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Centers and partial volumes of convex cones II. Advanced topics

Alberto Seeger · Mounir Torki

Abstract This is the second part of an extensive work on volumetric centers and least partial volumes of proper cones in $\mathbb{R}^n$. The first part [cf. Seeger and Torki (Beiträge Algebra Geom, 2014) Centers and partial volumes of convex cones. I: Basic theory] was devoted to presenting the general theory. We now treat some more specialized issues. The notion of least partial volume is a reasonable alternative to the classical concept of solid angle, whereas the concept of volumetric center is an alternative to the old notion of incenter.

Keywords Partial volume of a convex cone · Solid angle · Volumetric center · Incenter · Homogeneous cone · Blaschke-Santaló inequality

1 Introduction

This paper is a continuation of our previous work (Seeger and Torki 2014), therefore we keep exactly the same notation and terminology. Although this paper is essentially self-contained, we recommend the reader to get familiar first with Seeger and Torki (2014). Let Π_n denote the set of proper cones in $\mathbb{R}^n$. A closed convex cone is said to be proper if it is pointed and has nonempty interior.

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The least partial volume of $K \in \Pi_n$, denoted by $\mathrm{lpv}(K)$, is defined as the minimal value of

$$v_K(x) := \mathrm{vol}_n(K \cap H_x)$$

when x ranges on $\mathbb{S}_n$, the unit sphere of $\mathbb{R}^n$. Here, H_x refers to the closed half-space given by

$$H_x := \{u \in \mathbb{R}^n \,:\, \langle x, u\rangle \leq 1\}.$$

The least partial volume of the positive dual cone K^* is defined in a similar way, i.e.,

$$\mathrm{lpv}(K^*) := \min_{x \in \mathbb{S}_n} v_{K^*}(x). \tag{1}$$

The minimization problem (1) admits a unique solution, which is denoted by $\varrho(K)$ and called the *volumetric center* of K. Least partial volumes and volumetric centers are mathematical objects with a very appealing geometric meaning. From a purely analytic point of view, it is helpful to remember that $\varrho(K)$ is characterized as the unique solution to the fixed point problem

$$x \in \mathrm{int}(K), \quad x = \nabla\Gamma_K(x), \tag{2}$$

where $\Gamma_K : \mathrm{int}(K) \to \mathbb{R}$ is the strictly concave function given by

$$\Gamma_K(x) := -\frac{1}{n}\log\left[v_{K^*}(x)\right].$$

Such a characterization of $\varrho(K)$ has multiple consequences. In this paper we use intensively the gradient map $\nabla\Gamma_K : \mathrm{int}(K) \to \mathbb{R}^n$ and the fixed point equation in (2) for deriving a number of specialized results on volumetric centers. For the sake of convenience in the presentation, we start by considering the particular class of simplicial cones and then we move on to a more general setting.

2 Simplicial cones

Simplicial cones have a very simple structure, but they offer already some interesting challenges in relation to the computation of their volumetric centers. A simplicial cone in $\mathbb{R}^n$ is a proper cone that can be represented in the form $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$. The notation $\mathbb{GL}(n)$ refers to the set of nonsingular matrices of order n and the symbol $G^{-T} = (G^{-1})^T$ indicates the transpose of the inverse of G.

Proposition 2.1 *Let $\{g_1, \ldots, g_n\}$ and $\{f_1, \ldots, f_n\}$ be the columns of $G \in \mathbb{GL}(n)$ and $F = G^{-T}$, respectively. For $\bar{x} \in \mathbb{R}^n$, the following statements are equivalents:*

(a) *$\bar{x}$ is the volumetric center of $G(\mathbb{R}^n_+)$.*
(b) *$\bar{x}$ solves the fixed point problem*

$$x = \frac{1}{n}\sum_{i=1}^{n} \frac{f_i}{\langle f_i, x\rangle}, \quad F^T x \in \operatorname{int}(\mathbb{R}^n_+). \tag{3}$$

(c) *$\bar{x}$ solves the nonlinear system*

$$\left.\begin{aligned} \langle f_j, x\rangle\langle g_j, x\rangle &= 1/n \\ \langle f_j, x\rangle &\geq 0 \end{aligned}\right\} \quad \textit{for all} \quad j \in \{1, \ldots, n\}. \tag{4}$$

(d) *$\bar{x}$ solves the fixed point problem*

$$x = \frac{1}{n}\sum_{j=1}^{n} \frac{g_j}{\langle g_j, x\rangle}, \quad G^T x \in \operatorname{int}(\mathbb{R}^n_+). \tag{5}$$

(e) *$\bar{x}$ is the volumetric center of $F(\mathbb{R}^n_+)$.*

Proof Let $K = G(\mathbb{R}^n_+)$. Then $K^* = F(\mathbb{R}^n_+)$ and

$$\begin{aligned} \operatorname{int}(K) &= \{x \in \mathbb{R}^n : F^T x \in \operatorname{int}(\mathbb{R}^n_+)\}, \\ \operatorname{int}(K^*) &= \{x \in \mathbb{R}^n : G^T x \in \operatorname{int}(\mathbb{R}^n_+)\}. \end{aligned}$$

A matter of computation yields

$$\begin{aligned} v_K(x) &= \frac{1}{n!}\frac{|\det G|}{\Pi_{j=1}^n \langle g_j, x\rangle}, \\ \nabla\Gamma_{K^*}(x) &= \frac{1}{n}\sum_{j=1}^{n} \frac{g_j}{\langle g_j, x\rangle} \end{aligned} \tag{6}$$

for all $x \in \operatorname{int}(K^*)$. Similarly,

$$\begin{aligned} v_{K^*}(x) &= \frac{1}{n!}\frac{|\det F|}{\Pi_{i=1}^n \langle f_i, x\rangle}, \\ \nabla\Gamma_K(x) &= \frac{1}{n}\sum_{i=1}^{n} \frac{f_i}{\langle f_i, x\rangle} \end{aligned}$$

for all $x \in \operatorname{int}(K)$. The equivalences (a) $\Leftrightarrow$ (b) and (d) $\Leftrightarrow$ (e) are obtained by applying the fixed point characterization (2) to the cones K and K^*, respectively. For proving (b) $\Rightarrow$ (c), we multiply by g_j on each side of (3) and observe that

$$\langle f_i, g_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For proving the reverse implication (c) $\Rightarrow$ (b), we rely on the fact that

$$\sum_{i=1}^{n} f_i g_i^T = FG^T = I_n,$$

where I_n is the identity matrix of order n. If $\bar{x}$ solves (4), then $F^T\bar{x} \in \operatorname{int}(\mathbb{R}^n_+)$ and

$$\bar{x} = \left(\sum_{i=1}^{n} f_i g_i^T\right)\bar{x} = \sum_{i=1}^{n} \langle g_i, \bar{x}\rangle f_i = \frac{1}{n}\sum_{i=1}^{n} \frac{f_i}{\langle f_i, \bar{x}\rangle}.$$

The proof of the equivalence (d) $\Leftrightarrow$ (c) is similar to that of (b) $\Leftrightarrow$ (c). It is a matter of exchanging the roles of G and F. □

Corollary 2.2 *Let K be a simplicial cone in $\mathbb{R}^n$. Then*

$$\varrho(K^*) = \varrho(K), \tag{7}$$

$$\operatorname{lpv}(K^*)\operatorname{lpv}(K) = \left[\operatorname{lpv}(\mathbb{R}^n_+)\right]^2. \tag{8}$$

Proof Let $\bar{x} = \varrho(K)$. The equality (7) is due to the equivalence (a) $\Leftrightarrow$ (e) in Proposition 2.1 As we saw in Seeger and Torki (2014), the least partial volume of the nonnegative orthant is given by

$$\operatorname{lpv}\left(\mathbb{R}^n_+\right) = \frac{n^{n/2}}{n!}.$$

One has

$$\operatorname{lpv}(K^*) = v_{K^*}(\bar{x}) = \frac{1}{n!}\,\frac{|\det F|}{\Pi_{i=1}^n \langle f_i, \bar{x}\rangle}.$$

But $|\det F| = |\det G|^{-1}$ and $\left(\Pi_{i=1}^n \langle f_i, \bar{x}\rangle\right)^{-1} = n^n\,\Pi_{j=1}^n \langle g_j, \bar{x}\rangle$.Thus,

$$\operatorname{lpv}(K^*) \;=\; \frac{n^n}{n!}\left(\frac{|\det G|}{\Pi_{j=1}^n \langle g_j, \bar{x}\rangle}\right)^{-1} \;=\; \frac{n^n}{(n!)^2}\,(\operatorname{lpv}(K))^{-1}.$$

This proves (8). □

Except for some specially structured simplicial cones, it is not clear how to derive an explicit formula for the solution to any of the equivalent systems considered in Proposition 2.1. The next proposition concerns the particular class of equiangular simplicial cones. By definition, a simplicial cone is *equiangular* if all pairs of generators form the same angle.

Proposition 2.3 *Let $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$. Suppose that*

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ \lambda & \text{if } i \neq j, \end{cases} \tag{9}$$

with $-1 < \lambda < 1$. Then $1 + (n-1)\lambda > 0$ and

$$\varrho(K) = \left\| \sum_{i=1}^{n} g_i \right\|^{-1} \sum_{i=1}^{n} g_i, \tag{10}$$

$$\operatorname{lpv}(K) = \frac{n^{n/2}}{n!} \left(\frac{1-\lambda}{1+(n-1)\lambda} \right)^{(n-1)/2}. \tag{11}$$

Proof Let b_G denote the right-hand side of (10). Let $\mathcal{S}(K)$ be the intersection of all linear subspaces of $\mathbb{R}^n$ with respect to which K is symmetric. Symmetry with respect to a linear subspace means invariance under reflection through that subspace (cf. Barker and Carlson 1980, Definition 1). Thanks to the equiangularity condition (9), one has $\mathcal{S}(K) = \mathbb{R}b_G$. In particular, $\mathcal{S}(K)$ is a one dimensional subspace. Formula (10) is then obtained by applying (Seeger and Torki 2014, Theorem 4.2). In view of (6) and Corollary 2.2, one gets

$$\operatorname{lpv}(K) = \frac{1}{n!} \frac{|\det G|}{\Pi_{i=1}^{n} \langle g_i, b_G \rangle}. \tag{12}$$

As shown in (Iusem and Seeger 2008, Lemma 3), the equiangularity condition (9) implies that

$$|\det G| = (1-\lambda)^{(n-1)/2} (1+(n-1)\lambda)^{1/2}.$$

On the other hand, for all $i \in \{1, \ldots, n\}$, one has

$$\langle g_i, b_G \rangle = n^{-1} (1+(n-1)\lambda)^{1/2}.$$

By substituting this information into (12), one obtains the announced formula (11). □

Remark 2.4 Let K be an equiangular simplicial cone as in Proposition 2.1. Then K^* is also an equiangular simplicial cone. Furthermore,

$$\operatorname{lpv}\left(K^*\right) = \frac{n^{n/2}}{n!} \left(\frac{1+(n-1)\lambda}{1-\lambda} \right)^{(n-1)/2},$$

as one can see by combining (8) and (11).

2.1 Volumetric center versus incenter

A widely used concept of "center" for a proper cone is that of incenter, cf. (Henrion and Seeger 2010a, b; Sitarz 2013). The *incenter* of $K \in \Pi_n$, denoted by $\xi(K)$, is defined as the unique solution to the maximization problem

$$r(K) := \max_{x \in \mathbb{S}_n \cap K} \operatorname{dist}(x, \partial K),$$

where $\operatorname{dist}(x, \partial K)$ stands for the distance from x to the boundary of K.

Is there a link between the volumetric center and the incenter of a proper cone? For an equiangular simplicial cone K, one has $\varrho(K) = \xi(K)$. This equality is however unlikely to occur if K is a simplicial cone constructed at random.

Proposition 2.5 *Let $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$. Assume that*

$$\sum_{j=1}^{n} \langle g_i, g_j \rangle \|f_j\| \geq 0 \quad \textit{for all } i \in \{1, \ldots, n\}, \tag{13}$$

where f_j is the jth column of $F = G^{-T}$. Then $\varrho(K) = \xi(K)$ if and only if

$$\sum_{j=1}^{n} \|f_j\| g_j = \frac{1}{n} \left\| \sum_{j=1}^{n} \|f_j\| g_j \right\|^2 \sum_{j=1}^{n} \frac{f_j}{\|f_j\|}. \tag{14}$$

Proof As shown in (Henrion and Seeger 2011, Theorem 2.4), the assumption (13) implies that the incenter of K is equal to the vector

$$\bar{x} := \frac{\sum_{j=1}^{n} \|f_j\| g_j}{\left\| \sum_{j=1}^{n} \|f_j\| g_j \right\|}. \tag{15}$$

Furthermore,

$$\frac{\langle f_i, \bar{x} \rangle}{\|f_i\|} = \left\| \sum_{j=1}^{n} \|f_j\| g_j \right\|^{-1} = r(K)$$

for all $i \in \{1, \ldots, n\}$. In particular, $F^T \bar{x} \in \operatorname{int}(\mathbb{R}^n_+)$. For completing the proof of the proposition, it suffices to observe that (14) amounts to saying that (15) solves the fixed point problem (3). □

Remark 2.6 In Proposition 2.5, there is no loss of generality in choosing G so that the columns of F are unit vectors. In such a case, the condition (14) takes the simpler form

$$G\mathbf{1_n} = \frac{1}{n} \|G\mathbf{1_n}\|^2 F\mathbf{1_n},$$

where $\mathbf{1_n}$ is the n-dimensional vector of ones. In particular, $G\mathbf{1_n}$ and $F\mathbf{1_n}$ must be collinear.

For a general simplicial cone K, the volumetric center is usually different from the incenter. In fact, the angle

$$\theta(K) := \arccos\langle \varrho(K), \xi(K)\rangle$$

between these vectors could be fairly large. The next example shows that such angle could be almost 45 degrees.

Example 2.7 Let K be the simplicial cone in $\mathbb{R}^4$ generated by columns of

$$G = \begin{bmatrix} 1-\alpha & 1-\alpha & 0 & 0 \\ \sqrt{\alpha} & -\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1-\beta & 1-\beta \\ 0 & 0 & \sqrt{\beta} & -\sqrt{\beta} \end{bmatrix},$$

where $0 < \alpha, \beta < 1$. By relying on Proposition 2.1 one can check that $\varrho(K) = (1/\sqrt{2})\,(1, 0, 1, 0)^T$. On the other hand, a matter of computation shows that

$$\xi(K) = \left(\frac{\sqrt{m_\beta}}{\sqrt{m_\alpha + m_\beta}}, 0, \frac{\sqrt{m_\alpha}}{\sqrt{m_\alpha + m_\beta}}, 0 \right)^T$$

with $m_\alpha := \alpha[(1-\alpha)^2 + \alpha]^{-1}$ and m_β defined in a similar way. Note that

$$\theta(K) = \arccos\left(\frac{1}{\sqrt{2}} \frac{\sqrt{m_\alpha} + \sqrt{m_\beta}}{\sqrt{m_\alpha + m_\beta}} \right)$$

is near $\arccos(1/\sqrt{2}) = 0.25\,\pi$ when (α, β) is near $(0, 1)$.

The last row of Table 1 displays the expected value of the random variable $\theta(\mathbf{K})$, where $\mathbf{K}$ stands for a simplicial cone whose generators are stochastically independent random vectors following a uniform probability distribution on the unit sphere $\mathbb{S}_n$. For each n, the expected value of $\theta(\mathbf{K})$ is computed by working with a sample of 10^5 random simplicial cones. The second row displays not the expected value, but the largest observed value. As one can see from the last row of Table 1, it is very unlikely to get $\theta(\mathbf{K})$ near $0.25\,\pi$. For instance, when $n = 3$ the expected angle between the incenter and the volumetric center is $0.0528\,\pi$.

According to a terminology used in Henrion and Seeger (2011), a cone $K \in \Pi_n$ is called *non-eccentric* if $\xi(K) = \xi(K^*)$. Non-eccentricity is a geometric property

Table 1 Largest observed value (second row) and expected value (third row) of $\theta(\mathbf{K})$

$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$0.1988\,\pi$	$0.2239\,\pi$	$0.2409\,\pi$	$0.2479\,\pi$	$0.2492\,\pi$	$0.2497\,\pi$
$0.0528\,\pi$	$0.0720\,\pi$	$0.0829\,\pi$	$0.1025\,\pi$	$0.1089\,\pi$	$0.1127\,\pi$

studied in depth in the references Henrion and Seeger (2011) and Seeger and Torki (2013). The following proposition is a nontrivial result borrowed from (Seeger and Torki 2013, Theorem 2.2).

Proposition 2.8 *Let $G \in \mathbb{GL}(n)$ and $F = G^{-T}$. For $K = G(\mathbb{R}^n_+)$ to be non-eccentric it is necessary and sufficient that*

$$\frac{\sum_{k=1}^n \|f_k\| g_k}{\left\| \sum_{k=1}^n \|f_k\| g_k \right\|} = \frac{\sum_{k=1}^n \|g_k\| f_k}{\left\| \sum_{k=1}^n \|g_k\| f_k \right\|}. \tag{16}$$

In case of non-eccentricity, the vector on the left-hand side (respectively, right-hand side) of the Eq. (16) corresponds to the incenter of K (respectively, the incenter of K^*). The following theorem is a useful complement to Proposition 2.8.

Theorem 2.9 *A simplicial cone K in $\mathbb{R}^n$ satisfies the double equality*

$$\varrho(K) = \xi(K) = \xi(K^*) \tag{17}$$

if and only if one can write $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$ such that

$$\begin{cases} \text{the columns of } G \text{ are unit vectors,} \\ \text{the columns of } G^{-T} \text{ have equal length,} \\ G\mathbf{1_n} \text{ and } G^{-T}\mathbf{1_n} \text{ are collinear.} \end{cases} \tag{18}$$

Proof Let K be a simplicial cone satisfying (17). Without loss of generality, one may assume that $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$ such that $\|g_k\| = 1$ for all $k \in \{1, \ldots, n\}$. Let $F = G^{-T}$ and $\mu_F = (\|f_1\|, \ldots, \|f_n\|)^T$. Since K is non-eccentric, the vectors

$$\xi(K) = \frac{\sum_{k=1}^n \|f_k\| g_k}{\left\| \sum_{k=1}^n \|f_k\| g_k \right\|} = \frac{G\mu_F}{\|G\mu_F\|},$$
$$\xi(K^*) = \frac{\sum_{k=1}^n f_k}{\left\| \sum_{k=1}^n f_k \right\|} = \frac{F\mathbf{1_n}}{\|F\mathbf{1_n}\|}$$

are equal. On the other hand, by Proposition 2.1 one knows that

$$\langle \varrho(K), f_k \rangle \langle \varrho(K), g_k \rangle = 1/n$$

for all $k \in \{1, \ldots, n\}$. In view of (17), the above equality can be written as

$$\langle \xi(K), f_k \rangle \langle \xi(K^*), g_k \rangle = 1/n.$$

But $\langle \xi(K), f_k \rangle = \|G\mu_F\|^{-1} \|f_k\|$ and $\langle \xi(K^*), g_k \rangle = \|F\mathbf{1_n}\|^{-1}$. This proves that all the f_k's have the same length, namely

$$\|f_k\| = n^{-1}\|G\mu_F\| \, \|F\mathbf{1_n}\|.$$

It follows that $\mu_F = \gamma \mathbf{1_n}$ for some positive constant γ. Hence, the equality $\xi(K) = \xi(K^*)$ becomes

$$\frac{G\mathbf{1_n}}{\|G\mathbf{1_n}\|} = \frac{F\mathbf{1_n}}{\|F\mathbf{1_n}\|}.$$

This completes the proof of (18). Conversely, let $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$ as in (18). Let $F = G^{-T}$. By assumption, one knows that

$$\begin{cases} \|g_k\| = 1 \text{ for all } k \in \{1, \ldots, n\}, \\ \mu_F = \gamma \mathbf{1_n} \text{ for some } \gamma > 0, \\ G\mathbf{1_n} \text{ and } F\mathbf{1_n} \text{ are collinear.} \end{cases}$$

Hence,

$$\frac{G\mu_F}{\|G\mu_F\|} = \frac{G\mathbf{1_n}}{\|G\mathbf{1_n}\|} = \frac{F\mathbf{1_n}}{\|F\mathbf{1_n}\|}.$$

Since (16) holds, one deduces that $\xi(K) = \xi(K^*)$. Furthermore, for all $k \in \{1, \ldots, n\}$, one has

$$\begin{aligned} \langle \xi(K), f_k \rangle \, \langle \xi(K), g_k \rangle = \langle \xi(K), f_k \rangle \, \langle \xi(K^*), g_k \rangle &= \frac{\langle G\mathbf{1_n}, f_k \rangle}{\|G\mathbf{1_n}\|} \, \frac{\langle F\mathbf{1_n}, g_k \rangle}{\|F\mathbf{1_n}\|} \\ &= \frac{1}{\|G\mathbf{1_n}\|} \, \frac{1}{\|F\mathbf{1_n}\|} = \frac{1}{\langle G\mathbf{1_n}, F\mathbf{1_n} \rangle} = \frac{1}{n}. \end{aligned}$$

In other words, $\xi(K)$ solves the system (4). This proves that $\varrho(K) = \xi(K)$. □

The simplicial cone of Example 2.7 is eccentric. However, even for a non-eccentric simplicial cone, the volumetric center may be different from the incenter. The following example illustrates this fact.

Example 2.10 Consider the simplicial cone K generated by the columns of the matrix

$$G = \begin{bmatrix} 1 & 5/7 & 0 & 0 & 0 \\ 0 & \sqrt{24}/7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & \sqrt{3}/2 & \sqrt{3}/6 \\ 0 & 0 & 0 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

Each column of G is a unit vector and $F = G^{-T}$ is given by

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -5/\sqrt{24} & 7/\sqrt{24} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1/\sqrt{3} & 2/\sqrt{3} & 0 \\ 0 & 0 & -1/\sqrt{6} & -1/\sqrt{6} & 3/\sqrt{6} \end{bmatrix}.$$

By using Proposition 2.8, one can check that

$$\xi(K) = \xi(K^*) = \frac{1}{4}(\sqrt{6}, 1, \sqrt{6}, \sqrt{2}, 1)^T.$$

The above vector does not solve the fixed point problem (5). Hence, $\varrho(K)$ is different from $\xi(K)$.

The above example takes place in a space of dimension $n = 5$. Similar examples can be constructed in spaces of higher dimensions, but not in spaces of lower dimensions.

Proposition 2.11 *Let K be a non-eccentric simplicial cone in $\mathbb{R}^n$, with $n \in \{2, 3, 4\}$. Then $\varrho(K) = \xi(K)$.*

Proof The case $n = 2$ is obvious. The case $n = 3$ is because any non-eccentric simplicial cone in $\mathbb{R}^3$ is equiangular, cf. (Seeger and Torki 2013, Proposition 5.1). Finally, consider the case of a non-eccentric simplicial cone K in $\mathbb{R}^4$. As shown in the proof of Seeger and Torki (2013, Proposition 6.6), one can write $K = G(\mathbb{R}^4_+)$ with $G \in \mathbb{GL}(4)$ such that

$$\begin{cases} \text{the columns of } G \text{ are unit vectors,} \\ \text{the columns of } G^{-T} \text{ have equal length,} \\ G\mathbf{1_4} \text{ and } G^{-T}\mathbf{1_4} \text{ are collinear.} \end{cases}$$

It suffices then to apply Theorem 2.9. □

When it comes to compute numerically the volumetric center of a general simplicial cone, there are several options at our disposal. The most natural choice is to solve the smooth convex optimization program

$$\begin{cases} \text{maximize } \sum_{j=1}^n \ln \langle g_j, x \rangle \\ \|x\|^2 \leq 1, \\ \langle g_j, x \rangle > 0 \quad \text{for all } j \in \{1, \ldots, n\}. \end{cases}$$

This can be done by using a standard smooth convex optimization algorithm, so we do not need to further elaborate on this issue. In our numerical tests, we simply use the code "fmincon" of MATLAB.

An alternative option is to apply Newton's method to the system (5). The problem at hand is to find the coefficients $\lambda_1, \ldots, \lambda_n$ in the linear combination

$$\varrho(K) = \sum_{j=1}^{n} \lambda_j g_j. \tag{19}$$

If one substitutes (19) into (5), then one sees that $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ solves the system

$$\frac{1}{n\lambda_i} = \sum_{j=1}^{n} \langle g_i, g_j \rangle \lambda_j \quad \text{for } i = 1, \ldots, n, \tag{20}$$

$$\lambda \in \operatorname{int}(\mathbb{R}^n_+). \tag{21}$$

It is then natural to apply Newton's method to (20) with an initial point taken from $\operatorname{int}(\mathbb{R}^n_+)$. In our numerical tests, we have observed that the constraint (21) can be neglected if the vectors $\{g_1, \ldots, g_n\}$ are mutually acute in the sense that $\langle g_i, g_j \rangle \geq 0$ for all $i \neq j$.

2.2 Nonsimplicial polyhedral cones

Nonsimplicial proper polyhedral cones in $\mathbb{R}^n$ are proper cones of the form

$$K = G(\mathbb{R}^p_+) = \operatorname{cone}\{\mathrm{g}_1, \ldots, \mathrm{g}_\mathrm{p}\}, \tag{22}$$

where the number p of columns of G is greater than n. In the representation (22) one assumes that none of the g_i's can be written as positive linear combination of the others. This class of cones is more difficult to handle than the class of simplicial cones. It is worthwhile to observe that for a nonsimplicial polyhedral cone $K \in \Pi_n$, there is no guarantee that $\varrho(K^*)$ coincides with $\varrho(K)$. To see this, consider the following example.

Example 2.12 Let K be the proper polyhedral cone in $\mathbb{R}^3$ generated by the columns of

$$G = \begin{bmatrix} 3 & 2 & -3 & -2 \\ 1 & 5 & 1 & -3 \\ 4 & 3 & 2 & 3 \end{bmatrix}.$$

A matter of numerical computation leads to

$$\varrho(K^*) = (-0.1746, 0.1955, 0.9650)^T,$$
$$\varrho(K) = (-0.1662, 0.1999, 0.9656)^T,$$

where figures are rounded to four decimal places. In this example, $\varrho(K^*)$ and $\varrho(K)$ are very close to each other, but they are not equal.

Remark 2.13 When it comes to evaluate the function v_K for a polyhedral cone like (22), we use the formula

$$v_K(x) = \sum_{i \in I} v_{K_i}(x),$$

where $\{K_i\}_{i \in I}$ is a finite collection of simplicial cones in $\mathbb{R}^n$ such that

$$\begin{cases} K = \cup_{i \in I} K_i, \\ \mathrm{vol}_n(K_i \cap K_j) = 0 \text{ for } i \neq j. \end{cases} \tag{23}$$

Each function v_{K_i} is evaluated by using a determinantal formula like (6). Of course, the partition (23) is not unique and constructing the K_i's can be computationally expensive, especially if p and n are large.

3 Image of a proper cone under a nonsingular matrix

Two cones in Π_n are said to be *isomorphic* if one of them is the image of the other under a nonsingular matrix. Isomorphism between proper cones is an equivalence relation. The set

$$\mathrm{Isom}(K) := \{A(K) : A \in \mathbb{GL}(n)\}$$

corresponds to the class of all proper cones that are isomorphic to $K \in \Pi_n$.

As seen in Seeger and Torki (2014), the function $\mathrm{lpv} : \Pi_n \to \mathbb{R}$ is invariant under orthogonal transformations. However, a general nonsingular transformation may induce a substantial change in the least partial volume of a proper cone.

Example 3.1 The least partial volume of the n-dimensional Lorentz cone

$$\mathbb{L}_n := \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} \leq x_n \right\}$$

admits the explicit formula

$$\mathrm{lpv}\,(\mathbb{L}_n) = \frac{\pi^{(n-1)/2}}{n\,\Gamma((n+1)/2)}.$$

For instance, $\mathrm{lpv}\,(\mathbb{L}_3) = (1/3)\pi$. For each positive ε, consider the nonsingular matrix

$$A_\varepsilon := \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The least partial volume of the proper cone $A_\varepsilon(\mathbb{L}_3)$ goes to infinity as ε goes to 0.

The above example serves as basic model to derive the following result. The space $\mathbb{M}_n$ of square matrices of order n is equipped with an arbitrary norm $\|\cdot\|_{\mathbb{M}_n}$.

Proposition 3.2 *For all $K \in \Pi_n$, there are sequences $\{A_\nu\}_{\nu\in\mathbb{N}}$ and $\{B_\nu\}_{\nu\in\mathbb{N}}$ of unit norm nonsingular matrices such that* $\lim_{\nu\to\infty} \operatorname{lpv}(A_\nu(K)) = 0$ *and* $\lim_{\nu\to\infty} \operatorname{lpv}(B_\nu(K)) = \infty$.

Proof Let $\operatorname{rev}(c, s) := \{x \in \mathbb{R}^n : s\,\|x\| \le \langle c, x\rangle\}$ denote the revolution cone with central axis determined by $c \in \mathbb{S}_n$ and degree of aperture described by the parameter $s \in]0, 1[$. Let $K^\uparrow$ and $K^\downarrow$ be, respectively, the largest revolution cone contained in K and the smallest revolution cone containing K. In particular, one has

$$K^\uparrow \subseteq K \subseteq K^\downarrow. \tag{24}$$

From the general theory of incenters, one knows that

$$\begin{aligned} K^\uparrow &= \operatorname{rev}(\xi(K), s(K)), \\ K^\downarrow &= \operatorname{rev}\left(\xi(K^*), r(K^*)\right), \end{aligned}$$

where $s(K) := [1-[r(K)]^2]^{1/2}$. Clearly, there exist positive reals α, β and orthogonal matrices U, V such that

$$\begin{aligned} U^T(K^\downarrow) = \mathcal{R}(\alpha) &:= \left\{(z, t) \in \mathbb{R}^n : \alpha\,\|z\| \le t\right\}, \\ V^T(K^\uparrow) = \mathcal{R}(\beta) &:= \left\{(z, t) \in \mathbb{R}^n : \beta\,\|z\| \le t\right\}. \end{aligned}$$

Let $A_\nu := \left\|C_\nu U^T\right\|_{\mathbb{M}_n}^{-1} C_\nu U^T$ and $B_\nu := \left\|D_\nu V^T\right\|_{\mathbb{M}_n}^{-1} D_\nu V^T$, with

$$C_\nu := \begin{bmatrix} \nu^{-1} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad D_\nu := \begin{bmatrix} \nu I_{n-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A_ν, B_ν are unit norm nonsingular matrices and

$$\begin{aligned} A_\nu(K^\downarrow) = (C_\nu U^T)(K^\downarrow) &= C_\nu(\mathcal{R}(\alpha)) = \mathcal{R}(\nu\alpha), \\ B_\nu(K^\uparrow) = (D_\nu V^T)(K^\uparrow) &= D_\nu(\mathcal{R}(\beta)) = \mathcal{R}(\nu^{-1}\beta). \end{aligned}$$

Observe that

$$\begin{aligned} \lim_{\nu\to\infty} \operatorname{lpv}\left(A_\nu\left(K^\downarrow\right)\right) &= \lim_{\nu\to\infty} \operatorname{lpv}(\mathcal{R}(\nu\alpha)) = 0, \\ \lim_{\nu\to\infty} \operatorname{lpv}\left(B_\nu\left(K^\uparrow\right)\right) &= \lim_{\nu\to\infty} \operatorname{lpv}\left(\mathcal{R}(\nu^{-1}\beta)\right) = \infty. \end{aligned}$$

This and the sandwich (24) complete the proof. □

The lemma stated below will be useful in the sequel. Its proof is based on the fact that

$$\nabla\Gamma_{A(K)}(Ax) = A^{-T}\nabla\Gamma_K(x) \tag{25}$$

for all $x \in \mathrm{int}(K)$ and $A \in \mathbb{GL}(n)$.

Lemma 3.3 *Let $K \in \Pi_n$. Then*

$$\mathrm{int}(K) = \{A^{-1}\varrho(A(K)) : A \in \mathbb{GL}(n)\}.$$

Proof The inclusion $\supseteq$ is trivial, because

$$\varrho(A(K)) \in \mathrm{int}(A(K)) = A(\mathrm{int}(K)).$$

For proving the reverse inclusion, pick any $x \in \mathrm{int}(K)$. Since $y := \nabla\Gamma_K(x)$ belongs to $\mathrm{int}(K^*)$, there exists a positive definite symmetric matrix H of order n such that $y = Hx$. To see this, consider for instance the matrix

$$H = \frac{yy^T}{\langle y, x\rangle} + \left(I_n - \frac{yx^T}{\langle y, x\rangle}\right)\left(I_n - \frac{xy^T}{\langle y, x\rangle}\right).$$

Let $A \in \mathbb{GL}(n)$ be such that $A^TA = H$. Hence, $A^{-T}\nabla\Gamma_K(x) = Ax$. By using the identity (25), one gets $\nabla\Gamma_{A(K)}(Ax) = Ax$. Thus, $Ax = \varrho(A(K))$. In short, there exists $A \in \mathbb{GL}(n)$ such that $x = A^{-1}\varrho(A(K))$. □

The function $\varrho : \Pi_n \to \mathbb{R}^n$ is not injective, but it satisfies a sort of mild injectivity condition. The next proposition clarifies this point.

Proposition 3.4 *Let $K_1, K_2 \in \Pi_n$. If $\varrho(A(K_1)) = \varrho(A(K_2))$ for all $A \in \mathbb{GL}(n)$, then $K_1 = K_2$.*

Proof For all $A \in \mathbb{GL}(n)$, one has

$$A^{-1}\varrho(A(K_1)) = A^{-1}\varrho(A(K_2)).$$

From Lemma 3.3, it follows that $\mathrm{int}(K_1) = \mathrm{int}(K_2)$. Hence, $K_1 = K_2$. □

The next proposition compares the least partial volumes of two isomorphic proper cones.

Proposition 3.5 *Let $K \in \Pi_n$ and $A \in \mathbb{GL}(n)$. Then*

$$\mathrm{lpv}(A(K)) \;\le\; \|A^{-T}\varrho(K^*)\|^n |\det A|\,\mathrm{lpv}(K). \tag{26}$$

Furthermore, the relation (26) *holds as an equality if and only if the vector $A^{-T}\varrho(K^*)$ is a positive multiple of $\varrho(A^{-T}(K^*))$.*

Proof The dual cone of $A(K)$ is $A^{-T}(K^*)$. By using (Seeger and Torki 2014, Proposition 2.5), one gets

$$\mathrm{lpv}(A(K)) = \min_{x\in\mathbb{S}_n} v_{A(K)}(x) = |\det A| \min_{x\in\mathbb{S}_n} v_K(A^T x).$$

If one considers the particular choice

$$\bar{x} = \|A^{-T}\varrho(K^*)\|^{-1} A^{-T}\varrho(K^*), \tag{27}$$

then one obtains

$$\mathrm{lpv}(A(K)) \le |\det A|\, v_K(A^T\bar{x}) = \|A^{-T}\varrho(K^*)\|^n\, |\det A|\, v_K(\varrho(K^*)).$$

This proves (26). The inequality (26) becomes an equality if and only if (27) is a minimizer of $v_{A(K)}$ on $\mathbb{S}_n$, i.e., $\bar{x} = \varrho(A^{-T}(K^*))$. □

As seen in Proposition 3.5, the expression

$$\gamma(K_1, K_2) := \inf_{\substack{G\in\mathbb{GL}(n)\\ G(K_1)=K_2}} \|G^{-T}\varrho(K_1^*)\|^n |\det G| \tag{28}$$

relates somehow the least partial volumes of two isomorphic proper cones. We refer to (28) as the *volumetric transfer coefficient* of the pair (K_1, K_2).

Proposition 3.6 *Let $K_1, K_2 \in \Pi_n$ be a pair of isomorphic cones. Then the infimum in* (28) *is attained.*

Proof Let $x := \varrho(K_1^*)$. Take any $A \in \mathbb{GL}(n)$ such that $K_2 = A(K_1)$. Let $\mathrm{Aut}(K_1)$ denote the set of automorphisms of the cone K_1, i.e.,

$$\mathrm{Aut}(K_1) := \{H \in \mathbb{GL}(n) : H(K_1) = K_1\}.$$

Then

$$\begin{aligned}
\gamma(K_1, K_2) &= \inf_{H\in\mathrm{Aut}(K_1)} \|A^{-T}H^{-T}x\|^n |\det(AH)| \\
&= |\det A| \inf_{H\in\mathrm{Aut}(K_1)} \|A^{-T}H^{-T}x\|^n |\det H| \\
&= |\det A| \inf_{\substack{H\in\mathrm{Aut}(K_1)\\ |\det H|=1}} \|A^{-T}H^{-T}x\|^n \\
&= \zeta^n |\det A|,
\end{aligned}$$

where

$$\zeta := \inf_{\substack{B\in\mathrm{Aut}(K_1^*)\\ |\det B|=1}} \|A^{-T}Bx\|. \tag{29}$$

So, we must prove that the infimum in (29) is attained. Observe that the feasible set

$$\Omega := \{B \in \mathrm{Aut}(K_1^*) : |\det B| = 1\}$$

is closed. Let $\{B_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Ω such that

$$\lim_{\nu\to\infty} \|A^{-T} B_\nu x\| = \zeta. \tag{30}$$

Suppose that $\lim_{\nu\to\infty} \|B_\nu\| = \infty$, otherwise we are done. We must arrive to a contradiction. Without loss of generality, we may suppose that the normalized matrices $C_\nu := \|B_\nu\|^{-1} B_\nu$ converge to a nonzero matrix C. By using (30), one gets $A^{-T} Cx = 0$. Hence,

$$Cx = 0. \tag{31}$$

By passing to lower Painlevé–Kuratowski limits in $C_\nu(K_1^*) = K_1^*$, one gets the inclusion

$$C(K_1^*) \subseteq K_1^*.$$

The above inclusion shows that $C(K_1^*)$ is pointed. Note that $C(K_1^*)$ is different from the zero cone, because C is different from the zero matrix and K_1^* has nonempty interior. Hence,

$$0 \notin \mathrm{ri}[C(K_1^*)] = C(\mathrm{int}(K_1^*)).$$

This contradicts (31) and the fact that $x \in \mathrm{int}(K_1^*)$. □

The volumetric transfer coefficient $\gamma(K_1, K_2)$ is of special interest if K_1, K_2 are homogeneous. Recall that $K \in \Pi_n$ is said to be *homogeneous* if

$$\begin{cases} \text{for all } x, y \in \mathrm{int}(K), \text{ there exists } A \in \mathbb{GL}(n) \\ \text{such that } A(K) = K \text{ and } Ax = y. \end{cases}$$

General material concerning the theory of homogeneous cones can be found in Faraut and Korányi (1994), Güler (1996), Vinberg (1963).

Theorem 3.7 *Let $K_1, K_2 \in \Pi_n$ be a pair of isomorphic cones. Then*

$$\mathrm{lpv}(K_2) \;\le\; \gamma(K_1, K_2)\,\mathrm{lpv}(K_1). \tag{32}$$

Furthermore, if one of the cones is homogeneous, then the other cone is homogeneous as well, and the inequality (32) *becomes an equality.*

Proof The inequality (32) is a direct consequence of Proposition 3.5. It is well known that homogeneity in proper cones is preserved under nonsingular transformations. Suppose then that K_1 is homogeneous. We claim that there exists a matrix $G \in \mathbb{GL}(n)$ satisfying the following properties:

$$\begin{cases} K_2 = G(K_1), \\ \varrho(K_2^*) = G^{-T}\varrho(K_1^*), \\ \mathrm{lpv}(K_2) = |\det G|\,\mathrm{lpv}(K_1). \end{cases} \tag{33}$$

Such claim would prove that (32) holds as an equality. Let $K_2 = S(K_1)$ with $S \in \mathbb{GL}(n)$. Note that $K_2^* = S^{-T}(K_1^*)$ is homogeneous, because K_1 is homogeneous and homogeneity is preserved by passing to dual cones. Note also that $\varrho(K_2^*)$ and $S^{-T}\varrho(K_1^*)$ are both in the interior of K_2^*. Hence, there exists a matrix $B \in \mathbb{GL}(n)$ such that

$$B(K_2^*) = K_2^*, \tag{34}$$
$$BS^{-T}\varrho(K_1^*) = \varrho(K_2^*). \tag{35}$$

Let us check that $G = B^{-T}S$ satisfies the properties listed in (33). By using (34), one gets

$$G(K_1) = (B^{-T}S)(K_1) = B^{-T}(K_2) = (B(K_2^*))^* = (K_2^*)^* = K_2.$$

On the other hand, (35) yields

$$G^{-T}\varrho(K_1^*) = (B^{-T}S)^{-T}\varrho(K_1^*) = BS^{-T}\varrho(K_1^*) = \varrho(K_2^*).$$

The last equality in (33) is obtained by applying Proposition 3.5 with $K = K_1$ and $A = G$. □

The next proposition completes Theorem 3.7 with a sort of reverse statement. In fact, Proposition 3.8 gives a curious characterization of homogeneity.

Proposition 3.8 *For $K \in \Pi_n$, the following conditions are equivalent:*

(a) *K is homogeneous.*
(b) $\mathrm{lpv}(Q) = \gamma(K, Q)\,\mathrm{lpv}(K)$ *for all* $Q \in \mathrm{Isom}(K)$.

Proof That (a) implies (b) is taken care by Theorem 3.7. Conversely, suppose that (b) is true. Let $x, y \in \mathrm{int}(K^*)$. By Lemma 3.3, there exist $E, F \in \mathbb{GL}(n)$ such that

$$x = E^T\varrho([E(K)]^*) \quad \text{and} \quad y = F^T\varrho([F(K)]^*).$$

On the other hand, the hypothesis (b), together with Propositions 3.5 and 3.6, ensures the existence of $E_1, F_1 \in \mathbb{GL}(n)$ such that

$$E(K) = E_1(K), \quad \varrho([E(K)]^*) = E_1^{-T}\varrho(K^*),$$
$$F(K) = F_1(K), \quad \varrho([F(K)]^*) = F_1^{-T}\varrho(K^*).$$

If one defines $A := F^T F_1^{-T} E_1^T E^{-T}$, then one gets $y = Ax$ and $A(K^*) = K^*$. This proves that K^* is homogeneous. Hence, K is homogeneous. □

4 Involutory cones

Corollary 2.2 concerns the class of simplicial cones. Such corollary can be extended to a higher level of generality. By definition, $K \in \Pi_n$ is an *involutory cone* if

$$\nabla\Gamma_{K^*}(\nabla\Gamma_K(x)) = x \quad \text{for all } x \in \operatorname{int}(K). \tag{36}$$

The main motivation behind involutoriness is the following result.

Proposition 4.1 *If $K \in \Pi_n$ is an involutory cone, then $\varrho(K^*) = \varrho(K)$.*

Proof By applying $\nabla\Gamma_{K^*}$ to each side of the equality $\varrho(K) = \nabla\Gamma_K(\varrho(K))$, one gets

$$\nabla\Gamma_{K^*}(\varrho(K)) = \nabla\Gamma_{K^*}(\nabla\Gamma_K(\varrho(K))) = \varrho(K).$$

In other words, $\varrho(K)$ is a fixed point of $\nabla\Gamma_{K^*}$. Hence, $\varrho(K)$ is equal to $\varrho(K^*)$. □

There is a big gap between the hypothesis and the conclusion of Proposition 4.1. The next theorem closes that gap.

Theorem 4.2 *Let $K \in \Pi_n$. Then the following conditions are equivalent:*

(a) *K is an involutory cone.*
(b) *Each $Q \in \operatorname{Isom}(K)$ is an involutory cone.*
(c) *$\varrho(Q^*) = \varrho(Q)$ for all $Q \in \operatorname{Isom}(K)$.*
(d) *K^* is an involutory cone.*

Proof (a) ⇒ (b). Let K be an involutory cone. Let $Q = A(K)$ with $A \in \mathbb{GL}(n)$. Then

$$\nabla\Gamma_Q(x) = \nabla\Gamma_{A(K)}(x) = A^{-T}\nabla\Gamma_K(A^{-1}x) \quad \text{for all } x \in \operatorname{int}(Q),$$
$$\nabla\Gamma_{Q^*}(y) = \nabla\Gamma_{A^T(K^*)}(y) = A\nabla\Gamma_{K^*}(A^T y) \quad \text{for all } y \in \operatorname{int}(Q^*).$$

Hence, for all $x \in \operatorname{int}(Q)$, one has

$$\nabla\Gamma_{Q^*}\big(\nabla\Gamma_Q(x)\big) = A\nabla\Gamma_{K^*}\Big(A^T A^{-T}\nabla\Gamma_K(A^{-1}x)\Big) = AA^{-1}x = x.$$

Thus, Q is an involutary cone. (b) ⇒ (c). It suffices to apply Proposition 4.1 to each Q. (c) ⇒ (a). Let $x \in \operatorname{int}(K)$. We must prove (36). Lemma 3.3 implies that $x = A^{-1}\varrho(A(K))$ for some $A \in \mathbb{GL}(n)$. Let $Q := A(K)$. From (Seeger and Torki 2014, Theorem 3.3) and the general identity

$$\nabla\Gamma_K(x) = A^T\nabla\Gamma_Q(Ax),$$

one deduces that

$$\nabla\Gamma_K(x) = A^T\nabla\Gamma_Q(\varrho(Q)) = A^T\varrho(Q).$$

Hence,

$$\nabla\Gamma_{K^*}\left(\nabla\Gamma_K(x)\right) = \nabla\Gamma_{K^*}\left(A^T\varrho(Q)\right) = A^{-1}\nabla\Gamma_{Q^*}\left(\varrho(Q)\right).$$

But $\varrho(Q) = \varrho(Q^*)$, because Q is isomorphic to K. It follows that

$$\nabla\Gamma_{K^*}\left(\nabla\Gamma_K(x)\right) = A^{-1}\nabla\Gamma_{Q^*}\left(\varrho(Q^*)\right) = A^{-1}\varrho(Q^*) = A^{-1}\varrho(Q) = x.$$

(d) $\Leftrightarrow$ (a). Consider first the implication (d) $\Rightarrow$ (a). Pick any $x \in \operatorname{int}(K)$. Since $y := \nabla\Gamma_K(x)$ belongs to $\operatorname{int}(K^*)$, the involutory relation

$$\nabla\Gamma_K\left(\nabla\Gamma_{K^*}(y)\right) = y$$

yields

$$\nabla\Gamma_K\left(\nabla\Gamma_{K^*}(\nabla\Gamma_K(x))\right) = \nabla\Gamma_K(x).$$

The injectivity of $\nabla\Gamma_K$ completes the proof of (36). The reverse implication (a) $\Rightarrow$ (d) is obtained by exchanging the roles of K and K^*. □

By using either (36) or any of the equivalent characterizations of involutoriness given in Theorem 4.2, one sees that:

- simplicial cones are involutory cones,
- general ellipsoidal cones are involutory cones,
- the dual of an involutory cone is an involutory cone,
- the image of an involutory cone under a nonsingular matrix is an involutory cone,
- the Cartesian product of finitely many involutory cones is an involutory cone.

Remark 4.3 As mentioned in Vinberg (1963), every homogeneous cone is an involutory cone. It is not clear to us whether the reverse statement is true. We leave this item as an open question. A self-dual cone is a proper cone that coincides with its dual. Beware that a self-dual cone may not be involutory.

The cones $\mathbb{R}^n_+$ and $\mathbb{L}_n$ are involutory and self-dual. The "product formulas" established in Corollary 2.2 and (Seeger and Torki 2014, Corollary 4.7) can be stated in a much broader context.

Theorem 4.4 *Let $K \in \Pi_n$ be an involutory cone. Then*

$$\operatorname{lpv}(Q^*)\operatorname{lpv}(Q) = \operatorname{lpv}(K^*)\operatorname{lpv}(K) \tag{37}$$

for all $Q \in \operatorname{Isom}(K)$.

Proof For a better readability, the proof is divided into three steps. The first two steps are interesting results in their own right and can be found in Güler (1996). We give their proofs only for the sake of completeness.

Step 1 Let $K \in \Pi_n$. We show that

$$\nabla^2 \Gamma_K(x)x + \nabla \Gamma_K(x) = 0 \tag{38}$$

for all $x \in \mathrm{int}(K)$. Since v_{K^*} is positively homogeneous of degree $-n$, the Euler equation

$$\langle \nabla v_{K^*}(x), x \rangle = -n v_{K^*}(x)$$

holds. The above equation can be written as

$$\langle \nabla \Gamma_K(x), x \rangle = 1. \tag{39}$$

One obtains (38) by taking the gradient on each side of (39).

Step 2 We prove that $K \in \Pi_n$ is involutory if and only if there exists a constant c_K such that

$$v_{K^*}(x) v_K(\nabla \Gamma_K(x)) = c_K \tag{40}$$

for all $x \in \mathrm{int}(K)$. One knows that $\nabla^2 \Gamma_K(x)$ is invertible for all $x \in \mathrm{int}(K)$. Hence, K is involutory if and only if

$$\nabla^2 \Gamma_K(x)[\nabla \Gamma_{K^*}(\nabla \Gamma_K(x)) - x] = 0 \tag{41}$$

for all $x \in \mathrm{int}(K)$. In view of (38), the Eq. (41) can be written in the form

$$\nabla^2 \Gamma_K(x) \nabla \Gamma_{K^*}(\nabla \Gamma_K(x)) + \nabla \Gamma_K(x) = 0,$$

or equivalently,

$$\nabla(\Gamma_{K^*} \circ \nabla \Gamma_K + \Gamma_K)(x) = 0.$$

Hence, involutoriness of K is equivalent to the existence of a constant $d_K \in \mathbb{R}$ such that

$$\Gamma_{K^*}(\nabla \Gamma_K(x)) + \Gamma_K(x) = d_K \tag{42}$$

for all $x \in \mathrm{int}(K)$. Note that (42) is the same equation as (40) with $c_K = e^{-nd_K}$.

Step 3 We now prove (37). Let $Q = A(K)$ with $A \in \mathbb{GL}(n)$. One has

$$\begin{aligned}\mathrm{lpv}(Q^*)\mathrm{lpv}(Q) &= v_{Q^*}(\rho(Q))v_Q(\rho(Q^*)),\\ &= v_{Q^*}(\rho(Q))v_Q(\rho(Q)),\\ &= v_{K^*}(A^{-1}\rho(Q))v_K(A^T\rho(Q)),\end{aligned}$$

where the second equality is due to Theorem 4.2 and the third equality is due to (Seeger and Torki 2014, Proposition 2.5). If one sets $x := A^{-1}\rho(Q)$, then $x \in \mathrm{int}(K)$ and

$$\nabla\Gamma_K(x) = A^T\nabla\Gamma_Q(\rho(Q)) = A^T\rho(Q).$$

Hence,

$$\mathrm{lpv}(Q^*)\mathrm{lpv}(Q) = v_{K^*}(x)v_K(\nabla\Gamma_K(x)) = c_K.$$

We have shown in this way that

$$\mathrm{lpv}([A(K)]^*)\,\mathrm{lpv}(A(K)) = c_K$$

for all $A \in \mathbb{GL}(n)$. This proves (37) and yields also the characterization

$$c_K = \mathrm{lpv}(K^*)\mathrm{lpv}(K)$$

for the constant c_K in (40). □

5 A conic version of Blaschke–Santaló inequality

Duality for proper cones plays a similar role as polarity for convex bodies. The polar set associated to an arbitrary nonempty set C in $\mathbb{R}^n$ is defined by

$$C^\circ := \{y \in \mathbb{R}^n : \langle y, u\rangle \le 1 \text{ for all } u \in C\}.$$

It is well known that $C \mapsto C^\circ$ is an involution on the collection of all closed convex set containing the origin. The following theorem is a celebrated result by Blaschke (1923) and Santaló (1949).

Theorem 5.1 *Let C be a convex body in $\mathbb{R}^n$. Then*

$$\inf_{x\in C} \mathrm{vol}_n(C-x)\,\mathrm{vol}_n((C-x)^\circ) \le [\mathrm{vol}_n(\mathbb{B}_n)]^2. \tag{43}$$

The above inequality becomes an equality if and only if C is an ellipsoid.

For a modern discussion on Theorem 5.1, see for instance (Lutwak 1985; Meyer and Pajor 1990). Since $\mathrm{vol}_n(\cdot)$ is invariant under translations, the inequality (43) is usually written in the form

$$\mathrm{vol}_n(C) \inf_{x\in C} \mathrm{vol}_n((C-x)^\circ) \le [\mathrm{vol}_n(\mathbb{B}_n)]^2. \tag{44}$$

The infimum in (44) is attained at a unique point, denoted by $s(C)$ and called the Santaló point of C. It is known that $s(C)$ is equal to the unique point x of $\mathrm{int}(C)$ satisfying the equation

$$\int_{(C-x)^\circ} u\,du = 0.$$

Santaló points of convex bodies and volumetric centers of proper cones are related concepts. The next theorem can be seen as a conic version of Theorem 5.1. We start by providing a geometric characterization for the vector $\nabla\Gamma_K(x)$. The lemma stated below asserts that $\nabla\Gamma_K(x)$ is equal to the center of mass of $K^*\cap\partial H_x$.

Lemma 5.2 *Let $K\in\Pi_n$ and $x\in\mathrm{int}(K)$. Then*

$$\nabla\Gamma_K(x) = \frac{1}{\mathrm{vol}_{n-1}(K^*\cap\partial H_x)}\int_{K^*\cap\partial H_x} v\,dv.$$

Proof From the very definition of Γ_K one sees that

$$\nabla\Gamma_K(x) = \frac{1}{n}\,\frac{\int_{K^*} y\,e^{-\langle x,y\rangle}\,dy}{\int_{K^*} e^{-\langle x,y\rangle}\,dy}.$$

But

$$\int_{K^*} y\,e^{-\langle x,y\rangle}\,dy = \int_0^\infty \left[e^{-t}\int_{K^*\cap t\partial H_x} u\,du\right]dt = n!\int_{K^*\cap\partial H_x} v\,dv$$

and

$$\int_{K^*} e^{-\langle x,y\rangle}\,dy = \int_0^\infty e^{-t}\,\mathrm{vol}_{n-1}(K^*\cap t\partial H_x)dt = (n-1)!\,\mathrm{vol}_{n-1}(K^*\cap\partial H_x).$$

This proves the announced formula for $\nabla\Gamma_K(x)$. □

We now are ready to state:

Theorem 5.3 *Let K be a proper cone in $\mathbb{R}^n$. Then*

$$\mathrm{lpv}(K^*)\,\mathrm{lpv}(K) \le [\mathrm{lpv}(\mathbb{L}_n)]^2. \tag{45}$$

The above inequality becomes an equality if and only if K is an ellipsoidal cone.

Proof One knows that $x := \varrho(K)$ is a unit vector belonging to $\mathrm{int}(K)\cap\mathrm{int}(K^*)$. Hence,

$$K\cap\partial H_x = \{u\in K : \langle x,u\rangle = 1\} \tag{46}$$

is a convex compact subset of ∂H_x. The shifted set $C := K \cap \partial H_x - x$ is a convex body in the $(n-1)$-dimensional linear space

$$L := \partial H_x - x = \{h \in \mathbb{R}^n : \langle x, h\rangle = 0\}.$$

Let $\varepsilon > 0$ be such that $x + \varepsilon \mathbb{B}_n \subseteq K$. In such a case, $\varepsilon(\mathbb{B}_n \cap L) \subseteq C$, that is to say, 0 belongs to the interior of C relative to L. We claim that 0 is the Santaló point of C. For proving this claim we must check that

$$\int_{C^\circ} u\, du = 0, \tag{47}$$

where integration is taken with respect to the Lebesgue measure in L, and

$$C^\circ := \{y \in L : \langle y, u\rangle \leq 1 \text{ for all } u \in C\}$$

is the polar set of C relative to L. A matter of computation shows that

$$C^\circ = -[K^* \cap \partial H_x - x].$$

Hence,

$$\begin{aligned}\int_{C^\circ} u\, du &= \int_{-[K^*\cap\partial H_x - x]} u\, du = -\int_{K^*\cap\partial H_x} (v - x) dv \\ &= -\int_{K^*\cap\partial H_x} v\, dv + \mathrm{vol}_{n-1}(K^* \cap \partial H_x)\, x.\end{aligned}$$

This leads directly to (47), thanks to Lemma 5.2. Next, we observe that

$$\begin{aligned}n\,\mathrm{lpv}(K^*) = n v_{K^*}(x) \;&=\; \mathrm{vol}_{n-1}(K^* \cap \partial H_x) \\ = \mathrm{vol}_{n-1}(-[K^* \cap \partial H_x - x]) \;&=\; \mathrm{vol}_{n-1}(C^\circ).\end{aligned}$$

Similarly,

$$n\,\mathrm{lpv}(K) \leq n v_K(x) \;=\; \mathrm{vol}_{n-1}(K \cap \partial H_x) \;=\; \mathrm{vol}_{n-1}(C).$$

Thanks to Theorem 5.1, one gets in this way

$$\begin{aligned}n^2 \mathrm{lpv}(K^*)\mathrm{lpv}(K) &\leq \mathrm{vol}_{n-1}(C^\circ)\,\mathrm{vol}_{n-1}(C) \\ &\leq [\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})]^2 = n^2[\mathrm{lpv}(\mathbb{L}_n)]^2.\end{aligned}$$

This completes the proof of (45). If K is an ellipsoidal cone, then (45) holds as an equality (cf. [Seeger and Torki 2014, Corollary 4.7]). Conversely, if (45) holds as an equality, then

$$\mathrm{vol}_{n-1}(C^\circ)\,\mathrm{vol}_{n-1}(C) = [\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})]^2.$$

Hence, C is an ellipsoid in the linear space L and (46) is an ellipsoidal section of K. It follows that K is an ellipsoidal cone. □

Corollary 5.4 *Let K be a self-dual cone in $\mathbb{R}^n$. Then* $\mathrm{lpv}(K) \leq \mathrm{lpv}(\mathbb{L}_n)$, *with equality if and only if $K = U(\mathbb{L}_n)$ for some orthogonal matrix U.*

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