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# NORMALIZATION IN BANACH SCALE LIE ALGEBRAS VIA MOULD CALCULUS AND APPLICATIONS

THIERRY PAUL AND DAVID SAUZIN

ABSTRACT. We establish perturbative solutions for problems involving resonances of the unperturbed situation, and therefore the necessity of a normal form, in the general framework of Banach scale Lie algebras defined below. This situation covers the case of classical and quantum normal forms in a unified way which allows a direct comparison. In particular we prove a precise estimate for the difference between quantum and classical normal forms, proven to be of order of the square of the Planck constant. Our method uses heavily mould calculus and in particular a universal mould equation studied in [PS16]

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## 1. Introduction

Perturbation theory is a fascinating subject which appears to have been fundamental for the birth of dynamical systems through Poincaré and quantum mechanics in the Göttingen school. It is also of fundamental importance for the large computation in physics and chemistry, leading to a panel of different algorithms for computing perturbation series. Each such a method (e.g. generating functions for dynamical systems, functional analysis (expansion of the Neumann series) in quantum mechanics) is very well adapted to emblematic situations (small divisors and KAM theory in classical dynamics, Kato method and existence of dynamics in quantum mechanics), but each methodology seems to be strictly tied to the different underlying paradigms.

In the present article we will present in a unified way new results concerning the use of mould theory for (classical) Birkhoff normal forms (namely in presence of Hamiltonian resonances) and for quantum perturbation theory, this last topics having never met, to our knowledge, mould calculus.

As a by-product we also obtain a precise estimate of the difference between quantum normal forms and the classical ones corresponding to the underlying classical situation, Theorem D. Note that this estimate is of order of the square of the Planck constant and involve only the size of the perturbation.

Mould calculus has been introduced and was developed by Jean Écalle ([Eca81], [Eca93]) in the 80-90's in order to give powerful tools for handling problems in local dynamics, typically the normalization of vector fields or diffeomorphisms at a fixed point.

Beside the two topics already mentioned (classical and quantum normal forms), the large difference of paradigm between them has led us to formulate mould calculus in a kind of abstract operational setting able to include both classical and quantum dynamics, and probably many other situations.

This formulation leads to mould resolutions of general perturbation problems, that is problems where a perturbation is added to a bare problem already explicitly solved.

To put it in a nutshell, one of the key ideas of mould calculus can be phrased by saying that mould expansions are done on non-universal – namely related to the perturbation involved in the problem to be solved – objects (comould), with universal – namely dependent only on the unperturbed, solved problem – coefficients (mould). This is quite unfamiliar for people using standard perturbative tools (e.g. Taylor expansions) where universality is more placed on “active” objects. This might explain the poor penetration of the beautiful theory of moulds in other fields than local dynamics.

Thus, in the present article, we want to consider a general formalism that would include the following cases:

- the construction of the Birkhoff form for perturbations of integrable Hamiltonian systems  $h(I, \varphi) = h_0(I) + V(I, \varphi)$ ,  $I \in \mathbb{R}^d$ ,  $\varphi \in \mathbb{T}^d$ ,
- the unitary conjugation to a quantum Birkhoff form  $\mathcal{B} = \mathcal{B}(H_1, \dots, H_d)$  for perturbations of quantum “bare” operators  $H = H_1 + \dots + H_d$ ,  $[H_k, H_\ell] = 0$  (e.g.  $H_k = -\frac{1}{2}\hbar^2 \partial_{x_k}^2 + \frac{1}{2}\omega_k^2 x_k^2$  on  $L^2(\mathbb{R}, dx_k)$  or  $H_k = -i\hbar\omega_k \partial_{x_k}$  on  $L^2(\mathbb{T}, dx_k)$ ).

Let us notice that the following two situations have been already considered via mould theory in our companion article [PS16]:

- the formal linearization, or at least the formal normalization, of a vector field  $X = \sum_{i=1}^N \omega_i z_i \partial_{z_i} + B$  (where  $B$  represents higher order terms) in  $\mathbb{C}[[z_1, \dots, z_N]]$ ,
- the formal symplectic conjugation to a normal form of Hamiltonians  $h(z, \bar{z}) = \sum_{i=1}^d \frac{1}{2}\omega_i(x_i^2 + y_i^2) + V(x, y)$  near the origin.

Though these four situations are quite different and belong to different paradigms, we would like to emphasize that mould theory can provide a general formulation handling all of them.

Let us present this general framework. It consists of

- a Lie algebra  $\mathcal{L}$ , filtered or a Banach scale Lie algebra defined in Section 2 (the case of a filtered algebra has been treated in [PS16]),
- assumptions on  $\mathcal{L}$  insuring the existence of an exponential map defined on  $\mathcal{L}$ ,
- an element  $B$  of  $\mathcal{L}$ ,

- elements  $Y, Z$  of  $\mathcal{L}$  to be determined so that

$$[X_0, Z] = 0 \text{ and } e^{\text{ady}}(X_0 + B) = X_0 + Z, \quad (1.1)$$

for an “unperturbed”  $X_0 \in \mathcal{L}$ .

Let us present now briefly mould calculus.

Mould theory relies drastically on the notion of *homogeneity*, more precisely on the decomposition of the perturbation into homogeneous pieces. In the general setting we suppose that the starting point is an element  $X$  of a Lie algebra of the form

$$X = X_0 + B$$

where  $B$  is a “perturbation” of  $X_0$ , for which everything is supposed fully known.

The problem to solve consists in finding a Lie algebra automorphism  $\Theta$  such that, at any approximation of size any power of  $\|B\|$  for a certain norm  $\|\cdot\|$ ,

$$\Theta X = \Theta(X_0 + B) = X_0 + Z \quad (1.2)$$

where  $Z$  is a normal form, namely a 0-homogeneous element in a sense we will explain now.

We define an alphabet  $\Lambda \subset \mathbb{C}$  of letters  $\lambda$  through the decomposition

$$B = \sum_{\lambda \in \Lambda} B_\lambda$$

where  $B_\lambda$  satisfies

$$[X_0, B_\lambda] = \lambda B_\lambda. \quad (1.3)$$

An operator satisfying (1.3) is called  $\lambda$ -homogeneous and 0-homogeneous operators are called *resonant*.

To the alphabet  $\Lambda$  we can associate the set of *words*

$$\underline{\Lambda} := \{ \underline{\lambda} = \lambda_1 \lambda_2 \cdots \lambda_r \mid r \in \mathbb{N}, \lambda_i \in \Lambda \}. \quad (1.4)$$

If  $\underline{\lambda} = \lambda_1 \dots \lambda_r$ , then we use the notation  $r(\underline{\lambda}) := r$ , with the convention  $r = 0$  for the empty word  $\underline{\lambda} = \emptyset$ .

We can now define the *Lie comould* as the mapping

$$B_{[\bullet]}: \underline{\lambda} \in \underline{\Lambda} \mapsto B_{[\underline{\lambda}]} := [B_{\lambda_r}, [B_{\lambda_{r-1}}, \dots [B_{\lambda_2}, B_{\lambda_1}] \dots]] \in \mathcal{L} \quad (1.5)$$

with the convention  $B_{[\emptyset]} = 0$  and we call *mould* any mapping

$$M^\bullet: \underline{\lambda} \in \underline{\Lambda} \mapsto M^{\underline{\lambda}} = M^{\lambda_1 \cdots \lambda_r} \in \mathbb{C} \quad (1.6)$$

(in this article we use only complex-valued moulds, but [PS16] considers more generally  $\mathbf{k}$ -valued moulds, where  $\mathbf{k}$  is the field of scalars of  $\mathcal{L}$ , an arbitrary field of characteristic zero). To a mould  $M^\bullet$ , we associate an element of  $\mathcal{L}$  defined by

$$M^\bullet B_{[\bullet]} := \sum_{\underline{\lambda} \in \underline{\Lambda}} \frac{1}{r(\underline{\lambda})} M^\lambda B_{[\underline{\lambda}]}. \quad (1.7)$$

Returning to our problem of solving equation (1.2), the key idea will be to process a “mould ansatz”, that is looking to a solution of (1.2) of the form

$$Z = F^\bullet B_{[\bullet]}, \quad \Theta = \sum_{\underline{\lambda} \in \underline{\Lambda}} S^\lambda \text{ad}_{B_{\lambda_r}} \cdots \text{ad}_{B_{\lambda_1}}, \quad (1.8)$$

with two moulds  $F^\bullet$  and  $S^\bullet$  to be determined.

We will show that (1.2) is satisfied through (1.8) as soon as  $S^\bullet$  and  $F^\bullet$  are solution of the universal mould equation

$$\nabla S^\bullet = I^\bullet \times S^\bullet - S^\bullet \times F^\bullet, \quad (1.9)$$

universal because in (1.9) the perturbation  $B$  does not show up.

In (1.9) one has

$$\left\{ \begin{array}{l} \nabla M^\bullet: \underline{\lambda} \mapsto \Sigma(\underline{\lambda}) M^\lambda \text{ where } \Sigma(\underline{\lambda}) := \sum_{i=1}^r \lambda_i \\ M^\bullet \times N^\bullet: \underline{\lambda} \mapsto \sum_{\substack{\underline{a}=\underline{b} \\ r(\underline{a}), r(\underline{b}) < r(\underline{\lambda})}} M^{\underline{a}} N^{\underline{b}} \\ I^{\lambda_1 \dots \lambda_r} = \delta_{1r}, \text{ therefore } I^\bullet \times M^\bullet: \underline{\lambda} \mapsto M^{\prime \underline{\lambda}}, \text{ with } \prime \lambda_1 \dots \lambda_r := \lambda_2 \dots \lambda_r. \end{array} \right. \quad (1.10)$$

Constructing solutions of (1.9) process in a way familiar to any perturbative setting: first we note that, precisely because  $B$  is perturbation of  $X_0$ ,  $\Theta$  must be close to the identity and  $Z$  to zero. This entails that  $S^\emptyset = 1$  and  $F^\emptyset = 0$  from which it follows that  $S^\bullet \times F^\bullet = F^\bullet + S^\bullet \times' F^\bullet$  where  $M^\bullet \times' N^\bullet: \underline{\lambda} \mapsto \sum_{\substack{\underline{a}=\underline{b} \\ r(\underline{a}), r(\underline{b}) < r(\underline{\lambda})}} M^{\underline{a}} N^{\underline{b}}$ . Moreover willing  $Z$  to be 0-homogeneous is fulfilled by imposing  $F^\bullet$  to be *resonant*, i.e. that  $\nabla F^\bullet = 0$ . Putting all these properties together leads to the fact that  $F^\bullet$  and the non-resonant part of  $S^\bullet$  can be determined by induction on the length of letters. What is not determined because it disappears from the equation is the resonant part of  $S^\bullet$ , since it is “killed” by  $\nabla$ .

We showed in [PS16] that this ambiguity is removed – leading to uniqueness of the solution – by fixing a *gauge generator*, namely an arbitrary mould  $A^\bullet$ , resonant and alternal. More precisely, for any gauge  $A^\bullet$ , (1.9) has a unique solution  $(S^\bullet, F^\bullet)$ . Moreover it happens that  $S^\bullet = e^{G^\bullet}$  (where

$e$  has to be understood as the exponential in the algebra of moulds, that is  $e^{G^\bullet} = \sum_{k=0}^{\infty} \frac{G^{\bullet \times k}}{k!}$  where  $\times$  is defined in (1.10)) and  $G^\bullet$  and  $F^\bullet$  are *alternal*, a notion we define now.

The notion of alternality has to do with the notion of shuffling of two words  $\underline{a}$  and  $\underline{b}$  which is the set of words  $\underline{\lambda}$  obtained by interdigitating the letters of  $\underline{a}$  and those of  $\underline{b}$  while preserving their internal order in  $\underline{a}$  or  $\underline{b}$ . The number of different ways a word  $\underline{\lambda}$  can be obtained out of  $\underline{a}$  and  $\underline{b}$  is denoted by  $\text{sh}\left(\frac{\underline{a}, \underline{b}}{\underline{\lambda}}\right)$ . Saying that  $F^\bullet$  is *alternal* is nothing but saying that for all non-empty words  $\underline{a}, \underline{b}$ ,  $\sum_{\underline{\lambda} \in \underline{\Lambda}} \text{sh}\left(\frac{\underline{a}, \underline{b}}{\underline{\lambda}}\right) F^\lambda = 0$ .

To be more precise, in [PS16] was proven the following ‘‘existence-uniqueness’’ result for the mould equation. Let us define an operator  $\nabla_1: M^\bullet \mapsto \nabla_1 M^\bullet$  by the formula

$$\nabla_1 M^\bullet: \underline{\lambda} \in \underline{\Lambda} \mapsto r(\underline{\lambda}) M^\lambda \quad (1.11)$$

for an arbitrary mould  $M^\bullet$  (recall that  $r(\underline{\lambda})$  denotes the length of  $\underline{\lambda}$ ), and denote by  $M_0^\bullet$  the resonant part of the mould, defined by  $M_0^\lambda := \mathbb{1}_{\{\Sigma(\underline{\lambda})=0\}} M^\lambda$  for all  $\underline{\lambda} \in \underline{\Lambda}$  (where  $\Sigma(\underline{\lambda})$  is by defined in (1.10)).

**Proposition 1.1.** *Let  $\mathbf{k}$  be a field of characteristic zero and  $\Lambda$  a subset of  $\mathbf{k}$ . For any resonant alternal mould  $A^\bullet$ , there exists a unique pair  $(F^\bullet, G^\bullet)$  of alternal moulds such that*

$$\nabla F^\bullet = 0, \quad \nabla(e^{G^\bullet}) = I^\bullet \times e^{G^\bullet} - e^{G^\bullet} \times F^\bullet, \quad (1.12)$$

$$\left[ e^{-G^\bullet} \times \nabla_1 e^{G^\bullet} \right]_0 = A^\bullet. \quad (1.13)$$

The proof of Proposition 1.1 is constructive in the sense that we will obtain the following simple algorithm to compute the values of  $F^\bullet$  and  $S^\bullet := e^{G^\bullet}$  on any word  $\underline{\lambda}$  by induction on its length  $r(\underline{\lambda})$ : introducing an auxiliary alternal mould  $N^\bullet$ , one must take  $S^\emptyset = 1$ ,  $F^\emptyset = N^\emptyset = 0$  and, for  $r(\underline{\lambda}) \geq 1$ ,

$$\Sigma(\underline{\lambda}) \neq 0 \quad \Rightarrow \quad F^\lambda = 0, \quad S^\lambda = \frac{1}{\Sigma(\underline{\lambda})} \left( S^{\acute{\lambda}} - \sum_{\underline{a}=\underline{a} \underline{b}}^* S^{\underline{a}} F^{\underline{b}} \right), \quad N^\lambda = r(\underline{\lambda}) S^\lambda - \sum_{\underline{a}=\underline{a} \underline{b}}^* S^{\underline{a}} N^{\underline{b}}, \quad (1.14)$$

$$\Sigma(\underline{\lambda}) = 0 \quad \Rightarrow \quad F^\lambda = S^{\acute{\lambda}} - \sum_{\underline{a}=\underline{a} \underline{b}}^* S^{\underline{a}} F^{\underline{b}}, \quad S^\lambda = \frac{1}{r(\underline{\lambda})} \left( A^\lambda + \sum_{\underline{a}=\underline{a} \underline{b}}^* S^{\underline{a}} N^{\underline{b}} \right), \quad N^\lambda = A^\lambda, \quad (1.15)$$

where we have used the notation  $\acute{\lambda} := \lambda_2 \cdots \lambda_r$  for  $\underline{\lambda} = \lambda_1 \lambda_2 \cdots \lambda_r$  and the symbol  $\sum^*$  indicates summation over non-trivial decompositions (i.e.  $\underline{a}, \underline{b} \neq \emptyset$  in the above sums); the mould  $F^\bullet$  thus

inductively defined is alternal and

$$G^\emptyset = 0, \quad G^\lambda = \sum_{k=1}^{r(\lambda)} \frac{(-1)^{k-1}}{k} \sum_{\lambda=\underline{a^1} \dots \underline{a^k}}^* S^{a^1} \dots S^{a^k} \quad \text{for } \lambda \neq \emptyset \quad (1.16)$$

then defines the alternal mould  $G^\bullet$  which solves (1.12)–(1.13).

Finally we prove in [PS16], Propositions 3.8 and 3.9, the following result, crucial for the link between the mould equation and the original problem (1.2).

**Proposition 1.2.** *If  $M^\bullet$  and  $N^\bullet$  are two alternal moulds, then*

$$[M^\bullet, N^\bullet]B_{[\bullet]} = [N^\bullet B_{[\bullet]}, M^\bullet B_{[\bullet]}],$$

where  $[M^\bullet, N^\bullet] := M^\bullet \times N^\bullet - N^\bullet \times M^\bullet$ , and

$$e^{\text{ad}_{M^\bullet B_{[\bullet]}}} (N^\bullet B_{[\bullet]}) = \left( e^{-M^\bullet} \times N^\bullet \times e^{M^\bullet} \right) B_{[\bullet]}.$$

Moreover,

$$[X_0, M^\bullet B_{[\bullet]}] = (\nabla M^\bullet) B_{[\bullet]}, \quad e^{\text{ad}_{M^\bullet B_{[\bullet]}}} X_0 = X_0 - \left( e^{-M^\bullet} \times \nabla(e^{M^\bullet}) \right) B_{[\bullet]}.$$

This result shows that (1.2) is solved by  $Z = F^\bullet B_{[\bullet]}$  and  $\Theta = e^{\text{ad}_Y}$  with  $Y = G^\bullet B_{[\bullet]}$ , where  $F^\bullet$  and  $G^\bullet$  solve (1.12) (see [PS16] for the details).

The goal of the present article is twofold: first we want to show how we can solve perturbatively the normal form problem (1.1) in the general setting of an  $X_0$ -extended Banach scale Lie algebra – Theorem A – and second we want to show applications to the aforementioned dynamical problems – Theorems B and C. As a by-product we give also a quantitative estimate concerning the difference between classical and quantum normal forms – Theorem D.

The different situations in dynamics which can be realized as an  $X_0$ -extended Banach scale Lie algebra are displayed in the next table.



<b>Banach scale Lie algebra</b>	<i>Banach spaces (included in)</i>	<i>Element to be normalized</i>	<i>Normalizing transformation</i>
near-integrable Hamiltonians	$C_\rho^\omega(\mathbb{T}^n \times \mathbb{R}_+^n) =$ bounded functions analytic in the strip $ \Im z  < \rho$ $[\cdot, \cdot] = \text{Poisson bracket}$	Hamiltonian $H = H_0 + V$ $H_0 = \sum \omega_i I_i$ $V = \sum V_\lambda,$ $\lambda = ik \cdot \omega, k \in \mathbb{Z}^n$	$e^{\text{ad}_X^{\text{Ham}}} H = H \circ \Phi$ $\Phi = \text{formal}$ symplectomorphism, flow of the v.f. $\text{ad}_X^{\text{Ham}} = \{\chi, \cdot\}$
quantum perturbation theory	{pseudodifferential operators of Weyl symbols in $C_\rho^\omega \mathbb{T}^n \times \mathbb{R}_+^n\}$ $[\cdot, \cdot]_Q = \frac{\text{commutator}}{i\hbar}$	Hamiltonian $H = H_0 + V$ $H_0 = \sum E_n  \varphi_n\rangle\langle\varphi_n $ $V = \sum V_\lambda,$ $\lambda = E_m - E_n$	$e^{\text{ad}_X^Q} H = U H U^{-1}$ $U = e^{i\chi}$ unitary operator
quantum perturbation theory $\hbar \rightarrow 0$	{pseudodifferential operators of Weyl symbols in $C_\rho^\omega \mathbb{T}^n \times \mathbb{R}_+^n\}$ $[\cdot, \cdot]_Q = \frac{\text{commutator}}{i\hbar},$ $\hbar \rightarrow 0$	Hamiltonian $H = H_0 + V$ $H_0 =$ $\sum_i (-\hbar^2 \partial_{x_i}^2 + \omega_i^2 x_i^2),$ $V = \sum V_\lambda,$ $\lambda = k \cdot \omega, k \in \mathbb{Z}^n$	$e^{\text{ad}_X^Q} H = U H U^{-1}$ $U = e^{i\chi}$ unitary operator = quantization of $\Phi$

The paper is organized as follows. The first part is devoted to the result valid in any  $X_0$ -extended Banach scale Lie algebra whose definition is given in Section 2 and in Section 3 we state the general result of the article, proven in Section 4. The second part is devoted to applying the main result to classical dynamical situations, Section 5, the quantum ones, Section 6, and semiclassical approximation, Section 7. Appendix A gives the minimal setting in semiclassical analysis necessary to the present paper.

Let us finally mention that the present article is self-contained (it uses only Theorem B of [PS16], rephrased in Proposition 1.1 of the present article) and all the constants are explicit.

## NORMALIZATION IN $X_0$ -EXTENDED BANACH SCALE LIE ALGEBRAS

### 2. $X_0$ -extended Banach scale Lie algebras

Let  $(\mathcal{L}, [\cdot, \cdot])$  be a Lie algebra over  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ . We say that we have an “ $X_0$ -extended Banach scale Lie algebra ” if:

(1)  $\mathcal{L}$  contains a family  $(\mathcal{B}_\rho, \|\cdot\|_\rho)_{\rho \in \mathbb{R}_+^*}$  of Banach spaces over  $\mathbf{k}$  such that

$$0 < \rho' < \rho \quad \Rightarrow \quad \mathcal{B}_\rho \subset \mathcal{B}_{\rho'}, \quad \text{with } \|X\|_{\rho'} \leq \|X\|_\rho \text{ for all } X \in \mathcal{B}_\rho,$$

(2) there exists a constant  $\gamma > 0$  such that

$$0 < \rho' < \rho'' \leq \rho, \quad X \in \mathcal{B}_\rho, \quad Y \in \mathcal{B}_{\rho''} \quad \Rightarrow \quad \|[X, Y]\|_{\rho'} \leq \frac{\gamma}{e^2(\rho - \rho')(\rho'' - \rho')} \|X\|_\rho \|Y\|_{\rho''},$$

(3)  $\mathcal{L}$  contains an element  $X_0$  (which does not necessary belong to any of the  $\mathcal{B}_\rho$ 's) and there exists a function  $\chi : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  such that

$$0 < \rho' < \rho \quad \Rightarrow \quad \|[X_0, Y]\|_{\rho'} \leq \frac{1}{\chi(\rho - \rho')} \|Y\|_\rho \text{ for all } Y \in \mathcal{B}_\rho.$$

Let us denote the adjoint representation of  $\mathcal{L}$  by  $\text{ad}$ , i.e. for each  $Y \in \mathcal{L}$ ,  $\text{ad}_Y$  is the Lie algebra derivation defined by  $\text{ad}_Y X = [Y, X]$  for all  $X \in \mathcal{L}$ . One can check (see Corollary 4.5) that, for an  $X_0$ -extended Banach scale Lie algebra as above, if  $Y \in \mathcal{B}_\rho$  satisfies  $\|Y\|_\rho < \rho^2/\gamma$ , then  $e^{\text{ad}_Y} := \sum_{k \geq 0} \frac{1}{k!} (\text{ad}_Y)^k$  is a well-defined linear map

$$e^{\text{ad}_Y} : \mathcal{B}_\rho \rightarrow \mathcal{B}_{\rho'} \quad \text{for each } \rho' \text{ such that } 0 < \rho' < \rho - \sqrt{\gamma \|Y\|_\rho}$$

and

$$e^{\text{ad}_Y} [X_1, X_2] = [e^{\text{ad}_Y} X_1, e^{\text{ad}_Y} X_2] \quad \text{for all } X_1, X_2 \in \mathcal{B}_\rho.$$

Moreover,  $e^{\text{ad}_Y} X_0 := \sum_{k \geq 0} \frac{1}{k!} (\text{ad}_Y)^k X_0$  too is well-defined and  $e^{\text{ad}_Y} X_0 - X_0 \in \mathcal{B}_{\rho'}$  for each  $\rho'$  as above.

### 3. The general result

**Notation 3.1.** Let  $\Lambda$  be a nonempty subset of  $\mathbf{k}$ . For a word  $\underline{\lambda} = \lambda_1 \cdots \lambda_r \in \underline{\Lambda}$  of length  $r \geq 1$  and a subset  $\sigma$  of  $\{1, \dots, r\}$ , we set

$$\underline{\lambda}_\sigma := \sum_{\ell \in \sigma} \lambda_\ell. \tag{3.1}$$

For  $\tau \in \mathbb{R}^*$ , we define a function  $\beta_\tau: \underline{\Lambda} \rightarrow \mathbb{R}_+$  by the formula

$$\beta_\tau(\underline{\lambda}) := \sum_{\substack{\sigma \subset \{1, \dots, r(\underline{\lambda})\} \\ \text{such that } \underline{\lambda}_\sigma \neq 0}} \frac{1}{|\underline{\lambda}_\sigma|^{1/\tau}}. \quad (3.2)$$

**Theorem A.** *Let  $\mathcal{L}$  be an  $X_0$ -extended Banach scale Lie algebra and let  $\rho > 0$  and  $B \in \mathcal{B}_\rho$ . Suppose that there exist a subset  $\Lambda$  of  $\mathbf{k}$  and a decomposition*

$$B = \sum_{\lambda \in \Lambda} B_\lambda \quad \text{with } B_\lambda \in \mathcal{B}_\rho \text{ such that} \quad (3.3)$$

$$(i) [X_0, B_\lambda] = \lambda B_\lambda$$

(ii) for all  $r \in \mathbb{N}^*$ , there exist  $\eta_r > 0$  and  $\tau_r \geq 1$ , such that

$$\sum_{\underline{\lambda} = \lambda_1, \dots, \lambda_r \in \Lambda} \|B_{\lambda_1}\|_\rho \cdots \|B_{\lambda_r}\|_\rho e^{\eta_r \beta_{\tau_r}(\underline{\lambda})} := \epsilon_r < \infty. \quad (3.4)$$

Then, for all  $N \in \mathbb{N}^*$  and  $0 < \rho' < \rho$ , there exists  $\epsilon^* = \epsilon^*(N, \rho')$  and  $D = D(N, \rho')$ , expressed by (4.19)-(4.20) below, such that, if  $F^{\lambda_1, \dots, \lambda_r}, G^{\lambda_1, \dots, \lambda_r}$ ,  $\lambda_1, \dots, \lambda_r \in \Lambda$ , are the coefficients satisfying (1.12)-(1.13) with  $A^\bullet = 0$  and given recursively by (1.14)-(1.16),

(a) the two following expansions converge in  $\mathcal{B}_{\rho'}$ ,

$$\begin{aligned} \sum_{r=1}^N \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} F^{\lambda_1, \dots, \lambda_r} [B_{\lambda_r}, [\dots [B_{\lambda_2}, B_{\lambda_1}] \dots]] &:= R_N \in \mathcal{B}_{\rho'}, \\ \sum_{r=1}^N \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} G^{\lambda_1, \dots, \lambda_r} [B_{\lambda_r}, [\dots [B_{\lambda_2}, B_{\lambda_1}] \dots]] &:= W_N \in \mathcal{B}_{\rho'} \end{aligned}$$

(b) for  $\epsilon_1 + \dots + \epsilon_N < \epsilon^*$ ,

$$\begin{cases} e^{\text{ad}_{W_N}}(X_0 + B) = X_0 + R_N + Z_N, \\ [X_0, R_N] = 0, \\ \|Z_N\|_{\rho'} \leq D((\epsilon_1 + \dots + \epsilon_N)^{N+1} + \epsilon_{N+1} + \dots + \epsilon_{N^2}) \end{cases} \quad (3.5)$$

(see (4.18) for a more precise result).

**Remark 3.2.** If in Theorem A we take  $B = B(\epsilon)$  depending on a perturbation parameter  $\epsilon$  so that  $\|B_\lambda(\epsilon)\|_\rho \leq C_\lambda |\epsilon|$  for each  $\lambda \in \Lambda$ , with non-negative constants  $C_\lambda$ , then condition (3.4) factorises and  $\epsilon_r = O(\epsilon^r)$ . In this case, the final estimates reduces to  $\|\mathcal{E}_N\|_{\rho'} = O(\epsilon^{N+1})$ .

Moreover  $\epsilon_1 + \dots + \epsilon_N$  and  $\epsilon_{N+1} + \dots + \epsilon_{N^2}$  can be replaced obviously by  $\frac{N(N+1)}{2} \sup_{r=1 \dots N} \epsilon_r$  and  $\frac{N(N^3-1)}{2} \sup_{r=N+1 \dots N^2} \epsilon_r$  respectively in (3.4).

**Remark 3.3.** Below, in Sections 5 and 6, we will take  $\Lambda$  of the form  $\Lambda = \{i k \cdot \omega \mid k \in \mathbb{Z}^d\}$  for a given  $\omega \in \mathbb{R}^d$ . We shall see that, if there exist  $\alpha > 0$  and  $\tau \geq 1$  such that the Diophantine condition

$$\forall k \in \mathbb{Z}^d, \quad k \cdot \omega = 0 \text{ or } |k \cdot \omega| \geq \alpha |k|^{-\tau} \quad (3.6)$$

holds, then one can find  $X_0$ -extended Banach scale Lie algebras such that any  $B \in \mathcal{B}_\rho$  has a decomposition satisfying (3.4) provided  $\tau_r = \tau$  and  $\eta_r \leq \frac{\rho \alpha^{1/\tau}}{2^{r-1}}$ . Moreover,  $\epsilon_r = O(\|B\|_\rho^r)$  and  $\|\mathcal{E}_N\|_{\rho'} = O(\|B\|_\rho^{N+1})$  in (3.5) in this case.

**Remark 3.4.** There are alphabets for which there exists  $C > 0$  such that, for each  $\underline{\lambda} \in \underline{\Lambda}$  and  $\sigma \subset \{1, \dots, r(\underline{\lambda})\}$ , either  $\underline{\lambda}_\sigma = 0$  or  $|\underline{\lambda}_\sigma| \geq C$ . Then, condition (3.4) reduces to  $\|B_\lambda\|_\rho < \infty$  and entails  $\epsilon_r = O(\|B\|_\rho^r)$  and  $\|\mathcal{E}_N\|_{\rho'} = O(\|B\|_\rho^r)$ . This is the case for example in Remark 3.3 in dimension one, or when  $\omega$  is totally resonant.

## 4. Proof of Theorem A

**4.1. More about the mould equation.** We start by proving the following results concerning the solution of the mould equation (1.9) as expressed in Proposition 1.1.

**Lemma 4.1.** *Let us fix  $A^\bullet = 0$  in Theorem 1.1. Then, for the solution of the mould equation,  $F^{\lambda_1 \dots \lambda_r}$  (resp.  $G^{\lambda_1 \dots \lambda_r}$ ) is a linear combination of inverses of homogeneous monomials of order  $r-1$  (resp.  $r$ ) in the variables  $\{\lambda_i, \lambda_{i_1} + \lambda_{i_2}, \dots, \lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_r} \mid \lambda_i, \dots, \lambda_{i_r} \in \Lambda\}$ .*

*More precisely, setting  $\Lambda_r := \{\lambda_1 + \dots + \lambda_k \mid 1 \leq k \leq r, \lambda_1, \dots, \lambda_k \in \Lambda\}$  and making use of the notation (3.1),*

$$\underline{\lambda} = \lambda_1 \dots \lambda_r \quad \Rightarrow \quad F^\lambda = \sum_{\substack{\{\sigma_j\}_{j=1 \dots r-1} \\ \sigma_j \subset \{1, \dots, r\}}} \frac{C_r(\underline{\lambda}_{\sigma_1}, \dots, \underline{\lambda}_{\sigma_{r-1}})}{\prod_{j=1}^{r-1} \underline{\lambda}_{\sigma_j}}, \quad (4.1)$$

$$\underline{\lambda} = \lambda_1 \dots \lambda_r \quad \Rightarrow \quad G^\lambda = \sum_{\substack{\{\sigma_j\}_{j=1 \dots r} \\ \sigma_j \subset \{1, \dots, r\}}} \frac{D_r(\underline{\lambda}_{\sigma_1}, \dots, \underline{\lambda}_{\sigma_r})}{\prod_{j=1}^r \underline{\lambda}_{\sigma_j}}, \quad (4.2)$$

where  $C_r: (\Lambda_r)^{r-1} \rightarrow \mathbb{Q}$  and  $D_r: (\Lambda_r)^r \rightarrow \mathbb{Q}$  are piecewise constant (with finite number of values) symmetric functions such that

$$C_r(z_1, \dots, z_{r-1}) = 0 \text{ when } \prod_{j=1}^{r-1} z_j = 0, \quad D_r(z_1, \dots, z_r) = 0 \text{ when } \prod_{j=1}^r z_j = 0.$$

Note that the sum in (4.1) (resp. (4.2)) contains  $\binom{2^r}{r-1}$  (resp.  $\binom{2^r}{r}$ ) terms.

*Proof.* The fact of having evaluations of  $F^\bullet$ ,  $G^\bullet$  in the form of sums of inverse of monomials in the variables mentioned in the statement of Lemma 4.1 is a property obviously stable by mould multiplication. Therefore it is enough to prove it for  $S^\bullet$  in order to get it satisfied for  $G^\bullet$ . It is easily shown to be true by induction using (1.14) and (1.15) and the fact, easy to prove, that (once again we take  $A^\bullet = 0$ )

$$F^0 = 1, S^0 = N^0 = 0 \text{ and } F^\lambda = 0, S^\lambda = N^\lambda = \frac{1}{\lambda}, \text{ for } \lambda \neq 0.$$

The homogeneity property follows also easily from the induction generated by (1.14) and (1.15)<sup>1</sup>. Finally we get the fact that the functions  $C_r$  and  $D_r$  are piecewise constant with a finite number of values from the fact that the dichotomy which governs the use, for each  $\underline{\lambda}$ , of one of the two equations (1.14), (1.15) selects for each value of  $r$  a finite number of domains of  $\mathbb{C}^r$ , the resonance condition being linear.  $\square$

Since the functions  $C_r$  and  $D_r$  take only a finite number of values we can define

$$F_r = \sup_{z_1, \dots, z_{r-1} \in \Lambda_r} |C_r(z_1, \dots, z_{r-1})|, \quad G_r = \sup_{z_1, \dots, z_r \in \Lambda_r} |D_r(z_1, \dots, z_r)|.$$

**Corollary 4.2.** *Making use of the notation (3.2), we have*

$$|F^{\lambda_1, \dots, \lambda_r}| \leq F_r \left( \frac{\tau_r}{e\eta_r} \right)^{(r-1)\tau_r} e^{\eta_r \beta_{\tau_r}(\underline{\lambda})}, \quad |G^{\lambda_1, \dots, \lambda_r}| \leq G_r \left( \frac{\tau_r}{e\eta_r} \right)^{r\tau_r} e^{\eta_r \beta_{\tau_r}(\underline{\lambda})}.$$

*Proof.* We first remark that  $F^\bullet$  and  $G^\bullet$  are well defined for each words, so the denominators in each rational functions component don't contain any term of the form  $n \cdot \omega = 0$ . We finish using first the inequality

$$x < \frac{\tau}{e\eta} e^{\eta x^{1/\tau}} \quad \text{for all } \tau, \eta, x > 0 \quad (4.3)$$

with  $x = |\underline{\lambda}_\sigma|^{-1/\tau_r}$ ,  $\tau = \tau_r$  and  $\eta = \eta_r$ , and second the fact that  $|C_r(\underline{\lambda}_{\sigma_1}, \dots, \underline{\lambda}_{\sigma_{r-1}})| \leq F_r$  and  $|D_r(\underline{\lambda}_{\sigma_1}, \dots, \underline{\lambda}_{\sigma_r})| \leq G_r$  for all  $\sigma_1, \dots, \sigma_r \subset \{1, \dots, r\}$ .  $\square$

Using (4.5) in Lemma 4.4 below we get immediately the following result.

---

<sup>1</sup>We get also the homogeneity by a simple physical dimension reasoning: since the letters are defined by  $\{X_0, B_\lambda\} = \lambda B_\lambda$  we have that  $\lambda$  must have the dimension of  $\frac{\text{energy}}{\text{action}}$  (the dimension of action is the one of  $p \times q$ ). An evaluation of the comould on a word of length  $r$ ,  $\{B_{\lambda_r} \{B_{\lambda_{r-1}} \{ \dots, B_{\lambda_1} \} \} \dots \}$  has the dimension  $\frac{\text{energy}^r}{\text{action}^{r-1}}$ . Finally the dimension of the normal form is the one of an energy. Since all the constants in the mould equation (with zero gauge) are universal and therefore have no dimension, we conclude that the dimension of the evaluation of the mould  $F^\bullet$  on a word of length  $r$  is  $\text{energy} \times \frac{\text{action}^{r-1}}{\text{energy}^r} = \left( \frac{\text{action}}{\text{energy}} \right)^{r-1} = (\text{dimension of } \lambda)^{-(r-1)}$ . In the same way one sees that since one take the exponential of  $\text{ad}_G$ ,  $\text{ad}_G$  must have no dimension and therefore  $Y$  must have the dimension of an action and get the desired homogeneity.

**Corollary 4.3.** *Under the hypothesis (3.4) of Theorem A we have that*

$$\begin{aligned} \left\| \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} F^{\lambda_1, \dots, \lambda_r} [B_{\lambda_r}, [\dots [B_{\lambda_2}, B_{\lambda_1}] \dots]] \right\|_{\rho'} &\leq \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho - \rho')^2} \right)^{r-1} F_r \left( \frac{\tau_r}{e\eta_r} \right)^{\tau_r(r-1)} \epsilon_r \\ \left\| \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} G^{\lambda_1, \dots, \lambda_r} [B_{\lambda_r}, [\dots [B_{\lambda_2}, B_{\lambda_1}] \dots]] \right\|_{\rho'} &\leq \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho - \rho')^2} \right)^{r-1} G_r \left( \frac{\tau_r}{e\eta_r} \right)^{\tau_r r} \epsilon_r \end{aligned}$$

**4.2. More estimates.** The following Lemma is a slight generalization of [PS14, inequality (5.11)] (see also [GP12]).

**Lemma 4.4.** *Let us suppose that for  $0 < \rho' < \rho'' < \rho$  and  $i = 1 \dots d, d \in \mathbb{N}^*$*

$$\| [X_i, Y] \|_{\rho'} \leq \frac{\gamma}{e^2(\rho - \rho')(\rho'' - \rho')} \|X_i\|_{\rho} \|Y\|_{\rho''}, \quad \| [X_0, X_i] \|_{\rho'} \leq \frac{1}{\chi(\rho - \rho')} \|X_i\|_{\rho}. \quad (4.4)$$

Then,

$$\frac{1}{d!} \| [X_d, [X_{d-1}, \dots [X_1, Y]]] \|_{\rho'} \leq \frac{\gamma^d}{(\rho - \rho')^{2d}} \|Y\|_{\rho} \prod_{i=1}^d \|X_i\|_{\rho} \quad (4.5)$$

and

$$\frac{1}{d!} \| [X_d, [X_{d-1}, \dots [X_1, X_0]]] \|_{\rho'} \leq \frac{\gamma^d}{(\rho - \rho')^{2d}} \frac{(\rho - \rho')^2}{\chi(\rho - \rho')} \prod_{i=1}^d \|X_i\|_{\rho}. \quad (4.6)$$

Writing  $e^{\text{ad}_X} Y = \sum_{d=0}^{\infty} \frac{1}{d!} \underbrace{[X, [X, \dots [X, Y]]]}_{d \text{ times}}$  we get easily the following Corollary.

**Corollary 4.5.**

$$\| e^{\text{ad}_X} Y \|_{\rho'} \leq \frac{\|Y\|_{\rho}}{1 - \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho}}$$

and

$$\| e^{\text{ad}_X} X_0 - X_0 \|_{\rho'} \leq \frac{\gamma \|X\|_{\rho}}{\chi(\rho - \rho') \left( 1 - \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho} \right)}$$

Moreover

$$\| e^{\text{ad}_X} Y - \sum_{d=0}^N \frac{1}{d!} \underbrace{[X, [X, \dots [X, Y]]]}_{d \text{ times}} \|_{\rho'} \leq \|Y\|_{\rho} \frac{\left( \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho} \right)^{N+1}}{1 - \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho}}$$

and

$$\| e^{\text{ad}_X} X_0 - X_0 - \sum_{d=1}^N \frac{1}{d!} \underbrace{[X, [X, \dots [X, X_0]]]}_{d \text{ times}} \|_{\rho'} \leq \frac{(\rho - \rho')^2}{\chi(\rho - \rho')} \frac{\left( \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho} \right)^{N+1}}{1 - \frac{\gamma}{(\rho - \rho')^2} \|X\|_{\rho}}$$

*Proof of Lemma 4.4.* (following [GP12] and [PS14] where the case  $X_i = X$  is studied) (4.5) is easily obtained by iteration of the first part of (4.4). Consider the finite sequence of numbers  $\delta_s = \frac{d-s}{d}\delta$ . We have  $\delta_0 = \delta$ ,  $\delta_d = 0$  and  $\delta_{s-1} - \delta_s = \frac{\delta}{d}$ . Let us define  $G_0 := Y$  and  $G_{s+1} := [X_{s+1}, G_s]$ , for  $0 \leq s \leq d-1$ . According to (4.4), we have, denoting  $C = \frac{\gamma}{e^2}$ ,

$$\|G_s\|_{\rho-\delta_{d-s}} \leq \frac{C}{\delta_{d-s}(\frac{\delta}{d})} \|X_s\|_{\rho} \|G_{s-1}\|_{\rho-\delta_{d-s+1}} \quad \text{for } 1 \leq s \leq d.$$

Hence, by induction, we obtain, since  $\delta_0 = \delta$  and  $G_0 := Y$ ,

$$\begin{aligned} \frac{1}{d!} \|G_d\|_{\rho-\delta_0} &\leq \frac{C^d}{d! \delta_0 \cdots \delta_{d-1} (\frac{\delta}{d})^d} \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho} \\ &\leq \frac{C^d}{d! d! (\frac{\delta}{d})^d (\frac{\delta}{d})^d} \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho} \\ &\leq \left(\frac{Cd^2}{\delta^2}\right)^d \frac{1}{d! d!} \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho} \\ &= \frac{1}{2\pi d} \left(\frac{\gamma}{\delta^2}\right)^d \left(\frac{\sqrt{2\pi d} d^d e^{-d}}{d!}\right)^2 \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho} \\ &\leq \frac{1}{2\pi d} \left(\frac{\gamma}{\delta^2}\right)^d \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho} < \left(\frac{\gamma}{\delta^2}\right)^d \prod_{i=1}^d \|X_i\|_{\rho} \|Y\|_{\rho}. \end{aligned}$$

The proof of (4.6) follows exactly the same lines.  $\square$

The following Lemma is a direct consequence of Corollary 4.3 and the definition of  $Y_N$  in Theorem A.

**Lemma 4.6.**

$$\|Y_N\|_{\rho'} \leq \sum_{r=1}^N \frac{(r-1)!}{r} \left(\frac{\gamma}{(\rho-\rho')^2}\right)^{r-1} G_r \left(\frac{\tau_r}{e\eta_r}\right)^{\tau_r r} \epsilon_r := E_{N, \rho-\rho'}. \quad (4.7)$$

Let us now perform a truncation of  $e^{\text{ad}_{Y_N}}$  as in Corollary 4.5. We get, defining  $e_N^{\text{ad}_{Y_N}} = \sum_{d=0}^N \frac{1}{d!} \underbrace{[Y_N, [Y_N, \dots [Y_N, \cdot]]]}_{d \text{ times}}$ ,

$$\begin{aligned} & \|e^{\text{ad}_{Y_N}}(X_0 + B) - e_N^{\text{ad}_{Y_N}}(X_0 + B)\|_{\rho'} \\ & \leq \left( \frac{(\rho'' - \rho')^2}{\chi(\rho'' - \rho')} + \|B\|_{\rho''} \right) \frac{\left( \frac{\gamma}{(\rho'' - \rho')^2} \|Y_N\|_{\rho''} \right)^{N+1}}{\left( 1 - \frac{\gamma}{(\rho'' - \rho')^2} \|Y_N\|_{\rho''} \right)} \\ & = \left( \frac{(\rho - \rho')^2}{4\chi(\frac{\rho - \rho'}{2})} + \|B\|_{\rho} \right) \frac{\left( \frac{4\gamma}{(\rho - \rho')^2} \|Y_N\|_{\frac{\rho + \rho'}{2}} \right)^{N+1}}{\left( 1 - \frac{4\gamma}{(\rho - \rho')^2} \|Y_N\|_{\frac{\rho + \rho'}{2}} \right)} \end{aligned}$$

by taking  $\rho'' = \frac{\rho + \rho'}{2} \leq \rho$  and using  $\|\cdot\|_{\rho''} \leq \|\cdot\|_{\rho}$ .

**Lemma 4.7.** *Let*

$$E_{N, \frac{\rho - \rho'}{2}} \leq \frac{1}{2} \frac{(\rho - \rho')^2}{4\gamma}. \quad (4.8)$$

*Then*

$$\|e^{\text{ad}_{Y_N}}(X_0 + B) - e_N^{\text{ad}_{Y_N}}(X_0 + B)\|_{\rho'} \leq C_{N+1}^{\text{sg}} (E_{N, \frac{\rho - \rho'}{2}})^{N+1}$$

*with*

$$C_{N+1}^{\text{sg}} = 2 \left( \frac{(\rho - \rho')^2}{4\chi(\frac{\rho - \rho'}{2})} + \|B\|_{\rho} \right) \left( \frac{4\gamma}{(\rho - \rho')^2} \right)^{N+1} \quad (4.9)$$

**4.3. End of the proof.** Let us go back now to the mould equation:

$$\nabla F^\bullet = 0, \quad \nabla(e^{G^\bullet}) - I^\bullet \times e^{G^\bullet} - e^{G^\bullet} \times F^\bullet = 0.$$

Let us call  $F_N$  and  $G_N$  the moulds of  $Z_N$  and  $Y_N$ , that is  $F_N^\lambda = F^\lambda$  if  $r(\lambda) \leq N$  and  $F_N^\lambda = 0$  otherwise.

Let us define  $(e^{G^\bullet})_N$  as for  $F_N$  and  $G_N$ . Obviously  $(e^{G^\bullet})_N = (e^{G_N^\bullet})_N$ , and

$$\left( \nabla(e^{G^\bullet}) \right)_N = \left( \nabla(e^{G_N^\bullet}) \right)_N, \quad \left( I^\bullet \times e^{G^\bullet} \right)_N = \left( I^\bullet \times e^{G_N^\bullet} \right)_N, \quad \left( e^{G^\bullet} \times F^\bullet \right)_N = \left( e^{G_N^\bullet} \times F_N^\bullet \right)_N.$$

Moreover the mould equation reads

$$\left( \nabla(e^{G^\bullet}) - I^\bullet \times e^{G^\bullet} - e^{G^\bullet} \times F^\bullet \right)_N = 0, \quad \forall N$$

and therefore

$$\left( \nabla(e^{G_N^\bullet}) - I^\bullet \times e^{G_N^\bullet} - e^{G_N^\bullet} \times F_N^\bullet \right)_N = 0, \quad \forall N \quad (4.10)$$



We get

$$e^{\text{ad}_{Y_N}}(X_0 + B) \quad (4.11)$$

$$= e_N^{\text{ad}_{Y_N}}(X_0 + B) + \mathcal{E}_N^1 \quad (4.12)$$

$$= X_0 + \sum_{d=1}^N \frac{1}{d!} \underbrace{[G_N^\bullet, [G_N^\bullet, \dots, [G_N^\bullet, \nabla G_N^\bullet + [G_N^\bullet, I^\bullet]]]}_{d-1 \text{ times}} B_{[\bullet]} + \mathcal{E}_N^1 \quad (4.13)$$

$$= X_0 + \left( \sum_{d=1}^N \frac{1}{d!} \underbrace{[G_N^\bullet, [G_N^\bullet, \dots, [G_N^\bullet, \nabla G_N^\bullet + [G_N^\bullet, I^\bullet]]]}_{d-1 \text{ times}} \right)_N B_{[\bullet]} + \mathcal{E}_N^2 + \mathcal{E}_N^1 \quad (4.14)$$

$$= X_0 + \left( e^{-G_N^\bullet} \times (\nabla e^{G_N^\bullet}) + e^{-G_N^\bullet} \times I^\bullet \times e^{G_N^\bullet} \right)_N B_{[\bullet]} + \mathcal{E}_N^2 + \mathcal{E}_N^1 \quad (4.15)$$

$$= X_0 + F_N^\bullet B_{[\bullet]} + \mathcal{E}_N^2 + \mathcal{E}_N^1 \quad (4.16)$$

$$= X_0 + Z_N + \mathcal{E}_N$$

with  $\|\mathcal{E}_N^1\|_{\rho'} \leq C_{N+1}^{sg} E_{N,(\rho-\rho')/2}^{N+1}$  and

$$\begin{aligned} \|\mathcal{E}_N^2\|_{\rho'} &\leq 2 \sum_{r=N+1}^{N^2} \left( \sum_{d=1}^N \sum_{\substack{k_1+k_2+\dots+k_N=d \\ k_1+2k_2+\dots+Nk_N=r}} \frac{N!}{\prod_{i=1}^N k_i!} \right) \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho-\rho')^2} \right)^{r-1} G_r \left( \frac{\tau_r}{e\eta_r} \right)^{\tau_r r} \epsilon_r \\ &\leq 2 \sum_{r=N+1}^{N^2} \left( \sum_{d=1}^N N^d \right) \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho-\rho')^2} \right)^{r-1} G_r \left( \frac{\tau_r}{e\eta_r} \right)^{\tau_r r} \epsilon_r \\ &\leq 2N^N (E_{N^2, \rho-\rho'} - E_{N, \rho-\rho'}) \quad (4.17) \end{aligned}$$

where  $E_{N,\rho}$  is defined by (4.8). Let us explain how we derive the chain of inequalities after (4.12):

- (4.11) $\Rightarrow$ (4.12); is Lemma 4.7.
- (4.12) $\Rightarrow$ (4.13); writing the first part of (4.12) in mould calculus by Proposition 1.2.
- (4.13) $\Rightarrow$ (4.14) with (4.17): since the support of  $G_N^\bullet$  contains only words of length up to  $N$ , we can expand the mould in (4.13) up to words of length  $N$ , which appears in (4.14), plus the rest. The rest, whose support contains only words on length between  $N+1$  and  $N^2$ , gives (4.17) by combinatorial coefficients and Corollary 4.3.
- (4.14) $\Rightarrow$ (4.15); is obtained by developing  $e^{-G_N^\bullet} \times (\nabla e^{G_N^\bullet}) + e^{-G_N^\bullet} \times I^\bullet \times e^{G_N^\bullet}$  in the form  $\sum_{d=0}^{\infty} \frac{1}{d!} \underbrace{[G_N^\bullet, [G_N^\bullet, \dots, [G_N^\bullet, \nabla G_N^\bullet + [G_N^\bullet, I^\bullet]]]}_{d \text{ times}}$  and noticing that, since  $G_N^\bullet = 0$ , one has that  $\underbrace{[G_N^\bullet, [G_N^\bullet, \dots, [G_N^\bullet, \nabla G_N^\bullet + [G_N^\bullet, I^\bullet]]]}_{M \text{ times}, M > N}$  contains only word on length greater than  $N$ .

- (4.15) $\Rightarrow$ (4.16); is obtained by (4.10).
- (4.16) $\Rightarrow$ (4.17); by writing  $\mathcal{E}_N := \mathcal{E}_N^1 + \mathcal{E}_N^2$ .

Therefore

$$\|\mathcal{E}_N\|_{\rho'} \leq C_{N+1}^{sg} E_{N,(\rho-\rho')/2}^{N+1} + N^N (E_{N^2,\rho-\rho'} - E_{N+1,\rho-\rho'}) \text{ for } E_{N,\frac{\rho-\rho'}{2}} \leq \frac{1}{2} \frac{(\rho-\rho')^2}{4\gamma} \quad (4.18)$$

where  $C_{N+1}^{sg}$  and  $E_{N,(\rho-\rho')}$  are defined in (4.9) and (4.7). We now take

$$\epsilon^* = \frac{(\rho-\rho')^2}{32\gamma \sup_{r=1\dots N} \frac{(r-1)!}{r} \left(\frac{\gamma}{(\rho-\rho')^2}\right)^{r-1} \left(\frac{2^r \tau_r}{e \eta_r}\right)^r} = \frac{(\rho-\rho')^2}{32\gamma \frac{(N-1)!}{N} \left(\frac{\gamma}{(\rho-\rho')^2}\right)^{N-1} \left(\frac{2^N \tau_r}{e \inf_{r=1\dots N} \eta_r}\right)^N} \quad (4.19)$$

and

$$D = C_{N+1}^{sg} \left( 4 \frac{(N-1)!}{N} \left(\frac{\gamma}{(\rho-\rho')^2}\right)^{N-1} \left(\frac{2^N \tau_r}{e \inf_{r=1\dots N} \eta_r}\right)^N \right)^{N+1} + N^N \frac{(N^2-1)!}{N^2} \left(\frac{\gamma}{(\rho-\rho')^2}\right)^{N^2-1} \left(\frac{2^{N^2} \tau_r}{e \inf_{r=N+1\dots N^2} \eta_r}\right)^{N^2}. \quad (4.20)$$

Theorem A is proved.

## APPLICATIONS TO DYNAMICS

### 5. Quantitative classical formal normal forms

We denote the circle by

$$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}.$$

Let  $d \geq 1$  be integer. We are interested in two situations: the phase-space  $\mathcal{P}$  is either

$$T^*\mathbb{R}^d \simeq \mathbb{R}^d \times \mathbb{R}^d \text{ and then } X_0(x, \xi) = \frac{1}{2} \left( \sum_{j=1}^d \xi_j^2 + \sum_{j=1}^d \omega_j^2 x_j^2 \right),$$

or it is

$$T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d \text{ and } X_0(x, \xi) = \omega \cdot \xi.$$

In both cases, we suppose that  $\omega \in \mathbb{R}^d$  has components  $\omega_j > 0$ , and we will denote the variable in  $\mathcal{P}$  by  $(x, \xi)$ . The symplectic 2-form being  $\sum d\xi_j \wedge dx_j$ , the Hamiltonian vector field associated with  $X_0$  is  $\{X_0, \cdot\} = \sum (\xi_j \frac{\partial}{\partial x_j} - \omega_j^2 x_j \frac{\partial}{\partial \xi_j})$  in the first case, and  $\sum \omega_j \frac{\partial}{\partial x_j}$  in the second.

We want to perturb  $X_0$  by a “small” perturbation  $B$  and want to show that it is possible, after a symplectic change of coordinates, to put the new Hamiltonian  $X_0 + B$  into a *normal form*  $X_0 + Z$ ,  $\{X_0, Z\} = 0$ , modulo an arbitrarily small error.

The result will be expressed in Theorem B below, which will follow from Theorem A. We first have to show how our situation enters in the framework of the first part of this article.

Let

$$\widehat{\mathcal{P}} := \mathbb{R}^d \times \mathbb{R}^d \quad \text{if } \mathcal{P} = \mathbb{R}^d \times \mathbb{R}^d, \quad \widehat{\mathcal{P}} := \mathbb{R}^d \times \mathbb{Z}^d \quad \text{if } \mathcal{P} = T^*\mathbb{T}^d.$$

We define the (symplectic) Fourier transform  $\widehat{G}$  of a function  $G \in L^1(\mathcal{P}, dx d\xi)$  by

$$\widehat{G}(q, p) = \frac{1}{(2\pi)^d} \int_{\mathcal{P}} G(x, \xi) e^{-i(px - q\xi)} dx d\xi \quad \text{for } (q, p) \in \widehat{\mathcal{P}}. \quad (5.1)$$

Let  $d\mu$  denote either the Lebesgue measure  $dq dp$  on  $\mathbb{R}^d \times \mathbb{R}^d$  or the product of the Lebesgue measure  $dq$  by the Dirac mass on  $\mathbb{Z}^d \subset \mathbb{R}^d$ . If  $\widehat{G} \in L^1(\widehat{\mathcal{P}}, d\mu)$ , then

$$G(x, \xi) = \frac{1}{(2\pi)^d} \int \widehat{G}(q, p) e^{i(px - q\xi)} d\mu(q, p) \quad \text{for a.e. } (x, \xi) \in \mathcal{P}. \quad (5.2)$$

**By a slight abuse of notation, from now on, we will denote  $d\mu(q, p)$  by  $dq dp$  in both cases.**

Let us write  $X_0 = X_{0,1} + \dots + X_{0,d}$  with, for each  $j = 1, \dots, d$ ,

$$X_{0,j} := \frac{1}{2}(\xi_j^2 + \omega_j^2 x_j^2) \text{ on } \mathcal{P} = T^*\mathbb{R}^d, \quad X_{0,j} := \omega_j \xi_j \text{ on } \mathcal{P} = T^*\mathbb{T}^d.$$

Since the  $X_{0,j}$ 's Poisson-commute and since, for each  $j$ , all the solutions of the Hamiltonian vector field  $\{X_{0,j}, \cdot\}$  are  $\frac{2\pi}{\omega_j}$ -periodic, we get an action of  $\mathbb{T}^d$  on  $\mathcal{P}$  by defining

$$\Phi_0^t := \exp \left( \left\{ \frac{t_1}{\omega_1} X_{0,1} + \cdots + \frac{t_d}{\omega_d} X_{0,d}, \cdot \right\} \right) \quad \text{for } t = (t_1, \dots, t_d) \in \mathbb{T}^d.$$

Given  $k \in \mathbb{Z}^d$  and an integrable function  $G$ , we now define

$$G_{(k)}(x, \xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} G(\Phi_0^t(x, \xi)) e^{-ikt} dt \quad \text{for } (x, \xi) \in \mathcal{P}.$$

**Lemma 5.1.** *For any real-analytic  $G$ , one has  $\sum_{k \in \mathbb{Z}^d} G_{(k)} = G$  (pointwise convergence on  $\mathcal{P}$ ) and, for each  $k \in \mathbb{Z}^d$ ,*

$$\{X_0, G_{(k)}\} = ik \cdot \omega G_{(k)}. \quad (5.3)$$

*Proof.* For each  $(x, \xi) \in \mathcal{P}$ , the function  $t \in \mathbb{R}^d \mapsto G \circ \Phi_0^t(x, \xi)$  is analytic and  $2\pi$ -periodic in each  $t_j$ ; for each  $k \in \mathbb{Z}^d$ , its  $k$ th Fourier coefficient is  $G_{(k)}(x, \xi)$ . The first statement thus follows from the fact that  $G \circ \Phi_0^t(x, \xi)$  is the sum of its Fourier series.

For each  $j$ ,  $\{X_{0,j}, G_{(k)}\}$  is the  $k$ th Fourier coefficient of the function  $\{X_{0,j}, G \circ \Phi_0^t\}$ , and this function coincides with  $\omega_j \frac{\partial}{\partial t_j} (G \circ \Phi_0^t)$ , hence  $\{X_{0,j}, G_{(k)}\} = ik_j \omega_j G_{(k)}$ , and the second statement follows.  $\square$

For  $\rho > 0$  we will denote by  $\mathcal{J}_\rho$  the space of all integrable functions  $G$  whose associated family of functions  $G_{(k)}$  have a Fourier transform whose modulus is integrable with respect to  $dqdp$  weighted by  $e^{\rho(|q|+|p|)}$ , the family of integrals obtained that way being itself summable with the weight  $e^{\rho|k|}$ .

Namely,<sup>2</sup> we set  $\mathcal{J}_\rho := \{ G \in L^1(\mathcal{P}, dx d\xi) \mid \|G\|_\rho < \infty \}$ , with

$$\|G\|_\rho := \sum_{k \in \mathbb{Z}^d} \int_{\widehat{\mathcal{P}}} |\widehat{G}_{(k)}(q, p)| e^{\rho(|q|+|p|+|k|)} dq dp.$$

Note that each function in  $\mathcal{J}_\rho$  is real analytic and has a bounded holomorphic extension to the complex strip  $\{(x, \xi) \in \mathbb{C}^d \times \mathbb{C}^d \mid |\Im x_1|, \dots, |\Im x_d|, |\Im \xi_1|, \dots, |\Im \xi_d| < \rho\}$ .  $\mathcal{J}_\rho$  is obviously a Banach space satisfying  $\|G\|_{\rho'} \leq \|G\|_\rho$  whenever  $\rho' < \rho$ .

In the case  $\mathcal{P} = \mathbb{R}^d \times \mathbb{R}^d$  let us denote by  $\widehat{X}_{0,1}, \dots, \widehat{X}_{0,d}$  the functions defined on  $\widehat{\mathcal{P}}$  by  $\widehat{X}_{0,j}(q, p) = X_{0,j}(q, p)$  and let  $\widehat{\Phi}_0^t$  be the corresponding torus action on  $\widehat{\mathcal{P}}$ . It is easy to check that,

<sup>2</sup>Note that in the case of  $T^*\mathbb{T}^d$ ,  $G_{(k)}$  is nothing but the Fourier coefficient of  $G(\cdot, \xi)$  times  $e^{-ik\cdot}$ . Therefore in this case  $\widehat{G}_{(k)}(q, p) = \delta_{k,p} \widehat{G}_{(k)}(q, p)$  and so  $\|G\|_\rho = \int |\widehat{G}(q, p)| e^{\rho(|q|+2|p|)} dq dp$ . We present nevertheless the two cases ( $T^*\mathbb{T}^d$  and  $T^*\mathbb{R}^d$ ) in a unified way.

defining  $(x, \xi) \cdot (p, q) := px - q\xi$ , we have

$$\Phi_0^t(x, \xi) \cdot \widehat{\Phi}_0^t(q, p) = (x, \xi) \cdot (q, p). \quad (5.4)$$

Defining now

$$(\widehat{F})_{(k)}(q, p) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}_\omega^d} \widehat{F}(\widehat{\Phi}_0^t(q, p)) e^{-ikt} dt$$

we get by (5.4) and the conservation of the Liouville measure by symplectomorphisms that

$$\widehat{F}_{(k)} = (\widehat{F})_{(k)}. \quad (5.5)$$

In the case  $\mathcal{P} = T^*\mathbb{T}^d$  we get easily

$$\widehat{F}_{(k)}(q, p) = \widehat{F}(q, p) \delta_{k,p}. \quad (5.6)$$

**Lemma 5.2.**

$$\|\{F, G\}\|_{\rho'} \leq \frac{1}{e^2(\rho - \rho')(\rho'' - \rho')} \|F\|_{\rho} \|G\|_{\rho''}$$

whenever  $\rho' < \rho'' \leq \rho$ .

*Proof.* We will first prove, in the two cases  $\mathcal{P} = \mathbb{R}^d \times \mathbb{R}^d$  and  $\mathcal{P} = T^*\mathbb{T}^d$  the following identity.

$$\widehat{\{F, G\}}_{(k)}(q, p) = \sum_{k' \in \mathbb{Z}^d} \int dp' dq' ((q - q')p' - (p - p')q') \widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p'). \quad (5.7)$$

*Proof of (5.7).*  $\{F, G\}(x, \xi) = \partial_x F(x, \xi) \partial_\xi G(x, \xi) - \partial_\xi F(x, \xi) \partial_x G(x, \xi)$ . So

$$\widehat{\{F, G\}}(q, p) = \int ((q - q')p' - (p - p')q') \widehat{F}(q - q', p - p') \widehat{G}(q', p') dq' dp' \quad (5.8)$$

In the case  $\mathcal{P} = \mathbb{R}^d \times \mathbb{R}^d$ ,  $\widehat{\{F, G\}}_{(k)} = \widehat{F, G}_{(k)}$  so, since  $\Phi_0^t$  is linear symplectic and so preserves Liouville measure,

$$\begin{aligned}
& \widehat{\{F, G\}}_{(k)}(q, p) \\
&= \int dt dp' dq' ((\Phi_0^t(q) - q')(p') - (\Phi_0^t(p) - p')\Phi_0^t(q)) \widehat{F}(\Phi_0^t(q) - q', \Phi_0^t(p) - p') \widehat{G}(q', p') e^{ikt} \\
&= \int dt dp' dq' ((\Phi_0^t(q - q'))\Phi_0^t(p') - (\Phi_0^t(p - p'))\Phi_0^t(q)) \widehat{F}(\Phi_0^t(q - q'), \Phi_0^t(p - p')) \widehat{G}(\Phi_0^t(q'), \Phi_0^t(p')) e^{ikt} \\
&= \int dt dp' dq' ((q - q')p' - (p - p')q') \widehat{F}(\Phi_0^t(q - q'), \Phi_0^t(p - p')) \widehat{G}(\Phi_0^t(q'), \Phi_0^t(p')) e^{ikt} \\
&= \int dt dp' dq' dt' \sum_{k' \in \mathbb{Z}^d} e^{ik'(t'-t)} ((q - q')p' - (p - p')q') \widehat{F}(\Phi_0^t(q - q'), \Phi_0^t(p - p')) \widehat{G}(\Phi_0^{t'}(q'), \Phi_0^{t'}(p')) e^{ikt} \\
&= \sum_{k' \in \mathbb{Z}^d} \int dp' dq' ((q - q')p' - (p - p')q') \widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p') \\
&= \sum_{k' \in \mathbb{Z}^d} \int dp' dq' ((q - q')p' - (p - p')q') \widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p').
\end{aligned}$$

In the case  $\mathcal{P} = T^*\mathbb{T}^d$ ,  $\widehat{\{F, G\}}_{(k)}(q, p) = \widehat{F, G}(q, p)\delta_{k,p}$ , so

$$\begin{aligned}
& \widehat{\{F, G\}}_{(k)}(q, p) \\
&= \delta_{k,p} \int ((q - q')p' - (p - p')q') \widehat{F}(q - q', p - p') \widehat{G}(q', p') dq' dp' \\
&= \sum_{k' \in \mathbb{Z}^d} \delta_{p-p', k-k'} \delta_{p', k'} \int ((q - q')p' - (p - p')q') \widehat{F}(q - q', p - p') \widehat{G}(q', p') dq' dp' \\
&= \sum_{k' \in \mathbb{Z}^d} \int dp' dq' ((q - q')p' - (p - p')q') \widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p').
\end{aligned}$$

□

Using now  $|q| \leq |q - q'| + |q'|$ ,  $|p| \leq |p - p'| + |p'|$ ,  $|k| \leq |k - k'| + |k'|$  and  $xe^{(\rho')x} \leq \frac{1}{e^{(\rho'' - \rho')x}} e^{\rho''x}$ .  $x \geq 0$ , one get, by (5.7),

$$\begin{aligned}
& \|\{F, G\}\|_{\rho'} = \sum_{k \in \mathbb{Z}^d} \int dq dp |\widehat{\{F, G\}}_{(k)}(q, p)| e^{\rho'(|q|+|p|+|k|)} \quad (5.9) \\
& \leq \sum_{k, k'} \int dq dq' dp dp' (|q - q'| |p'| + |p - p'| |q'|) |\widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p')| e^{\rho'(|q|+|p|+|k|)} \\
& \leq \sum_{k, k'} \int (|q - q'| |p'| + |p - p'| |q'|) |\widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p')| \\
& \quad e^{\rho'(|q-q'|+|p-p'|+|k-k'|+|q'|+|p'|+|k'|)} dq dq' dp dp' \\
& \leq \sum_{k, k'} \int (|q - q'| + |p - p'|) |\widehat{F}_{(k-k')}(q - q', p - p') \widehat{G}_{(k')}(q', p')| \\
& \quad \frac{1}{e^{(\rho'' - \rho')}} e^{\rho'(|q-q'|+|p-p'|+|k-k'|)+\rho''(|q'|+|p'|+|k'|)} dq dq' dp dp' \\
& \leq \sum_{k, k'} \int (|q| + |p|) |\widehat{F}_{(k)}(q, p) \widehat{G}_{(k')}(q', p')| \\
& \quad \frac{1}{e^{(\rho'' - \rho')}} e^{\rho'(|q|+|p|+|k'|)+\rho''(|q'|+|p'|+|k'|)} dq dq' dp dp' \\
& \leq \sum_{k, k'} \int |\widehat{F}_{(k)}(q, p) \widehat{G}_{(k')}(q', p')| \\
& \quad \frac{1}{e^{(\rho'' - \rho')}} \frac{1}{e^{(\rho - \rho')}} e^{\rho'(|q|+|p|+|k'|)+\rho''(|q'|+|p'|+|k'|)} dq dq' dp dp'
\end{aligned}$$

since  $\rho' < \rho'' \leq \rho$  and one easily concludes.  $\square$

The same argument, used this time the weighted sum in  $k$ , leads to the next result.

**Lemma 5.3.**

$$\|\{X_0, G\}\|_{\rho'} \leq \frac{1}{e^{(\rho - \rho')}} \|G\|_{\rho}.$$

*Proof.* We first remark that

$$\{X_0, G\}_{(k)} = \int \{X_0, G\} \circ \Phi_0^t e^{ikt} dt = \int \{X_0, G \circ \Phi_0^t\} e^{ikt} dt = \{X_0, G_{(k)}\} = ik \cdot \omega G_{(k)}$$

by (5.3), and we easily concludes using again  $|k| e^{\rho'|k|} \leq \frac{1}{e^{(\rho - \rho')}} e^{\rho|k|}$ .  $\square$

Let  $\mathcal{L}$  be the space of real analytic functions on  $\mathcal{P}$ . We just proved that  $(\mathcal{L}, [\cdot, \cdot])$  endowed with  $\mathcal{J}_\rho = \mathcal{B}_\rho$ ,  $0 < \rho < \infty$  and with  $[\cdot, \cdot] = \{\cdot, \cdot\}$ ,  $\gamma = 1$  and  $\chi(\rho) = \frac{1}{e^\rho}$ , is an  $X_0$ -extended Banach scale Lie algebra.

Let us remark now that the homogeneous components of a perturbation  $B$  are easily deduced from the family  $(B_{(k)})_{k \in \mathbb{Z}^d}$ . Indeed, let us define

$$\Lambda = \{ i k \cdot \omega \mid k \in \mathbb{Z}^d \} \quad (5.10)$$

as in Remark 3.3. In view of Lemma 5.1, we see that, for each  $\lambda \in \Lambda$ ,

$$\{X_0, B_\lambda\} = \lambda B_\lambda \quad \text{with } B_\lambda := \sum_{k \in \mathbb{Z}^d | i k \cdot \omega = \lambda} B_{(k)}, \quad (5.11)$$

and  $B = \sum_{\lambda \in \Lambda} B_\lambda$ . Moreover, since

$$(F_{(k)})_{(k')} = \frac{1}{(2\pi)^{2d}} \int F \circ \Phi_0^t \circ \Phi_0^{t'} e^{-i(k t + k' t')} dt dt' = \frac{1}{(2\pi)^{2d}} \int F \circ \Phi_0^s e^{-i k s} ds \int e^{i(k' - k)t'} dt' = F_{(k)} \delta_{k k'},$$

we have that

$$\|B\|_\rho = \sum_{k \in \mathbb{Z}^d} \|B_{(k)}\|_\rho e^{\rho |k|}. \quad (5.12)$$

In particular,  $\sum_{k \in \mathbb{Z}^d} \|B_{(k)}\|_\rho e^{\eta |k|}$  is convergent for each  $\eta \leq \rho$ .

Let us assume that the Diophantine condition (3.6) is satisfied.

By (5.11),  $\|B_\lambda\|_\rho \leq \sum_{k \in \mathbb{Z}^d | i k \cdot \omega = \lambda} \|B_{(k)}\|_\rho$ , hence, we have, using the notation (3.1),

$$\epsilon_r \leq \sum_{k_1, \dots, k_r \in \mathbb{Z}^d} \|B_{(k_1)}\|_\rho \cdots \|B_{(k_r)}\|_\rho e^{\eta_r \beta_{\tau_r}((i k_1 \cdot \omega) \cdots (i k_r \cdot \omega))}.$$

Now, given  $k_1, \dots, k_r \in \mathbb{Z}^d$ , we have

$$\begin{aligned} \beta_{\tau_r}((i k_1 \cdot \omega) \cdots (i k_r \cdot \omega)) &= \sum_{\substack{\sigma \subset \{1, \dots, r\} \\ \text{such that } \underline{k}_\sigma \cdot \omega \neq 0}} \frac{1}{|\underline{k}_\sigma \cdot \omega|^{1/\tau}} \leq \frac{1}{\alpha^{1/\tau}} \sum_{\substack{\sigma \subset \{1, \dots, r\} \\ \text{such that } \underline{k}_\sigma \neq 0}} |\underline{k}_\sigma| \\ &\leq \frac{2^r}{\alpha^{1/\tau}} (|k_1| + \cdots + |k_r|). \end{aligned}$$

We get

$$\begin{aligned} \epsilon_r &\leq \sum_{k_1, \dots, k_r \in \mathbb{Z}^d} \|B_{(k_1)}\|_\rho \cdots \|B_{(k_r)}\|_\rho e^{\eta (|k_1| + \cdots + |k_r|)} \\ &\leq \sum_{k_1, \dots, k_r \in \mathbb{Z}^d} \|B_{(k_1)}\|_\rho \cdots \|B_{(k_r)}\|_\rho e^{\rho (|k_1| + \cdots + |k_r|)} = \|B\|_\rho^r < \infty, \quad \forall r \in \mathbb{N}^*. \end{aligned}$$

Therefore hypotheses (i) – (ii) are satisfied for all  $B \in \mathcal{J}_\rho$ , with  $\eta_r = \eta \alpha^{1/\tau} / 2^{r-1}$  and  $\tau_r = \tau$ . Theorem A applies.



Before to state it in the present setting, let us remark that since  $\epsilon_r \leq \|B\|_\rho^r$ , one can improve (3.5) (or rather (4.18)). To do so we first remark that in Lemma 4.6, if  $\|B\|_\rho \leq 1$ ,

$$E_{N,\rho-\rho'} \leq \|B\|_\rho \sum_{r=1}^N \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho-\rho')^2} \right)^{r-1} G_r \left( \frac{\tau}{e\eta_r} \right)^{\tau r} := \|B\|_\rho \Gamma_N.$$

Therefore for  $\epsilon = \inf(1, \frac{\rho-\rho'}{8\gamma\Gamma_N})$  we have that the second inequality of (4.18) is satisfied when  $\|B\|_\rho \leq \epsilon$ .

Under the same condition on  $\|B\|_\rho$  we find that

$$C_{N+1}^{sg} E_{N,(\rho-\rho')/2}^{N+1} \leq \|B\|_\rho^{N+1} C_{N+1}^{sg'} (\Gamma_N)^{N+1}$$

with

$$C_{N+1}^{sg'} = 2 \left( \frac{(\rho-\rho')^2}{4\chi(\frac{\rho-\rho'}{2})} + 1 \right) \left( \frac{4\gamma}{(\rho-\rho')^2} \right)^{N+1},$$

and

$$N^N (E_{N^2,\rho-\rho'} - E_{N+1,\rho-\rho'}) \leq \|B\|_\rho^{N+1} N^N \Gamma_{N^2,N}$$

with

$$\Gamma_{N^2,N} = \sum_{r=N+1}^{N^2} \frac{(r-1)!}{r} \left( \frac{\gamma}{(\rho-\rho')^2} \right)^{r-1} G_r \left( \frac{\tau}{e\eta_r} \right)^{\tau r},$$

Thus we define:

$$D = C_{N+1}^{sg'} (\Gamma_N)^{N+1} + 2N^N \Gamma_{N^2,N} \text{ and } \epsilon = \inf \left( 1, \frac{\rho-\rho'}{8\gamma\Gamma_N} \right). \quad (5.13)$$

Finally,  $Z_N$  and  $Y_N$  are real functions, and  $e^{\text{ad}Y_N}$  corresponds to a composition by a symplectic transform. We get the following rephrasing of Theorem A.

**Theorem B.** *Let  $\rho > 0$  such that  $\|B\|_\rho < \infty$ .*

*For all  $N \in \mathbb{N}^*$  and  $0 < \rho' < \rho$ , let  $D$  and  $\epsilon$  be given by (5.13). Then*

*(a) the following two expansions converge in  $\mathcal{B}_{\rho'}$*

$$Z_N := \sum_{r=1}^{N-1} \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} F^{\lambda_1, \dots, \lambda_r} \{B_{\lambda_r}, \{\dots \{B_{\lambda_2}, B_{\lambda_1}\} \dots\}\},$$

$$Y_N := \sum_{r=1}^{N-1} \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} G^{\lambda_1, \dots, \lambda_r} \{B_{\lambda_r}, \{\dots \{B_{\lambda_2}, B_{\lambda_1}\} \dots\}\}$$

(b) and, if moreover  $\|B\|_\rho \leq \epsilon$ , then

$$\begin{cases} (X_0 + B) \circ \Phi_N = X_0 + Z_N + \mathcal{E}_N, \\ \{X_0, Z_N\} = 0, \\ \|\mathcal{E}_N\|_{\rho'} \leq D\|B\|_\rho^{N+1}, \end{cases} \quad (5.14)$$

$\Phi_N$  being the Hamiltonian flow at time 1 of Hamiltonian  $Y_N$ .

## 6. Quantitative quantum formal normal forms

This section constitutes the quantum counterpart of the preceding section.

Let the Hilbert space  $\mathcal{H}$  be either  $L^2(\mathbb{R}^d)$  and in this case let  $X_0 = \frac{1}{2}(-\hbar^2\Delta + \sum_{i=1}^d \omega_i^2 x_i^2)$  or  $L^2(\mathbb{T}^d)$  and  $X_0 = -i\hbar\omega \cdot \nabla$ , corresponding indeed to the quantization of the two situations of Section 5. Let us recall that in both cases  $X_0$  is essentially self-adjoint on  $\mathcal{H}$ .

Here again we want to perturb  $X_0$  by a “small” perturbation  $B$  and want to show that it is possible, after a conjugation by a unitary operator on  $\mathcal{H}$ , to put the new quantum Hamiltonian  $X_0 + B$  into a *normal form*  $X_0 + Z$ ,  $[X_0, Z] := X_0Z - ZX_0 = 0$ , modulo an error we want to be as small as we wish.

The result will be expressed in Theorem C below but let us first see how the quantum situations just mentioned enter also in the framework of the first part of this article, though they seat on a very different paradigm than the one of the preceding section.

Let  $J_\rho$  be the set of all pseudo-differential operators whose Weyl symbols belong to  $\mathcal{J}_\rho$ . We define the norm of an operator belonging to  $J_\rho$  as the  $\|\cdot\|_\rho$  norm of its symbol and we denote it by the same expression  $\|\cdot\|_\rho$ .

There are different ways of defining Weyl quantization (see Appendix A below for elementary definitions). In the case  $\mathcal{P} = T^*\mathbb{R}^d$ , one of them, actually the historical one exposed in the book by Hermann Weyl [W28] consists in writing again the formula (5.2) for the inverse Fourier transform

$$G(x, \xi) = \frac{1}{(2\pi)^d} \int \widehat{G}(q, p) e^{i(px - q\xi)} d\mu(q, p,)$$

and replace in the right hand side  $x$  and  $\xi$  by  $\times x$  and  $-i\hbar\nabla$  respectively, in the case where  $\mathcal{P} = T^*\mathbb{R}^d$ . We get the operator  $B$  associated to the symbol  $\sigma_B$  by the formula

$$B = \frac{1}{(2\pi)^d} \int \widehat{\sigma}_B(q, p) e^{i(px + i\hbar q\nabla)} d\mu(q, p,).$$

The reader can check easily that when  $\sigma_B = x$  (resp.  $\xi$ ) on recover  $B = \times x$  (resp.  $-i\hbar\nabla$ ). Moreover, using the Campbell-Hausdorff formula, one gets that

$$e^{i(px+i\hbar q\nabla)} = e^{i\frac{pq}{2}} e^{ipx} e^{-q\hbar\nabla}.$$

This is this formulation that we use in the case where  $\mathcal{P} = T^*\mathbb{T}^d$  since  $px + i\hbar q\nabla$  doesn't make any sense on the torus, so we cannot use  $e^{-i(px+i\hbar q\nabla)}$ , but  $e^{-ipx}$  does (remember  $p$  is the dual variable of  $x$  so is discrete). Therefore we define in both cases

$$B = \frac{1}{(2\pi)^d} \int \widehat{\sigma}_B(q, p) e^{i\frac{pq}{2}} e^{-ipx} e^{-q\hbar\nabla} d\mu(q, p). \quad (6.1)$$

Note that a straightforward computation gives back the usual formula (A.2) or (A.3) of the Appendix:

$$Bf(x) = \int \sigma_B((x+y)/2, \xi) e^{-i\xi(x-y)/\hbar} f(y) dy \frac{d\xi}{(2\pi\hbar)^d}.$$

But the main interest of this formula for our purpose is the fact that  $e^{i\frac{pq}{2}} e^{-ipx} e^{-q\hbar\nabla}$  is unitary, since  $e^{i\frac{pq}{2}} e^{-ipx} e^{-q\hbar\nabla} \varphi(x) = e^{i\frac{pq}{2}} e^{-ipx} \varphi(x+q)$ , so  $\|e^{i\frac{pq}{2}} e^{-ipx} e^{-q\hbar\nabla}\|_{L^2 \rightarrow L^2} = 1^3$  and therefore

$$\|B\|_{L^2 \rightarrow L^2} \leq \|\widehat{\sigma}_B\|_{L^1(\mathcal{P})} = \|\sigma_B\|_0 \leq \|\sigma_B\|_\rho := \|B\|_\rho \text{ for all } \rho > 0. \quad (6.2)$$

Moreover it is straightforward to show that when  $\sigma_B$  is real valued,  $B$  is a symmetric operator so that, when bounded, is constitutes a symmetric bounded perturbation of  $X_0$ , therefore we just proved the following result.

**Lemma 6.1.** *Let  $B$  be defined by (6.1) with  $\sigma_B$  real valued and let  $\|B\|_\rho := \|\sigma_B\|_\rho < \infty$  for some  $\rho > 0$ . Then  $X_0 + B$  is essentially self-adjoint on  $\mathcal{H}$ .*

It has been proven in [BGP99] for  $L^2(\mathbb{R}^d)$  and [GP12, PS14] for  $L^2(\mathbb{T}^d)$  the following Lemma.

**Lemma 6.2.** *Suppose  $0 < \rho' < \rho$  and  $F, G \in J_\rho$ . Then*

$$\begin{aligned} \left\| \frac{1}{i\hbar} [F, G] \right\|_{\rho'} &\leq \frac{1}{e^2(\rho - \rho')(\rho'' - \rho')} \|F\|_\rho \|G\|_{\rho''} \\ \left\| \frac{1}{i\hbar} [X_0, G] \right\|_{\rho'} &\leq \frac{1}{e(\rho - \rho')} \|G\|_\rho \end{aligned}$$

*Proof.* Since the evolution by  $X_0$  commutes with quantization by Corollary A.2 (see the appendix), we get that  $\sigma_{[F,G]_k/i\hbar} = (\sigma_{[F,G]/i\hbar})_k$ ,  $\forall k \in \mathbb{Z}^d$ . Moreover, by Lemma A.1,  $\sigma_{[F,G]/i\hbar} = A(\sigma_F \otimes \sigma_G)$ . Therefore

$$\widehat{\sigma}_{[F,G]/i\hbar}(p, q) = \int \frac{1}{\hbar} \sin \hbar((q - q')p' - (p - p')q') \widehat{\sigma}_F(q - q', p - p') \widehat{\sigma}_G(q', p') dq' dp' \quad (6.3)$$

---

<sup>3</sup>We denote by  $\|\cdot\|_{L^2 \rightarrow L^2}$  the operator norm on  $\mathcal{H}$ .

a formula similar to (5.8) by the change  $((q - q')p' - (p - p')q') \rightarrow \frac{1}{\hbar} \sin \hbar((q - q')p' - (p - p')q')$ . The proof of the first inequality is identical to the one of Lemma 5.2 modulo this change up to (5.7), and the rest of the proof, after (5.9) is verbatim the same using the inequality

$$\left| \frac{1}{\hbar} \sin \hbar((q - q')p' - (p - p')q') \right| \leq |((q - q')p' - (p - p')q')|.$$

The proof of the second inequality is similar to the proof of Lemma 5.3.  $\square$

Let  $\mathcal{L}$  be the space of hermitian operators on  $\mathcal{H}$ . We just proved that  $(\mathcal{L}, [\cdot, \cdot]_q)$  endowed with  $J_\rho = \mathcal{B}_\rho$ ,  $0 < \rho < \infty$  and with  $[\cdot, \cdot]_q = \frac{1}{i\hbar}[\cdot, \cdot]$ ,  $\gamma = 1$  and  $\chi(\rho) = \frac{1}{e\rho}$ , is an  $X_0$ -extended Banach scale Lie algebra.

The decomposition into  $\lambda$ -homogeneous components of an arbitrary  $B \in J_\rho$  involves the letters of the same alphabet  $\Lambda$  defined by (5.10) as in Section 5. In fact, the homogeneous components of  $B$  can be obtained by Weyl quantization of the homogeneous components of the symbol  $\sigma_B$ , or directly as

$$B_\lambda = \sum_{k \in \mathbb{Z}^d | ik \cdot \omega = \lambda} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{\frac{1}{i\hbar} t \cdot \bar{X}_0} B e^{\frac{1}{i\hbar} \cdot \bar{X}_0} e^{-ik \cdot t} dt \quad \text{for each } \lambda \in \Lambda, \quad (6.4)$$

where  $\bar{X}_{0,j} = -\hbar^2 \partial_{x_j}^2 + \omega_j^2 x_j^2$  in the case of  $\mathbb{R}^d$  and  $\bar{X}_{0,j} = -i\hbar \omega_j \partial_{x_j}$  for  $\mathbb{T}^d$ , since linear Hamiltonian flows commute with quantization (see Lemma A.2 below).

Therefore the hypothesis (i)–(ii) of Theorem A are satisfied for the same values  $\eta_r = \eta \alpha^{1/\tau} / 2^{r-1}$  and  $\tau_r = \tau$  as in Section 5.

Moreover, if  $Y_N$  is a self-adjoint operator, then  $e^{\text{ad}_{Y_N}}$  corresponds to conjugation by the unitary transform  $e^{\frac{1}{i\hbar} Y_N}$ , hence the conclusions of Theorem A for this situation can be rephrased as:

**Theorem C.** *Let  $\rho > 0$  such that  $\|B\|_\rho < \infty$ .*

*For all  $N \in \mathbb{N}^*$  and  $0 < \rho' < \rho$ , let  $D$  and  $\epsilon$  be given by (5.13). Then*

*(a) the following two expansions converge in  $\mathcal{B}_{\rho'}$*

$$Z_N := \sum_{r=1}^{N-1} \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} F^{\lambda_1, \dots, \lambda_r} \frac{1}{i\hbar} [B_{\lambda_r}, \frac{1}{i\hbar} [\dots \frac{1}{i\hbar} [B_{\lambda_2}, B_{\lambda_1}] \dots]],$$

$$Y_N := \sum_{r=1}^{N-1} \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \frac{1}{r} G^{\lambda_1, \dots, \lambda_r} \frac{1}{i\hbar} [B_{\lambda_r}, \frac{1}{i\hbar} [\dots \frac{1}{i\hbar} [B_{\lambda_2}, B_{\lambda_1}] \dots]]$$

(b) and, if moreover  $\|B\|_\rho \leq \epsilon$ , then

$$\begin{cases} e^{\frac{1}{i\hbar}Y_N}(X_0 + B)e^{-\frac{1}{i\hbar}Y_N} = X_0 + Z_N + \mathcal{E}_N, \\ [X_0, Z_N] = 0, \\ \|\mathcal{E}_N\|_{\rho'} \leq D\|B\|_\rho^{N+1}. \end{cases} \quad (6.5)$$

## 7. Semiclassical approximation

In this final section we would like to link in a quantitative way the two preceding Section 5 and 6. Since the estimates in Section 6 are uniform in the Planck constant, it is natural to think that the quantum normal form should be “close” to the classical one when the Planck constant is close to zero. Since such a comparison involve objects of different nature (operators for quantum, functions for classical), it is natural to use the symbol “functor”  $\sigma$  to quantify this link. Let us use the superscripts  $C$  and  $Q$  for denoting the “same” object in Sections 5 and 6 respectively, so that  $X_0^C(x, \xi) = \frac{1}{2}(\xi^2 + \sum_{i=1}^d \omega_i^2 x_i^2)$  or  $X_0^C(x, \xi) = \omega \cdot \xi$  and  $X_0^Q = \frac{1}{2}(-\hbar^2 \Delta + \sum_{i=1}^d \omega_i^2 x_i^2)$  or  $X_0^Q = -i\hbar \omega \cdot \nabla$ . Note that in both cases,  $\sigma_{X_0^Q} = X_0^C$ . So it seems natural to link  $B^C$  and  $B^Q$  by  $\sigma_{B^Q} = B^C$ .

Expressing, for all  $N$ ,  $Z_N^Q$  in its mould-comould expansion, we see that, on one hand, the mould is independent of  $\hbar$ , and (therefore) is the same as the one in the mould-comould expansion of  $Z_N^C$ . On the other hand, by Lemma A.1, the symbol of any  $\frac{1}{i\hbar}[B_{\lambda_i}, B_{\lambda_j}]$  tends to the Poisson bracket between the two symbols of  $B_{\lambda_i}, B_{\lambda_j}$ . Moreover the symbol of  $B_{\lambda_i}$  is nothing else than the homogeneous parts of the symbol of  $B^Q$  (i.e.  $B^C$ ) by Corollary A.2. Finally, by iteration of Lemma A.1, iteration precisely estimated in Proposition A.3, we see that the symbol of the quantum normal form is, term by term in the mould-comould expansion graduated by the length of the words (i.e.  $N$ ), close to the classical normal form, as  $\hbar \rightarrow 0$ .

Our next result expresses quantitatively this fact, improving the results of [GP87] and [DGH91]. Let us recall  $Z_N^Q$  (resp.  $Z_N^C$ ) is the quantum (resp. classical) normal form in Theorem C (resp. theorem B).

**Theorem D.** *Let  $N \geq 1$ . Then*

$$\|\sigma(Z_N^Q - Z_{N-1}^Q) - (Z_N^C - Z_{N-1}^C)\|_{\rho'} \leq \hbar^2 C_N \|B\|_\rho^N$$

$$\text{where } C_N = \frac{1}{6N} \left( \frac{2^{2N-1} \tau}{\epsilon \rho \alpha^{1/N}} \right)^{N-1} \left( \frac{N+2}{\epsilon(\rho-\rho')} \right)^{N+2}.$$

The proof of Theorem D is easily obtained by using the mould expansion of  $Z_N^Q$ , in particular

$$Z_N^Q - Z_{N-1}^Q = \sum_{\lambda_1, \dots, \lambda_N \in \Lambda} \frac{1}{N} F^{\lambda_1, \dots, \lambda_N} \frac{1}{i\hbar} [B_{\lambda_N}, \frac{1}{i\hbar} [\dots \frac{1}{i\hbar} [B_{\lambda_2}, B_{\lambda_1}] \dots]],$$

the linearity of the “functor”  $\sigma$ , Corollary 4.2, Proposition A.3, and the arguments expressed just before the statement of the theorem.

Note that the correction is of order 2 in the Planck constant, which means that the classical perturbation theory incorporates the entire Bohr-Sommerfeld quantization, including the Maslov index.

## Appendix A. Weyl quantization and all that

Weyl quantization has been defined in Section 6. Defining the unitary operator  $U(q, p)$  where  $(q, p)$  are the Fourier variables of  $\mathcal{P} = T^*\mathbb{R}^d$  or  $T^*\mathbb{T}^d$  by:

$$U(q, p)\phi(x) = e^{i\frac{pq}{2}} e^{ipx} e^{-q\hbar\nabla} \varphi(x) = e^{i\frac{pq}{2\hbar}} e^{ipx} \phi(x - \hbar q), \quad \phi \in L^2(\mathbb{R}^d \text{ or } \mathbb{T}^d),$$

the Weyl quantization of a function  $\sigma_V$  on  $\mathcal{P}$  is the operator

$$V = \frac{1}{(2\pi)^d} \int dpdq \widehat{\sigma_V}(q, p) U(q, p) := \text{Op}^W(\sigma_V) \quad (\text{A.1})$$

where  $dpdq$  is used for  $d\mu(q, p)$  as in Section 5 page 18 and  $\widehat{\cdot}$  is the symplectic Fourier transform defined by (5.1)<sup>4</sup>.

Obviously, as mentioned earlier,  $\|U(q, p)\|_{L^2 \rightarrow L^2} = 1$  and therefore

$$\|V\|_{L^2 \rightarrow L^2} \leq \|\sigma_V\|_{\rho}, \quad \forall \rho > 0. \quad (\text{A.4})$$

Note that (A.1) makes also sense when  $\sigma_V$  is a polynomial on  $\mathcal{P}$  (polynomial in the variable  $\xi$  in the case  $\mathcal{P} = T^*\mathbb{T}^d$ ) since  $U(q, p)$  as defining an unbounded operator. One check easily that this is the case for  $X_0$  in the two examples of Section 5 and 6.

The following result is the fundamental one concerning the transition quantum-classical and, as presented here, is the only one we really need in the present article.

**Lemma A.1.** *Let  $V = \text{Op}^W(\sigma_V), V' = \text{Op}^W(\sigma_{V'})$  with  $\sigma_V, \sigma_{V'}$  either belong to  $\mathcal{J}_\rho$  for a certain  $\rho > 0$  or are polynomials on  $\mathcal{P}$ .*

Then

$$\frac{1}{i\hbar} [V, V'] = \text{Op}^W(\sigma_V *_{\hbar} \sigma_{V'})$$

---

<sup>4</sup>The goal of this appendix is not to give a crash course on pseudo-differential operators, but rather to recall the strict minimum used in the present paper. The reader is referred to [F89] for a general exposition. The reader not familiar with the presentation here can recognize easily the Weyl quantization of a symbol  $\sigma_V$  being, e.g. of the Schwartz class. That is to say that, when  $\sigma_V \in \mathcal{S}(\mathbb{R}^{2d})$ ,  $V$  defined by (A.1) acts on a function  $\varphi \in L^2(\mathbb{R}^d)$  through the formula

$$V\varphi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_V\left(\frac{x+y}{2}, \xi\right) e^{-i\frac{\xi(x-y)}{\hbar}} \varphi(y) \frac{d\xi dy}{(2\pi\hbar)^d}. \quad (\text{A.2})$$

and, in the case where  $\sigma_V \in C^\infty(\mathbb{T}^d) \otimes \mathcal{S}(\mathbb{R}^d)$ ,  $V$  acts on a function  $\varphi \in L^2(\mathbb{T}^d)$  by the same formula

$$V\varphi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_V\left(\frac{x+y}{2}, \xi\right) e^{-i\frac{\xi(x-y)}{\hbar}} \varphi(y) \frac{d\xi dy}{(2\pi\hbar)^d} \quad (\text{A.3})$$

where, in (A.3), it is understood that  $\sigma_V(\cdot, \xi)$  and  $\varphi$  are extended to  $\mathbb{R}^d$  by periodicity (see [PS14]). Note that (A.2) and (A.3) make sense thanks to the Schwartz property of  $V$  in  $\xi$  and that in (A.3) the r.h.s. depends only on the values of  $\sigma_V(x, \xi)$  for  $\xi \in \hbar\mathbb{Z}^d$ .

where  $*_{\hbar}$  is defined through the Fourier transform by

$$\widehat{\sigma *_{\hbar} \sigma'}(q, p) = \int_{\mathcal{P}} \frac{\sin [\hbar((q - q')p' - (p - p')q')]}{\hbar} \widehat{\sigma}(p - p', q - q') \widehat{\sigma'}(p', q') dp' dq'. \quad (\text{A.5})$$

In particular

$$\lim_{\hbar \rightarrow 0} \sigma *_{\hbar} \sigma' = \{\sigma, \sigma'\}.$$

and

$$\sigma_2 *_{\hbar} \sigma' = \{\sigma_2, \sigma'\}$$

when  $\sigma_2$  is a quadratic form. Therefore, in this case,

$$\exp(\text{ad}_{Op^W(\sigma_2)}) Op^W(\sigma_V) = Op^W(e^{\text{ad}_{\sigma_2}} \sigma_V). \quad (\text{A.6})$$

Using (6.4) and (A.6) we get the following result.

**Corollary A.2.** *Let  $X_0$  be as in Section 6. Then the homogeneous component  $B_\lambda$ ,  $\lambda = k \cdot \omega$ ,  $k \in \mathbb{Z}^d$  of any pseudodifferential operator  $B$  is the Weyl quantization of the  $\lambda$ -homogeneous part (with respect to  $\sigma_{X_0}$ ) of  $\sigma_B$ , that is*

$$B_\lambda = Op^W((\sigma_B)_\lambda).$$

Using the operator  $A$  of Lemma A.1 and similar arguments that the ones used in the proof of Lemma 4.4 we get the following result.

**Proposition A.3.** *Let  $0 < \rho' < \rho$ . Then for any  $d \geq 2$*

$$\begin{aligned} & \| \sigma_{B_d} *_{\hbar} (\sigma_{B_{d-1}} *_{\hbar} (\cdots *_{\hbar} (\sigma_{B_2} *_{\hbar} \sigma_{B_1}))) - \{\sigma_{B_d}, \{\sigma_{B_{d-1}}, \dots, \{\sigma_{B_2}, \sigma_{B_1}\}\}\} \|_{\rho'} \\ & \leq \frac{\hbar^2}{6} \left( \frac{d+2}{e(\rho - \rho')} \right)^{d+2} \prod_{k=1}^d \|B_k\|_{\rho}. \end{aligned}$$

Note that

$$\frac{[B_d, [B_{d-1}, \dots, [B_2, B_1]]]}{(i\hbar)^d} = \text{Op}^W(\sigma_{[B_d, [B_{d-1}, \dots, [B_2, B_1]]]/(i\hbar)^d})$$

with  $\sigma_{[B_d, [B_{d-1}, \dots, [B_2, B_1]]]/(i\hbar)^d} = \sigma_{B_d} *_{\hbar} (\sigma_{B_{d-1}} *_{\hbar} (\cdots *_{\hbar} (\sigma_{B_2} *_{\hbar} \sigma_{B_1})))$ .

*Proof.* The proof will be using the methods of the one of Lemma 6.2.



Iterating (A.7) we get

$$\begin{aligned}
\widehat{\sigma}_{[B_d, [B_{d-1}, \dots, [B_2, B_1]]]} / (i\hbar)^d (p_d, q_d) &= \int dp_1 dq_1 \dots dp_{d-1} dq_{d-1} \\
&\frac{1}{\hbar} \sin \hbar((q_d - q_{d-1})p_{d-1} - (p_d - p_{d-1})q_{d-1}) \widehat{\sigma}_{B_d}(p_d - p_{d-1}, q_d - q_{d-1}) \\
&\frac{1}{\hbar} \sin \hbar((q_{d-1} - q_{d-2})p_{d-2} - (p_{d-1} - p_{d-2})q_{d-2}) \widehat{\sigma}_{B_{d-1}}(p_{d-1} - p_{d-2}, q_{d-1} - q_{d-2}) \\
&\dots \\
&\frac{1}{\hbar} \sin \hbar((q_3 - q_2)p_2 - (p_3 - p_2)q_2) \widehat{\sigma}_{B_3}(p_3 - p_2, q_3 - q_2) \\
&\frac{1}{\hbar} \sin \hbar((q_2 - q_1)p_1 - (p_2 - p_1)q_1) \widehat{\sigma}_{B_2}(p_2 - p_1, q_2 - q_1) \\
&\widehat{\sigma}_{B_1}(p_1, q_1).
\end{aligned}$$

Expanding

$$\prod_{i=1}^d \frac{1}{\hbar} \sin \hbar x_i = \prod_{i=1}^d x_i - \frac{\hbar^2}{6} \sum_{k=1}^d x_k^3 \sin \hbar' x_k \prod_{l \neq k} \frac{1}{\hbar} \sin \hbar' x_l$$

for some  $0 \leq \hbar' \leq \hbar$ , one realizes that the first term gives precisely after integration the Fourier transform of  $\{\sigma_{B_d}, \{\sigma_{B_{d-1}}, \dots, \{\sigma_{B_2}, \sigma_{B_1}\}\}\}$ .

Using  $|\sin \hbar' x_k| \leq 1$  and  $|\frac{1}{\hbar} \sin \hbar' x_l| \leq |x_l|$  we get

$$\begin{aligned}
&\|\sigma_{[B_d, [B_{d-1}, \dots, [B_2, B_1]]]} / (i\hbar)^d - \{\sigma_{B_d}, \{\sigma_{B_{d-1}}, \dots, \{\sigma_{B_2}, \sigma_{B_1}\}\}\}\|_{\rho'} \\
&\leq \frac{\hbar^2}{6} \int e^{\rho'(|q_1| + |p_1| + \dots + |q_d| + |p_d|)} dp_d dq_d dp_{d-1} dq_{d-1} \dots dp_1 dq_1 \\
&\sum_{k=1}^d |x_k|^3 \prod_{l \neq k} |x_l| |\widehat{\sigma}_{B_k}(p_k - p_{k-1}, q_k - q_{k-1})| |\widehat{\sigma}_{B_l}(p_l - p_{l-1}, q_l - q_{l-1})|
\end{aligned}$$

with  $p_{-1} = q_{-1} := 0$  and  $x_k := (p_k - p_{k-1})q_{k-1} - (q_k - q_{k-1})p_{k-1} = p_k q_{k-1} - q_k p_{k-1}$ .

We have obviously that

$$\sum_{k=1}^d |x_k|^3 \prod_{l \neq k} |x_l| \leq \sum_{\substack{(a_1, \dots, a_m) \\ \in \{|p_1|, |q_1|, \dots, |p_d|, |q_d|\}}} \prod_{m=1}^{d+2} a_m \leq (|p_1| + |q_1| + \dots + |p_d| + |q_d|)^{d+2}$$

Therefore, using a last time the magic tool  $x^\beta e^{-\eta x} \leq \left(\frac{\beta}{e\eta}\right)^\beta$ ,  $\beta, \eta, x \geq 0$ , we get that

$$\sum_{k=1}^d |x_k|^3 \prod_{l \neq k} |x_l| e^{\rho'(|q_1| + |p_1| + \dots + |q_d| + |p_d|)} \leq \left(\frac{d+2}{e(\rho - \rho')}\right)^{d+2} e^{\rho'(|q_1| + |p_1| + \dots + |q_d| + |p_d|)}. \quad (\text{A.7})$$

Defining  $P_k = p_k - p_{k-1}$  and  $Q_k = q_k - q_{k-1}$  and using

$$\begin{aligned} |q_d| + |p_d| &\leq |q_d - q_{d-1}| + |q_{d-1} - q_{d-2}| + \dots + |q_2 - q_1| + |q_1| + |p_d - p_{d-1}| + \dots + |p_1| \\ &= \sum_{k=1}^d (|P_k| + |Q_k|) \end{aligned}$$

in  $e^{\rho(|q_1|+|p_1|+\dots+|q_d|+|p_d|)}$ , we get the result by (A.7) and the change of variables  $(p_k, q_k) \rightarrow (P_k, Q_k)$  (note that the covariance property with respect to the flow generated by  $X_0$  is exactly the same as explained in the beginning of the proof of Lemma 6.2).  $\square$

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