Spy-Game on graphs

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Abstract. We define and study the following two-player game on a graph G. Let $k \in \mathbb{N}^*$. A set of k *guards* is occupying some vertices of G while one *spy* is standing at some node. At each turn, first the spy may move along at most s edges, where $s \in \mathbb{N}^*$ is his *speed*. Then, each guard may move along one edge. The spy and the guards may occupy same vertices. The spy has to escape the surveillance of the guards, i.e., must reach a vertex at distance more than $d \in \mathbb{N}$ (a predefined distance) from every guard. Can the spy win against k guards? Similarly, what is the minimum distance d such that k guards may ensure that at least one of them remains at distance at most d from the spy? This game generalizes two well-studied games: Cops and robber games (when $s = 1$) and Eternal Dominating Set (when s is unbounded).
We consider the computational complexity of the problem, showing that it is NP-hard and that it is PSPACE-hard in DAGs. Then, we establish tight tradeoffs between the number of guards and the required distance d when G is a path or a cycle. Our main result is that there exists $\epsilon > 0$ such that $\Omega(n^{1+\epsilon})$ guards are required to win in any $n \times n$ grid.

1 Introduction

We consider the following two-player game on a graph G, called *Spy-game*. Let $k, d, s \in \mathbb{N}$ be three integers such that $k > 0$ and $s > 0$. One player uses a set of k *guards* occupying some vertices of G while the other player plays with one *spy* initially standing at some node. This is a full information game so any player has the full information about the positions and previous moves of the other player. Note that several guards and even the spy could occupy a same vertex.

Initially, the spy is placed at some vertex of G. Then, the k guards are placed at some vertices of G. Then, the game proceeds turn-by-turn. At each turn, first

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the spy may move along at most s edges (s is the *speed* of the spy). Then, each guard may move along one edge. The spy wins if, after a finite number of turns (after the guards' move), it reaches a vertex at distance greater than d from every guard. The guards win otherwise, in which case we say that the guards *control* the spy at distance d, i.e. that there is always at least one guard at distance at most d from the spy.

Given a graph G and two integers $d, s \in \mathbb{N}$, $s > 0$, let the *guard-number*, denoted by $gn_{s,d}(G)$, be the minimum number of guards required to control a spy with speed s at distance d, against any strategy from the spy. We also define the following dual notion. Given a graph G and two integers $k, s \in \mathbb{N}$, $s > 0$, $k > 0$, let $d_{s,k}(G)$, be the minimum distance d such that k guards can control a spy with speed s at distance d, whatever be the strategy of the spy.

1.1 Preliminary remarks

We could define the game by placing the guards first. In that case, since the spy could choose its initial vertex at distance greater than d from any guard, we need to slightly modify the rules of the game. If the guards are placed first, they win if, after a finite number of turns, they ensure that the spy always remains at distance at most d from at least one guard. Equivalently, the spy wins if it can reach infinitely often a vertex at distance greater than d from every guard. We show that both versions of the game are closely related. In what follows, we consider the spy-game against a spy with speed s that must be controlled at distance d for any fixed integers $s > 0$ and d.

Claim. If the spy wins in the game when it starts first, then it wins in the game when it is placed after the guards.

Proof of the claim. Assume that the spy has a winning strategy $\mathcal{S}$ when it is placed first. In particular, there is a vertex $v_0 \in V(G)$ such that, starting from v_0 and whatever be the strategy of the guards, the spy can reach a vertex at distance $> d$ from every guard. If the spy is placed after the guards, its strategy consists first at reaching v_0 and then at applying the strategy $\mathcal{S}$ until it is at distance $> d$ from every guard. The spy repeats this process infinitively often. $\diamond$

The converse is not necessary true, however we can prove a slightly weaker result which is actually tight. For this purpose, let us recall the definition of the well known *Cops and robber* game [15, 4]. In this game, first k cops occupy some vertices of the graph. Then, one robber occupies a vertex. Turn-by-turn, each player may move its token (the cops first and then the robber) along an edge. The cops win if one of them reach the same vertex as the robber after a finite number of turns. The robber wins otherwise. The *cop-number* $cn(G)$ of a graph G is the minimum number of cops required to win in G [1]. The proof of the following claim can be found in `https://hal.inria.fr/hal-01279339/file/RR-8869.pdf`.

Claim. If k guards win in the game when the spy is placed first in a graph G, then $k + cn(G) - 1$ guards win the game when they are placed first.

The bound of the previous claim is tight. Indeed, for any graph G, $gn_{1,0}(G) = 1$ since one guard can be placed at the initial position of the spy and then follows it. On the other hand, if the guards are placed first, the game (for $s = 1$ and $d = 0$) is equivalent to the classical Cops and robber game and, therefore, $cn(G)$ guards are required.

1.2 Related work

Further relationship with Cops and robber games. The Cops and robber game has been generalized in many ways [3, 8, 2, 5, 9]. In [3], Bonato *et al.* proposed a variant with *radius of capture*. That is , the cops win if one of them reaches a vertex at distance at most d (a fixed integer) from the robber. The version of our game when the guards are placed first and for $s = 1$ is equivalent to Cops and robber with radius of capture. Indeed, when the spy is not faster than the guards, capturing the spy (at any distance d) is equivalent to controling it at such distance: once a guard is at distance at most d from the spy, it can always maintain this distance (by following a shortest path toward the spy).

This equivalence is not true anymore as soon as $s > 1$. Indeed, one cop is always sufficient to capture one robber in any tree, whatever be the speed of the robber or the radius of capture. On the other hand, we prove below that $\Theta(n)$ cops are necessary to control a spy with speed at least 2 at some distance d in any n-node path. This is mainly due to the fact that, in the spy-game, the spy may cross (or even occupy) a vertex occupied by a guard. Therefore, in what follows, we only consider the case $s \geq 2$.

Note that the Cops and robber games when the robber is faster than the cops is far from being well understood. For instance, the exact number of cops with speed one required to capture a robber with speed two is unknown in grids [7]. One of our hopes when introducing the Spy-game is that it will lead us to a new approach to tackle this problem.

Generalization of Eternal Domination. A d-dominating set of a graph G is a set $D \subseteq V(G)$ of vertices such that any vertex $v \in V(G)$ is at distance at most d from a vertex in D. Let $\gamma_d(G)$ be the minimum size of a d-dominating set in G. Clearly, $gn_{s,d}(G) \leq \gamma_d(G)$ for any $s, d \in \mathbb{N}$. However these two parameters may differ arbitrary as shown by the following example. Let G be the graph obtained from a cycle C on n-vertices by adding a node x and, for any $v \in C$, adding a path of length $d + 1$ between v and x. It is easy to check that $\gamma_d(G) = \Omega(n/d)$ while $gn_{s,d}(G) = 2$ (the two guards moving on x and its neighbors).

In the *eternal domination* game [10, 11, 13, 14], a set of k *defenders* occupy some vertices of a graph G. At each turn, an *attacker* chooses a vertex $v \in V$ and the defenders may move to adjacent vertices in such a way that at least one defender is at distance at most d (a fixed predefined value) from v. Several variants of this game exist depending on whether exactly one or more defenders

may move at each turn [11, 13, 14]. It is easy to see that the spy-game, when the spy has unbounded speed (equivalently, speed at least the diameter of the graph) is equivalent to the Eternal Domination game when all defenders may move at each turn.

1.3 Our contributions

In this paper, we initiate the study of the spy-game for $s \geq 2$. In Section 2, we study the computational complexity of the problem of deciding the guard-number of a graph. We prove that computing $gn_{3,1}(G)$ is NP-hard in the class of graph G with diameter at most 5. Then, we show the problem is PSPACE-complete in the case of DAGs (where guards and spy have to follow the orientation of arcs, but distances are in the underlying graph). Then, we consider particular graph classes. In Section 3, we precisely characterize the cases of paths and cycles. Precisely, for any $k \geq 1$, $s \geq 2$, we prove that

$$\left\lfloor \frac{n(s-1)}{2ks} \right\rfloor \leq d_{s,k}(P_n) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil$$

for any path P_n on n vertices, and

$$\left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor \leq d_{s,k}(C_n) \leq \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor$$

for any cycle C_n on n vertices. Our most interesting result concerns the case of grids. In Section 4, we prove that there exists $\beta > 0$ such that $gn_{s,d}(G_{n\times n}) = \Omega(n^{1+\beta})$ in any $n \times n$ grid $G_{n\times n}$. For this purpose, we actually prove a lower bound on the number of guards required in a *fractional relaxation* of the game (the formal definition is given in the corresponding section).

Notations. As usual, we consider connected simple graphs. Given a graph $G = (V, E)$ and $v \in V$, let $N(v) = \{w \mid vw \in E\}$ denote the set of neighbors of v and let $N[v] = N(v) \cup \{v\}$.

2 Complexity

Theorem 1. *Given a graph G with diameter at most 5 and an integer k as inputs, deciding whether $gn_{3,1}(G) \leq k$ is NP-hard.*

Proof. The result is obtained by reducing the classical Set Cover Problem. In the Set Cover Problem the input is a set of elements $\mathcal{U}$, a family $\mathcal{S}$ of subsets of $\mathcal{U}$ such that $\cup_{S\in\mathcal{S}} S = \mathcal{U}$ and an integer k. The question is whether there exists a set $C \subseteq \mathcal{S}$ such that $|C| \leq k$ and $\cup_{S\in C} S = \mathcal{U}$, the set C is called a cover of $\mathcal{U}$.

Let $(\mathcal{U} = \{u_1, \ldots, u_n\}, \mathcal{S} = \{S_1, \ldots, S_m\}, k)$ be an instance of the Set Cover Problem. Note that, for any $i \leq n$, there exists $j \leq m$ such that $u_i \in S_j$ (since $\cup_{S\in\mathcal{S}} S = \mathcal{U}$). We create a graph G such that there is a cover $C \subseteq \mathcal{S}$ of $\mathcal{U}$ with size at most k if and only if $g_1^3(G) \leq k$.

The graph G is constructed in the following way. Abusing the notation, let us identify the elements in $\mathcal{U} \cup \mathcal{S}$ with some vertices of G. Let $V(G) = \mathcal{S} \cup \mathcal{U} \cup \mathcal{V}$ with $\mathcal{V} = \{v_1, \cdots, v_n\}$. Start with a complete graph with set of vertices $\mathcal{S} = \{S_1, \cdots, S_m\}$ and, for any $1 \leq i \leq n$, add an edge $\{u_i, v_i\}$. Finally, for any $i \leq n$ and $j \leq m$ such that $u_i \in S_j$, let us add an edge $\{u_i, S_j\}$.

First, let us prove that, if $\mathcal{U}$ admits a cover C of size at most k, then $g_1^3(G) \leq k$. For this purpose, we give a strategy for the guards that ensure that the spy is always at distance at most 1 from at least one guard. When the spy occupies a vertex in $C \cup \mathcal{U}$, the guards occupy all the vertices of C. When the spy occupies a vertex v_i for some $i \leq n$, let $j(i)$ be such that $u_i \in S_{j(i)} \in C$, then one guard occupies u_i and the other guards occupy the vertices of $C \setminus \{S_{j(i)}\}$. Because the speed of the spy is 3, from a vertex v_i, the spy can only reach a vertex in $C \cup \mathcal{U}$. Therefore, whatever be the initial position of the spy and its moves, the guards can always ensure the previously defined positions.

Suppose now that there is no cover C of $\mathcal{U}$ with size k, we show that $g_1^3(G) > k$. Let us assume at most k guards are occupying vertices in G, let us consider the following strategy for the spy. The spy starts at S_1. If there exists $i \leq n$ such that no guards dominate u_i, i.e., no guards occupy a vertex of $N[u_i]$, the spy goes at v_i (note that any vertex in $\{v_1, \cdots, v_n\}$ is at distance at most 3 from S_1). Then, no guard can reach a vertex at distance at most 1 from v_i (since u_i is the only neighbor of v_i) and the spy wins.

Let us show that such a vertex u_i exists by reverse induction on the number ℓ of guards occupying vertices in $\{S_1, \cdots, S_m\}$. That is, let $\mathcal{O}$ be the set of vertices occupied by the guards (note that $|\mathcal{O}| = k$) and let $\ell = |\mathcal{O} \cap \mathcal{S}|$. We show that there exists $i \leq n$ such that $\mathcal{O} \cap N[u_i] = \emptyset$. If $\ell = k$, i.e., $\mathcal{O} \subseteq \mathcal{S}$, then the result holds since there is no cover of $\mathcal{U}$ of size at most k. If $\ell < k$, there exists $j \leq n$ such that a guard is occupying u_j or v_j, i.e., there exists $x \in \{u_j, v_j\}$ such that $x \in \mathcal{O}$. Let $z \leq m$ such that $u_j \in S_z$ and let $\mathcal{O}' = \mathcal{O} \cup \{S_z\} \setminus \{x\}$. By induction and because $|\mathcal{O}' \cap \mathcal{S}| = \ell + 1$, there exists $i \leq n$ such that $\mathcal{O}' \cap N[u_i] = \emptyset$. Since $\mathcal{O} \cap N[u_p] \subseteq \mathcal{O}' \cap N[u_p]$ for any $p \leq n$, the result follows. $\square$

Note that the previous proof could be easily adapted for a speed $s > 2$ and distance $d = s - 2$ simply adjusting the size of the paths to $s - 1$. The question to generalize this result to any s and d is open. Moreover, since the set cover problem is not approximable within a factor of $(1 - o(1)) \ln n$ [6], our proof also implies the same result to the spy game.

Then, we consider a variant of our game played on digraphs. In this variant, both the guards and the spy can move only by following the orientation of the arcs. However, the distances are the ones of the underlying undirected graph.

Theorem 2. *The problem of computing $gn_{s,2}$ is PSPACE-hard in the class of DAGs, when the guards are placed first.*

The result is obtained by reducing the PSPACE-complete Quantified Boolean Formula in Conjunctive Normal Form (QBF) problem. Due to lack of space, the result can be found in `https://hal.inria.fr/hal-01279339/file/RR-8869.pdf`.

The question of the complexity of the spy game in undirected graphs is left open. Is it PSPACE-hard, or more probably EXPTIME-complete as Cops and Robber games [12]? The question of parameterized complexity is also open.

3 Case of paths and rings

In this section, we characterize optimal strategies in the case of two simple topologies: the path and the ring. For ease of readability, some proofs are given in the case $s = 2$. The general proofs (for any $s \geq 2$) are similar.

The following theorem directly follows from next two lemmas.

Theorem 3. *For any path P with $n+1$ nodes and for any $k \geq 1$ and $s \geq 2$,*

$$\left\lfloor \frac{n(s-1)}{2ks} \right\rfloor \leq d_{s,k}(P_n) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil$$

Lemma 1. *For any path P with $n+1$ nodes and for any $k \geq 1$ and $s \geq 2$, $d_{s,k}(P) \geq \lfloor \frac{n(s-1)}{2ks} \rfloor$.*

Proof. For ease of readability, we prove the lemma in the case $\frac{2d-1}{s-1} \in \mathbb{N}$.

Let $P = (v_0, v_1, \cdots, v_n)$. Let $d = \lfloor \frac{n(s-1)}{2ks} \rfloor$. We show that a spy with speed s playing against at most k guards can reach a vertex at distance at least d from any guard. Intuitively, the strategy of the spy simply consists in starting from one end of P and running at full speed toward the other end. We show that there must be a turn when the spy is at distance at least d from every guard and therefore $d_{s,k}(P) \geq d$.

More formally, let the strategy for the spy be the following. Initially, the spy is occupying an end of the path, say vertex v_0. Then, at each turn $i \geq 1$, the spy moves from $v_{i(s-1)}$ to v_{is}.

We prove by induction on $1 \leq i \leq k$, after turn $i\frac{2d-1}{s-1}$ (when the spy occupies $v_{si\frac{2d-1}{s-1}}$), either at least i guards are occupying vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}-d}\}$, or there is turn $0 \leq j < i\frac{2d-1}{s-1}$ such that, after Turn j, the distance between the spy and all guards was at least d.

Initially, there must be at least one guard, call it g_1, occupying some vertex in $\{v_0, \cdots, v_{d-1}\}$ because otherwise all guards are at distance at least d from the spy at Turn 0. Therefore, after Turn $\frac{2d-1}{s-1}$, Guard g_1 is occupying a vertex in $\{v_0, \cdots, v_{\frac{2d-1}{s-1}+d-1}\} = \{v_0, \cdots, v_{s\frac{2d-1}{s-1}-d}\}$ and the spy is occupying $v_{s\frac{2d-1}{s-1}}$. Hence, the induction hypothesis holds for $i = 1$. Note that the spy is at distance at least d from g_1.

Let $1 \leq i \leq k$ and let us assume by induction that, after Turn $i\frac{2d-1}{s-1}$, there are at least i guards occupying vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}-d}\}$. Moreover, by definition of the spy's strategy, the spy is occupying $v_{si\frac{2d-1}{s-1}}$. Note that, all these i guards are at distance at least d from the spy.

Then, after Turn $i\frac{2d-1}{s-1}$, there must be at least one guard, call it g_{i+1}, occupying some vertex in $\{v_{si\frac{2d-1}{s-1}-d+1}, \cdots, v_{si\frac{2d-1}{s-1}+d-1}\}$ because otherwise all

guards are at distance at least d from the spy at Turn i. Therefore, after Turn $(i+1)\frac{2d-1}{s-1}$, Guard g_{i+1} is occupying a vertex in $\{v_0, \cdots, v_{(si+1)\frac{2d-1}{s-1}+d-1}\}$, that is in $\{v_0, \cdots, v_{s(i+1)\frac{2d-1}{s-1}-d}\}$, and the spy is occupying $v_{(i+1)s\frac{2d-1}{s-1}}$. Similarly, all the i guards that were occupying some vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}}\}$ after Turn $i\frac{2d-1}{s-1}$ must occupy vertices in $\{v_0, \cdots, v_{s(i+1)\frac{2d-1}{s-1}-d}\}$ after Turn $(i+1)\frac{2d-1}{s-1}$. Hence, the induction hypothesis holds for $i+1$.

Therefore, after Turn $k\frac{2d-1}{s-1}$, either there has been a previous turn when the spy was at distance at least d from all guards, or all the k guards are occupying vertices in $\{v_0, \cdots, v_{sk\frac{2d-1}{s-1}-d}\}$ while the spy occupies $v_{ks\frac{2d-1}{s-1}}$ (note that this vertex exists since $ks\frac{2d-1}{s-1} \leq n$ by definition of d). In the latter case, the spy is at distance at least d from all guards at this turn. □

Lemma 2. *For any path P with $n+1$ nodes and any $k \geq 1$, $s \geq 2$,*

$$d_{s,k}(P) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil.$$

Proof. For ease of readability, we prove the lemma for $s = 2$.

It is clearly sufficient to prove the result in the case $d = \frac{n+1}{4k} \in \mathbb{N}$. Let $P = (v_0, \cdots, v_n)$ and, for any $1 \leq i \leq k$, let $P_i = (v_{4(i-1)d}, \cdots, v_{4di})$.

We design a strategy ensuring that k guards may maintain the spy at distance at most d from at least one guard. The ith guard is assigned to the subpath P_i (it moves only in P_i). Moreover, a guard i will move at some turn only if the move of the spy at this turn is along an edge of P_i (note that the subpaths P_i are edge-disjoint).

Let $i \leq k$ be such that the spy occupies the node $x = v_{(4i-2)d+\ell}$ with $-2d \leq \ell \leq 2d$. That is, $x \in P_i$. Let us assume that

- for any $1 \leq j < i$, the jth guard occupies $v_{(4j-1)d}$;
- for any $i < j \leq k$, the jth guard occupies $v_{(4j-3)d}$;
- the ith guard occupies $v_{(4i-2)d+\lfloor \ell/2 \rfloor}$ if $\ell \geq 0$ and $v_{(4i-2)d+\lceil \ell/2 \rceil}$ if $\ell \leq 0$.

Clearly, if these conditions are satisfied, the spy is at distance at most $\lceil |\ell|/2 \rceil \leq d$ from the ith guard. Moreover, such positions can be chosen by the guards once the spy has chosen its initial position.

We next show that, whatever be the move of the spy, we can maintain these conditions. Let y be the next vertex to be occupied by the spy. Note that $y = v_{(4i-2)d+\ell+a}$ with $a \in \{-2, -1, 0, +1, +2\}$.

We start with the case when x and y are not in the same subpath P_i. It may happen in only two cases: either $x = v_{4id-1}$ and $y = v_{4id+1}$ ($\ell = 2d-1$ and $a = +2$) or $x = v_{4(i-1)d+1}$ and $y = v_{4(i-1)d-1}$ ($\ell = -2d+1$ and $a = -2$). In the first case, the ith guard goes from $v_{(4i-1)d-1}$ to $v_{(4i-1)d}$ and the $(i+1)^{th}$ guard goes from $v_{(4(i+1)-3)d} = v_{(4i+1)d}$ to $v_{(4i+1)d+1}$. In the latter case, the ith guard goes from $v_{(4i-3)d+1}$ to $v_{(4i-3)d}$ and the $(i-1)^{th}$ guard goes from $v_{(4(i-1)-1)d}$ to $v_{(4(i-1)-1)d-1}$. In both cases, the conditions remain valid.

From now on, let us assume that x and y belong to P_i. In that case, only the ith guard may move. There are several cases depending on the value of $a \in \{-2, -1, 0, +1, +2\}$ and ℓ,

- if $\ell \geq 0$ and $\ell + a \geq 0$, then $v_{(4i-2)d+\lfloor(\ell+a)/2\rfloor} \in \{v_{(4i-2)d+\lfloor\ell/2\rfloor-1}, v_{(4i-2)d+\lfloor\ell/2\rfloor}; v_{(4i-2)d+\lfloor\ell/2\rfloor+1}\}$. Hence, whatever be the move of the spy, the ith guard can go from $v_{(4i-2)d+\lfloor\ell/2\rfloor}$ to $v_{(4i-2)d+\lfloor(\ell+a)/2\rfloor}$ either moving to one of its neighbor or staying idle.
- if $\ell \leq 0$ and $\ell + a \leq 0$ then $v_{(4i-2)d+\lceil(\ell+a)/2\rceil} \in \{v_{(4i-2)d+\lceil\ell/2\rceil-1}, v_{(4i-2)d+\lceil\ell/2\rceil}; v_{(4i-2)d+\lceil\ell/2\rceil+1}\}$. Hence, whatever be the move of the spy, the ith guard can go from $v_{(4i-2)d+\lceil\ell/2\rceil}$ to $v_{(4i-2)d+\lceil(\ell+a)/2\rceil}$ either moving to one of its neighbor or staying idle.
- finally, if $\ell * (\ell + a) < 0$, then $(\ell, a) = (-1, 2)$ or $(\ell, a) = (1, -2)$. In that case, the ith guard remains on $v_{(4i-2)d}$.

In all cases, all properties are satisfied after the move of the guards. □

We then consider the case of cycles. Due to lack of space, the proof is omitted and can be found in `https://hal.inria.fr/hal-01279339/file/RR-8869.pdf`.

Theorem 4. *For any cycle C with $n+1$ nodes and any $k \geq 1$,*

$$\left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor \leq d_{s,k}(C_n) \leq \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor.$$

4 Case of Grids

It is clear that, for any $n \times n$ grid G, $gn_{s,d}(G) = O(n^2)$. However, the exact order of magnitude of $gn_{s,d}(G)$ is not known. In this section, we prove that there exists $\beta > 0$, such that $\Omega(n^{1+\beta})$ guards are necessary to win against one spy in an $n \times n$-grid. Our lower bound actually holds for a relaxation of the game that we now define.

Fractional relaxation. In the *fractional relaxation* of the game, each guard can be *split* at any time, i.e., the guards are not required to be integral entities at any time but can be "fractions" of guards. More formally, let us assume that some *amount* $\alpha \in \mathbb{R}^+$ of guards occupies some vertex v at some step t, and let $N(v) = \{v_1, \cdots, v_{deg(v)}\}$. Then, at the its turn, the guards can choose any $deg(v) + 1$ nononegative reals $\alpha_0, \cdots, \alpha_{deg(v)} \in \mathbb{R}^+$ such that $\sum_i \alpha_i = \alpha$, and move an amount α_i of guards toward v_i, for any $0 \leq i \leq deg(v)$ (where $v = v_0$). Then, the guards must ensure that, at any step, the sum of the amount of guards occupying the nodes at distance at most d from the spy is at least one. That is, let $c_t(v) \in \mathbb{R}^+$ be the amount of guards occupying vertex v at step t. The guards wins if, for any step t, $\sum_{v \in B(R_t,d)} c_t(v) \geq 1$, where $B(R_t, d)$ denotes the ball of radius d centered into the position R_t of the spy at step t.

Let $g_{s,d}^{frac}(G)$ be the infimum total amount of guards (i.e., $\sum_{v\in V} c_0(v)$) required to win the fractional game at distance d and against a spy with speed s. Since any *integral strategy* (i.e. when guards cannot be split) is a fractional strategy, we get:

Proposition 1. *For any graph G and any integers d, s, $g_{s,d}^{frac}(G) \leq gn_{s,d}(G)$.*

Conversely, a fractional strategy can be to some extent represented by a variation of an integral strategy. Let G be a graph and d, s be two integers. Let also t, k be any two integers. In what follows, t and k will be arbitrary large and can be some function of n, the number of vertices of G. Let $g_{s,d}^{k,t}(G)$ be the minimum number of (integral) guards necessary to maintain at least k guards at distance $\leq d$ from a spy with speed s in G, during t turns. The next lemma will be used below to give a lower bound on $g_{s,d}^{frac}$.

Lemma 3. *Let G be a graph with n vertices and $d, s, t, k \in \mathbb{N}$ (t and k may be given by any function of n). Then,*

$$g_{s,d}^{k,t}(G) \leq k g_{s,d}^{frac}(G) + tn^2$$

Asymptotically, this yields a useful bound on $g_{s,d}^{frac}$: $\limsup_{k\to\infty} \frac{g_{s,d}^{k,t}(G)}{k} \leq g_{s,d}^{frac}(G)$.

Proof. From a fractional strategy using an amount c of guards, we produce an integer strategy keeping $\geq k$ guards around the spy. Initially, each vertex which has an amount x of guards receives $\lfloor xk \rfloor + tn$ guards, for total number of $\leq ck + tn^2$ guards.

We then ensure that, at step $i \in \{1, ..., t\}$, a vertex having an amount of x guards in the fractional strategy has $\geq xk + (t-i)n$ guards in the integer strategy. To this aim, whenever an amount x_{uv} of guards is to be transferred from u to v in the fractional strategy, we move $\lfloor x_{uv}k \rfloor + 1$ in the integer strategy.

As our invariant is preserved throughout the t steps, the spy which had an amount of ≥ 1 guards within distance d in the fractional strategy now has $\geq k$ guards around it, which proves the result. □

In what follows, we prove that $g_{s,d}^{frac}(G) = \Omega(n^{1+\beta})$ for some $\beta > 0$ in any $n \times n$-grid G. The next lemma is a key argument for this purpose.

Lemma 4. *Let $G = (V, E)$ be a graph and $d, s \in \mathbb{N}$ ($s \geq 2$), with $g_{s,d}^{frac}(G) > c \in \mathbb{Q}^*$ and the spy wins in at most t steps against c guards starting from $v \in V(G)$. For any strategy using a total amount $k > 0$ of guards, there exists a strategy for the spy (with speed $\leq s$) starting from $v \in V(G)$ such that after at most t steps, the amount of guards at distance at most d from the spy is less than k/c.*

Proof. For purpose of contradiction, assume that there is a strategy $\mathcal{S}$ using $k > 0$ guards that contradicts the lemma. Then consider the strategy $\mathcal{S}'$ obtained from $\mathcal{S}$ by multiplying the number of guards by c/k. That is, if $v \in V$ is initially occupied by $q > 0$ guards in $\mathcal{S}$, then $\mathcal{S}'$ places qc/k guards at v initially (note that $\mathcal{S}'$ uses a total amount of kc/k=c guards). Then, when $\mathcal{S}$ moves an amount q

of guards along an edge $e \in E$, $\mathcal{S}'$ moves qc/k guards along e. Since $\mathcal{S}$ contradicts the lemma, at any step $\leq t$, at least an amount k/c of guards is at distance at most d from the spy, whatever be the strategy of the spy. Therefore, $\mathcal{S}'$ ensures that an amount of at least 1 cop is at distance at most d from the spy during at least t steps. This contradicts that $g_{s,d}^{frac}(G) > c$ and that the spy wins after at most t steps. □

While it holds for any graph and its proof is very simple, we have not been able to prove a similar lemma in the classical (i.e., non-fractional) case.

The main technical lemma is the following. To prove it, we actually prove Lemma 6 which gives a lower bound on $g_{s,d}^{k,t}(G)$ in any grid G (this technical lemma is postponed at the end of the section). Then, it is sufficient to apply Lemmas 3 and 6 to obtain the following result.

Lemma 5. *Let G be a $n \times n$-grid and $a \in \mathbb{N}^*$ such that $d = 2n/a \in \mathbb{N}$. There exists $\gamma > 0$ such that $g_{s,d}^{frac}(G) \geq \gamma a H(a)$, where H is the harmonic function. Moreover, the spy wins after at most $2n$ steps starting from a corner of G.*

From Lemmas 4 and 5, we get

Corollary 1. *Let G be a $n \times n$-grid and $a \in \mathbb{N}^*$. For any strategy using a total amount of $k > 0$ guards, there exists a strategy for the spy (with speed $\leq s$) starting from a corner of G such that after at most $2n$ steps, the amount of guards at distance at most $2n/a$ from the spy is less than $k * (aH(a))^{-1}$.*

Theorem 5. *$\exists \beta, \gamma > 0$ such that, for any $n \times n$-grid $G_{n\times n}$ and $s, d \in \mathbb{N}$ ($s \geq 2$), the spy (with speed $\leq s$) can win (for distance d) in at most $2n$ steps against $< \gamma n^{1+\beta}$ guards.*

Proof. We actually prove that $\exists \beta > 0$ such that $\Omega(n^{1+\beta}) = g_{s,d}^{frac}(G_{n\times n})$ in any $n \times n$-grid $G_{n\times n}$ and the result follows from Proposition 1.

Let $a_0 \in \mathbb{N}$ be such that $H(a_0)^{-1} \leq 1/2$. Since $g_{s,d}^{frac}(G_{n\times n})$ is non-decreasing as a function of n, it is sufficient to prove the lemma for $n = (a_0)^i$ for any $i \in \mathbb{N}^*$.

We prove the result by induction on i. It is clearly true for $i = 1$ since a_0 is a constant. Assume by induction that there exists $\gamma, \beta > 0$, such that, for $i \geq 1$ with $n = (a_0)^i$, the spy (with speed $\leq s$) can win (for distance d) in at most $2n$ steps against $\gamma a_0^{i(1+\beta)}$ guards in any $n \times n$ grid.

Let G be a $n \times n$-grid with $n = (a_0)^{i+1}$. Let $k \leq \gamma n^{1+\beta}$. By Corollary 1, there exists a strategy for the spy (with speed $\leq s$) starting from a corner of G such that after $t \leq 2n$ steps, the amount of guards at distance at most $2n/a_0$ from the spy is less than $k * (a_0 H(a_0))^{-1} \leq k/(2a_0) \leq \gamma n^{1+\beta}/(2a_0)$.

Let v be the vertex reached by the spy at the step t of strategy $\mathcal{S}$. Let G' be any subgrid of G with side n/a_0 and corner G. By previous paragraph at most $\gamma n^{1+\beta}/(2a_0)$ can occupy the nodes at distance at most d from any node of G' during the next $2n/a_0$ steps of the strategy. So, by the induction hypothesis, the spy playing an optimal strategy in G' against at most $\gamma n^{1+\beta}/(2a_0)$ guards will win. □

Corollary 2. *$\exists \beta > 0$ such that, for any $n \times n$-grid $G_{n\times n}$ and $s, d \in \mathbb{N}$ ($s \geq 2$),*

$$g_{s,d}(G_{n\times n}) = \Omega(n^{1+\beta}).$$

To conclude, it remains to prove Lemma 5. As announced above, we actually prove a lower bound on $g^{k,t}_{s,d}(G)$. Since $g^{k,t}_{s,d}(G)$ is an nondecreasing function of s, it is sufficient to prove it for $s = 2$.

Lemma 6. *Let G be a $n \times n$ grid. $\exists \beta > 0$ such that for any $d, k > 0$, $g^{k,2n}_{2,d}(G) \geq \beta k \frac{n}{d} H(\frac{n}{d})$.*

Proof. Let G be a $n \times n$ grid and let us identify its vertices by their natural coordinates. That is, for any $(i_1, j_1), (i_2, j_2) \in [n]^2$, vertex (i_1, j_1) is adjacent to vertex (i_2, j_2) if $|i_1 - i_2| + |j_1 - j_2| = 1$.

In order to prove the result, we will consider a *family* of strategies for the spy. For every $r \in [n]$, the spy starts at position $(0, 0)$ and runs at full speed toward $(r, 0)$. Once there, it continues at full speed toward $(r, n-1)$. We name P_r the path it follows during this strategy, which is completed in $\lceil \frac{1}{2}(r + n - 1)\rceil$ tops.

Let us assume that there exists a strategy using an amount q of guards that maintains at least k guards at distance at most d from the spy during at least $2n$ turns. Moreover, the spy only plays the strategies described above.

Assuming that the guards are labelled with integers in $[q]$, we can name at any time of strategy P_r the labels of k guards that are at distance $\leq d$ of the spy. In this way, we write $c(2r, 2j)$ this set of k guards that are at distance $\leq d$ from the spy, when the spy is at position $(2r, 2j)$.

Claim. If $|j_2 - j_1| > 2d$, then $c(2r, 2j_1)$ and $c(2r, 2j_2)$ are disjoint.

Proof of the claim. Assuming $j_1 < j_2$, it takes $j_2 - j_1$ tops for the spy in strategy P_r to go from $(2r, 2j_1)$ to $c(2r, 2j_2)$. A cop cannot be at distance $\leq d$ from $(2r, 2j_1)$ and, $j_2 - j_1$ tops later, at distance $\leq d$ from $(2r, 2j_2)$. Indeed, to do so its speed must be $\geq 2(j_2 - j_1 - d)/(j_2 - j_1) > 1$, a contradiction. $\diamond$

Claim. If $|r_2 - r_1| > 2d + 2\min(j_1, j_2)$, then $c(2r_1, 2j_1)$ and $c(2r_2, 2j_2)$ are disjoint.

Proof of the claim. Assuming $r_1 < r_2$, note that strategies P_{2r_1} and P_{2r_2} are identical for the first r_1 tops. By that time, the spy is at position $(2r_1, 0)$. If $c(2r_1, 2j_1)$ intersects $c(2r_2, 2j_2)$, it means that at this instant some cop is simultaneously at distance $\leq d + j_1$ from $(2r_1, 2j_1)$ (strategy P_{2r_1}) and at distance $\leq d + |r_2 - r_1| + j_2$ from $(2r_2, 2j_2)$ (strategy P_{2r_2}). As those two points are at distance $2|r_2 - r_1| + 2|j_2 - j_1|$ from each other, we have:

$$\begin{aligned} 2|r_2 - r_1| + 2|j_2 - j_1| &\leq (d + j_1) + (d + |r_2 - r_1| + j_2) \\ |r_2 - r_1| + 2|j_2 - j_1| &\leq 2d + j_1 + j_2 \\ |r_2 - r_1| &\leq 2d + 2\min(j_1, j_2) \end{aligned}$$

$\diamond$

We can now proceed to prove that the number of guards is sufficiently large. To do so, we define a graph H on a subset of $V(G)$ and relate the distribution of the guards (as captured by c) with the independent sets of H. It is defined over $V(H) = \{(2r, 4dj) : 2r \in [n], 4dj \in [n]\}$, where:

- $(2r, 4dj_1)$ is adjacent with $(2r, 4dj_2)$ for $j_1 \neq j_2$ (see Claim 4).
- $(2r_1, 4dj_1)$ is adjacent with $(2r_2, 4dj_2)$ if $|r_2 - r_1| > 4d(1 + \min(j_1, j_2))$ (see Claim 4).

By definition, c gives k colors to each vertex of H, and any set of vertices of H receiving a common color is an independent set of H. If we denote by $\#c^{-1}(x)$ the number of vertices which received color x, and by $\alpha_{(2r_1,4dj_1)}(H)$ the maximum size of an independent set of H containing $(2r_1, 4dj_1)$, we have:

$$\begin{aligned} q &= \sum_{(2r_1,4dj_1)\in V(H)} \sum_{x\in c(2r_1,4dj_1)} \frac{1}{\#c^{-1}(x)} \\ &\geq \sum_{(2r_1,4dj_1)\in V(H)} \frac{k}{\alpha_{((2r_1,4dj_1))}(H)} \end{aligned}$$

It is easy, however, to approximate this lower bound.

Claim. $\alpha_{((2r_1,4dj_1))}(H) \leq 4d(j_1 + 1) + 1$

Proof of the claim. An independent set $S \subseteq V(H)$ containing $(2r_1, 4dj_1)$ cannot contain two vertices with the same first coordinate. Furthermore, $(2r_1, 4dj_1)$ is adjacent with any vertex $(2r_2, 4dj_2)$ if $|r_2 - r_1| > 4d(1 + j_1)$. $\diamond$

We can now finish the proof:

$$\begin{aligned} q &\geq \sum_{(2r_1,4dj_1)\in V(H)} \frac{k}{\alpha_{((2r_1,4dj_1))}(H)} \\ &\geq \sum_{(2r_1,4dj_1)\in V(H)} \frac{k}{4d(j_1+1)+1} \\ &\geq \frac{n}{2} \sum_{j_1\in\{0,\ldots,n/4d\}} \frac{k}{4d(j_1+1)+1} \\ &\geq \frac{kn}{16d} \sum_{j_1\in\{1,\ldots,n/4d+1\}} \frac{1}{j_1} \qquad \geq \frac{kn}{16d} H(n/4d) \end{aligned}$$

$\square$

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