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Distances between classes in $W^{1,1}(\Omega; \mathbb{S}^1)$

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1 Introduction

Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 2$. (Many of the results in this paper remain valid if Ω is replaced by a manifold \mathcal{M} , with or without boundary, and the case $\mathcal{M} = \mathbb{S}^1$ is already of interest (see [8, 9])). In some places we will assume in addition that Ω is simply connected (and this will be mentioned explicitly). Our basic setting is

$$W^{1,1}(\Omega; \mathbb{S}^1) = \{u \in W^{1,1}(\Omega; \mathbb{R}^2) \simeq W^{1,1}(\Omega; \mathbb{C}); |u| = 1 \text{ a.e.}\}.$$

We now introduce an equivalence relation in $W^{1,1}(\Omega; \mathbb{S}^1)$:

$$u \sim v \text{ if and only if } u = e^{i\varphi} v \text{ for some } \varphi \in W^{1,1}(\Omega; \mathbb{R}). \quad (1.1)$$

We denote by $\mathcal{E}(u)$ the equivalence class of an element $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. In particular,

$$\mathcal{E}(1) = \{u \in W^{1,1}(\Omega; \mathbb{S}^1); u = e^{i\varphi} \text{ for some } \varphi \in W^{1,1}(\Omega; \mathbb{R})\}.$$

It is easy to see (using (1.6)–(1.7) below) that $\mathcal{E}(u)$ is closed in $W^{1,1}(\Omega; \mathbb{S}^1)$.

Clearly

$$\mathcal{E}(1) \subset \overline{C^\infty(\overline{\Omega}; \mathbb{S}^1)}^{W^{1,1}}; \quad (1.2)$$

however equality in (1.2) fails in general. For example when $\Omega = \{x \in \mathbb{R}^2; 1 < |x| < 2\}$, the map $u(x) = x/|x|$ is smooth, but $u \notin \mathcal{E}(1)$ since it cannot be written as $u = e^{i\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$; the nonexistence of φ is an easy consequence of degree theory (such an argument has roots in [16], see also [2, 3, 6]). On the other hand, if Ω is simply connected, equality in (1.2) does hold since $C^\infty(\overline{\Omega}; \mathbb{S}^1) \subset \mathcal{E}(1)$ and $\mathcal{E}(1)$ is closed in $W^{1,1}(\Omega; \mathbb{S}^1)$.

A useful device for constructing maps in the same equivalence class is the following (see Lemma 2.1 below). Let $T \in \text{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ be a map of degree one. Then

$$T \circ u \sim u, \quad \forall u \in W^{1,1}(\Omega; \mathbb{S}^1). \quad (1.3)$$

We call the attention of the reader that maps u of the form $u = e^{i\varphi}$ with $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ belong to $W^{1,1}(\Omega; \mathbb{S}^1)$. However they do not exhaust $W^{1,1}(\Omega; \mathbb{S}^1)$: there exist maps in

$W^{1,1}(\Omega; \mathbb{S}^1)$ which *cannot* be written as $u = e^{i\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$. A typical example is the map $u(x) = x/|x|$ in $\Omega = \text{unit disc in } \mathbb{R}^2$; see above.

To each $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we associate a number $\Sigma(u) \geq 0$ defined by

$$\Sigma(u) = \inf_{v \in \mathcal{E}(u)} \int_{\Omega} |\nabla v|. \quad (1.4)$$

Note that

$$\Sigma(u) = \inf_{\psi \in W^{1,1}(\Omega; \mathbb{R})} \int_{\Omega} |u \wedge \nabla u - \nabla \psi|. \quad (1.5)$$

This follows from the identities

$$(uv) \wedge \nabla(uv) = u \wedge \nabla u + v \wedge \nabla v, \quad \forall u, v \in W^{1,1}(\Omega; \mathbb{S}^1), \quad (1.6)$$

$$e^{i\varphi} \wedge \nabla(e^{i\varphi}) = \nabla \varphi, \quad \forall \varphi \in W^{1,1}(\Omega; \mathbb{R}), \quad (1.7)$$

$$\bar{u} \wedge \nabla \bar{u} = -u \wedge \nabla u, \quad \forall u \in W^{1,1}(\Omega; \mathbb{S}^1). \quad (1.8)$$

The quantity $\Sigma(u)$ was originally introduced in [7] when $N = 2$. It plays an extremely important role in many questions involving $W^{1,1}(\Omega; \mathbb{S}^1)$ (see [6]). Clearly $\Sigma(u) = \Sigma(\bar{u})$, and $u \sim v \implies \Sigma(u) = \Sigma(v)$. It is straightforward that

$$u \sim 1 \iff \Sigma(u) = 0; \quad (1.9)$$

in particular,

$$u \sim v \iff \Sigma(u\bar{v}) = 0. \quad (1.10)$$

Note also that

$$\Sigma(uv) \leq \Sigma(u) + \Sigma(v). \quad (1.11)$$

Given $u_0, v_0 \in W^{1,1}(\Omega; \mathbb{S}^1)$ the following quantities will play a crucial role throughout the paper:

$$d_{W^{1,1}}(u_0, \mathcal{E}(v_0)) := \inf_{v \sim v_0} \int_{\Omega} |\nabla(u_0 - v)|, \quad (1.12)$$

$$\text{dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \inf_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) = \inf_{u \sim u_0} \inf_{v \sim v_0} \int_{\Omega} |\nabla(u - v)|, \quad (1.13)$$

$$\text{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) = \sup_{u \sim u_0} \inf_{v \sim v_0} \int_{\Omega} |\nabla(u - v)|, \quad (1.14)$$

so that $\text{dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0))$ is precisely the distance between the classes $\mathcal{E}(u_0)$ and $\mathcal{E}(v_0)$. On the other hand we will see below, as a consequence of (1.16), that $\text{Dist}_{W^{1,1}}$ is symmetric, a fact which is not clear from its definition. This implies that $\text{Dist}_{W^{1,1}}$ coincides with the Hausdorff distance

$$H - \text{dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \max(\text{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)), \text{Dist}_{W^{1,1}}(\mathcal{E}(v_0), \mathcal{E}(u_0)))$$

between $\mathcal{E}(u_0)$ and $\mathcal{E}(v_0)$. Our main result is

Theorem 1.1. *For every $u_0, v_0 \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have*

$$\text{dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \frac{2}{\pi} \Sigma(u_0 \bar{v}_0) \quad (1.15)$$

and

$$\text{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \Sigma(u_0 \bar{v}_0). \quad (1.16)$$

The two assertions in Theorem 1.1 look very simple but the proofs are quite tricky; they are presented in Sections 2 and 3. The proof of inequality “ \leq ” in (1.16) uses a version of the “dipole construction”:

Proposition 1.2. (*H. Brezis and P. Mironescu [6, Proposition 2.1]*) *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then there exists a sequence $\{u_n\} \subset \mathcal{E}(u)$ satisfying*

$$u_n \rightarrow 1 \text{ a.e.}, \text{ and } \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n| = \Sigma(u). \quad (1.17)$$

For completeness we present the proof of Proposition 1.2 in the Appendix.

A basic ingredient in the proof of inequality “ \geq ” in (1.16) is the following proposition which provides an explicit recipe for constructing “maximizing sequences” for $\text{Dist}_{W^{1,1}}$. In order to describe it we first introduce, for each $n \geq 3$, a map $T_n \in \text{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ with $\deg T_n = 1$ by $T_n(e^{i\theta}) = e^{i\tau_n(\theta)}$, with τ_n defined on $[0, 2\pi]$ by setting $\tau_n(0) = 0$ and

$$\tau_n'(\theta) = \begin{cases} n, & \theta \in [2j\pi/n^2, (2j+1)\pi/n^2] \\ -(n-2), & \theta \in ((2j+1)\pi/n^2, (2j+2)\pi/n^2] \end{cases}, \quad j = 0, 1, \dots, n^2 - 1. \quad (1.18)$$

Proposition 1.3. *For every $u_0, v_0 \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that $u_0 \not\sim v_0$ we have*

$$\lim_{n \rightarrow \infty} \frac{d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0))}{\Sigma(u_0 \bar{v}_0)} = 1 \quad (1.19)$$

and the limit is uniform over all such u_0 and v_0 . Consequently

$$\text{Dist}(\mathcal{E}(u_0), \mathcal{E}(v_0)) \geq \Sigma(u_0 \bar{v}_0). \quad (1.20)$$

Remark 1.4. When Ω is simply connected there is an alternative point of view on the equivalence relation $u \sim v$ using the Jacobian of u . For every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ consider the antisymmetric matrix with coefficients in \mathcal{D}' defined by

$$Ju := \frac{1}{2} \left[\frac{\partial}{\partial x_i} \left(u \wedge \frac{\partial u}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(u \wedge \frac{\partial u}{\partial x_i} \right) \right].$$

One can show (see [10] and [3, 6]) that $\forall u, v \in W^{1,1}(\Omega; \mathbb{S}^1)$,

$$u \sim v \text{ if and only if } J(u\bar{v}) = Ju - Jv = 0. \quad (1.21)$$

Remark 1.5. In order to have a more explicit description of the equivalence relation $u \sim v$ and of $\Sigma(u)$ it is instructive to understand what it means when $N = 2$ and Ω is simply connected, for $u, v \in \mathcal{R}$, where

$$\mathcal{R} = \{u \in W^{1,1}(\Omega; \mathbb{S}^1); u \text{ is smooth except at a finite number of points}\}.$$

The class \mathcal{R} plays an important role since it is dense in $W^{1,1}(\Omega; \mathbb{S}^1)$ (see [2, 6]). If $u \in \mathcal{R}$ then $Ju = \pi \sum k_j \delta_{a_j}$ where a_j are the singular points of u and $k_j := \deg(u, a_j)$. In particular, when $u, v \in \mathcal{R}$, then $u \sim v$ if and only if u and v have the same singularities and the same degree for each singularity. Moreover, $\Sigma(u) = 2\pi L(u)$ where $L(u)$ is the length of minimal connection between “positive points” and the “negative points”, allowing connection to the boundary (for more precise statements see [6, 7]).

A special case of interest is the distance of a given $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ to the class $\mathcal{E}(1)$. An immediate consequence of Theorem 1.1 is that for every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have

$$\frac{2}{\pi}\Sigma(u) \leq d_{W^{1,1}}(u, \mathcal{E}(1)) \leq \Sigma(u), \quad (1.22)$$

and the bounds are optimal in the sense that

$$\sup_{\substack{u \\ \Sigma(u) > 0}} \frac{d_{W^{1,1}}(u, \mathcal{E}(1))}{\Sigma(u)} = 1, \quad (1.23)$$

and

$$\inf_{\substack{u \\ \Sigma(u) > 0}} \frac{d_{W^{1,1}}(u, \mathcal{E}(1))}{\Sigma(u)} = \frac{2}{\pi}. \quad (1.24)$$

There are challenging problems concerning the question whether the supremum and the infimum in the above formulas are achieved (see §3.3).

Remark 1.6. It was established in [6, Sec. 11.6] that

$$\frac{1}{2}\Sigma(u) \leq d_{W^{1,1}}(u, \mathcal{E}(1)) \leq \Sigma(u), \quad \forall u \in W^{1,1}(\Omega; \mathbb{S}^1). \quad (1.25)$$

Earlier results concerning the upper bound in (1.25) can be found in [1, 13].

Finally, we turn in Section 4 to the classes in $W^{1,p}(\Omega; \mathbb{S}^1)$, $1 < p < \infty$, defined in an analogous way to the $W^{1,1}$ -case, i.e., using the equivalence relation

$$u \sim v \text{ if and only if } u = e^{i\varphi} v \text{ for some } \varphi \in W^{1,p}(\Omega; \mathbb{R}). \quad (1.26)$$

We point out that if $u, v \in W^{1,p}(\Omega; \mathbb{S}^1)$ are equivalent according to the equivalence relation in (1.1), then from the relation $e^{i\varphi} = u\bar{v}$ we deduce that

$$\nabla\varphi = (u\bar{v}) \wedge \nabla(u\bar{v}) \in L^p(\Omega; \mathbb{R}^N); \quad (1.27)$$

whence $u \sim v$ according to (1.26) as well. When $p \geq 2$ and Ω is simply connected we have $W^{1,p}(\Omega; \mathbb{S}^1) = \{u \in W^{1,1}(\Omega; \mathbb{S}^1); u = e^{i\varphi} \text{ for some } \varphi \in W^{1,p}(\Omega; \mathbb{R})\}$, see Remark 1.14 below. Therefore, the only cases of interest are:

- (a) general Ω and $1 < p < 2$,
- (b) multiply connected Ω and $p \geq 2$.

In all the theorems below we assume that we are in one of these situations. The distances between the classes are defined analogously to (1.13)–(1.14) by

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \inf_{u \sim u_0} \inf_{v \sim v_0} \|\nabla(u - v)\|_{L^p(\Omega)}. \quad (1.28)$$

and

$$\text{Dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \sup_{u \sim u_0} \inf_{v \sim v_0} \|\nabla(u - v)\|_{L^p(\Omega)}. \quad (1.29)$$

The next result establishes a lower bound for $\text{dist}_{W^{1,p}}$:

Theorem 1.7. For every $u_0, v_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$, $1 < p < \infty$, we have

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) \geq \left(\frac{2}{\pi}\right) \inf_{w \sim u_0 \bar{v}_0} \|\nabla w\|_{L^p(\Omega)}. \quad (1.30)$$

Remark 1.8. For $p > 1$ the infimum on the R.H.S. of (1.30) is actually a minimum; this follows easily from (1.27) and the fact that $W^{1,p}$ is reflexive.

Note that equality in (1.30) holds for $p = 1$ by (1.15). An example in [14, Section 4] shows that strict inequality “ $>$ ” may occur in (1.30) for a multiply connected domain in dimension two and $p = 2$. We will show in §4.4 that strict inequality may also occur for simply connected domains when $1 < p < 2$. On the positive side, we prove equality in (1.30) in the case of the distance to $\mathcal{E}(1)$:

Theorem 1.9. For every $u_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$, $1 < p < \infty$, we have

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(1)) = \left(\frac{2}{\pi}\right) \inf_{w \sim u_0} \|\nabla w\|_{L^p(\Omega)}. \quad (1.31)$$

Remark 1.10. When $p > 1$ we do not know general conditions on $u_0, v_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$ that guarantee equality in (1.30) (a sufficient condition in the case of multiply connected two dimensional domain and $p = 2$ is given in [14, Th. 4]).

On the other hand, when $p > 1$, $\text{Dist}_{W^{1,p}}$ between distinct classes is infinite:

Theorem 1.11. For every $u_0, v_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$, $1 < p < \infty$, such that $u_0 \not\sim v_0$ we have

$$\text{Dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \infty. \quad (1.32)$$

Remark 1.12. A simple but useful property of the equivalence relation that we defined in $W^{1,p}(\Omega; \mathbb{S}^1)$, $p \geq 1$, is the following: $u \sim u_0$ and $v \sim v_0$ imply $uv \sim u_0v_0$. This “group property” reflects the fact that our classes are the cosets $\{gH; g \in G\}$ of the subgroup

$$H = \{u \in W^{1,p}(\Omega; \mathbb{S}^1); u = e^{i\varphi} \text{ for some } \varphi \in W^{1,p}(\Omega; \mathbb{R})\}$$

of the group $G = W^{1,p}(\Omega; \mathbb{S}^1)$.

Remark 1.13. The quantity $\text{dist}_{W^{1,p}}$ for \mathbb{S}^1 -valued maps was studied in [14] (and in [12] for \mathbb{S}^N -valued maps, $N \geq 2$). In [14] distances between classes were investigated in two different settings: (i) $W^{1,p}(\mathbb{S}^1; \mathbb{S}^1)$, $p \geq 1$ and (ii) $W^{1,2}(\Omega; \mathbb{S}^1)$, where Ω is a multiply connected domain in \mathbb{R}^2 . A comprehensive study of distances in $W^{s,p}(\mathbb{S}^N; \mathbb{S}^N)$ ($sp \geq N$) is carried out in [8].

Remark 1.14. There is another natural equivalence relation in $W^{1,p}(\Omega; \mathbb{S}^1)$, $1 \leq p < \infty$, defined by the homotopy classes, i.e.,

$$u \stackrel{\mathcal{H}}{\sim} v \text{ if and only if } u = h(0) \text{ and } v = h(1) \text{ for some } h \in C([0, 1]; W^{1,p}(\Omega; \mathbb{S}^1)).$$

Homotopy classes have been well-studied (see [4, 5, 11, 15, 17]). Here are three basic facts valid for a general domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ (see [5]):

- a) If $1 \leq p < 2$, then $u \stackrel{\mathcal{H}}{\sim} v, \forall u, v \in W^{1,p}(\Omega; \mathbb{S}^1)$.
- b) If $2 \leq p < \infty$, then $u \stackrel{\mathcal{H}}{\sim} v \iff u \sim v, \forall u, v \in W^{1,p}(\Omega; \mathbb{S}^1)$.
- c) If $2 \leq p < \infty$, then $W^{1,p}(\Omega; \mathbb{S}^1)$ and $C^0(\bar{\Omega}; \mathbb{S}^1)$ have the same homotopy classes in the sense that any $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ is $W^{1,p}$ -homotopic to some $u_1 \in C^\infty(\bar{\Omega}; \mathbb{S}^1)$; moreover $u_1, v_1 \in$

$C^\infty(\bar{\Omega}; \mathbb{S}^1)$ are C^0 -homotopic if and only if u_1, v_1 are $W^{1,p}$ -homotopic. In particular, if Ω is simply connected and $p \geq 2$, then $u \stackrel{\mathcal{H}}{\sim} v$ and $u \sim v$, $\forall u, v \in W^{1,p}(\Omega; \mathbb{S}^1)$. Note that when $1 \leq p < 2$ the equivalence relation $u \sim v$ is *much more restrictive* than $u \stackrel{\mathcal{H}}{\sim} v$; for example let $\Omega = \text{unit disc in } \mathbb{R}^2$, $u(x) = x/|x|$ and $v(x) = (x - a)/|x - a|$ with $0 \neq a \in \mathbb{R}^2$, then $u \not\sim v$ (see Remark 1.5) while $u \stackrel{\mathcal{H}}{\sim} v$, e.g., via the homotopy $h(t) = (x - ta)/|x - ta|$, $0 \leq t \leq 1$.

Part of the results were announced in [9].

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2 Proof of (1.15) in Theorem 1.1

2.1 A basic lower bound inequality

We begin with a simple lemma about composition with Lipschitz maps; it provides a very useful device for constructing maps in the same equivalence class, or in the class $\mathcal{E}(1)$.

Lemma 2.1. *Let $T \in Lip(\mathbb{S}^1; \mathbb{S}^1)$ be a map of degree D . Then*

$$T \circ u \sim u^D, \quad \forall u \in W^{1,1}(\Omega; \mathbb{S}^1). \quad (2.1)$$

Proof. Since $T(z)/z^D$ is a Lipschitz self-map of \mathbb{S}^1 of zero degree, there exists $g \in Lip(\mathbb{S}^1; \mathbb{R})$ such that $T(z)/z^D = e^{ig(z)}$. The function $\varphi(x) = g(u(x))$ belongs to $W^{1,1}(\Omega; \mathbb{R})$ and satisfies $T(u(x)) = (u(x))^D e^{i\varphi(x)}$, and (2.1) follows by the definition of the equivalence relation. \square

The next simple lemma is essential for the proof of the lower bound in (1.22).

Lemma 2.2. *For any $w \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have*

$$\int_{\Omega} |\nabla(|w - 1|)| \geq \frac{2}{\pi} \Sigma(w). \quad (2.2)$$

Proof. As in [14], we define $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by

$$T(e^{i\varphi}) := e^{i\theta} \text{ with } \theta = \theta(\varphi) = \pi \sin(\varphi/2), \quad \forall \varphi \in (-\pi, \pi], \quad (2.3)$$

so that

$$|e^{i\varphi} - 1| = 2|\sin(\varphi/2)| = \frac{2}{\pi}|\theta|. \quad (2.4)$$

Clearly T is of class C^1 and its degree equals to one. We claim that

$$|\nabla(|w - 1|)| = \frac{2}{\pi} |\nabla(T \circ w)| \text{ a.e.} \quad (2.5)$$

This is a consequence of the standard fact that, if $F \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2) \cap C^1(\mathbb{S}^1 \setminus \{1\})$ and $w \in W^{1,1}(\Omega; \mathbb{S}^1)$, then $F \circ w \in W^{1,1}(\Omega; \mathbb{S}^1)$ and, moreover,

$$\nabla(F \circ w) = \begin{cases} \dot{F}(w) \nabla w & \text{a.e. in } [w \neq 1] \\ 0 & \text{a.e. in } [w = 1] \end{cases}.$$

Integration of (2.5) leads to

$$\int_{\Omega} |\nabla|w - 1|| = \frac{2}{\pi} \int_{\Omega} |\nabla(T \circ w)|. \quad (2.6)$$

By Lemma 2.1, we have $\Sigma(T \circ w) = \Sigma(w)$, and therefore (2.2) follows from (2.6). \square

Corollary 2.3. *For every $u, v \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have*

$$\int_{\Omega} |\nabla(u - v)| \geq \frac{2}{\pi} \Sigma(u\bar{v}). \quad (2.7)$$

Proof. Setting $w = u\bar{v}$ and applying (2.2) yields

$$\int_{\Omega} |\nabla(u - v)| \geq \int_{\Omega} |\nabla(|u - v|)| = \int_{\Omega} |\nabla(|w - 1|)| \geq \frac{2}{\pi} \Sigma(u\bar{v}). \quad \square$$

2.2 Proof of (1.15)

We begin by introducing some notation. For an open arc in \mathbb{S}^1 we use the notation

$$\mathcal{A}(\alpha, \beta) = \{e^{i\theta}; \theta \in (\alpha, \beta)\} \quad (2.8)$$

for any $\alpha < \beta$. We shall also use a specific notation for half-circles; for every $\zeta \in \mathbb{S}^1$ write $\zeta = e^{i\varphi}$ with $\varphi \in (-\pi, \pi]$ and denote $I(\zeta, -\zeta) = \mathcal{A}(\varphi, \varphi + \pi)$. Note that

$$z \in I(\zeta, -\zeta) \iff \zeta \in I(-z, z). \quad (2.9)$$

For each $\zeta = e^{i\varphi} \in \mathbb{S}^1$ define a map $P_{\zeta} : \mathbb{S}^1 \rightarrow \overline{I(\zeta, -\zeta)}$ by

$$P_{\zeta}(z) = \begin{cases} z, & \text{if } z = e^{i\theta} \in I(\zeta, -\zeta) \\ e^{i(2\varphi - \theta)} = \zeta^2 \bar{z}, & \text{if } z \notin I(\zeta, -\zeta) \end{cases}, \quad (2.10)$$

so that for $z \notin I(\zeta, -\zeta)$, $P_{\zeta}(z)$ is the reflection of z with respect to the line $\ell_{\zeta} = \{t\zeta; t \in \mathbb{R}\}$. Next we state

Proposition 2.4. *For every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have*

$$\int_{\mathbb{S}^1} \left(\int_{\Omega} |\nabla(u - P_{\zeta} \circ u)| dx \right) d\zeta = 4 \int_{\Omega} |\nabla u| dx. \quad (2.11)$$

Proof. For each $\zeta \in \mathbb{S}^1$ set $v_{\zeta} := P_{\zeta} \circ u$. By Lemma 2.1, $v_{\zeta} \in \mathcal{E}(1)$, since $\deg P_{\zeta} = 0$. We note that

$$z - P_{\zeta}(z) = \begin{cases} 0, & \text{if } z \in I(\zeta, -\zeta) \\ z - \zeta^2 \bar{z}, & \text{if } z \notin I(\zeta, -\zeta) \end{cases}. \quad (2.12)$$

Set $w_\zeta := u - v_\zeta$. Using (2.12), we find that for every $\zeta = e^{i\varphi}$ and a.e. $x \in \Omega$ we have

$$\nabla w_\zeta(x) = \begin{cases} 0, & \text{if } u(x) \in I(\zeta, -\zeta) \\ \nabla u(x) - \zeta^2 \nabla \bar{u}(x), & \text{if } u(x) \notin I(\zeta, -\zeta) \end{cases}. \quad (2.13)$$

Therefore, for a.e. $x \in \Omega$ we have

$$|\nabla w_\zeta(x)| = \begin{cases} 0, & \text{if } u(x) \in I(\zeta, -\zeta) \\ 2|\cos(\theta - \varphi)| |\nabla u(x)|, & \text{if } u(x) = e^{i\theta} \notin I(\zeta, -\zeta) \end{cases}. \quad (2.14)$$

Indeed, we justify (2.14) e.g. when $\zeta = 1$. In view of (2.13), we have to prove that

$$|\nabla \operatorname{Im} u(x)| = |\operatorname{Re} u(x)| |\nabla u(x)| \quad \text{for a.e. } x. \quad (2.15)$$

If we differentiate the identity $|u|^2 \equiv 1$, we obtain

$$\operatorname{Re} u \nabla(\operatorname{Re} u) + \operatorname{Im} u \nabla(\operatorname{Im} u) = 0 \quad \text{a.e.};$$

this easily implies (2.15).

Using (2.9) we find that, with $u(x) = e^{i\theta}$ and

$$A(x) = \{\varphi \in (-\pi, \pi]; u(x) \notin I(e^{i\varphi}, -e^{i\varphi})\},$$

we have

$$\begin{aligned} \int_{\mathbb{S}^1} \int_{\Omega} |\nabla w_\zeta(x)| dx d\zeta &= \int_{-\pi}^{\pi} \int_{\Omega} \chi_{A(x)}(\varphi) 2|\cos(\theta - \varphi)| |\nabla u(x)| dx d\varphi \\ &= \int_{\Omega} |\nabla u(x)| \left(\int_{\theta}^{\theta+\pi} 2|\cos(\theta - \varphi)| d\varphi \right) dx = 4 \int_{\Omega} |\nabla u(x)| dx, \end{aligned}$$

which is (2.11). Here we have used $\int_{\theta}^{\theta+\pi} 2|\cos(\theta - \varphi)| d\varphi = \int_0^\pi 2|\cos t| dt = 4$. \square

The identity (2.11) is a key tool in the proof of “ \leq ” in (1.15). For the convenience of the reader we shall present first the slightly simpler proof when $v_0 = 1$.

Proof of “ \leq ” in (1.15) for $v_0 = 1$. By Corollary 2.3 we have

$$\inf_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(1)) \geq \frac{2}{\pi} \Sigma(u_0).$$

Use (1.4) to choose a sequence $\{u_n\} \subset \mathcal{E}(u_0)$ with $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n| = \Sigma(u_0)$. Use Proposition 2.4 to choose $\zeta_n \in \mathbb{S}^1$ such that

$$\int_{\Omega} |\nabla(u_n - P_{\zeta_n} \circ u_n)| \leq \frac{2}{\pi} \int_{\Omega} |\nabla u_n|,$$

implying that $\lim_{n \rightarrow \infty} d_{W^{1,1}}(u_n, \mathcal{E}(1)) = \frac{2}{\pi} \Sigma(u_0)$. \square

Next we turn to the general case.

Proof of “ \leq ” in (1.15) for general v_0 . By (2.7) we have

$$\text{dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) \geq (2/\pi) \Sigma(u_0 \bar{v}_0),$$

so we need to prove that this is actually an equality. By Proposition 1.2 there exists a sequence $\{w_n\}$ satisfying $w_n \sim u_0 \bar{v}_0$ for all n , $\lim_{n \rightarrow \infty} w_n = 1$ a.e., and

$$\int_{\Omega} |\nabla w_n| = \Sigma(u_0 \bar{v}_0) + \varepsilon_n, \quad (2.16)$$

with $\varepsilon_n \searrow 0$. By Proposition 2.4 we get

$$\int_{\mathbb{S}^1} \int_{\Omega} |\nabla(w_n - P_{\zeta} \circ w_n)| dx d\zeta = 4 \int_{\Omega} |\nabla w_n| dx = 4(\Sigma(u_0 \bar{v}_0) + \varepsilon_n). \quad (2.17)$$

Hence, there exists $\zeta_n \in \mathbb{S}^1_- := \{z = e^{i\theta}; \theta \in [-\pi, 0]\}$ such that

$$\int_{\Omega} |\nabla(w_n - P_{\zeta_n} \circ w_n)| + \int_{\Omega} |\nabla(w_n - P_{-\zeta_n} \circ w_n)| \leq \frac{4}{\pi} (\Sigma(u_0 \bar{v}_0) + \varepsilon_n).$$

By (2.7) we have

$$\frac{2}{\pi} \Sigma(u_0 \bar{v}_0) \leq \min \left(\int_{\Omega} |\nabla(w_n - P_{\zeta_n} \circ w_n)|, \int_{\Omega} |\nabla(w_n - P_{-\zeta_n} \circ w_n)| \right),$$

and thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla(w_n - P_{\zeta_n} \circ w_n)| = \frac{2}{\pi} \Sigma(u_0 \bar{v}_0). \quad (2.18)$$

Passing to a subsequence, we may assume $\zeta_n \rightarrow \zeta \in \mathbb{S}^1_-$. Therefore, $P_{\zeta}(1) = 1$. Denote $F_n := P_{\zeta_n} \circ w_n$. Since $w_n \rightarrow 1$ a.e., we have $\lim_{n \rightarrow \infty} F_n = \lim_{n \rightarrow \infty} P_{\zeta} \circ w_n = 1$ a.e., and it follows that

$$F_n - w_n \rightarrow 0 \text{ a.e.} \quad (2.19)$$

For any v such that $v \sim v_0$ we have $vF_n \sim v_0$, $vw_n \sim u_0$ and

$$\frac{2}{\pi} \Sigma(u_0 \bar{v}_0) \leq \int_{\Omega} |\nabla(vF_n - vw_n)| \leq \int_{\Omega} |\nabla(F_n - w_n)| + \int_{\Omega} |\nabla v| |F_n - w_n|. \quad (2.20)$$

From (2.18)-(2.20) we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla(vF_n - vw_n)| = \frac{2}{\pi} \Sigma(u_0 \bar{v}_0),$$

and the result follows. \square

3 Proof of (1.16) in Theorem 1.1

3.1 An upper bound for $\text{Dist}_{W^{1,1}}$

This short subsection is devoted to the proof of the following

Proposition 3.1. *For every u_0, v_0 in $W^{1,1}(\Omega, \mathbb{S}^1)$ we have,*

$$\text{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) \leq \Sigma(u_0 \bar{v}_0). \quad (3.1)$$

Proof. We adapt an argument from [6]. By Proposition 1.2 there exists a sequence $\{w_n\} \subset W^{1,1}(\Omega; \mathbb{S}^1)$ satisfying $w_n \sim u_0 \bar{v}_0$, $w_n \rightarrow 1$ a.e., and $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla w_n| = \Sigma(u_0 \bar{v}_0)$. For a given $u \in \mathcal{E}(u_0)$ define $v_n = u \bar{w}_n$ for all n . Then, $v_n \sim v_0$ and

$$\begin{aligned} d_{W^{1,1}}(u, \mathcal{E}(v_0)) &\leq \int_{\Omega} |\nabla(u - v_n)| = \int_{\Omega} |\nabla(u(1 - \bar{w}_n))| \\ &\leq \int_{\Omega} |1 - w_n| |\nabla u| + \int_{\Omega} |\nabla w_n| \rightarrow \Sigma(u_0 \bar{v}_0). \quad \square \end{aligned}$$

3.2 A lower bound for $\text{Dist}_{W^{1,1}}$

We begin with the following elementary geometric lemma.

Lemma 3.2. *Let z_1 and z_2 two points in \mathbb{S}^1 satisfying, for some $\varepsilon \in (0, \pi/2)$,*

$$d_{\mathbb{S}^1}(z_1, z_2) \in (\varepsilon, \pi - \varepsilon). \quad (3.2)$$

If the vectors $v_1, v_2 \in \mathbb{R}^2$ satisfy

$$v_j \perp z_j, \quad j = 1, 2, \quad (3.3)$$

then

$$|v_1 - v_2| \geq (\sin \varepsilon) |v_j|, \quad j = 1, 2, \quad (3.4)$$

and in particular

$$|v_1 - v_2|^2 \geq \left(\frac{\sin^2 \varepsilon}{2} \right) (|v_1|^2 + |v_2|^2). \quad (3.5)$$

Note that the inequality (3.5) can be viewed as a “reverse triangle inequality”.

Proof. From the assumptions (3.2)–(3.3) it follows that

$$\langle v_1, v_2 \rangle \leq (\cos \varepsilon) |v_1| |v_2|,$$

and then

$$|v_1 - v_2|^2 \geq |v_1|^2 + |v_2|^2 - 2(\cos \varepsilon) |v_1| |v_2| \geq (\sin \varepsilon)^2 |v_j|^2, \quad j = 1, 2. \quad \square$$

An immediate consequence of Lemma 3.2 is

Lemma 3.3. *Let $v, \tilde{u} \in W^{1,1}(\Omega; \mathbb{S}^1)$ and denote, for $\varepsilon \in (0, \pi/2)$,*

$$\begin{aligned} A_{\varepsilon} &:= \{x \in \Omega; d_{\mathbb{S}^1}(\tilde{u}(x), v(x)) \in (\varepsilon, \pi - \varepsilon)\} \\ &= \{x \in \Omega; 2 \sin(\varepsilon/2) < |\tilde{u}(x) - v(x)| < 2 \cos(\varepsilon/2)\}. \end{aligned} \quad (3.6)$$

Then

$$|\nabla(\tilde{u} - v)| \geq (\sin \varepsilon) |\nabla \tilde{u}| \quad \text{a.e. in } A_{\varepsilon}. \quad (3.7)$$

Proof. Since $v \perp v_{x_i}$ and $\tilde{u} \perp \tilde{u}_{x_i}$ a.e. on Ω for $i = 1, \dots, N$, we may apply Lemma 3.2 with $z_1 = \tilde{u}(x)$, $z_2 = v(x)$, $v_1 = \tilde{u}_{x_i}(x)$ and $v_2 = v_{x_i}(x)$ to obtain

$$|\tilde{u}_{x_i} - v_{x_i}|^2 \geq (\sin \varepsilon)^2 |\tilde{u}_{x_i}|^2, \quad \text{a.e. in } A_\varepsilon, \quad i = 1, \dots, N.$$

Summing over i yields (3.7). □

The next lemma is the main ingredient in the proof of Proposition 1.3.

Lemma 3.4. *Let $u, \tilde{u}, v \in W^{1,1}(\Omega; \mathbb{S}^1)$, $\varepsilon \in (0, \pi/20)$ and A_ε as in (3.6). Assume that*

$$|u(x) - \tilde{u}(x)| \leq \varepsilon, \quad \forall x \in \Omega. \quad (3.8)$$

Then,

$$\int_{A_\varepsilon} |\nabla(v - \tilde{u})| \geq (1 - 6\varepsilon)\Sigma(v\bar{u}) - 2 \int_{A_\varepsilon} |\nabla u|. \quad (3.9)$$

Proof. Note first that (3.8) implies that $\tilde{u} \sim u$. Indeed, the image of the map $\tilde{u}\bar{u}$ is contained in an arc of \mathbb{S}^1 of length $\leq 2 \arcsin(\varepsilon/2)$, so there exists $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ such that $\tilde{u} = e^{i\varphi}u$. Hence, setting $w := v/u = v\bar{u}$ and $\tilde{w} := v/\tilde{u}$, we have also $\tilde{w} \sim w$. Consider the map

$$W := \bar{u}(v - \tilde{u}) + 1 = w + (1 - \tilde{u}/u). \quad (3.10)$$

By the triangle inequality,

$$|\nabla W| = |\nabla(\bar{u}(v - \tilde{u}))| \leq 2|\nabla \bar{u}| + |\nabla(v - \tilde{u})|,$$

whence

$$\int_{A_\varepsilon} |\nabla(v - \tilde{u})| \geq \int_{A_\varepsilon} |\nabla W| - 2 \int_{A_\varepsilon} |\nabla u|. \quad (3.11)$$

By (3.8), $|W - w| = |1 - \tilde{u}/u| = |u - \tilde{u}| \leq \varepsilon$ in Ω . Hence

$$||W| - 1| \leq |W - w| \leq \varepsilon \quad \text{in } \Omega, \quad (3.12)$$

and also

$$|\tilde{w} - w| = |\tilde{u} - u| \leq \varepsilon \quad \text{in } \Omega. \quad (3.13)$$

Consider the map $\widetilde{W} := W/|W|$, which thanks to (3.12) belongs to $W^{1,1}(\Omega; \mathbb{S}^1)$. Furthermore, again by (3.12),

$$|\widetilde{W} - w| \leq |\widetilde{W} - W| + |W - w| \leq 2\varepsilon \quad \text{in } \Omega, \quad (3.14)$$

implying in particular that

$$\widetilde{W} \in \mathcal{E}(w). \quad (3.15)$$

Combining (3.14) with (3.13) yields

$$|\widetilde{W} - \tilde{w}| \leq 3\varepsilon \quad \text{and} \quad d_{\mathbb{S}^1}(\widetilde{W}, \tilde{w}) \leq 6\varepsilon \quad \text{in } \Omega. \quad (3.16)$$

A direct consequence of (3.12) is the pointwise inequality in Ω

$$|\nabla W| \geq (1 - \varepsilon)|\nabla \widetilde{W}|,$$

which together with (3.11) yields

$$\int_{A_\varepsilon} |\nabla(v - \tilde{u})| \geq (1 - \varepsilon) \int_{A_\varepsilon} |\nabla \tilde{W}| - 2 \int_{A_\varepsilon} |\nabla u|. \quad (3.17)$$

Since

$$A_\varepsilon = \{x \in \Omega; \tilde{w}(x) \in \mathcal{A}(\varepsilon, \pi - \varepsilon) \cup \mathcal{A}(\pi + \varepsilon, 2\pi - \varepsilon)\} \quad (\text{see (3.6) and (2.8)}),$$

we deduce from (3.16) that

$$B_\varepsilon := \{x \in \Omega; \tilde{W}(x) \in \mathcal{A}(7\varepsilon, \pi - 7\varepsilon) \cup \mathcal{A}(\pi + 7\varepsilon, 2\pi - 7\varepsilon)\} \subseteq A_\varepsilon. \quad (3.18)$$

For each $\delta \in (0, \pi/2)$ consider the map $K_\delta : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined by

$$K_\delta(e^{i\theta}) := \begin{cases} 1, & \text{if } -\delta \leq \theta < \delta \\ e^{i\pi(\theta-\delta)/(\pi-2\delta)}, & \text{if } \delta \leq \theta < \pi - \delta \\ -1, & \text{if } \pi - \delta \leq \theta < \pi + \delta \\ -e^{i\pi(\theta-\pi-\delta)/(\pi-2\delta)}, & \text{if } \pi + \delta \leq \theta < 2\pi - 2\delta \end{cases}. \quad (3.19)$$

Clearly $K_\delta \in \text{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ with $\|\dot{K}_\delta\|_\infty = \pi/(\pi - 2\delta)$ and $\deg(K_\delta) = 1$. Therefore, by (3.15) and Lemma 2.1

$$w_1 := K_{7\varepsilon} \circ \tilde{W} \in \mathcal{E}(w). \quad (3.20)$$

Note that by definition, $\nabla w_1 = 0$ a.e. on $\Omega \setminus B_\varepsilon$, so by (3.18) and (3.20) we have

$$\int_{A_\varepsilon} |\nabla \tilde{W}| \geq \int_{B_\varepsilon} |\nabla \tilde{W}| \geq (1 - 5\varepsilon) \int_{B_\varepsilon} |\nabla w_1| = (1 - 5\varepsilon) \int_\Omega |\nabla w_1| \geq (1 - 5\varepsilon) \Sigma(w). \quad (3.21)$$

Plugging (3.21) in (3.17) yields

$$\begin{aligned} \int_{A_\varepsilon} |\nabla(v - \tilde{u})| &\geq (1 - \varepsilon) \int_{B_\varepsilon} |\nabla \tilde{W}| - 2 \int_{A_\varepsilon} |\nabla u| \\ &\geq (1 - \varepsilon)(1 - 5\varepsilon) \Sigma(w) - 2 \int_{A_\varepsilon} |\nabla u| \geq (1 - 6\varepsilon) \Sigma(w) - 2 \int_{A_\varepsilon} |\nabla u|, \end{aligned} \quad (3.22)$$

and (3.9) follows. \square

The next result is a direct consequence of Lemma 3.4.

Corollary 3.5. *There exists a universal constant C such that for every $\varepsilon > 0$ we have*

$$n \geq 1/\varepsilon^2 \implies \int_\Omega |\nabla(T_n \circ u) - v| \geq (1 - C\varepsilon) \Sigma(u\bar{v}), \quad \forall u, v \in W^{1,1}(\Omega; \mathbb{S}^1). \quad (3.23)$$

Proof. We shall use two basic properties of T_n :

$$d_{\mathbb{S}^1}(x, T_n(x)) \leq \frac{\pi(n-1)}{n^2}, \quad \forall x \in \mathbb{S}^1, \quad (3.24)$$

$$|\dot{T}_n| \geq n - 2 \quad \text{a.e. in } \mathbb{S}^1. \quad (3.25)$$

Clearly it suffices to consider $\varepsilon < \pi/20$. Hence for $n \geq 1/\varepsilon^2$ we can apply Lemma 3.4 with $\tilde{u} := T_n \circ u$ (thanks to (3.24)). By (3.25) we have

$$|\nabla(T_n \circ u)| \geq (n - 2)|\nabla u| \quad \text{a.e. on } \Omega, \quad (3.26)$$

so combining (3.7) and (3.9) gives (recall that A_ε is defined in (3.6)):

$$\begin{aligned} \int_{A_\varepsilon} |\nabla(T_n \circ u - v)| &\geq (1 - 6\varepsilon)\Sigma(u\bar{v}) - \frac{2}{(n-2)\sin\varepsilon} \int_{A_\varepsilon} |\nabla(T_n \circ u - v)| \\ &\geq (1 - 6\varepsilon)\Sigma(u\bar{v}) - \frac{3}{n\varepsilon} \int_{A_\varepsilon} |\nabla(T_n \circ u - v)|; \end{aligned}$$

this leads easily to (3.23). \square

Proof of Proposition 1.3. Recall (see (3.1)) that

$$\text{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) \leq \Sigma(u_0\bar{v}_0), \quad (3.27)$$

and in particular, $\forall n \geq 3$,

$$d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0)) \leq \Sigma(u_0\bar{v}_0). \quad (3.28)$$

On the other hand, from Corollary 3.5 we know that, $\forall \varepsilon > 0, \forall n \geq 1/\varepsilon^2$,

$$d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0)) \geq (1 - C\varepsilon)\Sigma(u_0\bar{v}_0). \quad (3.29)$$

We conclude combining (3.28) and (3.29). \square

Proof of (1.16). Use (1.20) and (3.27). \square

3.3 About equality cases in (1.22)

It is interesting to decide whether there exist maps $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ for which equality holds in any of the two inequalities in (1.22). Consider the following properties of a smooth bounded domain Ω in \mathbb{R}^N , $N \geq 2$:

(P₁) There exists $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that

$$\int_{\Omega} |\nabla u| = \Sigma(u) > 0. \quad (3.30)$$

(P₂) There exists $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that

$$d_{W^{1,1}}(u, \mathcal{E}(1)) = \Sigma(u) > 0. \quad (3.31)$$

(P₂^{*}) There exist $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ with $\Sigma(u) > 0$ and $v \in \mathcal{E}(1)$ for which

$$\int_{\Omega} |\nabla(u - v)| = d_{W^{1,1}}(u, \mathcal{E}(1)) = \Sigma(u). \quad (3.32)$$

(P₃) There exists $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that

$$d_{W^{1,1}}(u, \mathcal{E}(1)) = \frac{2}{\pi}\Sigma(u) > 0. \quad (3.33)$$

(P₃^{*}) There exist $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ with $\Sigma(u) > 0$ and $v \in \mathcal{E}(1)$ for which

$$\int_{\Omega} |\nabla(u - v)| = d_{W^{1,1}}(u, \mathcal{E}(1)) = \frac{2}{\pi}\Sigma(u). \quad (3.34)$$

Very little is known about domains satisfying any of the above properties. The unit disc $\Omega = B(0, 1)$ in \mathbb{R}^2 is an example of a domain for which (P_1) is satisfied. Indeed, for $u = x/|x|$ we have by Remark 1.5,

$$\Sigma\left(\frac{x}{|x|}\right) = 2\pi L\left(\frac{x}{|x|}\right) = 2\pi = \int_{\Omega} \left| \nabla\left(\frac{x}{|x|}\right) \right|,$$

whence (P_1) holds. In view of the following proposition we know that (P_3^*) is also satisfied for $\Omega = B(0, 1)$ in \mathbb{R}^2 .

Proposition 3.6. *Properties (P_1) and (P_3^*) are equivalent. More precisely, let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ with $\Sigma(u) > 0$. Then, the following are equivalent:*

(a) u satisfies (3.30).

(b) There exist $u_0 \in \mathcal{E}(u)$ and $v \in \mathcal{E}(1)$ such that $\int_{\Omega} |\nabla(u_0 - v)| = \frac{2}{\pi}\Sigma(u)$.

Proof of “(a) \implies (b)”. Use Proposition 2.4 to find $\zeta_0 \in \mathbb{S}^1$ such that $v = P_{\zeta_0} \circ u \in \mathcal{E}(1)$ satisfies

$$d_{W^{1,1}}(u, \mathcal{E}(1)) \leq \int_{\Omega} |\nabla(u - v)| \leq \frac{2}{\pi} \int_{\Omega} |\nabla u| = \frac{2}{\pi}\Sigma(u), \quad (3.35)$$

and the result follows, with $u_0 = u$, since by (1.24) we have

$$d_{W^{1,1}}(u, \mathcal{E}(1)) \geq \frac{2}{\pi}\Sigma(u). \quad (3.36)$$

Proof of “(b) \implies (a)”. Let u_0 and v be as in statement (b). Set $w_0 := u_0 \bar{v}$, so that $w_0 \sim u_0$. By assumption and (2.2) we have:

$$\frac{2}{\pi}\Sigma(u_0) \leq \int_{\Omega} |\nabla(|w_0 - 1|)| = \int_{\Omega} |\nabla(|u_0 - v|)| \leq \int_{\Omega} |\nabla(u_0 - v)| = \frac{2}{\pi}\Sigma(u_0). \quad (3.37)$$

Set $w_1 := T \circ w_0$, where $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is given by (2.3). By Lemma 2.1, $w_1 \sim w_0 \sim u_0$, and by (2.6) and (3.37) we obtain that

$$\int_{\Omega} |\nabla w_1| = \frac{\pi}{2} \int_{\Omega} |\nabla(|w_0 - 1|)| = \Sigma(u). \quad \square$$

We do not know any domain Ω in \mathbb{R}^2 for which (3.30) fails and we ask:

Open Problem 1. Is there a domain Ω in \mathbb{R}^N , $N \geq 2$, for which property (P_1) does not hold?

It seems plausible that if Ω is the interior of a non circular ellipse, then (P_1) fails. We also do not know whether properties (P_3) and (P_3^*) are equivalent.

Concerning properties (P_2) and (P_2^*) we know even less:

Open Problem 2. Is there a domain Ω for which (P_2) holds (respectively, fails)?

We suspect that (P_2) is satisfied in *every* domain, but we do not know *any* such domain. In particular, we do not know what happens when Ω is a disc in \mathbb{R}^2 .

4 Distances in $W^{1,p}(\Omega; \mathbb{S}^1)$, $1 < p < \infty$

Throughout this section we study classes in $W^{1,p}(\Omega; \mathbb{S}^1)$, where $1 < p < \infty$ and Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 2$. We give below the proofs of the results stated in the Introduction.

4.1 Proof of Theorem 1.7

Proof of Theorem 1.7. The result is a direct consequence of the following analog of Corollary 2.3: for every $u, v \in W^{1,p}(\Omega; \mathbb{S}^1)$ we have

$$\|\nabla(u - v)\|_{L^p(\Omega)} \geq \left(\frac{2}{\pi}\right) \inf_{w \sim u\bar{v}} \|\nabla w\|_{L^p(\Omega)}. \quad (4.1)$$

The proof of (4.1) uses an argument identical to the one used in the proofs of Lemma 2.2 and Corollary 2.3. Indeed, we first note that

$$\int_{\Omega} |\nabla(u - v)|^p \geq \int_{\Omega} |\nabla(|u - v|)|^p = \int_{\Omega} |\nabla(|u\bar{v} - 1|)|^p. \quad (4.2)$$

Next, by (2.5) we have

$$\int_{\Omega} |\nabla(|u\bar{v} - 1|)|^p = \left(\frac{2}{\pi}\right)^p \int_{\Omega} |\nabla(T \circ (u\bar{v}))|^p \geq \left(\frac{2}{\pi}\right)^p \inf_{w \sim u\bar{v}} \int_{\Omega} |\nabla w|^p. \quad (4.3)$$

The result clearly follows by combining (4.2) with (4.3). \square

4.2 Proof of Theorem 1.9

We shall need the following technical lemma.

Lemma 4.1. *For every $w_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$ we have*

$$\inf_{w \sim w_0} \|\nabla(|w - 1|)\|_{L^p(\Omega)} = \left(\frac{2}{\pi}\right) \inf_{w \sim w_0} \|\nabla w\|_{L^p(\Omega)}. \quad (4.4)$$

Proof. The inequality “ \geq ” follows from (4.3) (taking $v = 1$) so it remains to prove the reverse inequality. The argument is almost identical to the one used in the proof of [14, Prop 3.1]; we reproduce the argument for the convenience of the reader. We shall need the inverse $S := T^{-1}$ of $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that was defined in (2.3). It is given by:

$$S(e^{i\theta}) = e^{i\phi}, \quad \text{with } \phi = 2 \sin^{-1}(\theta/\pi), \quad \forall \theta \in (-\pi, \pi].$$

This map belongs to $C(\mathbb{S}^1; \mathbb{S}^1) \cap C^1(\mathbb{S}^1 \setminus \{-1\}; \mathbb{S}^1)$ but it is not Lipschitz. We therefore define, for each small $\varepsilon > 0$, an approximation S_ε by:

$$S_\varepsilon(e^{i\theta}) = e^{i\phi} \quad \text{with } \phi = 2 \sin^{-1}(J_\varepsilon(\theta/\pi)), \quad \forall \theta \in (-\pi, \pi], \quad (4.5)$$

where J_ε satisfies:

$$\begin{aligned} J_\varepsilon(\pm 1) &= \pm 1, \quad J'_\varepsilon(\pm 1) = 0, \\ J_\varepsilon(t) &= t, \quad \text{for } |t| \leq 1 - \varepsilon, \\ 0 < J'_\varepsilon(t) &< c_0, \quad \text{for } |t| < 1, \\ \frac{c_1}{\varepsilon} \leq |J''_\varepsilon(t)| &\leq \frac{c_2}{\varepsilon}, \quad \text{for } 1 - \frac{\varepsilon}{2} \leq |t| \leq 1, \end{aligned} \quad (4.6)$$

for some positive constants c_0, c_1, c_2 (independent of ε). Clearly $S_\varepsilon \in C^1(\mathbb{S}^1; \mathbb{S}^1)$ with $\deg(S_\varepsilon) = 1$, so by Lemma 2.1, for any $w \in \mathcal{E}(w_0)$ we have $w_\varepsilon := S_\varepsilon \circ w \in \mathcal{E}(w_0)$. Since $|S_\varepsilon(e^{i\theta}) - 1| = 2|J_\varepsilon(\theta/\pi)|$ it follows from (4.6) that

$$\left| \frac{d}{d\theta} (|S_\varepsilon(e^{i\theta}) - 1|) \right| \leq C, \quad \forall \theta, \forall \varepsilon. \quad (4.7)$$

Put $A_\varepsilon := \{x \in \Omega : w(x) \in \mathcal{A}(-\pi(1 - \varepsilon), \pi(1 - \varepsilon))\}$. By (2.5),

$$|\nabla|w_\varepsilon - 1|| = \frac{2}{\pi} |\nabla w| \text{ a.e. on } A_\varepsilon,$$

while, by (4.7),

$$|\nabla|w_\varepsilon - 1|| \leq C |\nabla w| \text{ a.e. on } \Omega.$$

Therefore, by dominated convergence,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla|w_\varepsilon - 1||^p = \left(\frac{2}{\pi}\right)^p \int_{\Omega} |\nabla w|^p,$$

and since the above is valid for any $w \in \mathcal{E}(w_0)$, the inequality “ \leq ” in (4.4) follows. \square

The next lemma is the main ingredient of the proof of Theorem 1.9.

Lemma 4.2. *For every $w \in W^{1,p}(\Omega; \mathbb{S}^1)$ and $0 < \delta < 1$ there exist a set $A = A(w, \delta) \subset \Omega$ and two functions $w_0, w_1 \in W^{1,p}(\Omega; \mathbb{S}^1)$ such that:*

- (i) $w_1 = \bar{w}_0$ in $\Omega \setminus A$;
- (ii) $w_0 = w_1$ in A ;
- (iii) $w_1 \in \mathcal{E}(w)$ and $w_0 \in \mathcal{E}(1)$;
- (iv) $\int_{\Omega} |\nabla(w_1 - w_0)|^p \leq (1 + C_p \delta) \int_{\Omega} |\nabla|w - 1||^p$;
- (v) $\int_{\Omega} |\nabla w_1|^p = \int_{\Omega} |\nabla w_0|^p \leq C(\delta, p) \int_{\Omega} |\nabla w|^p$.

Proof. Let I denote the open arc of \mathbb{S}^1 , $I := \mathcal{A}(2\pi - \delta, 2\pi) = \{e^{i\theta} : \theta \in (2\pi - \delta, 2\pi)\}$, and let $A := w^{-1}(I)$. Define $T_1 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by

$$T_1(e^{i\theta}) = \begin{cases} e^{i\pi\theta/(2\pi-\delta)}, & \text{if } 0 \leq \theta \leq 2\pi - \delta \\ (-1)e^{i\pi(\theta - (2\pi - \delta))/\delta}, & \text{if } 2\pi - \delta < \theta < 2\pi \end{cases}.$$

Note that the image of T_1 , restricted to the arc $\mathbb{S}^1 \setminus I$ is \mathbb{S}_+^1 , and that \mathbb{S}_+^1 is covered counterclockwise. Similarly, on the arc I , the image of T_1 is \mathbb{S}_-^1 , covered again counterclockwise. It follows that $\deg(T_1) = 1$. Next we define $T_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by $P_{-1} \circ T_1$ (see (2.10)), or explicitly by

$$T_0(e^{i\theta}) := \begin{cases} \overline{T_1(e^{i\theta})}, & \text{if } 0 \leq \theta \leq 2\pi - \delta \\ T_1(e^{i\theta}), & \text{if } 2\pi - \delta < \theta < 2\pi \end{cases}.$$

Clearly $\deg(T_0) = 0$. Define $w_0 := T_0 \circ w$ and $w_1 := T_1 \circ w$.

Properties (i)–(ii) are direct consequences of the definition of w_0, w_1 . The fact that $w_0 \in \mathcal{E}(1)$ and $w_1 \in \mathcal{E}(w)$ (i.e., property (iii)) follows from Lemma 2.1. Since T_0 and T_1 are Lipschitz maps (actually, piecewise smooth, with a single corner at $z = e^{i(2\pi - \delta)}$), the chain rule implies that

$$|\nabla w_0| = |\nabla w_1| \leq \begin{cases} (\pi/\delta) |\nabla w|, & \text{a.e. in } A \\ (\pi/(2\pi - 1)) |\nabla w|, & \text{a.e. in } \Omega \setminus A \end{cases},$$

whence property (v). Finally, in order to verify property (iv) we first notice that on $\Omega \setminus A$ we have

$$\tilde{w} := w_1 \bar{w}_0 = w_1^2 = Q \circ w,$$

where $Q(e^{i\theta}) := e^{2i\theta\pi/(2\pi-\delta)}$ for $\theta \in (0, 2\pi - \delta)$. Therefore,

$$\begin{aligned} \int_{\Omega} |\nabla(w_1 - w_0)|^p &= \int_{\Omega \setminus A} |\nabla(w_1 - w_0)|^p = \int_{\Omega \setminus A} |\nabla|w_1 - w_0||^p = \int_{\Omega \setminus A} |\nabla|\tilde{w} - 1||^p \\ &\leq (1 + C_p \delta) \int_{\Omega \setminus A} |\nabla|w - 1||^p \leq (1 + C_p \delta) \int_{\Omega} |\nabla|w - 1||^p. \end{aligned} \quad \square$$

We are now in a position to present the

Proof of Theorem 1.9. In view of Theorem 1.7 we only need to prove the inequality “ \leq ” in (1.31). For any $w \in \mathcal{E}(u_0)$ we may apply Lemma 4.2 with a sequence $\delta_n \rightarrow 0$ to obtain that

$$d_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(1)) \leq \inf_{w \sim u_0} \|\nabla|w - 1|\|_{L^p(\Omega)},$$

and the result follows from Lemma 4.1. \square

4.3 Proof of Theorem 1.11

We next turn to the unboundedness of the $\text{Dist}_{W^{1,p}}$ -distance between distinct classes.

Proof of Theorem 1.11 when $u_0 = 1$. For every $n \geq 1$ let

$$u_n := e^{inx_1} \quad (\text{we write } x = (x_1, \dots, x_N)),$$

so clearly $u_n \in C^\infty(\Omega; \mathbb{S}^1) \subset \mathcal{E}(1)$. We claim that

$$\lim_{n \rightarrow \infty} d_{W^{1,p}}(u_n, \mathcal{E}(v_0)) = \infty, \quad (4.8)$$

which implies of course (1.32) in this case. Fix a small $\varepsilon > 0$, e.g., $\varepsilon = \pi/8$. For each $v \in \mathcal{E}(v_0)$ let $w_n := \bar{u}_n v$ and define the set A_ε as in (3.6), with $\tilde{u} = u_n$. Note that $|\nabla u_n(x)| = n$, $x \in \Omega$, so by Lemma 3.3 we have

$$|\nabla(u_n - v)| \geq n \sin \varepsilon \text{ a.e. in } A_\varepsilon. \quad (4.9)$$

Therefore,

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq |A_\varepsilon| (\sin \varepsilon)^p n^p = c_1 |A_\varepsilon| n^p. \quad (4.10)$$

Using (3.5) instead of (3.4) in the computation leading to (4.9) yields

$$|\nabla(u_n - v)| \geq \left(\frac{\sin \varepsilon}{2} \right) (|\nabla u_n| + |\nabla v|) \geq \left(\frac{\sin \varepsilon}{2} \right) |\nabla w_n|, \quad \text{a.e. in } A_\varepsilon. \quad (4.11)$$

We set $\tilde{w}_n := K_\varepsilon \circ w_n$ (see (3.19)) and recall that $K_\varepsilon \in \text{Lip}(\mathbb{S}^1; \mathbb{S}^1)$, $\|K_\varepsilon\|_\infty = \pi/(\pi - 2\varepsilon)$ and $\deg(K_\varepsilon) = 1$. We have $\tilde{w}_n \in \mathcal{E}(v_0)$ and $\nabla \tilde{w}_n = 0$ a.e. in $\Omega \setminus A_\varepsilon$. By (1.4),

$$\int_{A_\varepsilon} |\nabla \tilde{w}_n| = \int_{\Omega} |\nabla \tilde{w}_n| \geq \Sigma(v_0). \quad (4.12)$$

Using Hölder inequality and (4.12) gives

$$\int_{A_\varepsilon} |\nabla \tilde{w}_n|^p \geq \frac{\left(\int_{A_\varepsilon} |\nabla \tilde{w}_n|\right)^p}{|A_\varepsilon|^{p-1}} \geq \frac{(\Sigma(v_0))^p}{|A_\varepsilon|^{p-1}}. \quad (4.13)$$

Since $|\nabla w_n| \geq (1 - 2\varepsilon/\pi)|\nabla \tilde{w}_n|$ on Ω we obtain by combining (4.11) and (4.13) that

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq \frac{c_2}{|A_\varepsilon|^{p-1}}, \quad (4.14)$$

whence,

$$|A_\varepsilon| \geq c_2^{1/(p-1)} \left(\int_{A_\varepsilon} |\nabla(u_n - v)|^p\right)^{-1/(p-1)}. \quad (4.15)$$

Plugging (4.15) in (4.10) finally yields

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq c_3 n^{p-1},$$

and (4.8) follows.

Proof of Theorem 1.11 in the general case. Consider an arbitrary $u_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$. We set $u_n := e^{inx_1} u_0 \in \mathcal{E}(u_0)$. By the triangle inequality,

$$|\nabla(e^{inx_1} - \bar{u}_0 v)| = |\nabla(\bar{u}_0(u_n - v))| \leq |\nabla(u_n - v)| + 2|\nabla \bar{u}_0|.$$

Therefore,

$$\|\nabla(u_n - v)\|_{L^p(\Omega)} \geq \|\nabla(e^{inx_1} - \bar{u}_0 v)\|_{L^p(\Omega)} - 2\|\nabla \bar{u}_0\|_{L^p(\Omega)},$$

and the result follows from the first part of the proof. \square

4.4 An example of strict inequality in (1.30)

Proposition 4.3. *There exist a smooth, bounded simply connected domain Ω in \mathbb{R}^2 and $u_0, v_0 \in \bigcap_{1 \leq p < 2} W^{1,p}(\Omega; \mathbb{S}^1)$ such that*

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) > \left(\frac{2}{\pi}\right) \inf_{w \sim u_0 \bar{v}_0} \|\nabla w\|_{L^p(\Omega)}, \quad 1 < p < 2. \quad (4.16)$$

Proof. The construction resembles the one used in the proof of [14, Proposition 4.1] (for a multiply connected domain and $p = 2$), but the details of the proof are quite different.

Step 1. Definition of Ω_ε and u_0, v_0

Consider the three unit discs with centers at the points $a_- := (-3, 0)$, $a := (0, 0)$ and $a_+ := (3, 0)$, respectively:

$$B_- := B(a_-, 1), \quad B := B(a, 1) \quad \text{and} \quad B_+ := B(a_+, 1).$$

For a small $\varepsilon \in (0, 1/4)$, to be determined later, define the domain Ω_ε by

$$\Omega_\varepsilon := B_- \cup B \cup B_+ \cup \{(x_1, x_2); x_1 \in (-3, 3), x_2 \in (-\varepsilon, \varepsilon)\}. \quad (4.17)$$

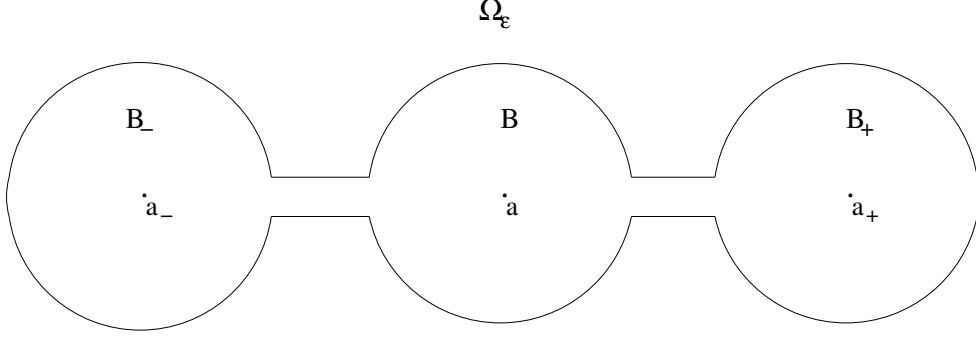


Figure 1: The domain Ω_ε (before smoothing)

Hence, B is connected to B_- and B_+ by two narrow tubes (see Figure 1). We can enlarge Ω_ε slightly near the “corners”=the contact points of the tubes with the circles, in order to have a smooth Ω_ε . But we do it keeping the following property:

$$\Omega_\varepsilon \text{ is symmetric with respect to reflections in both the } x \text{ and } y\text{-axis.} \quad (4.18)$$

For later use we denote by Ω_+ and Ω_- the two components of $\Omega_\varepsilon \setminus \overline{B}$ (with $\Omega_+ \subset \{z; \operatorname{Re} z > 0\}$). We define the maps $u_0, v_0 \in \bigcap_{1 \leq p < 2} W^{1,p}(\Omega_\varepsilon; \mathbb{S}^1)$ by

$$u_0 := \left(\frac{x - a_-}{|x - a_-|} \right) \left(\frac{x}{|x|} \right)^2 \left(\frac{x - a_+}{|x - a_+|} \right), \quad v_0 := \left(\frac{x - a_-}{|x - a_-|} \right) \left(\frac{x}{|x|} \right) \left(\frac{x - a_+}{|x - a_+|} \right), \quad (4.19)$$

and then

$$w_0 := u_0 \overline{v_0} = \frac{x}{|x|}. \quad (4.20)$$

Step 2. Properties of energy minimizers in $\mathcal{E}(w_0)$

We denote by $W_\varepsilon \in W^{1,p}(\Omega_\varepsilon; \mathbb{S}^1)$ a map realizing the minimum in

$$S_\varepsilon = \inf_{w \sim w_0} \|\nabla w\|_{L^p(\Omega_\varepsilon)}. \quad (4.21)$$

Note that the minimizer W_ε is unique, up to multiplication by a complex constant of modulo one. This follows from the strict convexity of the functional:

$$F(\varphi) = \int_{\Omega_\varepsilon} \left| \nabla \left(e^{i\varphi} \frac{x}{|x|} \right) \right|^p \quad \text{over } \varphi \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}). \quad (4.22)$$

We next claim that

$$\int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p \leq \int_B |\nabla W_\varepsilon|^p \leq \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^p \leq \int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p + C\varepsilon^2, \quad (4.23)$$

for some constant C . Here and in the sequel we denote by C different constants that are independent of ε and p . Indeed, the first inequality in (4.23) is clear since the restriction of W_ε to B belongs to the class of $x/|x|$ in B , and the latter map is a minimizer of the energy in its class (see Remark 4.4 below). For the proof of the last inequality in (4.23) it suffices to construct a comparison map $\tilde{w} \in \mathcal{E}(w_0)$ as follows. We first set $\tilde{w} = x/|x|$ in B . Then extend it to $\Omega_+ \cap \{x_1 \leq 1 + \varepsilon\}$ in such a way that $\tilde{w} \equiv \zeta$ (for some constant $\zeta \in \mathbb{S}^1$) on $\Omega_+ \cap \{x_1 = 1 + \varepsilon\}$. Such an extension can be constructed with $\|\nabla \tilde{w}\|_{L^\infty} \leq C$, whence

$$\int_{\Omega_+ \cap \{x_1 < 1 + \varepsilon\}} |\nabla \tilde{w}|^p \leq C\varepsilon^2.$$

In the remaining part of Ω_+ , namely $\Omega_+ \cap \{x_1 > 1 + \varepsilon\}$ we simply set $\tilde{w} \equiv \zeta$. We use a similar construction for \tilde{w} on Ω_- , and this completes the proof of (4.23).

We shall also use a certain symmetry property of W_ε . We claim that:

$$W_\varepsilon(x) = -W_\varepsilon(-x) \quad \text{in } \Omega_\varepsilon. \quad (4.24)$$

Indeed, since $W_\varepsilon(-x)$ is also a minimizer in (4.21), we must have

$$W_\varepsilon(-x) = e^{i\alpha} W_\varepsilon(x) \quad \text{for some constant } \alpha \in \mathbb{R}. \quad (4.25)$$

Write

$$W_\varepsilon = e^{i\Psi_\varepsilon} \left(\frac{x}{|x|} \right), \quad \text{with } \Psi_\varepsilon \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}). \quad (4.26)$$

Plugging (4.26) in (4.25) gives

$$-e^{i\Psi_\varepsilon(-x)} \left(\frac{x}{|x|} \right) = e^{i\alpha} e^{i\Psi_\varepsilon(x)} \left(\frac{x}{|x|} \right),$$

whence $e^{i(\Psi_\varepsilon(-x) - \Psi_\varepsilon(x))} = -e^{i\alpha}$. It follows that $\Psi_\varepsilon(-x) - \Psi_\varepsilon(x) \equiv \text{const}$ in Ω_ε . Since $\Psi_\varepsilon(-x) - \Psi_\varepsilon(x)$ is odd, it follows that the constant must be zero. Hence $\Psi_\varepsilon(-x) = \Psi_\varepsilon(x)$ a.e. in Ω_ε , $e^{i\alpha} = -1$ and (4.24) follows from (4.25).

The main property of W_ε that we need is the following: there exists $\zeta_\varepsilon \in \mathbb{S}^1$ such that

$$|W_\varepsilon - \zeta_\varepsilon| \leq c_0 \varepsilon^{2/p} \quad \text{on } B_+, \quad (4.27)$$

$$|W_\varepsilon + \zeta_\varepsilon| \leq c_0 \varepsilon^{2/p} \quad \text{on } B_-. \quad (4.28)$$

In order to verify (4.27)–(4.28) we first notice that we may write $W_\varepsilon = e^{i\Phi_\varepsilon}$ in $\Omega_\varepsilon \cap \{x_1 > 1\}$. Using (4.23) and Fubini Theorem we can find $t_\varepsilon \in (1, 3/2)$ such that the segment $I_\varepsilon = \{(t_\varepsilon, x_2); x_2 \in (-\varepsilon, \varepsilon)\}$ satisfies

$$\int_{I_\varepsilon} |\nabla \Phi_\varepsilon|^p = \int_{I_\varepsilon} |\nabla W_\varepsilon|^p \leq C\varepsilon^2.$$

By Hölder inequality it follows that $|\Phi_\varepsilon(z_1) - \Phi_\varepsilon(z_2)| \leq C\varepsilon^{2/p}$ for all $z_1, z_2 \in I_\varepsilon$. Hence, there exists $\alpha_\varepsilon \in \mathbb{R}$ satisfying

$$|\Phi_\varepsilon(z) - \alpha_\varepsilon| \leq C\varepsilon^{2/p}, \quad \forall z \in I_\varepsilon. \quad (4.29)$$

We claim that (4.29) continues to hold in $G_\varepsilon := \Omega_\varepsilon \cap \{x_1 > t_\varepsilon\}$, i.e.,

$$|\Phi_\varepsilon(x) - \alpha_\varepsilon| \leq C\varepsilon^{2/p}, \quad \forall x \in G_\varepsilon. \quad (4.30)$$

Indeed, defining

$$\tilde{\Phi}_\varepsilon(x) := \max(\alpha_\varepsilon - C\varepsilon^{2/p}, \min(\Phi_\varepsilon(x), \alpha_\varepsilon + C\varepsilon^{2/p})),$$

and then $\tilde{W}_\varepsilon := e^{i\tilde{\Phi}_\varepsilon}$, we clearly have $\int_{G_\varepsilon} |\nabla \tilde{W}_\varepsilon|^p \leq \int_{G_\varepsilon} |\nabla W_\varepsilon|^p$, with strict inequality, unless (4.30) holds. Setting $\zeta_\varepsilon := e^{i\alpha_\varepsilon}$, we deduce (4.27) from (4.30). Finally, using the symmetry properties, (4.18) of Ω_ε and (4.24) of Ψ_ε , we easily deduce (4.28) from (4.27).

Step 3. A basic estimate for maps in $W^{1,p}(\mathbb{S}^1; \mathbb{S}^1)$

The following claim provides a simple estimate which is essential for the proof. The case $p = 2$ was proved in [14, Lemma 4.1] and the generalization to any $p \geq 1$ is straightforward. We include the proof for the convenience of the reader.

Claim. For any $p \geq 1$, let $f, g \in W^{1,p}(\mathbb{S}^1; \mathbb{S}^1)$ satisfy:

$$\deg f = \deg g = k \neq 0 \text{ and } |(f - g)(\zeta)| = \eta > 0,$$

for some point $\zeta \in \mathbb{S}^1$. Then,

$$\int_{\mathbb{S}^1} |\dot{f} - \dot{g}|^p \geq \frac{2\eta^p}{\pi^{p-1}}. \quad (4.31)$$

Proof of Claim. Set $w := f - g$. We may assume without loss of generality that $w(1) = (f - g)(1) = \eta \iota$. There are two possibilities:

(i) $w(e^{i\theta_0}) = 0$ for some $\theta_0 \in (0, 2\pi)$. Hölder's inequality, and a straightforward computation yield

$$\int_{\mathbb{S}^1} |w'|^p \geq \int_{\mathbb{S}^1} ||w'|^p| \geq \frac{\eta^p}{\theta_0^{p-1}} + \frac{\eta^p}{(2\pi - \theta_0)^{p-1}} \geq \frac{2\eta^p}{(\theta_0(2\pi - \theta_0))^{(p-1)/2}} \geq \frac{2\eta^p}{\pi^{p-1}}.$$

(ii) $\gamma := \min\{|w(e^{i\theta})| : \theta \in [0, 2\pi)\} > 0$.

In this case the winding number of w with respect to 0 is also k . [This is proved using the homotopy $t \mapsto f - tg$, $t \in [0, 1]$.] In particular, there exists some $\theta_1 \in (0, 2\pi)$ for which $w(e^{i\theta_1}) = -t \iota$, with $t \geq \gamma$. Writing $w = w_1 + \iota w_2$ we have by the same computation as above

$$\int_{\mathbb{S}^1} |w'|^p \geq \int_{\mathbb{S}^1} |w_2'|^p \geq \frac{(\eta + t)^p}{\theta_1^{p-1}} + \frac{(\eta + t)^p}{(2\pi - \theta_1)^{p-1}} \geq 2 \frac{(\eta + t)^p}{\pi^{p-1}} > \frac{2\eta^p}{\pi^{p-1}},$$

and (4.31) follows in this case too. \square

Step 4. Conclusion

Consider two sequences $\{u_n\} \subset \mathcal{E}(u_0)$ and $\{v_n\} \subset \mathcal{E}(v_0)$ such that

$$\lim_{n \rightarrow \infty} \|\nabla(u_n - v_n)\|_{L^p(\Omega_\varepsilon)} = \text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)). \quad (4.32)$$

By a standard density argument we may assume that $u_n, v_n \in C^\infty(\overline{\Omega}_\varepsilon \setminus \{a_-, a, a_+\})$ for all n . Assume by contradiction that

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \left(\frac{2}{\pi}\right) S_\varepsilon = \left(\frac{2}{\pi}\right) \min_{w \sim w_0} \|\nabla w\|_{L^p(\Omega_\varepsilon)} \text{ (see (4.21)).} \quad (4.33)$$

By (4.33) and (4.23) there exists a constant C_0 such that

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \leq \left(\frac{2}{\pi}\right)^p \int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p + C_0 \varepsilon^2, \quad \forall n \geq n_0(\varepsilon). \quad (4.34)$$

Set $w_n := u_n \bar{v}_n$ and note that by the same computation as in (4.2)–(4.3) we have

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \geq \int_{\Omega_\varepsilon} |\nabla(|u_n - v_n|)|^p = \left(\frac{2}{\pi}\right)^p \int_{\Omega_\varepsilon} |\nabla(T \circ w_n)|^p \geq \left(\frac{2}{\pi}\right)^p S_\varepsilon^p \quad (4.35)$$

(recall that T is defined in (2.3)). Combining (4.32), (4.33) and (4.35) yields that $\tilde{w}_n := T \circ w_n$ satisfies

$$\lim_{n \rightarrow \infty} \|\nabla \tilde{w}_n\|_{L^p(\Omega_\varepsilon)} = S_\varepsilon,$$

and up to passing to a subsequence we have

$$W_\varepsilon = \lim_{n \rightarrow \infty} \tilde{w}_n \text{ in } W^{1,p}(\Omega; \mathbb{S}^1), \quad (4.36)$$

where W_ε is a minimizer in (4.21). Recall that W_ε is unique up to rotations; the particular W_ε in (4.36) is chosen by the subsequence. For any $\zeta \in \mathbb{S}^1$ we have $\max(|\zeta - 1|, |\zeta + 1|) \geq \sqrt{2}$. In particular, for ζ_ε associated with W_ε (see (4.27)–(4.28)) we may assume without loss of generality that

$$|\zeta_\varepsilon - 1| \geq \sqrt{2}. \quad (4.37)$$

By (4.36) and Egorov Theorem there exists $A_\varepsilon \subset \Omega_\varepsilon$ satisfying

$$|A_\varepsilon| \leq \varepsilon \text{ and } \tilde{w}_n \rightarrow W_\varepsilon \text{ uniformly on } \Omega_\varepsilon \setminus A_\varepsilon, \quad (4.38)$$

again, after passing to a subsequence. Combining (4.38) with (4.37) and (4.27) yields

$$|\tilde{w}_n - 1| \geq \sqrt{2} - 2c_0\varepsilon \text{ on } B_+ \setminus A_\varepsilon, \quad \forall n \geq n_1(\varepsilon),$$

and choosing $\varepsilon < (\sqrt{2} - 1)/(2c_0)$ guarantees that

$$|\tilde{w}_n - 1| \geq 1 \text{ on } B_+ \setminus A_\varepsilon, \quad \forall n \geq n_1(\varepsilon). \quad (4.39)$$

Going back to the definition of T in (2.3), we find by a simple computation the following equivalences for $e^{i\theta} = T(e^{i\varphi})$ (with $\theta \in (-\pi, \pi)$):

$$|T(e^{i\varphi}) - 1| \geq 1 \iff |\theta| = \pi |\sin(\varphi/2)| \geq \pi/3 \iff |e^{i\varphi} - 1| = 2 |\sin(\varphi/2)| \geq 2/3. \quad (4.40)$$

Using (4.40) we may rewrite (4.39) in terms of the original sequence $\{w_n\}$:

$$|u_n - v_n| = |w_n - 1| \geq 2/3 \text{ on } B_+ \setminus A_\varepsilon, \quad \forall n \geq n_1(\varepsilon). \quad (4.41)$$

Consider the set

$$\Lambda_\varepsilon = \{r \in (1/2, 1); \partial B(a_+, r) \subset A_\varepsilon\}. \quad (4.42)$$

By (4.38) we clearly have $\varepsilon \geq |A_\varepsilon| \geq (1/2)|\Lambda_\varepsilon| \cdot 2\pi$, whence

$$|\Lambda_\varepsilon| \leq \frac{\varepsilon}{\pi}. \quad (4.43)$$

For $n \geq n_1(\varepsilon)$ we have: on each circle $\partial B(a_+, r)$ with $r \in (1/2, 1) \setminus \Lambda_\varepsilon$ there exists at least one point where $|u_n - v_n| \geq 2/3$. Thus we may apply the Claim from Step 3 with $\eta := 2/3$, $f := u_n|_{\partial B(a_+, r)}$ and $g := v_n|_{\partial B(a_+, r)}$ to obtain by (4.31) (after a suitable rescaling):

$$\int_{\partial B(a_+, r)} |\nabla(u_n - v_n)|^p \geq 2(r\pi)^{1-p} \left(\frac{2}{3}\right)^p \geq (2\pi) \left(\frac{2}{3\pi}\right)^p := \gamma_p. \quad (4.44)$$

Integrating (4.44), taking into account (4.43), yields

$$\int_{B_+} |\nabla(u_n - v_n)|^p \geq \int_{(1/2, 1) \setminus \Lambda_\varepsilon} \int_{\partial B(a_+, r)} |\nabla(u_n - v_n)|^p \geq (1/2 - \varepsilon/\pi) \gamma_p. \quad (4.45)$$

In addition, by (1.30), applied to $u_n|_B, v_n|_B$, we clearly have

$$\int_B |\nabla(u_n - v_n)|^p \geq \left(\frac{2}{\pi}\right)^p \int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p,$$

which together with (4.45) gives

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \geq \left(\frac{2}{\pi}\right)^p \int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p + (1/2 - \varepsilon/\pi)\gamma_p, \quad \forall n \geq n_1(\varepsilon). \quad (4.46)$$

The inequality (4.46) clearly contradicts (4.34) for n large enough if ε is chosen sufficiently small. \square

Remark 4.4. In the course of the proof of Proposition 4.3 we used the following fact:

Let $1 \leq p < 2$ and let Ω be the unit disc. Set $u_0(x) := x/|x|$, $\forall x \in \Omega$. Then

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} |\nabla u_0|^p, \quad \forall u \in \mathcal{E}(u_0). \quad (4.47)$$

We sketch the proof for the convenience of the reader.

Let $u \in \mathcal{E}(u_0)$. Let $C_r := \{z; |z| = r\}$. Since we may write $u = e^{i\varphi} u_0$, with $\varphi \in W^{1,p}$, for a.e. $r \in (0, 1)$ we have $u|_{C_r} \in W^{1,p}(C_r; \mathbb{S}^1)$ and $\deg(u|_{C_r}) = 1$. This implies that for a.e. $r \in (0, 1)$ we have

$$\int_{C_r} |\nabla u| \geq \int_{C_r} |\dot{u}| = \int_{C_r} |u \wedge \dot{u}| \geq \int_{C_r} u \wedge \dot{u} = 2\pi = \int_{C_r} u_0 \wedge \dot{u}_0 = \int_{C_r} |\nabla u_0|. \quad (4.48)$$

In case $p = 1$ integration over $r \in (0, 1)$ of (4.48) yields (4.47). In case $1 < p < 2$ we use (4.48) and Hölder inequality, and then integration over r yields

$$\int_{\Omega} |\nabla u|^p \geq 2\pi \int_0^1 \frac{dr}{r^{p-1}} = \frac{2\pi}{2-p} = \int_{\Omega} |\nabla u_0|^p, \quad \text{and (4.47) follows.}$$

Examining the equality cases for the inequalities in (4.48) (and in Hölder inequality when $1 < p < 2$) we obtain in addition the following conclusion: equality holds in (4.47) if and only if

- (i) for $1 < p < 2$, $u = e^{i\alpha} u_0$ for some constant α ;
- (ii) for $p = 1$, $u(re^{i\theta}) = e^{i\varphi(\theta)}$ where $\varphi \in W^{1,1}([0, 2\pi]; \mathbb{R})$ satisfies $\varphi(2\pi) - \varphi(0) = 2\pi$ and $\varphi' \geq 0$ a.e. on $[0, 2\pi]$.

The difference between (i) and (ii) is the main reason why for the same u_0 and v_0 as in the proof of Proposition 4.3, we have the strict inequality (4.16) for $1 < p < 2$, while for $p = 1$ the equality (1.15) holds.

Remark 4.5. Consider the maps $u_1 := x/|x|$ and $v_1 := 1$ in Ω_ε (as in the proof of Proposition 4.3). By Theorem 1.9 we have $\text{dist}_{W^{1,p}}(\mathcal{E}(u_1), \mathcal{E}(v_1)) = (2/\pi) \inf_{w \sim u_1 \bar{v}_1} \|\nabla w\|_{L^p(\Omega_\varepsilon)}$. Therefore, we have

$$\text{dist}_{W^{1,p}}(\mathcal{E}(u_1), \mathcal{E}(v_1)) \neq \text{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0))$$

although $u_1 \bar{v}_1 = u_0 \bar{v}_0$. This shows that in general it is not even true that $\text{dist}_{W^{1,p}}(\mathcal{E}(u), \mathcal{E}(v))$ depends only on $\mathcal{E}(u\bar{v})$ when $1 < p < 2$. A similar phenomenon occurs when Ω is multiply connected and $p = 2$ (see [14, Remark 4.1]); a comparable argument works for $p > 2$.

Appendix. Proof of Proposition 1.2

Proof of Proposition 1.2. We fix a sequence $\varepsilon_n \searrow 0$ and use (1.4) to find a sequence $\{v_n\} \subset \mathcal{E}(u)$ such that

$$\int_{\Omega} |\nabla v_n| \leq \Sigma(u) + \varepsilon_n, \quad \forall n. \quad (\text{A.1})$$

For $\theta \in [0, 2\pi)$ define $\Psi_{n,\theta} \in \text{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ by

$$\Psi_{n,\theta}(z) := \begin{cases} e^{i\pi(1+2(\varphi-\theta)/\varepsilon_n)}, & \text{if } z = e^{i\varphi} \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \\ 1, & \text{if } z \notin \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \end{cases}. \quad (\text{A.2})$$

Clearly $\deg \Psi_{n,\theta} = 1$, so setting $w_{n,\theta} := \Psi_{n,\theta} \circ v_n$, we have, by Lemma 2.1, $w_{n,\theta} \sim v_n \sim u$. Moreover,

$$|\nabla w_{n,\theta}(x)| = \begin{cases} (2\pi/\varepsilon_n) |\nabla v_n(x)|, & \text{if } v_n(x) \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \\ 0, & \text{if } v_n(x) \notin \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \end{cases}. \quad (\text{A.3})$$

Set

$$\begin{aligned} A_n(x) &:= \{\theta \in [0, 2\pi); v_n(x) \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2)\}, \\ B_n(x) &:= \{\theta \in [0, 2\pi); w_{n,\theta}(x) \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2)\}. \end{aligned}$$

We have

$$\int_0^{2\pi} \int_{\Omega} |\nabla w_{n,\theta}| \, dx \, d\theta = \frac{2\pi}{\varepsilon_n} \int_{\Omega} |\nabla v_n(x)| |A_{n,\varepsilon}(x)| \, dx = 2\pi \int_{\Omega} |\nabla v_n|, \quad (\text{A.4})$$

and

$$\int_0^{2\pi} |\{w_{n,\theta} \neq 1\}| \, d\theta = \int_{\Omega} |B_n(x)| \, dx = \varepsilon_n |\Omega|. \quad (\text{A.5})$$

Combining (A.1) with (A.4)–(A.5) yields

$$\int_0^{2\pi} \left(|\{w_{n,\theta} \neq 1\}|/\varepsilon_n^{1/2} + \int_{\Omega} |\nabla w_{n,\theta}| \right) \, d\theta \leq |\Omega| \varepsilon_n^{1/2} + 2\pi(\Sigma(u) + \varepsilon_n). \quad (\text{A.6})$$

From (A.6) it follows that there exists $\theta_n \in [0, 2\pi)$ such that

$$|\{w_{n,\theta_n} \neq 1\}|/\varepsilon_n^{1/2} + \int_{\Omega} |\nabla w_{n,\theta_n}| \leq |\Omega| \varepsilon_n^{1/2}/(2\pi) + \Sigma(u) + \varepsilon_n;$$

so clearly $u_n := w_{n,\theta_n}$ satisfies (1.17). □

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