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Daniele Di Pietro, Jerome Droniou. $W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray-Lions problems. Mathematical Models and Methods in Applied Sciences, 2017, 27 (5), pp.879–908. 10.1142/S0218202517500191 . hal-01326818

HAL Id: hal-01326818

https://hal.science/hal-01326818

Submitted on 6 Jun 2016

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$W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray–Lions problems*

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June 6, 2016

Abstract

In this work we prove optimal $W^{s,p}$-approximation estimates (with $p \in [1, +\infty]$) for elliptic projectors on local polynomial spaces. The proof hinges on the classical Dupont–Scott approximation theory together with two novel abstract lemmas: An approximation result for bounded projectors, and an L^p-boundedness result for L^2-orthogonal projectors on polynomial subspaces. The $W^{s,p}$-approximation results have general applicability to (standard or polytopal) numerical methods based on local polynomial spaces. As an illustration, we use these $W^{s,p}$-estimates to derive novel error estimates for a Hybrid High-Order discretization of Leray–Lions elliptic problems whose weak formulation is classically set in $W^{1,p}(\Omega)$ for some $p \in (1, +\infty)$. This kind of problems appears, e.g., in the modelling of glacier motion, of incompressible turbulent flows, and in airfoil design. Denoting by h the meshsize, we prove that the approximation error measured in a $W^{1,p}$-like norm scales as $h^{\frac{k+1}{p-1}}$ when $p \geqslant 2$ and as $h^{(k+1)(p-1)}$ when $p < 2$.

2010 Mathematics Subject Classification: 65N08, 65N30, 65N12

Keywords: $W^{s,p}$-approximation properties of elliptic projector on polynomials, Hybrid High-Order methods, nonlinear elliptic equations, p-Laplacian, error estimates

1 Introduction

In this work we prove optimal $W^{s,p}$-approximation properties for elliptic projectors on local polynomial spaces, and use these results to derive novel a priori error estimates for a Hybrid High-Order discretisation of Leray–Lions elliptic equations.

Let $U \subset \mathbb{R}^d$, $d \geqslant 1$, be an open bounded set of diameter h_U. For all integers $s \in \mathbb{N}$ and $p \in [1, +\infty]$, we denote by $W^{s,p}(U)$ the space of functions having derivatives up to degree s in $L^p(U)$ with associated seminorm

$$|v|_{W^{s,p}(U)} := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d, \|\boldsymbol{\alpha}\|_1 = s} \|\partial^{\boldsymbol{\alpha}} v\|_{L^p(U)}, \tag{1}$$

where $\|\boldsymbol{\alpha}\|_1 := \alpha_1 + \ldots + \alpha_d$ and $\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ (this choice for the seminorm enables a seamless treatment of the case $p = +\infty$).

*This work was partially supported by ANR project HHOMM (ANR-15-CE40-0005)
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Let a polynomial degree $l \geqslant 0$ be fixed, and denote by $\mathbb{P}^l(U)$ the space of d-variate polynomials on U. The elliptic projector $\pi_U^{1,l} : W^{1,1}(U) \to \mathbb{P}^l(U)$ is defined as follows: For all $v \in W^{1,1}(U)$, $\pi_U^{1,l} v$ is the unique polynomial in $\mathbb{P}^l(U)$ that satisfies

$$\int_U \boldsymbol{\nabla}(\pi_U^{1,l} v - v) \cdot \boldsymbol{\nabla} w = 0 \text{ for all } w \in \mathbb{P}^l(U), \text{ and } \int_U (\pi_U^{1,l} v - v) = 0. \tag{2}$$

As a result of the Poincaré–Wirtinger inequality, the quantity $\pi_U^{1,l} v$ is well-defined. Moreover, we have the following characterisation:

$$\pi_U^{1,l} v = \underset{w \in \mathbb{P}^l(U), \int_U (w-v)=0}{\arg\min} \|\boldsymbol{\nabla}(w - v)\|^2_{L^2(U)^d}.$$

The first main result of this work is summarised in the following theorem.

Theorem 1 ($W^{s,p}$-approximation for $\pi_U^{1,l}$)**.** *Assume that U is star-shaped with respect to every point in a ball of radius ϱh_U for some $\varrho > 0$. Let $s \in \{1, \dots, l+1\}$ and $p \in [1, +\infty]$. Then, there exists a real number $C > 0$ depending only on d, ϱ, l, s, and p such that, for all $m \in \{0, \dots, s\}$ and all $v \in W^{s,p}(U)$,*

$$|v - \pi_U^{1,l} v|_{W^{m,p}(U)} \leqslant C h_U^{s-m} |v|_{W^{s,p}(U)}. \tag{3}$$

The proof of Theorem 1 is based on the classical Dupont–Scott approximation theory [26] (cf. also [7, Chapter 4]) and hinges on two novel abstract lemmas for projectors on polynomial spaces: A $W^{s,p}$-approximation result for projectors that satisfy a suitable boundedness property, and an L^p-boundedness result for L^2-orthogonal projectors on polynomial subspaces. Both results make use of the reverse Lebesgue and Sobolev embeddings for polynomial functions proved in [13] (cf., in particular Lemma 5.1 and Remark A.2 therein). Following similar arguments as in [26, Section 7], the results of Theorem 1 still hold if U is a finite union of domains that are star-shaped with respect to balls of radius comparable to h_U.

The second main result concerns the approximation of traces, and therefore requires more assumptions on the domain U.

Theorem 2 ($W^{s,p}$-approximation of traces for $\pi_U^{1,l}$)**.** *Assume that U is a polytope which admits a partition $\mathcal{S}_U$ into disjoint simplices S of diameter h_S and inradius r_S, and that there exists a real number $\varrho > 0$ such that, for all $S \in \mathcal{S}_U$,*

$$\varrho^2 h_U \leqslant \varrho h_S \leqslant r_S.$$

Let $s \in \{1, \dots, l+1\}$, $p \in [1, +\infty]$, and denote by $\mathcal{F}_U$ the set of hyperplanar faces of U. Then, there exists a real number C depending only on d, ϱ, l, s and p such that, for all $m \in \{0, \dots, s-1\}$ and all $v \in W^{s,p}(U)$,

$$h_U^{\frac{1}{p}} |v - \pi_U^{1,l} v|_{W^{m,p}(\mathcal{F}_U)} \leqslant C h_U^{s-m} |v|_{W^{s,p}(U)}. \tag{4}$$

Here, $W^{m,p}(\mathcal{F}_U)$ denotes the set of functions that belong to $W^{m,p}(F)$ for all $F \in \mathcal{F}_U$, and $|\cdot|_{W^{m,p}(\mathcal{F}_U)}$ the corresponding broken seminorm.

The proof of Theorem 2 is obtained combining the results of Theorem 1 with a continuous L^p-trace inequality.

The approximation results of Theorems 1 and 2 are used to prove novel error estimates for the Hybrid High-Order (HHO) method of [13] for nonlinear Leray–Lions elliptic problems of the form: Find a potential $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div}(\mathbf{a}(\boldsymbol{x}, \boldsymbol{\nabla} u)) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where Ω is a bounded polytopal subset of $\mathbb{R}^d$ with boundary $\partial\Omega$, while the source term $f : \Omega \to \mathbb{R}$ and the function $\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the requirements detailed in Eq. (20) below. This

equation, which contains the p-Laplace equation (cf. (21) below), appears in the modelling of glacier motion [30], of incompressible turbulent flows in porous media [20], and in airfoil design [29].

In the context of conforming Finite Element (FE) approximations of problems which can be traced back to the general form (5), a priori error estimates were derived in [4, 30]. For nonconforming (Crouzeix–Raviart) FE approximations, error estimates are proved in [33], with convergence rates consistent with the ones presented in this work (concerning the link between the HHO method and nonconforming FE, cf. [18, Remark 1]). Error estimates for a nodal Mimetic Finite Difference (MFD) method for a particular kind of operator $\mathbf{a}$ and with $p = 2$ are proved in [2]. Finite volume methods, on the other hand, are considered in [1], where error estimates similar to the ones obtained here are derived under the assumption that the source term f vanishes on the boundary (additional error terms are present when this is not the case). Finally, we also cite here [21], where the convergence study of a Mixed Finite Volume (MFV) scheme inspired by [22] is carried out using a compactness argument under minimal regularity assumptions on the exact solution.

The HHO method analysed here is based on meshes composed of general polytopal elements and its formulation hinges on degrees of freedom (DOFs) that are polynomials of degree $k \geqslant 0$ on mesh elements and faces; cf. [14–17] for an introduction to HHO methods and and [9,13] for applications to nonlinear problems. Based on such DOFs, a gradient reconstruction operator $\boldsymbol{G}_T^k$ of degree k and a potential reconstruction operator p_T^{k+1} of degree $(k+1)$ are devised by solving local problems inside each mesh element T. By construction, the composition of the potential reconstruction p_T^{k+1} with the interpolator on the DOF space coincides with the elliptic projector $\pi_T^{1,k+1}$. The gradient and potential reconstruction operators are then used to formulate a local contribution composed of a consistent and a stabilisation term. The $W^{s,p}$-approximation properties for $\pi_T^{1,k+1}$ play a crucial role in estimating the error associated with the latter. Denoting by h the meshsize, we prove in Theorem 7 below that, for smooth enough exact solutions, the approximation error measured in a discrete $W^{1,p}$-like norm converges as $h^{\frac{k+1}{p-1}}$ when $p \geqslant 2$ and as $h^{(k+1)(p-1)}$ when $p < 2$.

As noticed in [17], the lowest-order version of the HHO method corresponding to $k = 0$ is essentially analogous (up to equivalent stabilisation) to the SUSHI scheme of [27] when face unknowns are not eliminated by interpolation. This method, in turn, has been proved in [24] to be equivalent to the MFV method of [22] and the mixed-hybrid MFD method [8,32] (cf. also [6] for an introduction to MFD methods). As a consequence, our results extend the analysis conducted in [21], by providing in particular error estimates for the MFV scheme applied to Leray–Lions equations.

To conclude, it is worth mentioning that the tools of Theorems 1 and 2, alongside the optimum $W^{s,p}$-estimates of [13] for L^2-projectors on polynomial spaces (see Lemma 13), are potentially of interest also for the study of other polytopal methods. Elliptic projections on polynomial spaces appear, e.g., in the conforming and nonconforming Virtual Element Methods (cf. [5, Eq. (4.18)] and [3, Eqs. (3.18)–(3.20)], respectively). They also play a role in determining the high-order part of some post-processings of the potential used in the context of Hybridizable Discontinuous Galerkin methods; cf., e.g., the variation proposed in [10] of the post-processing considered in [11, 12].

The rest of the paper is organised as follows. In Section 2 we provide the proofs of Theorems 1 and 2 preceeded by the required preliminary results. In Section 3 we use these results to derive error estimates for the Hybrid High-Order discretization of problem 5. Appendix A collects some useful inequalities for Leray–Lions operators.

2 $W^{s,p}$-approximation properties of the elliptic projector on polynomial spaces

This section contains the proofs of Theorems 1 and 2 preceeded by two abstract lemmas for projectors on polynomials subspaces. Throughout the paper, to alleviate the notation, when

writing integrals we omit the dependence on the integration variable $\boldsymbol{x}$ as well as the differential with the exception of those integrals involving the function $\mathbf{a}$ (cf. (5)).

2.1 Two abstract results for projectors on polynomial subspaces

Our first lemma is an abstract approximation result valid for any projector on a polynomial space that satisfies a suitable boundedness property.

Lemma 3 ($W^{s,p}$-approximation for W-bounded projectors)**.** *Assume that U is star-shaped with respect to every point of a ball of radius ϱh_U for some $\varrho > 0$. Let five integers $l \geqslant 0$, $s \in \{1, \dots, l+1\}$, $p \in [1, +\infty]$, and $q, m \in \{0, \dots, s\}$ be fixed. Let $\Pi_U^{q,l} : W^{q,1}(U) \to \mathbb{P}^l(U)$ be a projector such that there exists a real number $C > 0$ depending only on d, ϱ, l, q, and p such that for all $v \in W^{q,p}(U)$,*

$$\text{If } m < q: \quad |\Pi_U^{q,l} v|_{W^{m,p}(U)} \leqslant C \sum_{r=m}^{q} h_U^{r-m} |v|_{W^{r,p}(U)}, \tag{6a}$$

$$\text{If } m \geqslant q: \quad |\Pi_U^{q,l} v|_{W^{q,p}(U)} \leqslant C |v|_{W^{q,p}(U)}, \tag{6b}$$

Then, there exists a real number $C > 0$ depending only on d, ϱ, l, q, m, s, and p such that, for all $v \in W^{s,p}(U)$,

$$|v - \Pi_U^{q,l} v|_{W^{m,p}(U)} \leqslant C h_U^{s-m} |v|_{W^{s,p}(U)}. \tag{7}$$

Proof. Here $A \lesssim B$ means $A \leqslant MB$ with real number $M > 0$ having the same dependencies as C in (7). Since smooth functions are dense in $W^{s,p}(U)$, we can assume $v \in C^\infty(U) \cap W^{s,p}(U)$. We consider the following representation of v proposed in [7, Chapter 4]:

$$v = Q^s v + R^s v, \tag{8}$$

where $Q^s v \in \mathbb{P}^{s-1}(U) \subset \mathbb{P}^l(U)$ is the averaged Taylor polynomial, while the remainder $R^s v$ satisfies, for all $r \in \{0, \dots, s\}$ (cf. [7, Lemma 4.3.8]),

$$|R^s v|_{W^{r,p}(U)} \lesssim h_U^{s-r} |v|_{W^{s,p}(U)}. \tag{9}$$

Since $\Pi_U^{q,l}$ is a projector, it holds $\Pi_U^{q,l}(Q^s v) = Q^s v$ so that, taking the projection of (8), it is inferred

$$\Pi_U^{q,l} v = Q^s v + \Pi_U^{q,l}(R^s v).$$

Subtracting this equation from (8), we arrive at $v - \Pi_U^{q,l} v = R^s v - \Pi_U^{q,l}(R^s v)$. Hence, the triangle inequality yields

$$|v - \Pi_U^{q,l} v|_{W^{m,p}(U)} \leqslant |R^s v|_{W^{m,p}(U)} + |\Pi_U^{q,l}(R^s v)|_{W^{m,p}(U)}. \tag{10}$$

For the first term in the right-hand side, the estimate (9) with $r = m$ readily yields

$$|R^s v|_{W^{m,p}(U)} \lesssim h_U^{s-m} |v|_{W^{s,p}(U)}. \tag{11}$$

Let us estimate the second term. If $m < q$, using the boundedness assumption (6a) followed by the estimate (9), it is inferred

$$|\Pi_U^{q,l}(R^s v)|_{W^{m,p}(U)} \lesssim \sum_{r=m}^{q} h_U^{r-m} |R^s v|_{W^{r,p}(U)} \lesssim \sum_{r=m}^{q} h_U^{r-m} h_U^{s-r} |v|_{W^{s,p}(U)} \lesssim h_U^{s-m} |v|_{W^{s,p}(U)}.$$

If, on the other hand, $m \geqslant q$, using the reverse Sobolev embeddings on polynomial spaces of [13, Remark A.2] followed by assumption (6b) and the estimate (9) with $r = q$, it is inferred that

$$|\Pi_U^{q,l}(R^s v)|_{W^{m,p}(U)} \lesssim h_U^{q-m} |\Pi_U^{q,l}(R^s v)|_{W^{q,p}(U)} \lesssim h_U^{q-m} |R^s v|_{W^{q,p}(U)} \lesssim h_U^{s-m} |v|_{W^{s,p}(U)}.$$

In conclusion we have, in either case $m < q$ or $m \geqslant q$,

$$|\Pi_U^{q,l}(R^s v)|_{W^{m,p}(U)} \lesssim h_U^{s-m}|v|_{W^{s,p}(U)}. \tag{12}$$

Using (11) and (12) to estimate the first and second term in the right-hand side of (10), respectively, the conclusion follows. □

Our second technical result concerns the L^p-boundedness of L^2-orthogonal projectors on polynomial subspaces, and will be central to prove property (6) (with $q = 1$) for the elliptic projector $\pi_U^{1,l}$. This result generalises [13, Lemma 3.2], which corresponds to $\mathcal{P} = \mathbb{P}^l(U)$.

Lemma 4 (L^p-boundeness of L^2-orthogonal projectors on polynomial subspaces)**.** *Let two integers $l \geqslant 0$ and $n \geqslant 1$ be fixed, and let $\mathcal{P}$ be a subspace of $\mathbb{P}^l(U)^n$. We consider the L^2-orthogonal projector $\Pi_{\mathcal{P}} : L^1(U)^n \to \mathcal{P}$ such that, for all $\Phi \in L^1(U)^n$,*

$$\int_T (\Pi_{\mathcal{P}}\Phi - \Phi)\cdot\Psi = 0 \text{ for all } \Psi \in \mathcal{P}. \tag{13}$$

Let $p \in [1, +\infty]$. Let r_U be the inradius of U and assume that there is a real number δ such that

$$\frac{r_U}{h_U} \geqslant \delta > 0.$$

Then there exists a real number $C > 0$ depending only on n, d, δ, l, and p such that

$$\forall \Phi \in L^p(U)^n \ : \ \|\Pi_{\mathcal{P}}\Phi\|_{L^p(U)^n} \leqslant C\|\Phi\|_{L^p(U)^n}. \tag{14}$$

Proof. We abridge as $A \lesssim B$ the inequality $A \leqslant MB$ with real number $M > 0$ having the same dependencies as C. Since $\Pi_{\mathcal{P}}$ is an L^2-orthogonal projector, (14) trivially holds with $C = 1$ if $p = 2$. On the other hand, if $p > 2$, we have, using the reverse Lebesgue embeddings on polynomial spaces of [13, Lemma 3.2] followed by (14) for $p = 2$,

$$\|\Pi_{\mathcal{P}}\Phi\|_{L^p(U)^n} \lesssim |U|_d^{\frac{1}{p}-\frac{1}{2}}\|\Pi_{\mathcal{P}}\Phi\|_{L^2(U)^n} \lesssim |U|_d^{\frac{1}{p}-\frac{1}{2}}\|\Phi\|_{L^2(U)^n}.$$

Here, $|U|_d$ is the d-dimensional measure of U. Using the Hölder inequality to infer $\|\Phi\|_{L^2(U)^n} \lesssim |U|_d^{\frac{1}{2}-\frac{1}{p}}\|\Phi\|_{L^p(U)^n}$ concludes the proof for $p > 2$. It only remains to treat the case $p < 2$. We first observe that, using the definition (13) of $\Pi_{\mathcal{P}}$ twice, for all $\Phi, \Psi \in L^1(U)^n$,

$$\int_U (\Pi_{\mathcal{P}}\Phi)\cdot\Psi = \int_U (\Pi_{\mathcal{P}}\Phi)\cdot(\Pi_{\mathcal{P}}\Psi) = \int_U \Phi\cdot(\Pi_{\mathcal{P}}\Psi).$$

Hence, with p' such that $1/p + 1/p' = 1$, it holds

$$\begin{aligned}
\|\Pi_{\mathcal{P}}\Phi\|_{L^p(U)^n} &= \sup_{\Psi\in L^{p'}(U)^n, \|\Psi\|_{L^{p'}(U)^n}=1} \int_U (\Pi_{\mathcal{P}}\Phi)\cdot\Psi \\
&= \sup_{\Psi\in L^{p'}(U)^n, \|\Psi\|_{L^{p'}(U)^n}=1} \int_U \Phi\cdot(\Pi_{\mathcal{P}}\Psi) \\
&\leqslant \sup_{\Psi\in L^{p'}(U)^n, \|\Psi\|_{L^{p'}(U)^n}=1} \|\Phi\|_{L^p(U)^n}\|\Pi_{\mathcal{P}}\Psi\|_{L^{p'}(U)^n},
\end{aligned} \tag{15}$$

where we have used the Hölder inequality to conclude. Using (14) for $p' > 2$, we have $\|\Pi_{\mathcal{P}}\Psi\|_{L^{p'}(U)^n} \lesssim \|\Psi\|_{L^{p'}(U)^n} = 1$. Plugging this bound into (15) concludes the proof for $p < 2$. □

2.2 Proof of the main results

We are now ready to prove Theorems 1 and 2. Inside the proofs, $A \lesssim B$ means $A \leqslant MB$ with M having the same dependencies as the real number C in the corresponding statement.

Proof of Theorem 1. The proof consists in verifying the boundedness property (6), with $q = 1$, for the elliptic projector first with $m = 1$ (**Step 1**) then with $m = 0$ (**Step 2**). The conclusion then follows applying Lemma 3 to $\Pi_U^{1,l} = \pi_U^{1,l}$.

Step 1. $|\cdot|_{W^{1,p}(U)}$*-boundedness.* We start by proving that

$$\forall v \in W^{1,p}(U) \,:\, |\pi_U^{1,l} v|_{W^{1,p}(U)} \lesssim |v|_{W^{1,p}(U)}. \tag{16}$$

By definition (2) of $\pi_U^{1,l}$, it holds, for all $v \in W^{1,1}(T)$,

$$\boldsymbol{\nabla} \pi_U^{1,l} v = \Pi_{\boldsymbol{\nabla}\mathbb{P}^l(U)} \boldsymbol{\nabla} v, \tag{17}$$

where $\Pi_{\boldsymbol{\nabla}\mathbb{P}^l(U)}$ denotes the L^2-orthogonal projector on $\boldsymbol{\nabla}\mathbb{P}^l(U) \subset \mathbb{P}^{l-1}(U)^d$. Then, (16) is proved observing that, by definition (1) of the $|\cdot|_{W^{1,p}(U)}$-seminorm, and invoking (17) and the $(L^p)^d$-boundedness of $\Pi_{\boldsymbol{\nabla}\mathbb{P}^l(U)}$ resulting from (14) with $\mathcal{P} = \boldsymbol{\nabla}\mathbb{P}^l(U)$, we have

$$|\pi_U^{1,l} v|_{W^{1,p}(U)} \lesssim \|\boldsymbol{\nabla} \pi_U^{1,l} v\|_{L^p(U)^d} = \|\Pi_{\boldsymbol{\nabla}\mathbb{P}^l(U)} \boldsymbol{\nabla} v\|_{L^p(U)^d} \lesssim \|\boldsymbol{\nabla} v\|_{L^p(U)^d} \lesssim |v|_{W^{1,p}(U)}.$$

Step 2. $\|\cdot\|_{L^p(U)}$*-boundedness.* We next prove that

$$\forall v \in W^{1,p}(U) \,:\, \|\pi_U^{1,l} v\|_{L^p(U)} \lesssim h_U |v|_{W^{1,p}(U)} + \|v\|_{L^p(U)}. \tag{18}$$

Let $v \in W^{1,p}(U)$ and denote by $\overline{v} \in \mathbb{P}^0(U)$ the L^2-orthogonal projection of v on $\mathbb{P}^0(U)$ such that

$$\int_U (v - \overline{v}) = 0, \text{ that is, } \overline{v} = \frac{1}{|U|_d} \int_U v.$$

By definition (2) of the elliptic projector, $\overline{v}$ is also the L^2-orthogonal projection on $\mathbb{P}^0(U)$ of $\pi_U^{1,l} v$. The $W^{s,p}$-approximation of the L^2-projector (63) (applied with $m = 0$ and $s = 1$ to $\pi_U^{1,l} v$ instead of v) therefore gives $\|\pi_U^{1,l} v - \overline{v}\|_{L^p(U)} \lesssim h_U |\pi_U^{1,l} v|_{W^{1,p}(U)}$. This yields

$$\begin{aligned}
\|\pi_U^{1,l} v\|_{L^p(U)} &\leqslant \|\pi_U^{1,l} v - \overline{v}\|_{L^p(U)} + \|\overline{v}\|_{L^p(U)} \\
&\lesssim h_U |\pi_U^{1,l} v|_{W^{1,p}(U)} + \|\overline{v}\|_{L^p(U)} \\
&\lesssim h_U |v|_{W^{1,p}(U)} + \|v\|_{L^p(U)},
\end{aligned}$$

where we have introduced $\pm\overline{v}$ inside the norm and used the triangle inequality in the first line, and the terms in the second line are have been estimated using (16) for the first one and the Jensen inequality for the second one. □

Proof of Theorem 2. Under the assumptions on U, we have the following L^p-trace inequality (cf. [13, Lemma 3.6] for a proof): For all $w \in W^{1,p}(U)$,

$$h_U^{\frac{1}{p}} \|w\|_{L^p(\partial U)} \lesssim \|w\|_{L^p(U)} + h_U \|\boldsymbol{\nabla} w\|_{L^p(U)}. \tag{19}$$

For $m \leqslant s-1$, by applying (19) to $w = \partial^{\boldsymbol{\alpha}}(v - \pi_U^{1,l} v) \in W^{1,p}(U)$ for all $\boldsymbol{\alpha} \in \mathbb{N}^d$ such that $\|\boldsymbol{\alpha}\|_1 = m$, we find

$$h_U^{\frac{1}{p}} |v - \pi_U^{1,l} v|_{W^{m,p}(\mathcal{F}_U)} \lesssim |v - \pi_U^{1,l} v|_{W^{m,p}(U)} + h_U |v - \pi_U^{1,l} v|_{W^{m+1,p}(U)}.$$

To conclusion follows using (3) for m and $m+1$ to bound the two terms in the right-hand side. □

3 Error estimates for a Hybrid High-Order discretisation of Leray–Lions problems

In this section we use the approximation results for the elliptic projector to derive new error estimates for the HHO discretisation of Leray–Lions problems introduced in [13] (where convergence to minimal regularity solutions is proved using a compactness argument).

3.1 Continuous model

We consider problem (5) under the following assumptions for a fixed $p \in (1, +\infty)$ with $p' := \frac{p}{p-1}$:

$$f \in L^{p'}(\Omega), \tag{20a}$$

$$\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \text{ is a Caratheodory function,} \tag{20b}$$

$$\begin{aligned} &\mathbf{a}(\cdot, \mathbf{0}) \in L^{p'}(\Omega)^d \text{ and} \\ &\exists \beta_{\mathbf{a}} \in (0, +\infty) \,:\, |\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \mathbf{0})| \leqslant \beta_{\mathbf{a}} |\boldsymbol{\xi}|^{p-1} \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \end{aligned} \tag{20c}$$

$$\exists \lambda_{\mathbf{a}} \in (0, +\infty) \,:\, \mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant \lambda_{\mathbf{a}} |\boldsymbol{\xi}|^p \text{ for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \tag{20d}$$

$$\begin{aligned} &\exists \gamma_{\mathbf{a}} \in (0, +\infty) \,:\, |\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant \gamma_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}| (|\boldsymbol{\xi}|^{p-2} + |\boldsymbol{\eta}|^{p-2}) \\ &\qquad \text{for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d, \end{aligned} \tag{20e}$$

$$\begin{aligned} &\exists \zeta_{\mathbf{a}} \in (0, +\infty) \,:\, [\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}] \geqslant \zeta_{\mathbf{a}} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 (|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2} \\ &\qquad \text{for a.e. } \boldsymbol{x} \in \Omega, \text{ for all } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d, \end{aligned} \tag{20f}$$

Assumptions (20b)–(20d) are the pillars of Leray–Lions operators and stipulate, respectively, the regularity for $\mathbf{a}$, its growth, and its coercivity. Assumptions (20e) and (20f) additionally require the Lipschitz continuity and uniform monotonicity of $\mathbf{a}$ in an appropriate form.

Remark 5 (p-Laplacian). A particularly important example of Leray–Lions problem is the p-Laplace equation, which corresponds to the function

$$\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2} \boldsymbol{\xi}. \tag{21}$$

Properties (20b)–(20d) are trivially verified for this choice, which additionally verifies (20e) and (20f); cf. [4] for a proof of the former and [23] for a proof of both.

As usual, problem (5) is understood in the following weak sense:

$$\text{Find } u \in W_0^{1,p}(\Omega) \text{ such that, for all } v \in W_0^{1,p}(\Omega), \int_\Omega \mathbf{a}(\boldsymbol{x}, \boldsymbol{\nabla} u(\boldsymbol{x})) \cdot \boldsymbol{\nabla} v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v, \tag{22}$$

where $W_0^{1,p}(\Omega)$ is spanned by the elements of $W^{1,p}(\Omega)$ that vanish on $\partial\Omega$ in the sense of traces.

3.2 The Hybrid High-Order (HHO) method

We briefly recall here the construction of the HHO method and a few known results that will be needed in the analysis.

3.2.1 Mesh and notations

Let us start by the notion of mesh, and some associated notations. A mesh $\mathcal{T}_h$ is a finite collection of nonempty disjoint open polytopal elements T such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$, with h_T standing for the diameter of T. A face F is defined as a hyperplanar closed connected subset of $\overline{\Omega}$ with positive $(d-1)$-dimensional Hausdorff measure and such that (i) either there exist $T_1, T_2 \in \mathcal{T}_h$ such that $F \subset \partial T_1 \cap \partial T_2$ and F is called an interface or (ii) there exists $T \in \mathcal{T}_h$ such that $F \subset \partial T \cap \partial\Omega$ and F is called a boundary face. Interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$, boundary faces in $\mathcal{F}_h^{\mathrm{b}}$, and we let $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$. The diameter of a face $F \in \mathcal{F}_h$ is denoted by h_F. For all $T \in \mathcal{T}_h$, $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denotes the set of faces contained in ∂T (with ∂T denoting the boundary of T) and, for all $F \in \mathcal{F}_T$, $\boldsymbol{n}_{TF}$ is the unit normal to F pointing out of T. Throughout the rest of the paper, we assume the following regularity for $\mathcal{T}_h$.

Assumption 6 (Regularity assumption on $\mathcal{T}_h$)**.** The mesh $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ and there exists a real number $\varrho > 0$ such that: (i) For all simplices $S \in \mathfrak{T}_h$ of diameter h_S and inradius r_S, $\varrho h_S \leqslant r_S$, and (ii) for all $T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_h$ such that $S \subset T$, $\varrho h_T \leqslant h_S$.

When working on refined mesh sequences, all the (explicit or implicit) constants we consider below remain bounded provided that ϱ remains bounded away from 0 in the refinement process. Additionally, mesh elements satisfy the geometric regularity assumptions that enable the use of both Theorems 1 and 2 (as well as Lemma 13 below).

3.2.2 Degrees of freedom and interpolation operators

Let a polynomial degree $k \geqslant 0$ and an element $T \in \mathcal{T}_h$ be fixed. The local space of degrees of freedom (DOFs) is

$$\underline{\mathsf{U}}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F) \right), \tag{23}$$

where $\mathbb{P}^k(F)$ denotes the set of $(d-1)$-variate polynomials on F. We use the underlined notation $\underline{\mathsf{v}}_T = (\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T})$ for a generic element $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$. If $U = T \in \mathcal{T}_h$ or $U = F \in \mathcal{F}_h$, we define the L^2-projector $\pi_U^{0,l} : L^1(U) \to \mathbb{P}^l(U)$ such that, for any $v \in L^1(U)$, $\pi_U^{0,l} v$ is the unique element of $\mathbb{P}^l(U)$ satisfying

$$\forall w \in \mathbb{P}^l(U) \,:\, \int_U (\pi_U^{0,l} v - v)\, w = 0. \tag{24}$$

When applied to vector-valued function, it is understood that $\pi_U^{0,l}$ acts component-wise. The local interpolation operator $\underline{\mathsf{I}}_T^k : W^{1,1}(T) \to \underline{\mathsf{U}}_T^k$ is then given by

$$\forall v \in W^{1,1}(T) \,:\, \underline{\mathsf{I}}_T^k v := (\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F\in\mathcal{F}_T}). \tag{25}$$

Local DOFs are collected in the following global space obtained by patching interface values:

$$\underline{\mathsf{U}}_h^k := \left(\bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F\in\mathcal{F}_h} \mathbb{P}^k(F) \right).$$

A generic element of $\underline{\mathsf{U}}_h^k$ is denoted by $\underline{\mathsf{v}}_h = ((\mathsf{v}_T)_{T\in\mathcal{T}_h}, (\mathsf{v}_F)_{F\in\mathcal{F}_h})$ and, for all $T \in \mathcal{T}_h$, $\underline{\mathsf{v}}_T = (\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T})$ is its restriction to T. We also introduce the notation v_h for the broken polynomial function in $\mathbb{P}^k(\mathcal{T}_h) := \left\{ v \in L^1(\Omega) \,:\, v_{|T} \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h \right\}$ obtained from element-based DOFs by setting $\mathsf{v}_{h|T} = \mathsf{v}_T$ for all $T \in \mathcal{T}_h$. The global interpolation operator $\underline{\mathsf{I}}_h^k : W^{1,1}(\Omega) \to \underline{\mathsf{U}}_h^k$ is such that

$$\forall v \in W^{1,1}(\Omega) \,:\, \underline{\mathsf{I}}_h^k v := ((\pi_T^{0,k} v)_{T\in\mathcal{T}_h}, (\pi_F^{0,k} v)_{F\in\mathcal{F}_h}). \tag{26}$$

3.2.3 Gradient and potential reconstructions

For $U = T \in \mathcal{T}_h$ or $U = F \in \mathcal{F}_h$, we denote henceforth by $(\cdot,\cdot)_U$ the L^2- or $(L^2)^d$-inner product on U. The HHO method hinges on the local discrete gradient operator $\boldsymbol{G}_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}^k(T)^d$ such that, for all $\underline{\mathsf{v}}_T = (\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k$, $\boldsymbol{G}_T^k \underline{\mathsf{v}}_T$ is the unique solution of the following problem: For all $\boldsymbol{\phi} \in \mathbb{P}^k(T)^d$,

$$(\boldsymbol{G}_T^k \underline{\mathsf{v}}_T, \boldsymbol{\phi})_T := -(\mathsf{v}_T, \operatorname{div} \boldsymbol{\phi})_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F, \boldsymbol{\phi}\cdot\boldsymbol{n}_{TF})_F. \tag{27}$$

In (27), the right-hand side mimicks an integration by parts formula where the role of the scalar function inside volumetric and boundary integrals is played by element-based and face-based DOFs, respectively. This recipe for the gradient reconstruction is justified observing that, as a consequence of the definitions (25) of $\underline{\mathsf{I}}_T^k$ and (24) of the L^2-projector, we have the following commuting property: For all $v \in W^{1,1}(T)$,

$$\boldsymbol{G}_T^k \underline{\mathsf{I}}_T^k v = \pi_T^{0,k}(\boldsymbol{\nabla} v). \tag{28}$$

For further use, we note the following formula inferred from (27) integrating by parts the first term in the right-hand side: For all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$ and all $\boldsymbol{\phi} \in \mathbb{P}^k(T)^d$,

$$(\boldsymbol{G}_T^k \underline{\mathsf{v}}_T, \boldsymbol{\phi})_T = (\boldsymbol{\nabla}\mathsf{v}_T, \boldsymbol{\phi})_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F - \mathsf{v}_T, \boldsymbol{\phi}\cdot\boldsymbol{n}_{TF})_F. \tag{29}$$

We also define the local potential reconstruction operator $p_T^{k+1} : \underline{\mathsf{U}}_T^k \to \mathbb{P}^{k+1}(T)$ such that, for all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$,

$$\int_T (\boldsymbol{\nabla} p_T^{k+1}\underline{\mathsf{v}}_T - \boldsymbol{G}_T^k \underline{\mathsf{v}}_T)\cdot\boldsymbol{\nabla} w = 0 \text{ for all } w \in \mathbb{P}^{k+1}(T) \text{ and } \int_T (p_T^{k+1}\underline{\mathsf{v}}_T - \mathsf{v}_T) = 0. \tag{30}$$

As already noticed in [17] (cf., in particular, Eq. (17) therein), we have the following relation which establishes a link between the potential reconstruction p_T^{k+1} composed with the interpolation operator $\underline{\mathsf{I}}_T^k$ defined by (25) and the elliptic projector $\pi_T^{1,k+1}$ defined by (2):

$$p_T^{k+1} \circ \underline{\mathsf{I}}_T^k = \pi_T^{1,k+1}. \tag{31}$$

The local gradient and potential reconstructions give rise to the global gradient operator $\boldsymbol{G}_h^k : \underline{\mathsf{U}}_h^k \to \mathbb{P}^k(\mathcal{T}_h)^d$ and potential reconstruction $p_h^{k+1} : \underline{\mathsf{U}}_h^k \to \mathbb{P}^{k+1}(\mathcal{T}_h)$ such that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$(\boldsymbol{G}_h^k \underline{\mathsf{v}}_h)_{|T} = \boldsymbol{G}_T^k \underline{\mathsf{v}}_T \text{ and } (p_h^{k+1}\underline{\mathsf{v}}_h)_{|T} = p_T^{k+1}\underline{\mathsf{v}}_T \text{ for all } T \in \mathcal{T}_h. \tag{32}$$

3.2.4 Discrete problem

For all $T \in \mathcal{T}_h$, we define the local function $\mathrm{A}_T : \underline{\mathsf{U}}_T^k \times \underline{\mathsf{U}}_T^k \to \mathbb{R}$ such that

$$\mathrm{A}_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := \int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{\mathsf{u}}_T(\boldsymbol{x})) \cdot \boldsymbol{G}_T^k \underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T), \tag{33a}$$

with $s_T : \underline{\mathsf{U}}_T^k \times \underline{\mathsf{U}}_T^k \to \mathbb{R}$ stabilisation term such that

$$s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := \sum_{F\in\mathcal{F}_T} h_F^{1-p} \int_F \left|\delta_{TF}^k \underline{\mathsf{u}}_T\right|^{p-2} \delta_{TF}^k \underline{\mathsf{u}}_T\ \delta_{TF}^k \underline{\mathsf{v}}_T, \tag{33b}$$

where the scaling factor h_F^{1-p} ensures the dimensional homogeneity of the terms composing A_T, and the face-based residual operator $\delta_{TF}^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}^k(F)$ is defined such that, for all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$,

$$\delta_{TF}^k \underline{\mathsf{v}}_T := \pi_F^{0,k}(\mathsf{v}_F - p_T^{k+1}\underline{\mathsf{v}}_T) - \pi_T^{0,k}(\mathsf{v}_T - p_T^{k+1}\underline{\mathsf{v}}_T). \tag{33c}$$

A global function $\mathrm{A}_h : \underline{\mathsf{U}}_h^k \times \underline{\mathsf{U}}_h^k \to \mathbb{R}$ is assembled element-wise from local contributions setting

$$\mathrm{A}_h(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{A}_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T). \tag{33d}$$

Boundary conditions are strongly enforced by considering the following subspace of $\underline{\mathsf{U}}_h^k$:

$$\underline{\mathsf{U}}_{h,0}^k := \left\{ \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k \mid \mathsf{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}. \tag{33e}$$

The HHO approximation of problem (22) reads:

$$\text{Find } \underline{\mathsf{u}}_h \in \underline{\mathsf{U}}_{h,0}^k \text{ such that, for all } \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k,\ \mathrm{A}_h(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) = \int_\Omega f \mathsf{v}_h. \tag{33f}$$

For a discussion on the existence and uniqueness of a solution to (33) we refer the reader to [13, Theorem 4.5 and Remark 4.7].

3.3 Error estimates

We state in this section an error estimate in terms of the following discrete $W^{1,p}$-seminorm on $\underline{\mathsf{U}}_h^k$:

$$\|\underline{\mathsf{v}}_h\|_{1,p,h} := \left(\sum_{T \in \mathcal{T}_h} \|\underline{\mathsf{v}}_T\|_{1,p,T}^p \right)^{\frac{1}{p}}, \text{ where } \|\underline{\mathsf{v}}_T\|_{1,p,T} := \left(\|\boldsymbol{\nabla} p_T^{k+1} \underline{\mathsf{v}}_T\|_{L^p(T)^d}^p + s_T(\underline{\mathsf{v}}_T, \underline{\mathsf{v}}_T) \right)^{\frac{1}{p}}. \tag{34}$$

It is a simple matter to realise that the map $\|\cdot\|_{1,p,h}$ defines a norm on $\underline{\mathsf{U}}_{h,0}^k$. The regularity assumptions on the exact solution are expressed in terms of the broken $W^{s,p}$-spaces defined by

$$W^{s,p}(\mathcal{T}_h) := \{ v \in L^p(\Omega) \,:\, \forall T \in \mathcal{T}_h \,,\ v \in W^{s,p}(T) \},$$

which we endow with the norm

$$\|v\|_{W^{s,p}(\mathcal{T}_h)} := \left(\sum_{T \in \mathcal{T}_h} \|v\|_{W^{s,p}(T)}^p \right)^{\frac{1}{p}}.$$

Notice that, if $v \in W^{s,p}(\mathcal{T}_h)$ for a certain mesh $\mathcal{T}_h$, then $\|v\|_{W^{s,p}(\mathcal{T}_h)}$ depends only on v, not on $\mathcal{T}_h$. Our main result is summarised in the following theorem, whose proof makes use of the approximation results for the elliptic projector stated in Theorems 1 and 2; cf. Remark 12 for further insight into their role.

Theorem 7 (Error estimate)**.** *Let the assumptions in* (20) *hold, and let u solve* (22)*. Let a polynomial degree $k \geqslant 0$ and a mesh $\mathcal{T}_h$ be fixed, and let $\underline{\mathsf{u}}_h$ solve* (33)*. Assume the additional regularity $u \in W^{k+2,p}(\mathcal{T}_h)$ and $\mathbf{a}(\cdot, \boldsymbol{\nabla} u) \in W^{k+1,p'}(\mathcal{T}_h)^d$ (with $p' = \frac{p}{p-1}$), and define the quantity $\mathrm{E}_h(u)$ as follows:*

- *If $p \geqslant 2$,*

$$\mathrm{E}_h(u) := h^{k+1} |u|_{W^{k+2,p}(\mathcal{T}_h)} + h^{\frac{k+1}{p-1}} \left(|u|_{W^{k+2,p}(\mathcal{T}_h)}^{\frac{1}{p-1}} + |\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}^{\frac{1}{p-1}} \right); \tag{35a}$$

- *If $p < 2$,*

$$\mathrm{E}_h(u) := h^{(k+1)(p-1)} |u|_{W^{k+2,p}(\mathcal{T}_h)}^{p-1} + h^{k+1} |\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}. \tag{35b}$$

Then, there exists a real number $C > 0$ depending only on Ω, k, the mesh regularity parameter ϱ defined in Assumption 6, the coefficients p, $\beta_{\mathbf{a}}$, $\lambda_{\mathbf{a}}$, $\gamma_{\mathbf{a}}$, $\zeta_{\mathbf{a}}$ defined in (20)*, and an upper bound of $\|f\|_{L^{p'}(\Omega)}$ such that*

$$\|\underline{\mathsf{I}}_h^k u - \underline{\mathsf{u}}_h\|_{1,p,h} \leqslant C \mathrm{E}_h(u). \tag{36}$$

Proof. See Section 3.4. □

Some remarks are of order.

Remark 8 (Order of convergence). From (36), it is inferred that the approximation error in the discrete $W^{1,p}$-norm scales as the dominant terms in E_h, namely $h^{\frac{k+1}{p-1}}$ if $p \geqslant 2$ and $h^{(k+1)(p-1)}$ if $p < 2$.

Remark 9 (Role of the various terms). There is a nice parallel between the various error terms in (35) and the error estimate obtained for gradient schemes in [23]. In the gradient schemes framework [25, 28], the accuracy of a scheme is essentially assessed through two quantities: a measure $W_{\mathcal{D}}$ of the default of conformity of the scheme, and a measure $S_{\mathcal{D}}$ of the consistency of the scheme. In (35), the terms involving $|\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}$ estimate the contribution to the error of the default of conformity of the method, and the terms involving $|u|_{W^{k+2,p}(\mathcal{T}_h)}$ come from the consistency error of the method.

From the convergence result in Theorem 7, we can infer an error estimate on the potential reconstruction $p_h^{k+1}\underline{\mathsf{u}}_h$ and on its jumps measured through the stabilisation function s_T.

Corollary 10 (Convergence of the potential reconstruction)**.** *Under the notations and assumptions in Theorem 7, and denoting by $\boldsymbol{\nabla}_h$ the broken gradient on $\mathcal{T}_h$, we have*

$$\left(\|\boldsymbol{\nabla}_h(u - p_h^{k+1}\underline{\mathsf{u}}_h)\|^p_{L^p(\Omega)^d} + \sum_{T\in\mathcal{T}_h} s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{u}}_T)\right)^{1/p} \leqslant C\left(\mathrm{E}_h(u) + h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)}\right), \tag{37}$$

where C has the same dependencies as in Theorem 7.

Proof. See Section 3.4. □

Remark 11 (Variations). Following [13, Remark 4.4], variations of the HHO scheme (33) are obtained replacing the space $\underline{\mathsf{U}}_T^k$ defined by (23) by

$$\underline{\mathsf{U}}_T^{l,k} := \mathbb{P}^l(T) \times \left(\bigtimes_{F\in\mathcal{F}_h} \mathbb{P}^k(F)\right),$$

for $k \geqslant 0$ and $l \in \{k-1, k, k+1\}$. For the sake of simplicity, we consider the case $l = k-1$ only when $k \geqslant 1$ (technical modifications, not detailed here, are required for $k = 0$ and $l = k-1$ owing to the absence of element DOFs). The interpolant $\underline{\mathsf{I}}_T^k$ naturally has to be replaced with $\underline{\mathsf{I}}_T^{l,k} v := (\pi_T^{0,l} v, (\pi_F^{0,k} v)_{F\in\mathcal{F}_T})$. The definitions (27) of $\boldsymbol{G}_T^k$ and (30) of p_T^{k+1} remain formally the same (only the domain of the operators changes), and a close inspection shows that both key properties (28) and (31) remain valid for all the proposed choices for l –replacing, of course, $\underline{\mathsf{I}}_T^k$ with $\underline{\mathsf{I}}_T^{l,k}$ in (31). In the expression (33b) of the penalization bilinear form s_T, we replace the face-based residual δ_{TF}^k defined by (33c) with a new operator $\delta_{TF}^{l,k} : \underline{\mathsf{U}}_T^{l,k} \to \mathbb{P}^k(F)$ such that, for all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^{l,k}$,

$$\delta_{TF}^{l,k}\underline{\mathsf{v}}_T := \pi_F^{0,k}\left(\mathsf{v}_F - p_T^{k+1}\underline{\mathsf{v}}_T - \pi_T^{0,l}(\mathsf{v}_T - p_T^{k+1}\underline{\mathsf{v}}_T)\right).$$

Up to minor modifications, the proof of Theorem 7 remains valid, and therefore so is the case for the error estimates (36) and (37).

3.4 Proof of the error estimates

In this section, we write $A \lesssim B$ for $A \leqslant MB$ with M having the same dependencies as C in Theorem 7. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

Proof of Theorem 7. The proof is split into several steps. In **Step 1** we obtain an initial estimate involving, on the left-hand side, $\mathbf{a}$ and s_T, and, on the right-hand side, a sum of four terms. In **Step 2** we prove that the left-hand side of this estimate provides an upper bound of the approximation error $\|\underline{\mathsf{I}}_h^k u - \underline{\mathsf{u}}_h\|_{1,p,h}$. Then, in **Steps 3–5**, we estimate each of the four terms in the right-hand side of the original estimate. Combined with the result of **Step 2**, these estimates prove (36).

Throughout the proof, to alleviate the notation, we write $\mathcal{O}(X)$ for a quantity that satisfies $|\mathcal{O}(X)| \lesssim X$, and we abridge $\underline{\mathsf{I}}_h^k u$ into $\widehat{\underline{\mathsf{u}}}_h$.

We will need the following equivalence of local seminorms, established in [13, Lemma 5.2]: For all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$,

$$\|\underline{\mathsf{v}}_T\|_{1,p,T} \approx \left(\|\boldsymbol{\nabla}\mathsf{v}_T\|_{L^p(T)^d}^p + \sum_{F\in\mathcal{F}_T} h_F^{1-p}\|\mathsf{v}_F - \mathsf{v}_T\|_{L^p(F)}^p\right)^{\frac{1}{p}} \approx \left(\|\boldsymbol{G}_T^k\underline{\mathsf{v}}_T\|_{L^p(T)^d}^p + s_T(\underline{\mathsf{v}}_T,\underline{\mathsf{v}}_T)\right)^{\frac{1}{p}}. \quad (38)$$

Step 1. *Initial estimate.* Let $\underline{\mathsf{v}}_h$ be a generic element of $\underline{\mathsf{U}}_{h,0}^k$, and denote by $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$ its restriction to a generic mesh element $T \in \mathcal{T}_h$. In this step, we estimate the error made when using $\widehat{\underline{\mathsf{u}}}_h$, instead of $\underline{\mathsf{u}}_h$, in the scheme, namely

$$\mathcal{E}_h(\underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h}\int_T \left[\mathbf{a}(\boldsymbol{x},\boldsymbol{G}_T^k\widehat{\underline{\mathsf{u}}}_T) - \mathbf{a}(\boldsymbol{x},\boldsymbol{G}_T^k\underline{\mathsf{u}}_T)\right]\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T + \sum_{T\in\mathcal{T}_h}(s_T(\widehat{\underline{\mathsf{u}}}_T,\underline{\mathsf{v}}_T) - s_T(\underline{\mathsf{u}}_T,\underline{\mathsf{v}}_T)). \quad (39)$$

Let $T \in \mathcal{T}_h$ be fixed. Setting

$$\mathfrak{T}_{1,T} := \|\mathbf{a}(\cdot,\boldsymbol{G}_T^k\widehat{\underline{\mathsf{u}}}_T) - \mathbf{a}(\cdot,\boldsymbol{\nabla}u)\|_{L^{p'}(T)^d}, \quad (40)$$

by the Hölder inequality we infer

$$\int_T \mathbf{a}(\boldsymbol{x},\boldsymbol{G}_T^k\widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x}))\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_T \mathbf{a}(\boldsymbol{x},\boldsymbol{\nabla}u(\boldsymbol{x}))\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}(\mathfrak{T}_{1,T})\|\boldsymbol{G}_T^k\underline{\mathsf{v}}_T\|_{L^p(T)^d}.$$

To benefit from the definition (29) of $\boldsymbol{G}_T^k\underline{\mathsf{v}}_T$, we approximate $\mathbf{a}(\cdot,\boldsymbol{\nabla}u)$ by its L^2-orthogonal projetion on the polynomial space $\mathbb{P}^k(T)^d$. We therefore introduce

$$\mathfrak{T}_{2,T} := \|\mathbf{a}(\cdot,\boldsymbol{\nabla}u) - \pi_T^{0,k}\mathbf{a}(\cdot,\boldsymbol{\nabla}u)\|_{L^{p'}(T)^d}, \quad (41)$$

and we have

$$\int_T \mathbf{a}(\boldsymbol{x},\boldsymbol{G}_T^k\widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x}))\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_T \pi_T^{0,k}\mathbf{a}(\boldsymbol{x},\boldsymbol{\nabla}u(\boldsymbol{x}))\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}(\mathfrak{T}_{1,T}+\mathfrak{T}_{2,T})\|\boldsymbol{G}_T^k\underline{\mathsf{v}}_T\|_{L^p(T)^d}. \quad (42)$$

Using (29) with $\boldsymbol{\phi} = \pi_T^{0,k}\mathbf{a}(\cdot,\boldsymbol{\nabla}u)$, the first term in the right-hand side rewrites

$$\int_T \pi_T^{0,k}\mathbf{a}(\boldsymbol{x},\boldsymbol{\nabla}u(\boldsymbol{x}))\cdot\boldsymbol{G}_T^k\underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (\pi_T^{0,k}\mathbf{a}(\cdot,\boldsymbol{\nabla}u),\boldsymbol{\nabla}\mathsf{v}_T)_T + \sum_{F\in\mathcal{F}_T}(\pi_T^{0,k}\mathbf{a}(\cdot,\boldsymbol{\nabla}u)\cdot\boldsymbol{n}_{TF},\mathsf{v}_F-\mathsf{v}_T)_F.$$

We now want to eliminate the projectors $\pi_T^{0,k}$, in order to utilise the fact that u is a solution to (5). In the first term, the projector $\pi_T^{0,k}$ can be cancelled simply by observing that $\boldsymbol{\nabla}\mathsf{v}_T \in \mathbb{P}^{k-1}(T)^d \subset \mathbb{P}^k(T)^d$, whereas for the second term we introduce an error controlled by

$$\mathfrak{T}_{3,T} := \left(\sum_{F\in\mathcal{F}_T} h_F\|\mathbf{a}(\cdot,\boldsymbol{\nabla}u) - \pi_T^{0,k}\mathbf{a}(\cdot,\boldsymbol{\nabla}u)\|_{L^{p'}(F)}^{p'}\right)^{\frac{1}{p'}} \quad (43)$$

(this quantity is well defined since $\mathbf{a}(\cdot, \boldsymbol{\nabla} u) \in W^{1,p'}(T)^d$ by assumption). We therefore have, using the Hölder inequality,

$$\int_T \pi_T^{0,k}\mathbf{a}(\boldsymbol{x}, \boldsymbol{\nabla} u(\boldsymbol{x}))\cdot \boldsymbol{G}_T^k \underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (\mathbf{a}(\cdot, \boldsymbol{\nabla} u), \boldsymbol{\nabla}\mathsf{v}_T)_T + \sum_{F\in\mathcal{F}_T} (\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{TF}, \mathsf{v}_F - \mathsf{v}_T)_F$$
$$+ \mathcal{O}(\mathfrak{T}_{3,T})\left(\sum_{F\in\mathcal{F}_T} h_F^{1-p}\|\mathsf{v}_F - \mathsf{v}_T\|_{L^p(F)}^p\right)^{\frac{1}{p}}.$$

We plug this expression into (42) and use the equivalence of seminorms (38) to obtain

$$\int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x}))\cdot \boldsymbol{G}_T^k \underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (\mathbf{a}(\cdot, \boldsymbol{\nabla} u), \boldsymbol{\nabla}\mathsf{v}_T)_T + \sum_{F\in\mathcal{F}_T} (\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{TF}, \mathsf{v}_F - \mathsf{v}_T)_F$$
$$+ \mathcal{O}(\mathfrak{T}_{1,T} + \mathfrak{T}_{2,T} + \mathfrak{T}_{3,T})\|\underline{\mathsf{v}}_T\|_{1,p,T}.$$

Integrating by parts the first term in the right-hand side and writing $-\operatorname{div}(\mathbf{a}(\cdot, \boldsymbol{\nabla} u)) = f$ in T, we arrive at

$$\int_T \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x}))\cdot \boldsymbol{G}_T^k \underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} =$$
$$(f, \mathsf{v}_T)_T + \sum_{F\in\mathcal{F}_T} (\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{TF}, \mathsf{v}_F)_F + \mathcal{O}(\mathfrak{T}_{1,T} + \mathfrak{T}_{2,T} + \mathfrak{T}_{3,T})\|\underline{\mathsf{v}}_T\|_{1,p,T}.$$

We then sum over $T \in \mathcal{T}_h$, use $\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{T_1F} = -\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{T_2F}$ on F whenever $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$ (this is because $-\operatorname{div}(\mathbf{a}(\cdot, \boldsymbol{\nabla} u)) \in L^{p'}(\Omega)$) together with $\mathsf{v}_F = 0$ whenever $F \in \mathcal{F}_h^{\mathrm{b}}$ to infer

$$\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} (\mathbf{a}(\cdot, \boldsymbol{\nabla} u)\cdot \boldsymbol{n}_{TF}, \mathsf{v}_F)_F = 0,$$

invoke the scheme (33), and use the Hölder inequality on the $\mathcal{O}$ terms to write

$$\sum_{T\in\mathcal{T}_h}\int_T \left[\mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x})) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{\mathsf{u}}_T(\boldsymbol{x}))\right]\cdot \boldsymbol{G}_T^k \underline{\mathsf{v}}_T(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \sum_{T\in\mathcal{T}_h} s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T)$$
$$= \mathcal{O}(\mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3)\|\underline{\mathsf{v}}_h\|_{1,p,h}$$

where, for $i \in \{1, 2, 3\}$, we have set

$$\mathfrak{T}_i := \left(\sum_{T\in\mathcal{T}_h} \mathfrak{T}_{i,T}^{p'}\right)^{\frac{1}{p'}}. \tag{44}$$

Finally, introducing the last error term

$$\mathfrak{T}_4 := \sup_{\underline{\mathsf{v}}_h\in\underline{\mathsf{U}}_h^k, \underline{\mathsf{v}}_h\neq\underline{0}_h} \frac{\sum_{T\in\mathcal{T}_h} s_T(\widehat{\underline{\mathsf{u}}}_T, \underline{\mathsf{v}}_T)}{\|\underline{\mathsf{v}}_h\|_{1,p,h}}, \tag{45}$$

we have

$$\mathcal{E}_h(\underline{\mathsf{v}}_h) = \mathcal{O}(\mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3 + \mathfrak{T}_4)\|\underline{\mathsf{v}}_h\|_{1,p,h}. \tag{46}$$

Step 2. *Lower bound for $\mathcal{E}_h(\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h)$.*

Let, for the sake of conciseness, $\underline{\mathsf{e}}_h := \widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h$. The goal of this step is to find a lower bound for $\mathcal{E}_h(\underline{\mathsf{e}}_h)$ in terms of the error measure $\|\underline{\mathsf{e}}_h\|_{1,p,h}$. To this end, we let $\underline{\mathsf{v}}_h = \underline{\mathsf{e}}_h$ in the definition (39) of $\mathcal{E}_h$ and distinguish two cases.

Case $p \geqslant 2$: Using for all $T \in \mathcal{T}_h$ the bound (70) below with $\boldsymbol{\xi} = \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T$ and $\boldsymbol{\eta} = \boldsymbol{G}_T^k \underline{\mathsf{u}}_T$ for the first term in the right-hand side of (39), the definition (33b) of s_T and, for all $F \in \mathcal{F}_T$, the bound (72) below with $t = \delta_{TF}^k \widehat{\underline{\mathsf{u}}}_T$ and $r = \delta_{TF}^k \underline{\mathsf{u}}_T$ for the second, and concluding by the norm equivalence (38), we have

$$\mathcal{E}_h(\underline{\mathsf{e}}_h) \gtrsim \sum_{T \in \mathcal{T}_h} \left(\|\boldsymbol{G}_T^k \underline{\mathsf{e}}_T\|_{L^p(T)^d}^p + \sum_{F \in \mathcal{F}_T} h_F^{1-p} \|\delta_{TF}^k \underline{\mathsf{e}}_T\|_{L^p(F)}^p \right) \gtrsim \|\underline{\mathsf{e}}_h\|_{1,p,h}^p. \tag{47}$$

Case $p < 2$: Let an element $T \in \mathcal{T}_h$ be fixed. Applying (69) below to $\boldsymbol{\xi} = \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T$ and $\boldsymbol{\eta} = \boldsymbol{G}_T^k \underline{\mathsf{u}}_T$, integrating over T and using the Hölder inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we get

$$\|\boldsymbol{G}_T^k \underline{\mathsf{e}}_T\|_{L^p(T)^d}^p \lesssim \left(\int_T [\mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T(\boldsymbol{x})) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{G}_T^k \underline{\mathsf{u}}_T(\boldsymbol{x}))] \cdot \boldsymbol{G}_T^k \underline{\mathsf{e}}_T(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\frac{p}{2}} \times \left(\|\boldsymbol{G}_T^k \widehat{\underline{\mathsf{u}}}_T\|_{L^p(T)^d}^p + \|\boldsymbol{G}_T^k \underline{\mathsf{u}}_T\|_{L^p(T)^d}^p \right)^{\frac{2-p}{2}}.$$

Summing over $T \in \mathcal{T}_h$ and using the discrete Hölder inequality, we obtain

$$\|\boldsymbol{G}_h^k \underline{\mathsf{e}}_h\|_{L^p(\Omega)^d}^p \lesssim \mathcal{E}_h(\underline{\mathsf{e}}_h)^{\frac{p}{2}} \times \left(\|\boldsymbol{G}_h^k \widehat{\underline{\mathsf{u}}}_h\|_{L^p(\Omega)^d}^p + \|\boldsymbol{G}_h^k \underline{\mathsf{u}}_h\|_{L^p(\Omega)^d}^p \right)^{\frac{2-p}{2}}. \tag{48}$$

A similar reasoning starting from (71) with $t = h_F^{\frac{1-p}{p}} \delta_{TF}^k \widehat{\underline{\mathsf{u}}}_T$ and $r = h_F^{\frac{1-p}{p}} \delta_{TF}^k \underline{\mathsf{u}}_T$, integrating over F, summing over $F \in \mathcal{F}_T$ and using the Hölder inequality gives

$$s_T(\underline{\mathsf{e}}_T, \underline{\mathsf{e}}_T) \lesssim \left(s_T(\widehat{\underline{\mathsf{u}}}_T, \underline{\mathsf{e}}_T) - s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{e}}_T) \right)^{\frac{p}{2}} \left(s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T) + s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{u}}_T) \right)^{\frac{2-p}{2}}.$$

Summing over $T \in \mathcal{T}_h$ and using the discrete Hölder inequality, we get

$$\sum_{T \in \mathcal{T}_h} s_T(\underline{\mathsf{e}}_T, \underline{\mathsf{e}}_T) \lesssim \mathcal{E}_h(\underline{\mathsf{e}}_h)^{\frac{p}{2}} \times \left(\sum_{T \in \mathcal{T}_h} s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T) + \sum_{T \in \mathcal{T}_h} s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{u}}_T) \right)^{\frac{2-p}{2}}. \tag{49}$$

Combining (48) and (49), and using the seminorm equivalence (38) leads to

$$\|\underline{\mathsf{e}}_h\|_{1,p,h}^p \lesssim \mathcal{E}_h(\underline{\mathsf{e}}_h)^{\frac{p}{2}} \times \left(\|\widehat{\underline{\mathsf{u}}}_h\|_{1,p,h}^p + \|\underline{\mathsf{u}}_h\|_{1,p,h}^p \right)^{\frac{2-p}{2}}.$$

From the $W^{1,p}$-boundedness of $\underline{\mathsf{I}}_T^k$ and the a priori bound on $\|\underline{\mathsf{u}}_h\|_{1,p,h}$ proved in [13, Proposition 7.1 and Proposition 6.1], respectively, we infer that

$$\|\widehat{\underline{\mathsf{u}}}_h\|_{1,p,h} \lesssim \|u\|_{W^{1,p}(\Omega)} \lesssim 1 \text{ and } \|\underline{\mathsf{u}}_h\|_{1,p,h} \lesssim \|f\|_{L^{p'}(\Omega)}^{1/(p-1)} \lesssim 1, \tag{50}$$

so that

$$\|\underline{\mathsf{e}}_h\|_{1,p,h}^2 \lesssim \mathcal{E}_h(\underline{\mathsf{e}}_h). \tag{51}$$

In conclusion, combining the initial estimate (46) with $\underline{\mathsf{v}}_h = \underline{\mathsf{e}}_h$ with the bounds (47) (if $p \geqslant 2$) and (51) (if $p < 2$), we obtain

$$\begin{aligned} &\text{If } p \geqslant 2: \quad \|\underline{\mathsf{e}}_h\|_{1,p,h} \lesssim \mathcal{O}\left(\mathfrak{T}_1^{\frac{1}{p-1}} + \mathfrak{T}_2^{\frac{1}{p-1}} + \mathfrak{T}_3^{\frac{1}{p-1}} + \mathfrak{T}_4^{\frac{1}{p-1}} \right), \\ &\text{If } p < 2: \quad \|\underline{\mathsf{e}}_h\|_{1,p,h} \lesssim \mathcal{O}\left(\mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3 + \mathfrak{T}_4 \right). \end{aligned} \tag{52}$$

Step 3. *Estimate of* $\mathfrak{T}_1$.

Recall that, by (44) and (40),

$$\mathfrak{T}_1 = \left(\sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot, \boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T) - \mathbf{a}(\cdot, \boldsymbol{\nabla} u)\|_{L^{p'}(T)^d}^{p'} \right)^{\frac{1}{p'}}.$$

Notice also that, by (28), $\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T = \boldsymbol{G}_T^k \mathfrak{I}_T^k u = \pi_T^{0,k}(\boldsymbol{\nabla} u)$. Thus, using the approximation properties of $\pi_T^{0,k}$ summarised in Lemma 13 below (with $v = \partial_i u$ for $i = 1, \ldots, d$), we infer

$$\|\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T - \boldsymbol{\nabla} u\|_{L^p(T)^d} \lesssim h_T^{k+1} |u|_{W^{k+2,p}(T)}. \tag{53}$$

<u>Case $p \geqslant 2$</u>: Assume first $p > 2$. Recalling (20e), and using the generalised Hölder inequality with exponents (p', p, r) such that $\frac{1}{p'} = \frac{1}{p} + \frac{1}{r}$ (that is $r = \frac{p}{p-2}$) together with (53) yields, for all $T \in \mathcal{T}_h$,

$$\begin{aligned}\|\mathbf{a}(\cdot, \boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T) - \mathbf{a}(\cdot, \boldsymbol{\nabla} u)\|_{L^{p'}(T)^d} &\lesssim \|\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T - \boldsymbol{\nabla} u\|_{L^p(T)^d} \left(\|\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T\|_{L^p(T)^d}^{p-2} + \|\boldsymbol{\nabla} u\|_{L^p(T)^d}^{p-2} \right) \\ &\lesssim h_T^{k+1} |u|_{W^{k+2,p}(T)} \left(\|\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T\|_{L^p(T)^d}^{p-2} + \|\boldsymbol{\nabla} u\|_{L^p(T)^d}^{p-2} \right).\end{aligned}$$

This relation is obviously also valid if $p = 2$. We then sum over $T \in \mathcal{T}_h$ and use, as before, the generalised Hölder inequality, and (50) to infer

$$\mathfrak{T}_1 \lesssim h^{k+1} |u|_{W^{k+2,p}(\mathcal{T}_h)} \left(\|\boldsymbol{G}_h^k \widehat{\underline{\mathfrak{u}}}_h\|_{L^p(\Omega)^d}^{p-2} + \|u\|_{W^{1,p}(\Omega)}^{p-2} \right) \lesssim h^{k+1} |u|_{W^{k+2,p}(\mathcal{T}_h)}.$$

<u>Case $p < 2$</u>: By (66) below, $\|\mathbf{a}(\cdot, \boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T) - \mathbf{a}(\cdot, \boldsymbol{\nabla} u)\|_{L^{p'}(T)^d} \lesssim \|\boldsymbol{G}_T^k \widehat{\underline{\mathfrak{u}}}_T - \boldsymbol{\nabla} u\|_{L^p(T)^d}^{p-1}$. Use then (53) and sum over $T \in \mathcal{T}_h$ to obtain $\mathfrak{T}_1 \lesssim h^{(k+1)(p-1)} |u|_{W^{k+2,p}(\mathcal{T}_h)}^{p-1}$.

In conclusion, we obtain the following estimates on $\mathfrak{T}_1$:

$$\begin{aligned}&\text{If } p \geqslant 2: \quad \mathfrak{T}_1 \lesssim h^{k+1} |u|_{W^{k+2,p}(\mathcal{T}_h)}, \\ &\text{If } p < 2: \quad \mathfrak{T}_1 \lesssim h^{(k+1)(p-1)} |u|_{W^{k+2,p}(\mathcal{T}_h)}^{p-1}.\end{aligned} \tag{54}$$

Step 4. *Estimate of* $\mathfrak{T}_2 + \mathfrak{T}_3$. Owing to (44) together with the definitions (41) and (43) of $\mathfrak{T}_{2,T}$ and $\mathfrak{T}_{3,T}$, we have

$$\mathfrak{T}_2^{p'} + \mathfrak{T}_3^{p'} = \sum_{T\in\mathcal{T}_h} \left(\|\mathbf{a}(\cdot, \boldsymbol{\nabla} u) - \pi_T^{0,k}(\mathbf{a}(\cdot, \boldsymbol{\nabla} u))\|_{L^{p'}(T)^d}^{p'} + \sum_{F\in\mathcal{F}_T} h_F \|\mathbf{a}(\cdot, \boldsymbol{\nabla} u) - \pi_T^{0,k}(\mathbf{a}(\cdot, \boldsymbol{\nabla} u))\|_{L^{p'}(F)^d}^{p'} \right).$$

Using the approximation properties (63) and (64) of $\pi_T^{0,k}$ with v replaced by the components of $\mathbf{a}(\cdot, \boldsymbol{\nabla} u)$, p' instead of p, and $m = 0$, $s = k+1$, we get

$$\mathfrak{T}_2^{p'} + \mathfrak{T}_3^{p'} \lesssim h^{(k+1)p'} |\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}^{p'}.$$

Taking the power $1/p'$ of this inequality and using $(a+b)^{\frac{1}{p'}} \leqslant 2^{\frac{1}{p'}} a^{\frac{1}{p'}} + 2^{\frac{1}{p'}} b^{\frac{1}{p'}}$ leads to

$$\mathfrak{T}_2 + \mathfrak{T}_3 \lesssim h^{k+1} |\mathbf{a}(\cdot, \boldsymbol{\nabla} u)|_{W^{k+1,p'}(\mathcal{T}_h)^d}. \tag{55}$$

Step 5. *Estimate of* $\mathfrak{T}_4$.

Recall that $\mathfrak{T}_4$ is defined by (45). Using the Hölder inequality, we have for all $T \in \mathcal{T}_h$,

$$s_T(\widehat{\underline{\mathsf{u}}}_T, \underline{\mathsf{v}}_T) \lesssim s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T)^{\frac{1}{p'}} s_T(\underline{\mathsf{v}}_T, \underline{\mathsf{v}}_T)^{\frac{1}{p}}.$$

Hence, using again the Hölder inequality, since $\sum_{T\in\mathcal{T}_h} s_T(\underline{\mathsf{v}}_T, \underline{\mathsf{v}}_T) \leqslant \|\underline{\mathsf{v}}_h\|_{1,p,h}^p$,

$$\mathfrak{T}_4 \lesssim \left(\sum_{T\in\mathcal{T}_h} s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T)\right)^{\frac{1}{p'}}. \tag{56}$$

We proceed in a similar way as in [17, Lemma 4] to estimate $s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T)$. Let $F \in \mathcal{F}_T$. We use the definition (33c) of the face-based residual operator δ_{TF}^k together with the triangle inequality, the relation $\pi_F^{0,k}\pi_T^{0,k} = \pi_T^{0,k}$, the $L^p(F)$-boundedness (65) of $\pi_F^{0,k}$, the equality $p_T^{k+1}\widehat{\underline{\mathsf{u}}}_T = p_T^{k+1}\underline{\mathsf{I}}_T^k u = \pi_T^{1,k+1}u$ (cf. (31)), the trace inequality (19), and the $L^p(T)$- and $W^{1,p}(T)$-boundedness (65) of $\pi_T^{0,k}$ to write

$$\begin{aligned}
&\|\delta_{TF}^k \widehat{\underline{\mathsf{u}}}_T\|_{L^p(F)} \\
&\quad\leqslant \|\pi_F^{0,k}(u - p_T^{k+1}\widehat{\underline{\mathsf{u}}}_T)\|_{L^p(F)} + \|\pi_T^{0,k}(u - p_T^{k+1}\widehat{\underline{\mathsf{u}}}_T)\|_{L^p(F)} \\
&\quad\leqslant \|u - \pi_T^{1,k+1}u\|_{L^p(F)} + h_T^{-\frac{1}{p}}\|\pi_T^{0,k}(u - \pi_T^{1,k+1}u)\|_{L^p(T)} + h_T^{1-\frac{1}{p}}|\pi_T^{0,k}(u - \pi_T^{1,k+1}u)|_{W^{1,p}(T)} \\
&\quad\leqslant \|u - \pi_T^{1,k+1}u\|_{L^p(F)} + h_T^{-\frac{1}{p}}\|u - \pi_T^{1,k+1}u\|_{L^p(T)} + h_T^{1-\frac{1}{p}}|u - \pi_T^{1,k+1}u|_{W^{1,p}(T)}.
\end{aligned} \tag{57}$$

The optimal $W^{s,p}$-estimates on the elliptic projector (3) and (4) therefore give, for all $F \in \mathcal{F}_T$,

$$\|\delta_{TF}^k \widehat{\underline{\mathsf{u}}}_T\|_{L^p(F)} \lesssim h_T^{k+2-\frac{1}{p}}|u|_{W^{k+2,p}(T)}.$$

Raise this inequality to the power p, multiply by h_F^{1-p}, use $h_F^{1-p}h_T^{(k+2)p-1} \lesssim h_F^{1-p+(k+2)p-1} = h_F^{(k+1)p} \lesssim h^{(k+1)p}$, and sum over $F \in \mathcal{F}_T$ to obtain

$$s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T) \lesssim h^{(k+1)p}|u|_{W^{k+2,p}(T)}^p. \tag{58}$$

Substituted into (56), this gives

$$\mathfrak{T}_4 \lesssim h^{(k+1)(p-1)}|u|_{W^{k+2,p}(\mathcal{T}_h)}^{p-1}. \tag{59}$$

Conclusion. Use (54), (55) and (59) in (52). □

Remark 12 (Role of Theorems 1 and 2). Theorems 1 and 2 are used in **Step 5** of the proof of Theorem 7 below to derive a bound on the stabilisation term s_T when its arguments are the interpolate of the exact solution.

Proof of Corollary 10. Let an element $T \in \mathcal{T}_h$ be fixed and set, as in the proof of Theorem 7, $\widehat{\underline{\mathsf{u}}}_T := \underline{\mathsf{I}}_T^k u$. Recalling the definition (33b) of s_T, and using the inequality

$$(a+b)^p \leqslant 2^{p-1}a^p + 2^{p-1}b^p, \tag{60}$$

it is inferred

$$\begin{aligned}
s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{u}}_T) &= \sum_{F\in\mathcal{F}_T} h_F^{1-p}\int_F |\delta_{TF}^k \underline{\mathsf{u}}_T|^p = \sum_{F\in\mathcal{F}_T} h_F^{1-p}\int_F |\delta_{TF}^k \widehat{\underline{\mathsf{u}}}_T + \delta_{TF}^k(\underline{\mathsf{u}}_T - \widehat{\underline{\mathsf{u}}}_T)|^p \\
&\lesssim s_T(\widehat{\underline{\mathsf{u}}}_T, \widehat{\underline{\mathsf{u}}}_T) + s_T(\underline{\mathsf{u}}_T - \widehat{\underline{\mathsf{u}}}_T, \underline{\mathsf{u}}_T - \widehat{\underline{\mathsf{u}}}_T).
\end{aligned} \tag{61}$$

On the other hand, inserting $p_T^{k+1}\widehat{\underline{\mathbf{u}}}_T - \pi_T^{1,k+1}u = 0$ (cf. (31)), and using again (60), we have

$$\|\boldsymbol{\nabla}(u - p_T^{k+1}\underline{\mathbf{u}}_T)\|_{L^p(T)^d}^p \lesssim \|\boldsymbol{\nabla}(u - \pi_T^{1,k+1}u)\|_{L^p(T)^d}^p + \|\boldsymbol{\nabla}p_T^{k+1}(\widehat{\underline{\mathbf{u}}}_T - \underline{\mathbf{u}}_T)\|_{L^p(T)^d}^p. \tag{62}$$

Summing (61) and (62), and recalling the definition (34) of $\|\cdot\|_{1,p,T}$, we obtain

$$\|\boldsymbol{\nabla}(u - p_T^{k+1}\underline{\mathbf{u}}_T)\|_{L^p(T)^d}^p + s_T(\underline{\mathbf{u}}_T, \underline{\mathbf{u}}_T) \lesssim \|\boldsymbol{\nabla}(u - \pi_T^{1,k+1}u)\|_{L^p(T)^d}^p + s_T(\widehat{\underline{\mathbf{u}}}_T, \widehat{\underline{\mathbf{u}}}_T) + \|\widehat{\underline{\mathbf{u}}}_T - \underline{\mathbf{u}}_T\|_{1,p,T}^p.$$

The result follows by summing this estimate over $T \in \mathcal{T}_h$ and invoking Theorem 1 for the first term in the right-hand side, (58) for the second, and (36) for the third. □

The following optimal approximation properties for the L^2-orthogonal projector were used in **Step 4** of the proof of Theorem 7 with $U = T \in \mathcal{T}_h$.

Lemma 13 ($W^{s,p}$-approximation for $\pi_U^{0,l}$)**.** *Let U be as in Theorem 2. Let $s \in \{0, \dots, l+1\}$ and $p \in [1, +\infty]$. Then, there exists C depending only on d, ϱ, l, s and p such that, for all $v \in W^{s,p}(U)$,*

$$\forall m \in \{0, \dots, s\} \,:\, |v - \pi_U^{0,l}v|_{W^{m,p}(U)} \leqslant C h_U^{s-m}|v|_{W^{s,p}(U)} \tag{63}$$

and, if $s \geqslant 1$,

$$\forall m \in \{0, \dots, s-1\} \,:\, h_U^{\frac{1}{p}}|v - \pi_U^{0,l}v|_{W^{m,p}(\mathcal{F}_U)} \leqslant C h_U^{s-m}|v|_{W^{s,p}(U)}, \tag{64}$$

with $\mathcal{F}_U$, $W^{m,p}(\mathcal{F}_U)$ and corresponding seminorm as in Theorem 2.

Proof. This result is a combination of [13, Lemmas 3.4 and 3.6]. We give here an alternative proof based on the abstract results of Section 2.1. By Lemma 4 with $\mathcal{P} = \mathbb{P}^l(U)$, we have the following boundedness property for $\pi_U^{0,l}$: For all $v \in L^1(U)$, $\|\pi_U^{0,l}v\|_{L^p(U)} \leqslant C\|v\|_{L^p(U)}$ with real number $C > 0$ depending only on d, ϱ, and l. The estimate (63) is then an immediate consequence of Lemma 3 with $q = 0$ and $\Pi_U^{0,l} = \pi_U^{0,l}$. To prove (64), proceed as in Theorem 2 using (63) in place of (3). □

Corollary 14 ($W^{s,p}$-boundedness of $\pi_U^{0,l}$)**.** *With the same notation as in Theorem 13, it holds, for all $v \in W^{s,p}(U)$,*

$$|\pi_U^{0,l}v|_{W^{s,p}(U)} \leqslant C|v|_{W^{s,p}(U)}. \tag{65}$$

Proof. Use the triangle inequality to write $|\pi_U^{0,l}v|_{W^{s,p}(U)} \leqslant |\pi_U^{0,l}v - v|_{W^{s,p}(U)} + |v|_{W^{s,p}(U)}$ and conclude using (63) with $m = s$ for the first term. □

3.5 Numerical examples

For the sake of completeness, we present here some new numerical examples that demonstrate the orders of convergence achieved by the HHO method in practice. The test were run using the hho software platform[1]. We solve on the unit square domain $\Omega = (0,1)^2$ the homogeneous p-Laplace Dirichlet problem corresponding to the exact solution

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2),$$

with $p \in \{2, 3, 4\}$ and source term inferred from u (cf. (21) for the expression of $\mathbf{a}$ in this case). We consider the matching triangular, Cartesian, locally refined, and (predominantly) hexagonal mesh families depicted in Figure 1 and polynomial degrees ranging from 0 to 3. The three former mesh families are taken from the FVCA5 benchmark [31], whereas the latter is taken from [19]. The local refinement in the third mesh family has no specific meaning for the problem considered here: its purpose is to demonstrate the seamless treatment of nonconforming interfaces.

[1] *Agence pour la Protection des Programmes* deposit number IDDN.FR.001.220005.000.S.P.2016.000.10800

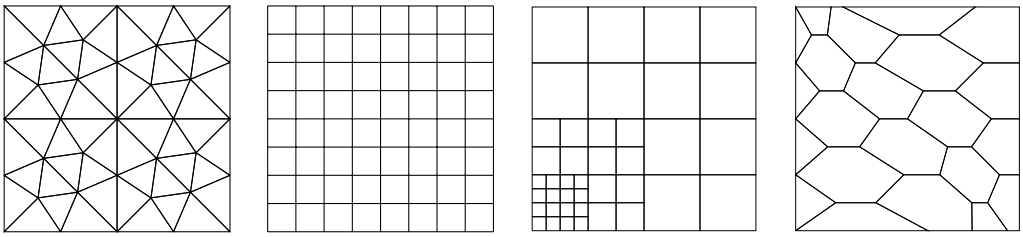


Figure 1: Matching triangular, Cartesian, locally refined and hexagonal mesh families used in the numerical examples of Section 3.5.

We report in Figure 2 the error $\|\underline{\mathsf{I}}_h^k u - \underline{\mathsf{u}}_h\|_{1,p,h}$ versus the meshsize h. From the leftmost column, we see that the error estimates are sharp for $p = 2$, which confirms the results of [17] (a known superconvergence phenomenon is observed on the Cartesian mesh for $k = 0$). For $p = 3, 4$, better orders of convergence than the asymptotic ones (cf. Remark 8) are observed in most of the cases. One possible explanation is that the lowest-order terms in the right-hand side of (36) are not yet dominant for the specific problem data and mesh at hand. Another possibility is that compensations occur among lowest-order terms that are separately estimated in the proof of Theorem 7. For $k = 3$ and $p = 3$, the observed orders of convergence in the last refinement steps are inferior to the predicted value for smooth solutions, which can likely be ascribed to the violation of the regularity assumption on $\mathbf{a}(\cdot, \boldsymbol{\nabla} u)$ (cf. Theorem 7), due to the lack of smoothness of $\mathbf{a}$ for that p.

A Inequalities involving the Leray–Lions operator

This section collects inequalities involving the Leray–Lions operator adapted from [23].

Lemma 15. *Assume* (20c), (20e), *and* $p \leqslant 2$. *Then, for a.e.* $\boldsymbol{x} \in \Omega$ *and all* $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant (2\gamma_{\mathbf{a}} + 2^{p-1}\beta_{\mathbf{a}} + \beta_{\mathbf{a}})|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p-1}. \tag{66}$$

Proof. Let $r > 0$. If $|\boldsymbol{\xi}| \geqslant r$ and $|\boldsymbol{\eta}| \geqslant r$ then, using (20e) and $p - 2 \leqslant 0$, we have

$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant \gamma_{\mathbf{a}}|\boldsymbol{\xi} - \boldsymbol{\eta}|(|\boldsymbol{\xi}|^{p-2} + |\boldsymbol{\eta}|^{p-2}) \leqslant 2\gamma_{\mathbf{a}} r^{p-2}|\boldsymbol{\xi} - \boldsymbol{\eta}|. \tag{67}$$

Otherwise, assume for example that $|\boldsymbol{\eta}| < r$. Then $|\boldsymbol{\xi}| \leqslant |\boldsymbol{\xi} - \boldsymbol{\eta}| + r$ and thus, owing to (20c),

$$\begin{aligned} |\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| &\leqslant |\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \mathbf{0})| + |\mathbf{a}(\boldsymbol{x}, \mathbf{0}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \\ &\leqslant \beta_{\mathbf{a}}(|\boldsymbol{\xi}|^{p-1} + |\boldsymbol{\eta}|^{p-1}) \\ &\leqslant \beta_{\mathbf{a}}(|\boldsymbol{\xi} - \boldsymbol{\eta}| + r)^{p-1} + \beta_{\mathbf{a}} r^{p-1}. \end{aligned} \tag{68}$$

Combining (67) and (68) shows that, in either case,

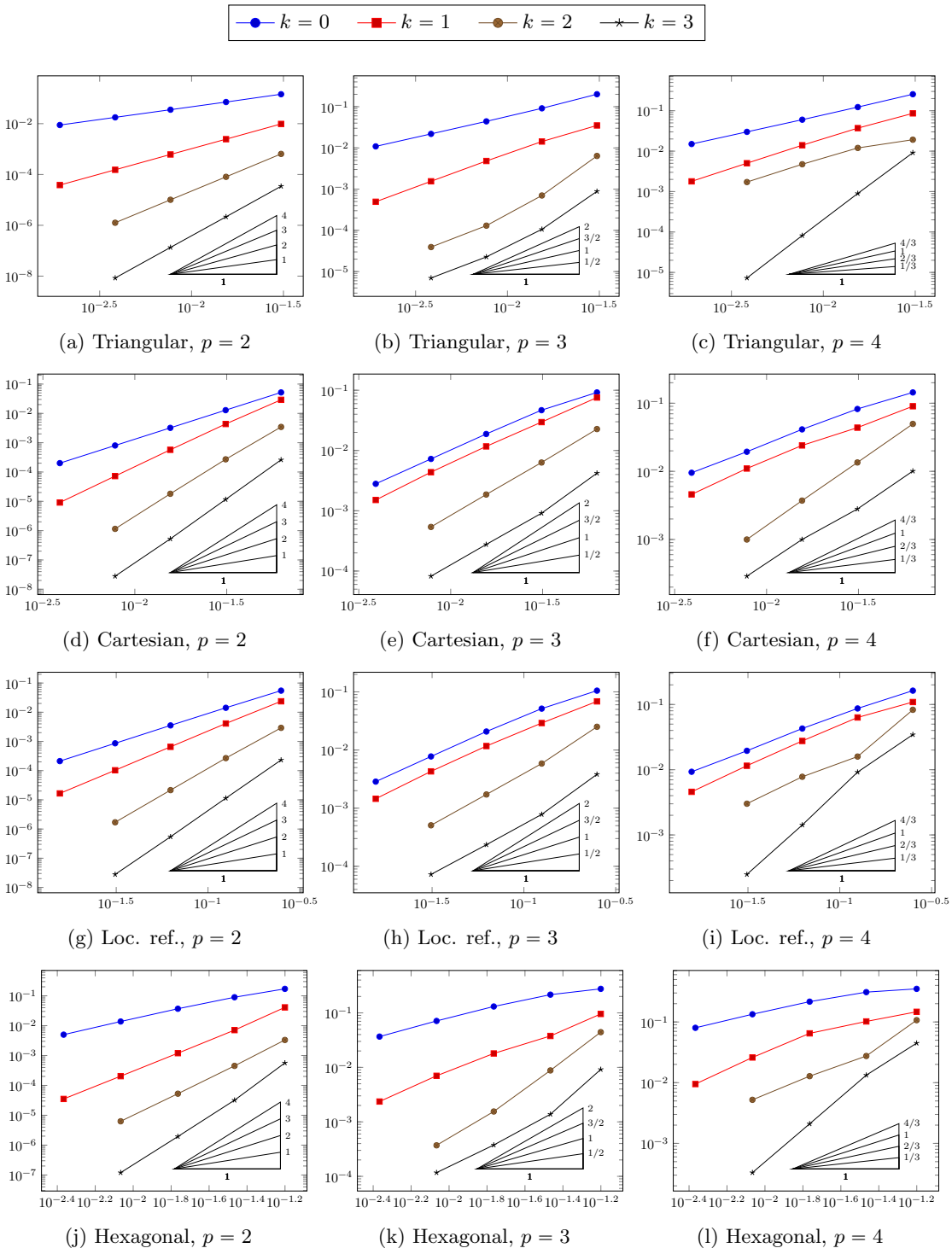
$$|\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant 2\gamma_{\mathbf{a}} r^{p-2}|\boldsymbol{\xi} - \boldsymbol{\eta}| + \beta_{\mathbf{a}}(|\boldsymbol{\xi} - \boldsymbol{\eta}| + r)^{p-1} + \beta_{\mathbf{a}} r^{p-1}.$$

Taking $r = |\boldsymbol{\xi} - \boldsymbol{\eta}|$ concludes the proof of (66). □

Lemma 16. *Under Assumption* (20f) *we have, for a.e.* $\boldsymbol{x} \in \Omega$ *and all* $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d$,

- *If* $p < 2$,

$$|\boldsymbol{\xi} - \boldsymbol{\eta}|^p \leqslant \zeta_{\mathbf{a}}^{-\frac{p}{2}} 2^{(p-1)\frac{2-p}{2}} \Big([\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}]\Big)^{\frac{p}{2}} \Big(|\boldsymbol{\xi}|^p + |\boldsymbol{\eta}|^p\Big)^{\frac{2-p}{2}}; \tag{69}$$



(a) Triangular, $p = 2$ (b) Triangular, $p = 3$ (c) Triangular, $p = 4$

(d) Cartesian, $p = 2$ (e) Cartesian, $p = 3$ (f) Cartesian, $p = 4$

(g) Loc. ref., $p = 2$ (h) Loc. ref., $p = 3$ (i) Loc. ref., $p = 4$

(j) Hexagonal, $p = 2$ (k) Hexagonal, $p = 3$ (l) Hexagonal, $p = 4$

Figure 2: $\|\mathbf{I}_h^k u - \underline{\mathbf{u}}_h\|_{1,p,h}$ versus h for the mesh families of Figure 1. The slopes represent the orders of convergence expected from Theorem 7, i.e. $\frac{k+1}{p-1}$ for $k \in \{0, \dots, 3\}$ and $p \in \{2, 3, 4\}$.

- *If* $p \geqslant 2$,

$$|\boldsymbol{\xi} - \boldsymbol{\eta}|^p \leqslant \zeta_{\mathbf{a}}^{-1}[\mathbf{a}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathbf{a}(\boldsymbol{x}, \boldsymbol{\eta})] \cdot [\boldsymbol{\xi} - \boldsymbol{\eta}]. \tag{70}$$

Proof. Estimate (69) is obtained by raising (20f) to the power $p/2$ and using $(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^p \leqslant 2^{p-1}(|\boldsymbol{\xi}|^p + |\boldsymbol{\eta}|^p)$. To prove (70), we simply write $|\boldsymbol{\xi} - \boldsymbol{\eta}|^p \leqslant |\boldsymbol{\xi} - \boldsymbol{\eta}|^2(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2}$. □

Remark 17. The (real-valued) mapping $a : t \mapsto |t|^{p-2}t$ corresponds to the p-Laplace operator in dimension 1, and it therefore satisfies (20f). Hence, by Lemma 16,

$$\text{If } p < 2: \quad |t - r|^p \leqslant C \left(\left[|t|^{p-2}t - |r|^{p-2}r\right][t - r]\right)^{\frac{p}{2}} \left(|t|^p + |r|^p\right)^{\frac{2-p}{p}}, \tag{71}$$

$$\text{If } p \geqslant 2: \quad |t - r|^p \leqslant C \left[|t|^{p-2}t - |r|^{p-2}r\right][t - r], \tag{72}$$

where C depends only on p.

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