2d}} |f(x+h) - f(x)|^p dx \right)^{2/p}.$$

We have with $\epsilon = \frac{s'-s}{2} > 0$

$$\begin{aligned} & \int_{|h| \leq 1} \frac{1}{|h|^{2d+2s}} \left(\int_{\mathbb{R}^{2d}} |f(x+h) - f(x)|^p dx \right)^{2/p} dh \\ &= \int_{|h| \leq 1} \frac{1}{|h|^{2d(\frac{p-2}{p})-\epsilon}} \left(\int_{\mathbb{R}^{2d}} \frac{|f(x+h) - f(x)|^p}{|h|^{(\frac{2}{p}2d+2s+\epsilon)\frac{p}{2}}} dx \right)^{2/p} dh \\ &\leq c \left(\int_{|h| \leq 1} dh \int_{\mathbb{R}^{2d}} \frac{|f(x+h) - f(x)|^p}{|h|^{2d+(s+\epsilon/2)p}} dx \right)^{2/p} \cdot \left(\int_{|h| \leq 1} \frac{1}{|h|^{2d-\frac{\epsilon p}{p-2}}} dh \right)^{\frac{p-2}{p}} \\ &\leq C [f]_{W^{s+\frac{\epsilon}{2},p}}^2 \end{aligned}$$

Similarly, we have with $0 < \epsilon < \min(2s, \frac{s'-s}{2})$

$$\begin{aligned} & \int_{|h| > 1} \frac{1}{|h|^{2d+2s}} \left(\int_{\mathbb{R}^{2d}} |f(x+h) - f(x)|^p dx \right)^{2/p} dh \\ &= \int_{|h| > 1} \frac{1}{|h|^{2d(\frac{p-2}{p})+\epsilon}} \left(\int_{\mathbb{R}^{2d}} \frac{|f(x+h) - f(x)|^p}{|h|^{(\frac{2}{p}2d+2s-\epsilon)\frac{p}{2}}} dx \right)^{2/p} dh \\ &\leq c \left(\int_{|h| > 1} dh \int_{\mathbb{R}^{2d}} \frac{|f(x+h) - f(x)|^p}{|h|^{2d+(s-\epsilon/2)p}} dx \right)^{2/p} \cdot \left(\int_{|h| > 1} \frac{1}{|h|^{2d+\frac{\epsilon p}{p-2}}} dh \right)^{\frac{p-2}{p}} \\ &\leq C [f]_{W^{s-\frac{\epsilon}{2},p}}^2 \leq C' \|f\|_{W^{s+\frac{\epsilon}{2},p}}^2 \end{aligned}$$

(cf. (16) and (14)). The proof is complete. ■

Proof of Lemma 26. Remark that, due to (99),

$$\left(\varepsilon + |Z_t^z - Z_t^y|^2 \right)^a \leq C \left(\varepsilon + |\gamma(Z_t^z) - \gamma(Z_t^y)|^2 \right)^a.$$

Therefore, we can prove (111) for $\gamma(Z_t)$ instead of Z_t .

We proceed as in [Ku84, Lemma II.2.4] or [FF13a, Lemma 5.4]. Fix any $t \in [0, T]$ and set for $z, y \in \mathbb{R}^{2d}$: $g(z) := f^a(z)$, $f(z) := (\varepsilon + |z|^2)$ and $\eta_t := \gamma(Z_t^z) - \gamma(Z_t^y)$. Then, applying Itô formula we obtain as in the proof of Lemma 23

$$\begin{aligned} g(\eta_t) - g(\eta_0) &= 2a \int_0^t f^{a-1}(\eta_s) \eta_s \cdot \left[\tilde{b}(Z_s^z) - \tilde{b}(Z_s^y) \right] ds \\ &\quad + 2a \int_0^t f^{a-1}(\eta_s) \eta_s \cdot \left[\tilde{\sigma}(Z_s^z) - \tilde{\sigma}(Z_s^y) \right] \cdot dW_s \\ &\quad + a \sum_{i,j} \int_0^t f^{a-2}(\eta_s) \left[f(\eta_s) \delta_{i,j} + 2(a-1) \eta_s^i \eta_s^j \right] \\ &\quad \quad \quad \times \left[\left(\tilde{\sigma}(Z_s^z) - \tilde{\sigma}(Z_s^y) \right) \left(\tilde{\sigma}(Z_s^z) - \tilde{\sigma}(Z_s^y) \right)^{t-1} \right]^{i,j} ds. \end{aligned}$$

Recall that $|z| \leq f^{1/2}(z)$ and that the coefficient \tilde{b} is Lipschitz continuous:

$$|\tilde{b}(z) - \tilde{b}(y)| \leq L|z - y| \leq C|\gamma(z) - \gamma(y)| \leq C f^{1/2}(|\gamma(z) - \gamma(y)|).$$

We can continue with the estimates and obtain

$$\begin{aligned} g(\eta_t) - g(\eta_0) &\leq 2C|a| \int_0^t f^a(\eta_s) ds + 2a \int_0^t f^{a-1}(\eta_s) \eta_s \left[\tilde{\sigma}(Z_s^z) - \tilde{\sigma}(Z_s^y) \right] dW_s \quad (130) \\ &\quad + C_{a,d}|a| \int_0^t f^{a-1}(\eta_s) |\eta_s|^2 dA_s. \end{aligned}$$

Here, A_t is the process introduced and studied in Lemma 18:

$$\int_0^t \|\tilde{\sigma}(Z_s^z) - \tilde{\sigma}(Z_s^y)\|_{HS}^2 ds = \int_0^t |Z_s^z - Z_s^y|^2 dA_s.$$

The stochastic integral in (130) is a martingale with zero mean ($\tilde{\sigma}$ is bounded). Proceeding as in (103), we get

$$\mathbb{E} \left[e^{-A_t} g(\eta_t) \right] - e^{-A_0} g(\eta_0) \leq C_{a,d} \int_0^t \mathbb{E} \left[e^{-A_s} g(\eta_s) \right] ds.$$

By Grönwall's inequality applied to the function $h(t) := \mathbb{E} \left[e^{-A_t} g(\eta_t) \right]$, it follows

$$\begin{aligned} \mathbb{E} \left[e^{-A_t} \left(\varepsilon + |Z_t^z - Z_t^y|^2 \right)^a \right] &\leq C \mathbb{E} \left[e^{-A_t} g(\eta_t) \right] \leq C_{a,d} g(\eta_0) = C_{a,d} (\varepsilon + |\gamma(z) - \gamma(y)|^2)^a \\ &\leq C_a (\varepsilon + |z - y|^2)^a. \end{aligned} \quad (131)$$

To complete the proof of the lemma, we manipulate (131) using Hölder's inequality and we conclude invoking Lemma 18 to bound the term $\mathbb{E}[e^{2A_T}]$:

$$\mathbb{E} \left[\left(\varepsilon + |Z_t^z - Z_t^y|^2 \right)^a \right]^2 \leq \mathbb{E} \left[e^{2A_t} \right] \mathbb{E} \left[e^{-2A_t} g^2(\eta_t) \right] \leq C_{a,d} (\varepsilon + |z - y|^2)^{2a}.$$

■

Proof of Lemma 28. We have set here $\eta_t(z, y) := |Z_t^z - Z_t^y|$. To ease notation in the following computations we will write η and η' for $\eta_t(z, y)$ and $\eta_{t'}(z', y')$ respectively and set $\xi = \eta^{-1}$ and $\xi' = \eta'^{-1}$. Observe that

$$\begin{aligned} \left| \eta_t(z, y) - \eta_{t'}(z', y') \right|^a &= \left| \frac{1}{\xi} - \frac{1}{\xi'} \right|^a = \left| \frac{\xi' - \xi}{\xi \xi'} \right|^a = |\eta|^a |\eta'|^a |\xi' - \xi|^a \\ &= |\eta|^a |\eta'|^a \left| |Z_{t'}^{z'} - Z_{t'}^{y'}| - |Z_t^z - Z_t^y| \right|^a \\ &\leq |\eta|^a |\eta'|^a \left(|Z_{t'}^{z'} - Z_t^z| + |Z_{t'}^{y'} - Z_t^y| \right)^a \\ &\leq C_a |\eta|^a |\eta'|^a \left(|Z_{t'}^{z'} - Z_t^z|^a + |Z_{t'}^{y'} - Z_t^y|^a \right). \end{aligned}$$

The first inequality above follows from the triangular inequality as in [Ku84, Lemma II.4.1]: every side of a quadrilateral is shorter than the sum of the other three. We now take expectations and use Hölder's inequality:

$$\begin{aligned} &\mathbb{E} \left[\left| \eta_t(z, y) - \eta_{t'}(z', y') \right|^a \right] \\ &\leq C_a \mathbb{E} [|\eta|^{4a}]^{1/4} \mathbb{E} [|\eta'|^{4a}]^{1/4} \left(\mathbb{E} \left[|Z_{t'}^{z'} - Z_t^z|^{2a} \right]^{1/2} + \mathbb{E} \left[|Z_{t'}^{y'} - Z_t^y|^{2a} \right]^{1/2} \right) \\ &\leq C_{a,d,\lambda,T} |z - y|^{-a} |z' - y'|^{-a} \left(|z - z'|^a + |y - y'|^a \right. \\ &\quad \left. + (1 + |z|^a + |z'|^a + |y|^a + |y'|^a) |t - t'|^{a/2} \right). \end{aligned}$$

For the last inequality we have used (112) to estimate the first two terms and Proposition 24 for the last ones. ■

Proof of Lemma 30. Consider first the case $z, y \neq \infty$ ($\widehat{z}, \widehat{y} \neq 0$). In this case, proceeding just as in the proof of Lemma 28 we obtain the estimate

$$\left| \eta_t(\widehat{z}) - \eta_{t'}(\widehat{y}) \right|^a = \left| \frac{1 + |Z_{t'}^y| - 1 - |Z_t^z|}{(1 + |Z_t^z|)(1 + |Z_{t'}^y|)} \right|^a \leq |\eta_t(\widehat{z}) \eta_{t'}(\widehat{y})|^a |Z_t^z - Z_{t'}^y|^a.$$

Use again Hölder's inequality, Lemma 23 and Proposition 24:

$$\begin{aligned} \mathbb{E} \left[\left| \eta_t(\widehat{z}) - \eta_{t'}(\widehat{y}) \right|^a \right] &\leq \mathbb{E} \left[|\eta_t(\widehat{z})|^{4a} \right]^{1/4} \mathbb{E} \left[|\eta_{t'}(\widehat{y})|^{4a} \right]^{1/4} \mathbb{E} \left[|Z_t^z - Z_{t'}^y|^{2a} \right]^{1/2} \\ &\leq C(1 + |z|)^{-a} (1 + |y|)^{-a} \left(|z - y|^a + (1 + |z|^a + |y|^a) |t - t'|^{a/2} \right) \\ &\leq C \left(|\widehat{z} - \widehat{y}|^a + |t - t'|^{a/2} \right). \end{aligned}$$

The last inequality above holds if z and y are both finite and is a consequence of the inequality

$$(1 + |z|)^{-1} (1 + |y|)^{-1} |z - y| \leq |\widehat{z} - \widehat{y}| = \left| \frac{z}{|z|^2} - \frac{y}{|y|^2} \right|.$$

The case $z = \infty$ (or $y = \infty$) is even easier, since $\eta_t(\widehat{z}) = 0$ and Lemma 23 again implies

$$\mathbb{E} \left[\left| \eta_{t'}(\widehat{y}) \right|^a \right] \leq C_{a,d} (1 + |y|)^{-a} \leq C_{a,d} |\widehat{y}|^a.$$

The lemma is proved. ■

Proof of Theorem 31. Take $a > 2(2d + 3)$ in Lemma 30. Then, by Kolmogorov's theorem, $\eta_t(\widehat{z})$ is continuous at $\widehat{z} = 0$. Therefore, Z_t^z can be extended to a continuous map from $\widehat{\mathbb{R}^{2d}}$ into itself for any $t \in [0, T]$ almost surely and the extension $\widehat{Z}_t^z(\omega)$ is continuous in (t, z) almost surely. For all ω such that \widehat{Z} is continuous, the map $\widehat{Z}_t(\omega) : \widehat{\mathbb{R}^{2d}} \rightarrow \widehat{\mathbb{R}^{2d}}$ is homotopically equivalent to the identity map $\widehat{Z}_0(\omega)$. Proceeding by contradiction, assume that $\widehat{Z}_t(\omega)$ is not surjective. Then it takes values in $\widehat{\mathbb{R}^{2d}}$ without one point, which is a contractible space, so that it must be homotopically equivalent to a constant. This implies that also the map $Id_{\widehat{\mathbb{R}^{2d}}} = \widehat{Z}_0(\omega)$ is homotopically equivalent to a constant, and the space $\widehat{\mathbb{R}^{2d}}$ would be contractible, which is absurd (because, for example, $\pi_{2d}(\widehat{\mathbb{R}^{2d}}) = \mathbb{Z}$). The contradiction found shows that the function $\widehat{Z}_t(\omega)$ needs to be an onto map. Since $\widehat{Z}_t^\infty(\omega) = \infty$, the restriction of $\widehat{Z}_t(\omega)$ to \mathbb{R}^{2d} is again onto. The theorem is proved. ■

We now present some results on the convergence and regularity of approximations $\phi_{0,n}^t$ of the inverse flow ϕ_0^t associated to the SDE (73). Note that $\phi_{0,n}^t$ are solutions of SDEs with regular coefficients, see the proof of Theorem 38. These results are adapted from [FF13b] and based on the following lemma on the stability of the PDE (76), which is of independent interest.

Lemma 40 (Stability of the PDE (76)) *Let U_n be the unique solutions provided by Theorem 10 to the PDE (76) with smooth approximations $B_n(z) = (0, F_n(z))$ of $B(z) = (0, F(z))$ and some λ large enough for (133) to hold. If $F_n(z) \rightarrow F(z)$ in $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$, with s, p as in Hypothesis 1, then U_n and $D_v U_n$ converge pointwise and locally uniformly to the respective limits. In particular, for any $r > 0$ there exists a function $g(n) \rightarrow 0$ as $n \rightarrow \infty$ s.t.*

$$\begin{aligned} \sup_{z \in B_r} |U_n(z) - U(z)| &\leq g(n), \\ \sup_{z \in B_r} |D_v U_n(z) - D_v U(z)| &\leq g(n). \end{aligned} \tag{132}$$

Moreover, there exists a λ_0 s.t. for all $\lambda > \lambda_0$

$$\|D_v U_n\|_\infty \leq 1/2. \tag{133}$$

Proof. Setting $V_n = (U_n - U)$ we write for λ large enough (cf. (76))

$$\begin{aligned} \lambda V_n(z) - \frac{1}{2} \text{Tr}(QD^2 V_n(z)) - \langle Az, DV_n(z) \rangle - \langle B(z), DV_n(z) \rangle \\ = B_n(z) - B(z) + \langle B_n(z) - B(z), D_v U_n(z) \rangle. \end{aligned}$$

By (66) we know that

$$\sqrt{\lambda} \|D_v U_n\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \leq C \|B_n\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))} \leq C \|B\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}, \quad n \geq 1,$$

with $C = C(s, p, d, \|F\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}) > 0$. Hence applying (67), (68), (69) and Sobolev embedding we obtain (132) with $g(n) = C \|B - B_n\|_{L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))}$. On the other hand the last assertion follows from (69). ■

Lemma 41 ([FF13b, Lemma 3]) *For every $R > 0$, $a \geq 1$ and $z \in B_R$,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \sup_{z \in B_R} \mathbb{E} \left[|\phi_{0,n}^t(z) - \phi_0^t(z)|^a \right] = 0.$$

Proof. To ease notation, we shall prove the convergence result for the forward flows $\phi_{t,n} \rightarrow \phi_t$. This is enough since the backward flow solves the same equation with a drift of opposite sign. Since the flow ϕ_t is jointly continuous in (t, z) , the image of $[0, T] \times B_R$ is contained in $[0, T] \times B_r$ for some $r < \infty$. Thus for $z \in B_R$, from Lemma 40 we get $|U_n(\phi_{t,n}) - U(\phi_t)| \leq g(n) + 1/2|\phi_{t,n} - \phi_t|$ and $|D_v U_n(\phi_{t,n}) - D_v U(\phi_t)| \leq g(n) + |D_v U_n(\phi_{t,n}) - D_v U_n(\phi_t)|$. Extending the definition (98) to $\gamma_n(z) = z + U_n(z)$ we have the approximate equivalence

$$\frac{2}{3} \left(|\gamma_n(\phi_{t,n}) - \gamma(\phi_t)| - g(n) \right) \leq |\phi_{t,n} - \phi_t| \leq 2 \left(|\gamma_n(\phi_{t,n}) - \gamma(\phi_t)| + g(n) \right).$$

Therefore, it is enough prove the convergence result for the transformed flows $\gamma_{t,n} = \gamma_n(\phi_{t,n}) \rightarrow \gamma(\phi_t) = \gamma_t$. Proceeding as in the proof of Theorem 21 we get, for any $a \geq 2$

$$\begin{aligned} \frac{1}{a} d|\gamma_{t,n} - \gamma_t|^a &\leq |\gamma_{t,n} - \gamma_t|^{a-2} \left\{ (\gamma_{t,n} - \gamma_t) \cdot \left[\lambda(U_n(\phi_{t,n}) - U(\phi_t)) + A(\phi_{t,n} - \phi_t) \right] dt \right. \\ &\quad + (\gamma_{t,n} - \gamma_t) \cdot (DU_n(\phi_{t,n}) - DU(\phi_t)) R \cdot dW_t \\ &\quad \left. + C_{a,d} \| (DU_n(\phi_{t,n}) - DU(\phi_t)) R \|_{HS}^2 dt \right\}. \end{aligned} \quad (134)$$

The stochastic integral is a martingale. Since

$$\frac{|\phi_{t,n} - \phi_t|}{|\gamma_{t,n} - \gamma_t|} \leq C \left(1 + \frac{g(n)}{|\gamma_{t,n} - \gamma_t|} \right),$$

the term on the last line in (134) can be bounded using (132) by a constant times $|\gamma_{t,n} - \gamma_t|^a dB_{t,n} + |\gamma_{t,n} - \gamma_t|^{a-2} g^2(n) (dB_{t,n} + dt)$, where for every n the process $B_{t,n}$ is defined as in (102) but with $DU_n(\phi_{t,n})$ and $DU_n(\phi_t)$ in the place of $DU(Z_t)$ and $DU(Y_t)$ respectively. One can show that $B_{t,n}$ share the same integrability properties of the process A_t studied in Lemma 18, uniformly in n , see [FF13b, Lemma 14]. Computing $\mathbb{E}[e^{-B_{t,n}} |\gamma_{t,n} - \gamma_t|^a]$ using Itô formula and taking the supremum over $t \in [0, T]$ leads to

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} \left[e^{-B_{t,n}} |\gamma_{t,n} - \gamma_t|^a \right] &\leq C \mathbb{E} \left[\int_0^T e^{-B_{s,n}} |\gamma_{s,n} - \gamma_s|^a ds \right] \\ &\quad + Cg(n) \mathbb{E} \left[\int_0^T e^{-B_{s,n}} (|\gamma_{s,n} - \gamma_s|^{a-1} + g(n) |\gamma_{s,n} - \gamma_s|^{a-2}) ds \right] \\ &\quad + g^2(n) \mathbb{E} \left[\int_0^T e^{-B_{s,n}} |\gamma_{s,n} - \gamma_s|^{a-2} dB_{s,n} \right]. \end{aligned}$$

Using the integrability properties of $\phi_t, \phi_{t,n}, U(\phi_t), U_n(\phi_{t,n})$ one can see that all terms are bounded, uniformly in n . To conclude the proof we can pass to the limit

$$\limsup_n \sup_{t \in [0, T]} \mathbb{E} \left[e^{-B_{t,n}} |\gamma_{t,n} - \gamma_t|^a \right] \leq C \int_0^T \limsup_n \sup_{t \in [0, s]} \mathbb{E} \left[e^{-B_{t,n}} |\gamma_{t,n} - \gamma_t|^a \right] ds,$$

apply Grönwall's lemma and proceed as in Corollary 22 to get rid of the exponential term. ■

Lemma 42 ([FF13b, Lemma 5]) *For every $a \geq 1$, there exists $C_{a,d,T} > 0$ such that*

$$\sup_{t \in [0, T]} \sup_{z \in \mathbb{R}^{2d}} \mathbb{E}[|D\phi_{0,n}^t(z)|^a] \leq C_{a,d,T} \quad (135)$$

uniformly in n .

Proof. Let us show the bound for the forward flows $\phi_{t,n}$. These are regular flows: let $\theta_{t,n}$ and $\xi_{t,n}$ denote the weak derivative of $D\phi_{t,n}$ and $D\gamma_{t,n} = D\gamma_n(\phi_{t,n})$, respectively. They are equivalent in the sense of (114), so we shall prove the bound for $\xi_{t,n}$ instead of $\theta_{t,n}$. Proceeding as in the proof of Theorem 33 we obtain as in (116)

$$de^{-C_1 B_{t,n}} |\xi_{t,n}|^a \leq e^{-C_1 B_{t,n}} \left[C_2 |\xi_{t,n}|^a dt + dM_t \right],$$

where the process $B_{t,n}$ is simply given by $\int_0^t |DD_v U_n(\phi_{s,n})|^2 ds$. We can integrate, take expected values, the supremum over $t \in [0, T]$ and apply Grönwall's inequality to get

$$\sup_{t \in [0, T]} \mathbb{E}[e^{-C_1 B_{t,n}} |\xi_{t,n}|^a] \leq C_T |\xi_{0,n}|^a = C_{a,d,T}.$$

Observe that this bound is uniform in n and $z \in \mathbb{R}^{2d}$. Proceeding as in Corollary 22 we can get rid of the exponential term and obtain the desired uniform bound on $\xi_{t,n}$. ■

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