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## EULER AND THE STRONG LAW OF SMALL NUMBERS

KARL DILCHER AND CHRISTOPHE VIGNAT

In addition to modern computer algebra systems and bibliographic databases, old-fashioned tables of integrals, sums and products remain important research tools in mathematics. This is especially the case for the present authors whose research is in the classical areas of analysis, combinatorics, and number theory.

What follows is a cautionary tale about the use of tables, along with a reminder, as much to ourselves as to the reader, to go back to the sources if at all possible. But before we continue, an important caution to the reader of this note: *Some of the identities that follow are incorrect, while others may be misleading.* Also, some notations will be inconsistent, to preserve the historical context.

In the process of looking up some infinite products in the excellent and very useful handbook by E. R. Hansen [11], we came across a curious identity for the Euler numbers  $E_n$ . These numbers can be defined by the exponential generating function

$$(1) \quad \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Since the left-hand side of (1) is an even function, we have  $E_{2k+1} = 0$  for all  $k \geq 0$ . Furthermore, it can be shown that the even-index Euler numbers are integers, and  $E_0 = 1$ ,  $E_2 = -1$ ,  $E_4 = 5$ ,  $E_6 = -61$ ,  $E_8 = 1385, \dots$  For further properties, see, e.g., [1] or its successor volume [13]. The identity in question is (89.8.3) on p. 486 of [11], namely

$$(2) \quad \prod_{k=1}^{\infty} \left[ 1 - \frac{(-1)^k}{(2k+1)^{2n+1}} \right] = \frac{1}{2(2n)!} |E_{2n}| \left( \frac{\pi}{2} \right)^{2n+1}.$$

A quick numerical check with *Maple* and *Mathematica* for a few small values of  $n$  revealed that the identity is definitely *incorrect*. Before we realized what was going on, we found the identity, in the same form, in the first volume of the well-known multi-volume tables by Prudnikov et al. [14, p. 754]. No reference is given, and the identity also appears in the Russian original [15], again in the same form.

One of the useful features of Hansen's handbook [11] is the fact that most entries come with one or more references to the literature, or at least to other entries in the table. The reference associated to identity (2), i.e., Hansen's (89.8.3), is given as identity (1131) in the smaller and older collection of formulas by Jolley [12]. The identity in question is listed on pp. 238–239 as

$$(3) \quad 2(2n)! \left( \frac{2}{\pi} \right)^{2n+1} \left( 1 + \frac{1}{3^{2n+1}} \right) \left( 1 - \frac{1}{5^{2n+1}} \right) \left( 1 + \frac{1}{7^{2n+1}} \right) \dots = E_n^* = E_{2n},$$

where  $E_n^*$  and  $E_{2n}$  are notations of the Euler numbers that differ from the almost universally accepted modern usage in (1) and (2), namely  $E_0^* = E_1^* = 1$ ,  $E_2^* = 5$ ,  $E_3^* = 61$ ,  $E_4^* = 1385, \dots$

Jolley also provides references, and for his identity (1131), i.e., (3) above, the reader is referred to p. 365 of the famous old Algebra book by Chrystal [3]; it is actually part II of this two-volume work, a fact that is not mentioned in [12]. The identity in question turns out to be (15) on p. 365 of [3], namely

$$(4) \quad E_m = 2(2m)! \left(\frac{2}{\pi}\right)^{2m+1} \bigg/ \left(1 + \frac{1}{3^{2m+1}}\right) \left(1 - \frac{1}{5^{2m+1}}\right) \left(1 + \frac{1}{7^{2m+1}}\right) \cdots,$$

where yet another notation for the Euler numbers is used, namely  $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, \dots$  (see [3, p. 342]). Note that the large fraction sign in (4) is missing in (3).

In addition to hinting at a proof of this last identity, Chrystal refers the reader to Euler by writing, “See again Euler, *Introd. in Anal. Inf.*, §284” in a footnote on p. 365. Euler’s influential book [5] is available in English translation [6], and §284 can be found on pp. 239–241 of [5], or on pp. 244–245 of [6], where we find

$$(5) \quad A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \&c.$$

and at the end of §284, Euler writes,

$$(6) \quad A = \frac{3^n}{3^n + 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n + 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \frac{17^n}{17^n - 1} \cdot \&c,$$

where the powers of all the prime numbers in the numerators occur and the denominators are increased or decreased by 1 depending on whether the number has the form  $4m - 1$  or  $4m + 1$ . [6, p. 245].

The identity (6) already gives us a glimpse of what might have gone wrong on the way towards the infinite product (2). But first we need to establish a connection between Euler’s expressions (5) and (6) and what were later called the Euler numbers.

In another well-known book [7], published a few years after [5], Euler expressed the numbers  $A$  in (5) in terms of the Taylor coefficients of the secant function, which by (1) are closely related to the Euler numbers. For instance, one of the explicit identities that can be found in §224 in [7, p. 542], or in German translation in [8, p. 259], is

$$(7) \quad 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \&c. = \frac{\varepsilon}{1 \cdot 2 \dots 8} \cdot \frac{\pi^9}{2^{10}},$$

where  $\varepsilon = 1385$ . Since this note deals with incorrect identities, we must mention in passing that there are typographical errors in the original identity (and neighboring ones) on p. 542 of [7]; however, only two pages further they are correctly printed; it is also correct in [8].

The rest of the story is now easy to piece together. The numbers  $\alpha, \beta, \dots, \varepsilon, \dots$  used by Euler in [7, p. 542] correspond to  $1, E_1, \dots, E_4, \dots$  in Chrystal’s notation, and (4) is indeed the general form of Euler’s identity — if the sequence  $3, 5, 7, \dots$  is interpreted as the beginning of the sequence of odd primes, rather than the sequence of odd integers greater than 1. Therefore, the origin of the incorrect formula (2) is quite likely Chrystal’s identity (4). Showing just one more term, along the lines of Euler’s identity (6), would have avoided all this. However, whether or not Jolley misinterpreted the sequence  $3, 5, 7, \dots$ , the identity (3) still contains the mistake of the missing fraction sign which then made it into Hansen’s identity (2).

In the end, we shouldn't blame Chrystal too much. Given that his book is written in great detail, even a moderately attentive reader would realize that the product in (4) had to be over the odd primes. This is all the more so as Chrystal remarks that a previous identity is transformed into his (15), i.e., identity (4) above, “*in the same way as before.*” He apparently refers to identity (8) in [3, p. 364], namely

$$(8) \quad B_m = 2(2m)! / (2\pi)^{2m} (1 - 1/2^{2m}) (1 - 1/3^{2m}) (1 - 1/5^{2m}) \dots$$

(once again in Chrystal's notation), where  $B_m$  is the  $m$ th Bernoulli number in the historical notation that has  $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, B_4 = 1/30, B_5 = 5/66, \dots$ ; for different notations see, e.g., [13, Ch. 24]. This last identity (8) is closely related to the Euler product for the Riemann zeta function, especially if we compare it with the following famous formula named after Euler:

$$(9) \quad B_m = \frac{2(2m)!}{(2\pi)^{2m}} \left\{ \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \dots \right\},$$

again reproduced as in [3, p. 363]. While there is much less danger of the product in (8) to be misunderstood, Euler himself showed more terms in the analogs of (8) and (9); see §283 in [5] or [6].

We already mentioned that the product in (8) is, essentially, the Euler product for the Riemann zeta function. Similarly, the product in (4) is the Euler product of an appropriate  $L$ -series (in fact, the series (5)), and both are special cases of Euler products of Dirichlet  $L$ -series; see, e.g., [4, p. 162ff.].

Before we close, let us reiterate that some of the identities in this note are incorrect or misleading. Indeed, the reader will have realized that (2) and (3) are *false*, and (4) is correct only when interpreted as a product over the odd primes. Along with the incorrect entry (89.8.3) in [11], i.e., (2) above, two consequences are mentioned, namely (89.4.11) for  $n = 0$ , and (89.6.12) for  $n = 1$ . The first one of these is correct since it is in a somewhat different form, taken from an identity in the well-known classical book by Bromwich [2, p. 224, Ex. 9]. The special case of (89.4.11), namely  $w = -1$ , that is relevant here is

$$(10) \quad \prod_{k=1}^{\infty} \left[ 1 - \frac{(-1)^k}{2k+1} \right] = \frac{\pi\sqrt{2}}{4}.$$

However, the second identity, (89.6.12), is indeed false, the corrected version being

$$(11) \quad \prod_{k=1}^{\infty} \left[ 1 - \frac{(-1)^k}{(2k+1)^3} \right] = \frac{\pi}{12} + \frac{\pi\sqrt{2}}{12} \cosh \left( \frac{\pi\sqrt{2}}{4} \right).$$

In an attempt to prove (11), we tried to use Hansen's formula (89.6.2) which, interestingly, also turned out to be incorrect. In this case, however, there was only a small typographical error (it should read  $t = e^{\pi i/3}$ ). We first obtained the correct version (11) symbolically with the computer algebra system *Mathematica*. We were then able to prove its extension to arbitrary (even or odd) powers of the denominators  $2k+1$ , using properties of the Gamma function. But this is another story.

The mistake that has been the main subject of this note is a good example of the Strong Law of Small Numbers which Richard K. Guy [10] formulated as follows: “*There aren't enough small numbers to meet the many demands made of them.*” One of the corollaries stated in [10] captures the situation even better: “*Superficial similarities spawn spurious statements.*”

However, we would like to give the last word to Pierre-Simon Laplace and his famous dictum: “*Lisez Euler, lisez Euler, c’est notre maître à tous.*” (“*Read Euler, read Euler, he is the master of us all.*”) [16]. And to do so, the best place to begin is the wonderful Euler Archive [9].

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