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# On highly eccentric cones

Alberto Seeger · Mounir Torki

**Abstract** This paper addresses the issue of estimating the largest possible eccentricity in the class of proper cones of  $\mathbb{R}^n$ . The eccentricity of a proper cone is defined as the angle between the incenter and the circumcenter of the cone. This work establishes also various geometric and topological results concerning the concept of eccentricity.

**Keywords** Convex cone · Incenter · Circumcenter · Eccentricity of a proper cone

## 1 Introduction

Let  $n$  be an integer greater than or equal to two and  $\Pi_n$  be the collection of proper cones in  $\mathbb{R}^n$ . A closed convex cone is proper if it is pointed (i.e., it contains no line) and solid (i.e., it has nonempty interior). There are many ways to define the “center” of a proper cone  $K$ , but perhaps the two most natural choices are:

$$\begin{aligned}\xi(K) &:= \text{incenter of } K, \\ \eta(K) &:= \text{circumcenter of } K.\end{aligned}$$

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The formal definitions of these concepts are recalled in Sect. 1.1. A wealth of information on incenters and circumcenters of proper cones can be found in Henrion and Seeger (2010a,b, 2011). There are other concepts of center for specially structured cones, see Kelly et al. (1990), but we do not need to elaborate on this point.

A proper cone  $K$  is eccentric if the unit vectors  $\xi(K)$  and  $\eta(K)$  are distinct. From the general theory of incenters and circumcenters one knows that  $\langle \xi(K), \eta(K) \rangle$  is positive, where  $\langle y, x \rangle = y^T x$  stands for the usual inner product of  $\mathbb{R}^n$ . Henrion and Seeger (2011) suggest to measure the degree of eccentricity of  $K$  in terms of the angle

$$\psi(K) := \arccos \langle \xi(K), \eta(K) \rangle$$

between incenter and circumcenter. A large value of  $\psi(K)$  indicates that  $K$  is highly eccentric. On the contrary,  $K$  is non-eccentric when  $\psi(K) = 0$ .

An interesting challenge raised in the last page of Henrion and Seeger (2011) is that of computing the number

$$\varrho_n := \sup_{K \in \Pi_n} \psi(K), \tag{1}$$

which corresponds the largest possible eccentricity of a proper cone in  $\mathbb{R}^n$ . The two-dimensional case is of no interest because any proper cone in the plane is non-eccentric. The three-dimensional case is structurally different and, to some extent, more difficult to handle than the higher dimensional case  $n \geq 4$ . We have obtained the following rather surprising result.

**Theorem 1.1** *Depending on the dimension  $n$ , there are two cases for consideration:*

- (a) *If  $n \geq 4$ , then there exists a sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  in  $\Pi_n$  that is eventually degenerate (ED) in the sense that it satisfies the asymptotic orthogonality condition*

$$\lim_{\nu \rightarrow \infty} \langle \xi(K_\nu), \eta(K_\nu) \rangle = 0.$$

*In particular,  $\varrho_n = \pi/2$ .*

- (b) *ED sequences do not exist in  $\Pi_3$ . In fact, one has  $(1/4)\pi \leq \varrho_3 < (1/2)\pi$ .*

The proof of Theorem 1.1 is given in Sect. 4 for the part (a) and in Sect. 5 for the part (b). The purpose of this paper is to analyze the maximization problem (1) and, in tandem, to state various results concerning the concept of eccentricity.

## 1.1 Notation and preliminary material

Let  $\Xi_n$  be the collection of nontrivial closed convex cones in  $\mathbb{R}^n$ . That a closed convex cone is nontrivial means that is different from the zero cone and different from the whole space. Topological and continuity issues on  $\Xi_n$  are relative to the gap metric

$$\delta(K_1, K_2) := \max \left\{ \max_{x \in K_1 \cap \mathbb{S}_n} \text{dist}(x, K_2), \max_{x \in K_2 \cap \mathbb{S}_n} \text{dist}(x, K_1) \right\},$$

where  $\mathbb{S}_n$  is the unit sphere of  $\mathbb{R}^n$  and  $\text{dist}(x, K)$  stands for the distance from  $x$  to  $K$ . Convergence relative to the metric  $\delta$  turns out to be equivalent to convergence in the Painlevé–Kuratowski sense. The metric space  $(\Xi_n, \delta)$  is known to be compact, see Iusem and Seeger (2004, Proposition 2.1) or Rockafellar and Wets (1998, Chapter 4). Since

$$\begin{aligned} \Xi_n^{\text{ptd}} &:= \{K \in \Xi_n : K \text{ is pointed}\}, \\ \Xi_n^{\text{sol}} &:= \{K \in \Xi_n : K \text{ is solid}\} \end{aligned}$$

are open sets in  $\Xi_n$ , so is their intersection  $\Pi_n = \Xi_n^{\text{ptd}} \cap \Xi_n^{\text{sol}}$ .

Revolution cones play an ubiquitous role in this work. A revolution cone in  $\mathbb{R}^n$  is a closed convex cone of the form

$$\Gamma(c, s) := \{x \in \mathbb{R}^n : s \|x\| \leq \langle c, x \rangle\}, \quad (2)$$

where  $c \in \mathbb{S}_n$  determines the revolution axis and  $s \in [0, 1]$  is a parameter reflecting the degree of aperture of the cone. Note that (2) is proper if and only if  $s$  belongs to the open interval  $]0, 1[$ .

If  $K \in \Xi_n^{\text{sol}}$ , then the minimization problem

$$\begin{cases} \text{minimize } s \\ (\xi, s) \in \mathbb{S}_n \times [0, 1] \\ \Gamma(\xi, s) \subseteq K \end{cases} \quad (3)$$

admits a unique solution, which we denote by  $(\xi(K), s(K))$ . The vector  $\xi(K)$  is called the incenter of  $K$ . Similarly, if  $K \in \Xi_n^{\text{ptd}}$ , then the maximization problem

$$\begin{cases} \text{maximize } r \\ (\eta, r) \in \mathbb{S}_n \times [0, 1] \\ K \subseteq \Gamma(\eta, r) \end{cases} \quad (4)$$

admits a unique solution, which we denote by  $(\eta(K), r(K))$ . The vector  $\eta(K)$  is called the circumcenter of  $K$ . Any  $K \in \Pi_n$  can be sandwiched between a pair of proper revolution cones, the tightest sandwich

$$K^\uparrow \subseteq K \subseteq K^\downarrow$$

being obtained with

$$\begin{aligned} K^\uparrow &:= \Gamma(\xi(K), s(K)), \\ K^\downarrow &:= \Gamma(\eta(K), r(K)). \end{aligned}$$

One refers to  $K^\uparrow$  and  $K^\downarrow$  as the inner and outer revolution envelopes of  $K$  (cf. Gourion and Seeger 2012; Henrion and Seeger 2010a). Historically speaking, the optimization problems (3) and (4) have been studied under different points of view by several authors. The very formulation of these problems goes back at least to three decades ago, cf. Goffin (1980). The following facts are known and can be found disseminated in the literature on convex cones:

- i) If  $K \in \Xi_n^{\text{sol}}$ , then  $\xi(K)$  belongs to  $\text{int}(K)$  and it is equal to the unique solution to the maximization problem

$$\rho(K) = \max_{x \in K \cap \mathbb{S}_n} \text{dist}(x, \partial K), \quad (5)$$

where  $\partial K$  denotes the boundary of  $K$ . The coefficient  $\rho(K)$  is called the inradius of  $K$ . One has the general identity

$$[s(K)]^2 + [\rho(K)]^2 = 1. \quad (6)$$

- ii) If  $K \in \Xi_n^{\text{ptd}}$ , then  $\eta(K)$  belongs to  $K$ , but not necessarily to  $\text{int}(K)$ . One has

$$\eta(K) = \xi(K^*) \quad \text{and} \quad r(K) = \rho(K^*), \quad (7)$$

where  $K^*$  stands for the positive dual cone of  $K$ .

It follows from the first equality in (7) that the eccentricity of a proper cone is invariant with respect to dualization, i.e.,  $\psi(K^*) = \psi(K)$  for all  $K \in \Pi_n$ . Of course, eccentricity is also invariant under orthogonal transformations, i.e.,  $\psi(U(K)) = \psi(K)$  whenever  $U$  is an orthogonal matrix.

## 1.2 On osculating points

By definition, an osculating point of  $K \in \Pi_n$  is an element of the intersection

$$K \cap \partial K^\downarrow = \{x \in K : \langle \eta(K), x \rangle = r(K) \|x\|\}.$$

As shown in the next proposition, a nonzero osculating point of  $K$  is necessarily a generator of  $K$ . A nonzero vector  $x \in K$  is called a generator of  $K$  if the ray  $\mathbb{R}_+x$  is a face of  $K$ .

**Proposition 1.2** *Let  $K \in \Pi_n$ . If  $x$  is a nonzero vector in  $K \cap \partial K^\downarrow$ , then  $x$  is a generator of  $K$ .*

*Proof* Suppose that one can write  $x = u_1 + u_2$  as sum of two non-collinear vectors in  $K$ . One has

$$\begin{aligned} \langle \eta(K), u_1 + u_2 \rangle &= r(K) \|u_1 + u_2\|, \\ \langle \eta(K), u_k \rangle &\geq r(K) \|u_k\| \quad \text{for } k \in \{1, 2\}, \end{aligned}$$

the above inequality being due to the fact that  $u_1, u_2 \in K^\downarrow$ . This leads directly to

$$r(K)\|u_1 + u_2\| \geq r(K)(\|u_1\| + \|u_2\|).$$

Hence,  $\|u_1 + u_2\| = \|u_1\| + \|u_2\|$ , contradicting the non-collinearity assumption.  $\square$

The circumcenter of a proper cone can be represented as positive linear combination of at least two normalized osculating points.

**Proposition 1.3** *For each  $K \in \Pi_n$  there exist an integer  $p \in \{2, \dots, n\}$ , positive scalars  $\lambda_1, \dots, \lambda_p$  adding up to 1, and distinct unit vectors  $z_1, \dots, z_p$  in  $K \cap \partial K^\downarrow$  such that*

$$\begin{aligned} r(K) &= \|\lambda_1 z_1 + \dots + \lambda_p z_p\|, \\ \eta(K) &= [r(K)]^{-1}(\lambda_1 z_1 + \dots + \lambda_p z_p). \end{aligned}$$

*Proof* It is not difficult to check that

$$(\eta(K), r(K)) = \left( \|x_*\|^{-1} x_*, \|x_*\|^{-1} \right), \quad (8)$$

where  $x_*$  is the unique solution to the least norm problem

$$\begin{cases} \text{minimize } \|x\| \\ \langle z, x \rangle \geq 1 \text{ for all } z \in K \cap \mathbb{S}_n. \end{cases} \quad (9)$$

In order to write down the KKT-optimality conditions for  $x_*$ , we reformulate (9) as a smooth semi-infinite program

$$\begin{cases} \text{minimize } f(x) \\ g_z(x) \geq 0 \text{ for all } z \in Z \end{cases}$$

with data  $f(x) := (1/2)\|x\|^2$ ,  $g_z(x) := \langle z, x \rangle - 1$ , and  $Z := K \cap \mathbb{S}_n$ . If one takes  $h$  in the interior of  $K^*$ , then one sees that

$$\langle \nabla g_z(x_*), h \rangle > 0 \text{ for all } z \in Z,$$

i.e., the Mangasarian–Fromovitz constraint qualification holds at  $x_*$ . In view of López and Still (2007, Theorem 2), there exist an integer  $p \in \{1, \dots, n\}$ , Karush–Kuhn–Tucker multipliers  $\mu_1, \dots, \mu_p > 0$ , and distinct vectors  $z_1, \dots, z_p \in Z$  such that

$$\begin{aligned} \nabla f(x_*) &= \mu_1 \nabla g_{z_1}(x_*) + \dots + \mu_p \nabla g_{z_p}(x_*), \\ g_{z_1}(x_*) &= 0, \dots, g_{z_p}(x_*) = 0. \end{aligned}$$

The above system takes here the more explicit form

$$\begin{aligned} x_* &= \mu_1 z_1 + \dots + \mu_p z_p, \\ \langle z_1, x_* \rangle &= 1, \dots, \langle z_p, x_* \rangle = 1. \end{aligned}$$

In particular, the  $z_k$ 's are normalized osculating points of  $K$  and  $\mu_1 + \dots + \mu_p = [r(K)]^{-2}$ . The remaining part of the proof is a matter of introducing the change of variables  $\lambda_k = [r(K)]^2 \mu_k$  and exploiting the formula (8). The case  $p = 1$  must be ruled out, because  $\eta(K)$  is not in the boundary of  $K^\downarrow$ .  $\square$

*Remark 1.4* Implicit in the statement of Proposition 1.3 is the fact that the boundary of the outer revolution envelope  $K^\downarrow$  touches the cone  $K$  in at least two non-collinear points. This fact was already observed in Astorino et al. (2013, Lemma 2.21) for the case in which  $K$  is polyhedral.

## 2 Eccentricity in simplicial cones

By a simplicial cone in  $\mathbb{R}^n$  one understands a proper cone of the form

$$K = G(\mathbb{R}_+^n), \quad (10)$$

where  $G$  is an invertible matrix of order  $n$ . The next lemma provides a necessary and sufficient condition for a simplicial cone  $K$  to have its circumcenter in  $\text{int}(K)$ . In such a situation, the circumcenter of  $K$  can be computed by using a very simple algebraic formula. In the sequel we use the notation

$$\xi_\diamond := \sum_{k=1}^n \|f_k\| g_k, \quad \eta_\diamond := \sum_{k=1}^n \|g_k\| f_k,$$

where  $g_k$  and  $f_k$  stand for the  $k$ th column of  $G$  and  $F := G^{-T}$ , respectively.

**Lemma 2.1** *Let  $K$  be a simplicial cone represented as in (10). Consider the following statements:*

- (a)  $\eta(K) \in \text{int}(K)$ .
- (b)  $\langle f_i, \eta_\diamond \rangle > 0$  for all  $i \in \{1, \dots, n\}$ .
- (c)  $\langle f_i, \eta_\diamond \rangle \geq 0$  for all  $i \in \{1, \dots, n\}$ .
- (d)  $g_1, \dots, g_n$  are osculating points of  $K$ .
- (e)  $\eta(K) = \|\eta_\diamond\|^{-1} \eta_\diamond$ .

Then (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e).

*Proof* (b)  $\Rightarrow$  (c). Obvious.

(c)  $\Leftrightarrow$  (e). This equivalence is established in Henrion and Seeger (2011, Corollary 2.5). By the way, under the condition (e) one automatically has  $r(K) = \|\eta_\diamond\|^{-1}$ .

(a)  $\Rightarrow$  (d). Let  $\eta(K) \in \text{int}(K)$ . Then the combination of Propositions 1.2 and 1.3 yields

$$K \cap \partial K^\downarrow \cap \mathbb{S}_n = \left\{ \|g_1\|^{-1} g_1, \dots, \|g_n\|^{-1} g_n \right\}.$$

Hence, each  $g_k$  is an osculating point of  $K$ .

**Table 1** How often  $\eta(\mathbb{K}_n)$  belongs to  $\partial\mathbb{K}_n$

$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
50.08 %	72.72 %	85.11 %	91.89 %	95.65 %	97.69 %	98.78 %	99.37 %

(d)  $\Leftrightarrow$  (e). One has  $\eta_\diamond = F\mu_G$  with  $\mu_G := (\|g_1\|, \dots, \|g_n\|)^T$ . Hence,

$$\|\eta_\diamond\| = [\langle \mu_G, F^T F \mu_G \rangle]^{1/2} = \sup_{\beta \neq 0} \frac{\beta_1 \|g_1\| + \dots + \beta_n \|g_n\|}{\|\beta_1 g_1 + \dots + \beta_n g_n\|}.$$

From the above supremum one sees that  $\|\eta_\diamond\|$  is greater than 1. The system

$$\begin{cases} \langle \eta, g_k \rangle = r \|g_k\| & \text{for } k \in \{1, \dots, n\} \\ \|\eta\| = 1, & 0 < r < 1 \end{cases}$$

admits exactly one solution, namely

$$(\bar{\eta}, \bar{r}) = \left( \|\eta_\diamond\|^{-1} \eta_\diamond, \|\eta_\diamond\|^{-1} \right).$$

Thus, the condition (d) amounts to saying that  $(\eta(K), r(K))$  is equal to  $(\bar{\eta}, \bar{r})$ . (a)  $\Leftrightarrow$  (b). The condition (b) says that  $\eta_\diamond \in \text{int}(K)$ . If (a) holds or (b) holds, then  $\eta(K)$  is equal to  $\eta_\diamond$  up to normalization. Hence, (a) and (b) are both true or both false.  $\square$

Intensive numerical testing with random simplicial cones reveals that  $\eta(K)$  belongs very frequently to the boundary of  $K$ . The percentages reported in Table 1 are estimated by using a sample of  $10^6$  Gaussian simplicial cones. That a random simplicial cone is Gaussian means that the entries of the associated matrix  $G$  are stochastically independent standard Gaussian variables. For convenience, we use the symbol  $\mathbb{K}_n$  to refer to an  $n$ -dimensional Gaussian simplicial cone.

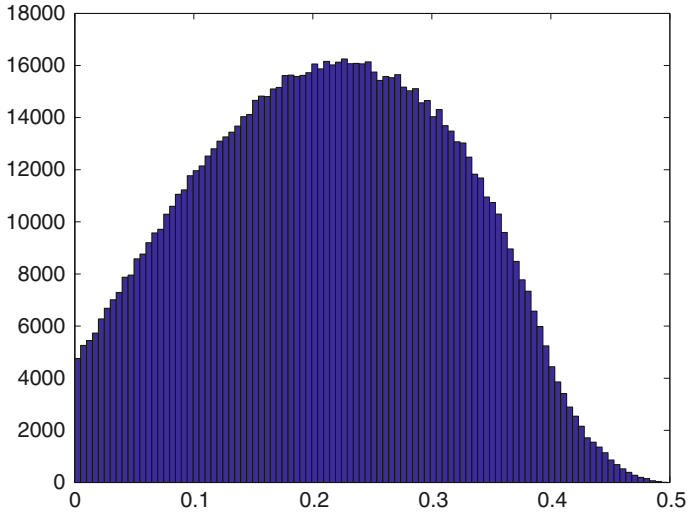
An explicit formula for the density function of the random variable  $\psi(\mathbb{K}_n)$  is hard to obtain. The histogram shown in Fig. 1 is an empirical model for the shape of such density function in dimension  $n = 10$ . The histogram has been constructed by using a sample of  $10^6$  Gaussian simplicial cones. The interval  $[0, \pi/2]$  is divided into 100 subintervals of equal length. For easy of visualization, the horizontal axis indicates values of  $(2/\pi) \psi(\mathbb{K}_{10})$ .

The histogram for the three dimensional case is not quite the same. The horizontal axis in Fig. 2 indicates values of  $(2/\pi) \psi(\mathbb{K}_3)$ .

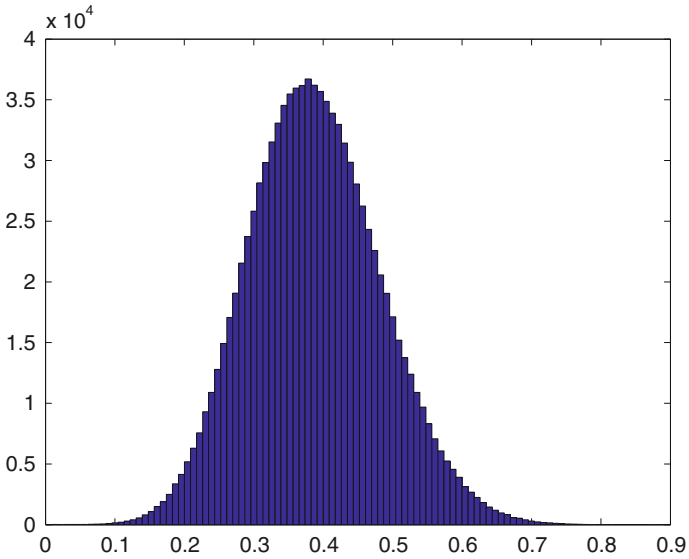
Our last numerical test with Gaussian simplicial cones concerns  $E[\psi(\mathbb{K}_n)]$ , i.e., the expected value of the random variable  $\psi(\mathbb{K}_n)$ . The experimental outcome reported in Table 2 suggests that  $E[\psi(\mathbb{K}_n)]$  is an increasing function of  $n$ .

The next theorem provides a necessary and sufficient condition for a simplicial cone to be non-eccentric. The proof relies on Lemma 2.1 and the first duality relationship mentioned in (7).





**Fig. 1** Histogram for the density function of  $(2/\pi)\psi(\mathbb{K}_{10})$  obtained with a sample of  $10^6$  Gaussian simplicial cones



**Fig. 2** Histogram for the density function of  $(2/\pi)\psi(\mathbb{K}_3)$  obtained with a sample of  $10^6$  Gaussian simplicial cones

**Table 2** Expected value of  $\psi(\mathbb{K}_n)$  as function of  $n$

$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
0.214	0.280	0.316	0.341	0.357	0.370	0.381	0.388

Figures are rounded to three decimal places and are to be multiplied by  $\pi/2$

**Theorem 2.2** *Let  $K$  be a simplicial cone represented as in (10). Then  $K$  is non-eccentric if and only if the baricentric equation*

$$\frac{\sum_{k=1}^n \|f_k\|g_k}{\left\| \sum_{k=1}^n \|f_k\|g_k \right\|} = \frac{\sum_{k=1}^n \|g_k\|f_k}{\left\| \sum_{k=1}^n \|g_k\|f_k \right\|} \quad (11)$$

holds.

*Proof* Suppose that  $K$  is non-eccentric. One has

$$\xi(K^*) = \eta(K) = \xi(K) = \eta(K^*).$$

Hence,  $\eta(K) \in \text{int}(K)$  and  $\eta(K^*) \in \text{int}(K^*)$ . By applying Lemma 2.1 first to the simplicial cone  $K = G(\mathbb{R}_+^n)$  and then to the simplicial cone  $K^* = F(\mathbb{R}_+^n)$ , one gets

$$\|\eta_\diamond\|^{-1}\eta_\diamond = \eta(K) = \eta(K^*) = \|\xi_\diamond\|^{-1}\xi_\diamond. \quad (12)$$

Conversely, suppose that  $\xi_\diamond$  and  $\eta_\diamond$  are equal up to normalization. Then

$$\text{int}(K) \ni \|\eta_\diamond\|^{-1}\eta_\diamond = \|\xi_\diamond\|^{-1}\xi_\diamond \in \text{int}(K^*).$$

In such a case, Lemma 2.1 yields (12). Hence,  $K$  is non-eccentric.  $\square$

A simplicial cone is equiangular if all pairs of generators form the same angle. Not surprisingly, equiangular simplicial cones are non-eccentric.

**Proposition 2.3** *Let  $K$  be a simplicial cone represented as in (10). Suppose that*

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ c & \text{if } i \neq j \end{cases}$$

for some constant  $c \in ]-1, 1[$ . Then

$$\xi(K) = \frac{G\mathbf{1}_n}{\|G\mathbf{1}_n\|} = \frac{F\mathbf{1}_n}{\|F\mathbf{1}_n\|} = \eta(K), \quad (13)$$

where  $\mathbf{1}_n$  is the  $n$ -dimensional vector of ones.

*Proof* Since the  $g_k$ 's are unit vectors, one has  $\mu_G = \mathbf{1}_n$ . Note also that

$$G^T G = (1 - c)I_n + c\mathbf{1}_n\mathbf{1}_n^T, \quad (14)$$

where  $I_n$  is the identity matrix of order  $n$ . The Gramian matrix  $G^T G$  being positive definite, one has  $1 + (n - 1)c > 0$ . As a direct consequence of (14) one gets

$$G\mathbf{1}_n = (1 + (n - 1)c)F\mathbf{1}_n.$$

On the other hand, by taking the inverse on each side of (14) one gets

$$F^T F = \frac{1}{1-c} \left( I_n - \frac{c}{1+(n-1)c} \mathbf{1}_n \mathbf{1}_n^T \right).$$

Hence,  $\mu_F := (\|f_1\|, \dots, \|f_n\|)^T = \gamma \mathbf{1}_n$  for some positive constant  $\gamma$ , namely

$$\gamma := \left[ \frac{1+(n-2)c}{(1-c)(1+(n-1)c)} \right]^{1/2}.$$

We have shown in this way that

$$\xi_\diamond = G\mu_F = \gamma G\mathbf{1}_n = \gamma(1+(n-1)c)F\mathbf{1}_n = \gamma(1+(n-1)c)\eta_\diamond.$$

Lemma 2.1 leads to the formulas announced in (13). □

### 3 Eccentricity in Cartesian products of proper cones

Calculus rules for computing incenters and circumcenters have been developed mainly in Henrion and Seeger (2010b) and Seeger and Torki (2013). To the best of our knowledge, the next result is new. Lemma 3.1 explains how to compute the inradius, the incenter, and the circumcenter of a Cartesian product of two proper cones.

**Lemma 3.1** *Let  $p, q \geq 2$ . Let  $P \in \Pi_p$  and  $Q \in \Pi_q$ . Then  $K := P \times Q$  belongs to  $\Pi_{p+q}$  and*

$$\begin{aligned} \rho(K) &= \rho(P)\rho(Q) \left( [\rho(P)]^2 + [\rho(Q)]^2 \right)^{-1/2}, \\ \xi(K) &= \left( [\rho(P)]^2 + [\rho(Q)]^2 \right)^{-1/2} (\rho(Q)\xi(P), \rho(P)\xi(Q)), \\ \eta(K) &= \left( [r(P)]^2 + [r(Q)]^2 \right)^{-1/2} (r(Q)\eta(P), r(P)\eta(Q)). \end{aligned} \tag{15}$$

*Proof* We must solve the maximization problem

$$\rho(K) = \max_{\substack{(u,v) \in P \times Q \\ \|u\|^2 + \|v\|^2 = 1}} \text{dist}[(u, v), \partial(P \times Q)].$$

A matter of computation yields

$$\begin{aligned} \rho(K) &= \max_{(u,v) \in P \times Q} \min \{ \text{dist}(u, \partial P), \text{dist}(v, \partial Q) \} \\ &\quad \|u\|^2 + \|v\|^2 = 1 \\ &= \max_{t \in [0,1]} \max_{\substack{u \in P \cap t\mathbb{S}_p \\ v \in Q \cap (1-t^2)^{1/2}\mathbb{S}_q}} \min \{ \text{dist}(u, \partial P), \text{dist}(v, \partial Q) \} \end{aligned} \tag{16}$$

$$\begin{aligned}
&= \max_{t \in [0,1]} \max_{\substack{u \in P \cap \mathbb{S}_p \\ v \in Q \cap \mathbb{S}_q}} \min \left\{ t \operatorname{dist}(u, \partial P), (1-t^2)^{1/2} \operatorname{dist}(v, \partial Q) \right\} \\
&= \max_{t \in [0,1]} \min \left\{ t \rho(P), (1-t^2)^{1/2} \rho(Q) \right\}.
\end{aligned}$$

The above maximum is attained with  $t_0 \in [0, 1]$  such that  $t_0 \rho(P) = (1-t_0^2)^{1/2} \rho(Q)$ , i.e.,

$$t_0 = \rho(Q) \left( [\rho(P)]^2 + [\rho(Q)]^2 \right)^{-1/2}.$$

This leads to the formula (15). The incenter of  $K$  is equal to

$$(u_0, v_0) = \left( t_0 \xi(P), (1-t_0^2)^{1/2} \xi(Q) \right),$$

because this vector achieves the maximum in (16). The circumcenter of  $K$  is obtained by a simple duality argument.  $\square$

**Corollary 3.2** *Let  $P, Q, K$  be as in Lemma 3.1. Then*

$$\begin{aligned}
\langle \xi(K), \eta(K) \rangle &\leq \gamma(P, Q) \max\{\langle \xi(P), \eta(P) \rangle, \langle \xi(Q), \eta(Q) \rangle\}, \\
\langle \xi(K), \eta(K) \rangle &\geq \gamma(P, Q) \min\{\langle \xi(P), \eta(P) \rangle, \langle \xi(Q), \eta(Q) \rangle\},
\end{aligned}$$

where  $\gamma : \Pi_p \times \Pi_q \rightarrow [0, 1]$  is given by

$$\gamma(P, Q) := \frac{\rho(P)r(P) + \rho(Q)r(Q)}{([\rho(P)]^2 + [\rho(Q)]^2)^{1/2} ([r(P)]^2 + [r(Q)]^2)^{1/2}}.$$

In particular,  $\psi(K) \geq \min\{\psi(P), \psi(Q)\}$ .

*Proof* The corollary follows directly from Lemma 3.1. That  $\gamma(P, Q) \leq 1$  is a consequence of the Cauchy–Schwarz inequality.  $\square$

For the sake of completeness we state below the case of a Cartesian product of several proper cones. The obtained formulas are easy to remember.

**Lemma 3.3** *Let  $n := n_1 + \dots + n_\ell$  with  $n_1, \dots, n_\ell \geq 2$ . Let  $K_1 \in \Pi_{n_1}, \dots, K_\ell \in \Pi_{n_\ell}$ . Then  $K := K_1 \times \dots \times K_\ell$  belongs to  $\Pi_n$  and*

$$\frac{1}{[\rho(K)]^2} = \frac{1}{[\rho(K_1)]^2} + \dots + \frac{1}{[\rho(K_\ell)]^2}, \quad (17)$$

$$\frac{1}{[r(K)]^2} = \frac{1}{[r(K_1)]^2} + \dots + \frac{1}{[r(K_\ell)]^2}, \quad (18)$$

$$\frac{\xi(K)}{\rho(K)} = \left( \frac{\xi(K_1)}{\rho(K_1)}, \dots, \frac{\xi(K_\ell)}{\rho(K_\ell)} \right), \quad (19)$$

$$\frac{\eta(K)}{r(K)} = \left( \frac{\eta(K_1)}{r(K_1)}, \dots, \frac{\eta(K_\ell)}{r(K_\ell)} \right). \quad (20)$$

*Proof* The proof of (19) is by induction on the integer  $\ell$ . The case  $\ell = 2$  is taken care by Lemma 3.1. The formula (20) is obtained by dualization. The formula (17) is obtained by taking norms on (19). Similarly, (18) is obtained by taking norms on (20).  $\square$

Beware that the Cartesian product of non-eccentric cones may be eccentric. One has in fact the following result.

**Theorem 3.4** *Let  $K$  be a Cartesian product as in Lemma 3.3. Then  $K$  is non-eccentric if and only if  $K_1, \dots, K_\ell$  are non-eccentric and*

$$\frac{\rho(K_1)}{r(K_1)} = \dots = \frac{\rho(K_\ell)}{r(K_\ell)}. \quad (21)$$

*Proof* In view of Lemma 3.3,  $K$  is non-eccentric if and only if

$$\frac{\rho(K)}{\rho(K_j)} \xi(K_j) = \frac{r(K)}{r(K_j)} \eta(K_j)$$

for all  $j \in \{1, \dots, \ell\}$ . This is in turn equivalent to the combination of (21) and the non-eccentricity of each  $K_j$ .  $\square$

The next example shows that the Cartesian product of two revolution cones can be highly eccentric.

*Example 3.5* Let  $p, q \geq 2$ . Let  $K$  be the Cartesian product of the proper revolution cones

$$\begin{aligned} \Gamma(a, s_1) &= \{u \in \mathbb{R}^p : s_1 \|u\| \leq \langle a, u \rangle\}, \\ \Gamma(b, s_2) &= \{v \in \mathbb{R}^q : s_2 \|v\| \leq \langle b, v \rangle\}. \end{aligned}$$

Regardless of the choice of  $a \in \mathbb{S}_p$  and  $b \in \mathbb{S}_q$ , one has

$$\begin{aligned} \rho(\Gamma(a, s_1)) &= (1 - s_1^2)^{1/2}, & r(\Gamma(a, s_1)) &= s_1, \\ \rho(\Gamma(b, s_2)) &= (1 - s_2^2)^{1/2}, & r(\Gamma(b, s_2)) &= s_2. \end{aligned}$$

By applying Lemma 3.1 one gets

$$\langle \xi(K), \eta(K) \rangle = \frac{s_1(1 - s_1^2)^{1/2} + s_2(1 - s_2^2)^{1/2}}{[2 - s_1^2 - s_2^2]^{1/2}(s_1^2 + s_2^2)^{1/2}}. \quad (22)$$

Note that (22) goes to 0 if one lets  $s_1 \rightarrow 0$  and  $s_2 \rightarrow 1$ .

## 4 Construction of ED sequences

Example 3.5 gives the clue for the proof of Theorem 1.1(a). In fact, Example 3.5 serves to prove the following stronger result concerning the levels sets

$$\text{Lev}(\theta) := \{K \in \Pi_n : \psi(K) = \theta\} \quad (23)$$

of the eccentricity function  $\psi$ . By an obvious reason one refers to (23) as an iso-eccentricity set.

**Theorem 4.1** *Let  $n \geq 4$ . Then  $\text{Lev}(\theta)$  is nonempty for all  $\theta \in [0, \pi/2[$ .*

*Proof* Let  $\theta \in [0, \pi/2[$ . Write  $n = p + q$  with  $p, q \geq 2$ . Let  $K$  be as in Example 3.5. If one chooses

$$s_1 = t := \left[ \frac{1 + \sin \theta}{2} \right]^{1/2} \quad \text{and} \quad s_2 = (1 - t^2)^{1/2} = \left[ \frac{1 - \sin \theta}{2} \right]^{1/2},$$

then one gets  $\langle \xi(K), \eta(K) \rangle = 2t(1 - t^2)^{1/2} = \cos \theta$ , i.e.,  $K$  belongs to  $\text{Lev}(\theta)$ .  $\square$

The proof of Theorem 4.1 indicates how to construct an ED sequence in  $\Pi_n$  when  $n \geq 4$ . One may consider for instance

$$K_\nu = \Gamma(a, t_\nu) \times \Gamma\left(b, (1 - t_\nu^2)^{1/2}\right), \quad (24)$$

where  $\{t_\nu\}_{\nu \in \mathbb{N}}$  is a sequence in  $]0, 1[$  converging to 0. The example (24) is by no means unique. We next explain how to construct a wider category of proper cones that eventually degenerate. We start by introducing a useful definition.

**Definition 4.2** A sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  in  $\Pi_n$  is solidity maximizing if  $\lim_{\nu \rightarrow \infty} \rho(K_\nu) = 1$ .

The reason that justifies the above terminology is that  $\rho(K)$  is a coefficient in  $[0, 1]$  that measures the degree of solidity of the cone  $K$ ; see the recent papers Gourion and Seeger (2012) and Iusem and Seeger (2005a) for a long discussion on solidity coefficients. Solidity maximizing sequences exist in any dimension  $n \geq 2$  and, in fact, they are quite easy to construct. One may rely for instance on the following auxiliary lemma, where the notation

$$\theta_{\max}(K) := \max_{u, v \in K \cap \mathbb{S}_n} \arccos \langle u, v \rangle$$

refers to the maximal angle of  $K \in \Xi_n$ . Computing a maximal angle is often times easier than computing an inradius.

**Lemma 4.3** *Let  $n \geq 2$ . A sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  in  $\Pi_n$  is solidity maximizing if and only if*

$$\lim_{\nu \rightarrow \infty} \theta_{\max}(K_\nu^*) = 0. \quad (25)$$

*Proof* By applying Henrion and Seeger (2011, Theorem 2.10) one gets

$$[\rho(K_\nu)]^2 \geq \frac{1}{n} + \left(1 - \frac{1}{n}\right) \cos[\theta_{\max}(K_\nu^*)].$$

This proves the sufficiency of the condition (25). Suppose now that  $\lim_{\nu \rightarrow \infty} \rho(K_\nu) = 1$ . Let  $\vartheta_\nu := \theta_{\max}(K_\nu^*)$ . We must prove that  $\bar{\vartheta} := \limsup_{\nu \rightarrow \infty} \vartheta_\nu$  is equal to zero. Take a subsequence  $\{\vartheta_{\varphi(\nu)}\}_{\nu \in \mathbb{N}}$  such that  $\lim_{\nu \rightarrow \infty} \vartheta_{\varphi(\nu)} = \bar{\vartheta}$ . By a compactness argument, one may assume that

$$\lim_{\nu \rightarrow \infty} \delta(K_{\varphi(\nu)}, K) = 0$$

for some  $K \in \Xi_n$ . The Walkup-Wets isometry theorem (cf. Walkup and Wets 1967, Theorem 1) yields

$$\lim_{\nu \rightarrow \infty} \delta(K_{\varphi(\nu)}^*, K^*) = 0.$$

The continuity of the function  $\theta_{\max} : \Xi_n \rightarrow \mathbb{R}$  ensures that  $\bar{\vartheta} = \theta_{\max}(K^*)$ . On the other hand, the coefficient (5) is well defined on  $\Xi_n$ , and not just on  $\Pi_n$ . Furthermore,  $\rho : \Xi_n \rightarrow \mathbb{R}$  is a continuous function (cf. Iusem and Seeger 2005a, Proposition 6.3). Hence,

$$\rho(K) = \lim_{\nu \rightarrow \infty} \rho(K_{\varphi(\nu)}) = \lim_{\nu \rightarrow \infty} \rho(K_\nu) = 1.$$

This proves that  $K$  is a half-space. Therefore,  $K^*$  is a ray and  $\bar{\vartheta} = 0$ . □

*Example 4.4* For each  $\nu \in \mathbb{N}$ , let  $K_\nu \subseteq \mathbb{R}^n$  be the simplicial cone generated by the columns of

$$G_{n,\nu} := \begin{bmatrix} I_{n-1} & -\nu \mathbf{1}_{n-1} \\ 0^T & 1 \end{bmatrix}.$$

The dual cone  $K_\nu^*$  is then generated by the columns of

$$F_{n,\nu} := G_{n,\nu}^{-T} = \begin{bmatrix} I_{n-1} & 0 \\ \nu \mathbf{1}_{n-1}^T & 1 \end{bmatrix}.$$

By using Iusem and Seeger (2005b, Proposition 6.2) one sees that

$$\theta_{\max}(K_\nu^*) = \arccos\left(\frac{\nu^2}{1 + \nu^2}\right).$$

Since (25) holds, it follows that  $\{K_\nu\}_{\nu \in \mathbb{N}}$  is solidity maximizing.

We now come back to the main stream of the exposition.

**Proposition 4.5** *Let  $p, q \geq 2$ . Let  $\{P_v\}_{v \in \mathbb{N}}$  and  $\{Q_v\}_{v \in \mathbb{N}}$  be solidity maximizing sequences in  $\Pi_p$  and  $\Pi_q$ , respectively. Then each  $K_v := P_v \times Q_v^*$  belongs to  $\Pi_{p+q}$  and the sequence  $\{K_v\}_{v \in \mathbb{N}}$  is ED.*

*Proof* By applying Lemma 3.1 one obtains

$$\begin{aligned} \langle \xi(K_v), \eta(K_v) \rangle &= \alpha_v \langle \xi(P_v), \eta(P_v) \rangle + \beta_v \langle \xi(Q_v^*), \eta(Q_v^*) \rangle \\ &= \alpha_v \langle \xi(P_v), \eta(P_v) \rangle + \beta_v \langle \xi(Q_v), \eta(Q_v) \rangle, \end{aligned}$$

where  $\alpha_v := \gamma_v^{-1} \rho(Q_v) r(Q_v)$ ,  $\beta_v := \gamma_v^{-1} \rho(P_v) r(P_v)$ , and

$$\gamma_v := \left( [\rho(P_v)]^2 + [r(Q_v)]^2 \right)^{1/2} \left( [\rho(Q_v)]^2 + [r(P_v)]^2 \right)^{1/2}.$$

One knows already that  $\lim_{v \rightarrow \infty} \rho(P_v) = 1$  and  $\lim_{v \rightarrow \infty} \rho(Q_v) = 1$ . By using (6), one gets

$$[\rho(P_v)]^2 + [r(P_v)]^2 \leq 1, \quad (26)$$

$$[\rho(Q_v)]^2 + [r(Q_v)]^2 \leq 1. \quad (27)$$

Hence, one also has  $\lim_{v \rightarrow \infty} r(P_v) = 0$  and  $\lim_{v \rightarrow \infty} r(Q_v) = 0$ . It follows that  $\lim_{v \rightarrow \infty} \alpha_v = 0$  and  $\lim_{v \rightarrow \infty} \beta_v = 0$ . This completes the proof of the proposition. By the way, the assumption  $p \geq 2$  is essential for the validity of (26). Similarly, the assumption  $q \geq 2$  is required for (27).  $\square$

**Corollary 4.6** *Let  $n \geq 4$ . Then there exists a sequence  $\{K_v\}_{v \in \mathbb{N}}$  of simplicial cones in  $\mathbb{R}^n$  such that  $\lim_{v \rightarrow \infty} \psi(K_v) = \pi/2$ . In particular,*

$$\sup_{\substack{K \in \Pi_n \\ K \text{ simplicial}}} \psi(K) = \pi/2.$$

*Proof* Write  $n = p + q$  with  $p, q \geq 2$ . Let  $P_v \subseteq \mathbb{R}^p$  and  $Q_v \subseteq \mathbb{R}^q$  be the simplicial cones generated by the columns of

$$G_{p,v} := \begin{bmatrix} I_{p-1} & -v \mathbf{1}_{p-1} \\ 0^T & 1 \end{bmatrix} \quad \text{and} \quad G_{q,v} := \begin{bmatrix} I_{q-1} & -v \mathbf{1}_{q-1} \\ 0^T & 1 \end{bmatrix},$$

respectively. As we saw in Example 4.4,  $\{P_v\}_{v \in \mathbb{N}}$  and  $\{Q_v\}_{v \in \mathbb{N}}$  are solidity maximizing sequences. Note that  $K_v := P_v \times Q_v^*$  is a simplicial cone in  $\mathbb{R}^n$ . Indeed,  $K_v$  is generated by the columns of the block structured matrix

$$G_v := \begin{bmatrix} G_{p,v} & \\ & G_{q,v}^{-T} \end{bmatrix},$$

where the void part is filled with zeros. Proposition 4.5 ensures that  $\{K_v\}_{v \in \mathbb{N}}$  is an ED sequence.  $\square$



The next proposition concerns limits and, more generally, cluster points of ED sequences. From any sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  in  $\Xi_n$  one can extract a subsequence that converges to some  $K_\infty \in \Xi_n$ . Such element  $K_\infty$  is called a cluster point of  $\{K_\nu\}_{\nu \in \mathbb{N}}$ .

**Proposition 4.7** *Let  $n \geq 4$  and  $\{K_\nu\}_{\nu \in \mathbb{N}}$  be an ED sequence in  $\Pi_n$ . Then*

$$\lim_{\nu \rightarrow \infty} \rho(K_\nu) = 0 \quad \text{and} \quad \lim_{\nu \rightarrow \infty} r(K_\nu) = 0. \quad (28)$$

*In particular, the cluster points of  $\{K_\nu\}_{\nu \in \mathbb{N}}$  are neither solid nor pointed.*

*Proof* The eccentricity inequality of Henrion and Seeger (2011, Theorem 4.2) yields

$$\langle \xi(K_\nu), \eta(K_\nu) \rangle \geq \frac{\rho(K_\nu) + r(K_\nu)}{1 + \rho(K_\nu)r(K_\nu)}$$

for all  $\nu \in \mathbb{N}$ . This proves (28). Let  $K_\infty \in \Xi_n$  be a cluster point of  $\{K_\nu\}_{\nu \in \mathbb{N}}$ . Since a subsequence of an ED sequence is ED, there is no loss of generality in assuming that the whole sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  converges to  $K_\infty$ . Since  $\rho : \Xi_n \rightarrow \mathbb{R}$  and  $r : \Xi_n \rightarrow \mathbb{R}$  are continuous functions, one deduces that  $\rho(K_\infty) = 0$  and  $r(K_\infty) = 0$ . Hence,  $K_\infty$  is neither solid nor pointed.  $\square$

The next proposition deals with the asymptotic behavior of extremal angles in ED sequences. The smallest angle of  $K \in \Xi_n$  is defined by

$$\theta_{\min}(K) := \pi - \theta_{\max}(K^*). \quad (29)$$

Information on the geometric interpretation of (29) can be found in Iusem and Seeger (2005b, 2009).

**Proposition 4.8** *Let  $n \geq 4$  and  $\{K_\nu\}_{\nu \in \mathbb{N}}$  be an ED sequence in  $\Pi_n$ . Then*

$$\lim_{\nu \rightarrow \infty} \theta_{\max}(K_\nu) = \pi \quad \text{and} \quad \lim_{\nu \rightarrow \infty} \theta_{\min}(K_\nu) = 0. \quad (30)$$

*Proof* We must prove that the lower limit

$$\underline{\vartheta} := \liminf_{\nu \rightarrow \infty} \theta_{\max}(K_\nu)$$

is equal to  $\pi$ . By a compactness argument, there exists a subsequence  $\{K_{\varphi(\nu)}\}_{\nu \in \mathbb{N}}$  converging to some  $K_\infty \in \Xi_n$  and satisfying

$$\lim_{\nu \rightarrow \infty} \theta_{\max}(K_{\varphi(\nu)}) = \underline{\vartheta}.$$

Since  $K_\infty$  is not pointed by Proposition 4.7, one has  $\theta_{\max}(K_\infty) = \pi$ . This fact and the continuity of  $\theta_{\max} : \Xi_n \rightarrow \mathbb{R}$  lead to  $\underline{\vartheta} = \pi$ . The second equality in (30) follows by a duality argument. Indeed,  $\{K_\nu^*\}_{\nu \in \mathbb{N}}$  is also an ED sequence in  $\Pi_n$ , and therefore

$$\lim_{\nu \rightarrow \infty} \theta_{\max}(K_\nu^*) = \pi.$$

This completes the proof.  $\square$

Recall that  $K \in \Xi_n$  is infra-dual if  $K \subseteq K^*$  and supra-dual if  $K \supseteq K^*$ . The angular width of  $K$  is defined as the nonnegative number  $\text{aw}(K) := \theta_{\max}(K) - \theta_{\min}(K)$ . The combination of both limits in (30) is equivalent to

$$\lim_{\nu \rightarrow \infty} \text{aw}(K_\nu) = \pi. \quad (31)$$

This observation leads to the following corollary.  $\square$

**Corollary 4.9** *Let  $n \geq 4$ . Let  $\{K_\nu\}_{\nu \in \mathbb{N}}$  be a sequence in  $\Pi_n$  such that*

$$\text{for all } \nu \in \mathbb{N}, \text{ either } K_\nu \subseteq K_\nu^* \text{ or } K_\nu \supseteq K_\nu^*. \quad (32)$$

*Then  $\{K_\nu\}_{\nu \in \mathbb{N}}$  is not ED.*

*Proof* The inclusion  $K_\nu \subseteq K_\nu^*$  yields  $\theta_{\max}(K_\nu) \leq \pi/2$ , whereas  $K_\nu \supseteq K_\nu^*$  yields  $\theta_{\min}(K_\nu) \geq \pi/2$ . Thus, the assumption (32) implies that

$$\sup_{\nu \in \mathbb{N}} \text{aw}(K_\nu) \leq \pi/2.$$

Such upper bound on the angular width of  $K_\nu$  prevents the sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  from being ED.  $\square$

*Remark 4.10* The condition (31) is necessary for a sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$  to be ED, but it is not sufficient. In fact, (31) may hold for a sequence of non-eccentric proper cones. Think for instance of an ill-conditioned ellipsoidal cone like

$$K_\nu := \left\{ x \in \mathbb{R}^n : \left[ \nu(x_1^2 + \cdots + x_{n-2}^2) + (1/\nu)x_{n-1}^2 \right]^{1/2} \leq x_n \right\}.$$

Such proper cone is non-eccentric, but its angular width

$$\text{aw}(K_\nu) = \arccos\left(\frac{1-\nu}{1+\nu}\right) - \arccos\left(\frac{\nu-1}{\nu+1}\right)$$

goes to  $\pi$  as  $\nu \rightarrow \infty$ .

## 5 Eccentricity results valid only in dimension three

Proposition 2.3 admits a sort of converse statement when the underlying space is three dimensional. Beware that the next proposition is false in higher dimensional spaces.

**Proposition 5.1** *Any non-eccentric simplicial cone in  $\mathbb{R}^3$  is equiangular.*

*Proof* The proof relies on geometric arguments that are exclusive to a three dimensional space. Let  $K$  be a non-eccentric simplicial cone in  $\mathbb{R}^3$ . Without loss of

generality, one may assume that the generators  $\{g_1, g_2, g_3\}$  are unit vectors. Let  $a := \xi(K) = \eta(K)$ . If one cuts  $\partial K^\uparrow$  and  $\partial K^\downarrow$  with the plane

$$H := \{x \in \mathbb{R}^3 : \langle a, x \rangle = 1\},$$

then one gets a pair

$$\begin{aligned} C_1 &:= H \cap \partial K^\uparrow = \{x \in H : s(K)\|x\| = 1\}, \\ C_2 &:= H \cap \partial K^\downarrow = \{x \in H : r(K)\|x\| = 1\} \end{aligned}$$

of concentric circles in that plane. The radius of the inner circle  $C_1$  is of course smaller than the radius of the outer circle  $C_2$ . Consider the polytope  $\Delta := \text{co}\{w_1, w_2, w_3\}$ , where the  $k$ th vertex  $w_k := \langle a, g_k \rangle^{-1} g_k$  is a certain positive multiple of  $g_k$ . Geometrically speaking,  $\Delta$  is a triangle on the plane  $H$ . Since the  $g_k$ 's are osculating points of  $K$ , the three vertices of  $\Delta$  are on the outer circle  $C_2$ , i.e.,

$$\|w_1\| = \|w_2\| = \|w_3\| = [r(K)]^{-1}. \quad (33)$$

On the other hand, by using Henrion and Seeger (2011, Theorem 2.4) one deduces that  $a$  is equidistant from each facet of  $K$ . Hence, the inner circle  $C_1$  touches the three sides of  $\Delta$ . These observations, together with the fact that  $C_1$  and  $C_2$  are concentric, imply that the triangle  $\Delta$  is equilateral, i.e., there exists a positive constant  $\ell$  such that

$$\|w_1 - w_2\| = \|w_1 - w_3\| = \|w_2 - w_3\| = \ell. \quad (34)$$

By combining (33) and (34) one gets

$$\langle g_i, g_j \rangle = \frac{\langle w_i, w_j \rangle}{\|w_i\| \|w_j\|} = 1 - (\ell^2/2)[r(K)]^2$$

for  $i \neq j$ . This proves not only that  $K$  is equiangular, but it provides also an explicit formula for the common angle between generators.  $\square$

The next example displays a non-eccentric simplicial cone in  $\mathbb{R}^4$  that is not equiangular. Similar examples can be constructed in yet higher dimensional spaces.

*Example 5.2* Let  $K$  be the simplicial cone generated by the columns of the block structured matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

The first column is orthogonal to the last column, but not to the second one. Hence,  $K$  is not equiangular. On the other hand,  $K$  is non-eccentric as one can check by working out the barycentric equation (11).

We now take care of Theorem 1.1(b). The lower bound  $\varrho_3 \geq (1/4)\pi$  has been already established in Henrion and Seeger (2011), so it remains to prove the following statement.

**Proposition 5.3** *ED sequences do not exist in  $\Pi_3$ .*

*Proof* Suppose, to the contrary, that  $\Pi_3$  admits an ED sequence  $\{K_\nu\}_{\nu \in \mathbb{N}}$ . Taking a subsequence if necessary, one may assume that

$$\lim_{\nu \rightarrow \infty} \delta(K_\nu, K) = 0, \quad \lim_{\nu \rightarrow \infty} \|\xi(K_\nu) - a\| = 0, \quad \lim_{\nu \rightarrow \infty} \|\eta(K_\nu) - b\| = 0,$$

with  $K \in \Xi_3$  and  $a, b \in \mathbb{R}^3$ . Since

$$\xi(K_\nu), \eta(K_\nu) \in K_\nu \cap K_\nu^* \cap \mathbb{S}_3$$

for all  $\nu \in \mathbb{N}$ , the vectors  $a$  and  $b$  belong to the set

$$M := K \cap K^* \cap \mathbb{S}_3. \tag{35}$$

But the proof of Proposition 4.7 also works for  $n = 3$ . Hence,  $K$  is a closed convex cone in  $\mathbb{R}^3$  that is neither solid nor pointed. It follows that  $K$  is a half-plane. In particular, the set (35) is a singleton and  $a = b$ . One gets in this way

$$\lim_{\nu \rightarrow \infty} \langle \xi(K_\nu), \eta(K_\nu) \rangle = \langle a, b \rangle = 1,$$

contradicting the fact that  $\{K_\nu\}_{\nu \in \mathbb{N}}$  is an ED sequence. □

## 6 Further comments on eccentricity

Non-eccentricity can be obtained as a consequence of self-duality. The next result follows straightforwardly from the first equality mentioned in (7).

**Proposition 6.1** *Let  $n \geq 3$ . Every self-dual cone in  $\mathbb{R}^n$  is non-eccentric.*

The class of self-dual cones is very narrow and do not deserve further comments. Another way of ensuring non-eccentricity is to ask the cone to be symmetric enough. The symmetry rank of a cone  $K \in \Pi_n$  is defined as the integer

$$\text{sr}(K) := \dim[\mathcal{L}(K)],$$

where  $\mathcal{L}(K)$  denotes the intersection of all linear subspaces with respect to which  $K$  is symmetric. Symmetry relative to a linear subspace is understood in the classical sense, i.e., invariance with respect to reflections through that subspace (cf. Barker and Carlson 1979, Definition 1).

**Proposition 6.2** *Let  $n \geq 3$ . Then any element in the family*

$$\mathfrak{L}_1 := \{K \in \Pi_n : \text{sr}(K) = 1\}$$

*is non-eccentric.*

*Proof* A general symmetry principle stated in Seeger and Torki (2013, Theorem 2.4) asserts that

$$\xi(K), \eta(K) \in \mathfrak{L}(K)$$

for all  $K \in \Pi_n$ . So, if  $\mathfrak{L}(K)$  is one-dimensional, then  $\xi(K)$  and  $\eta(K)$  must coincide.  $\square$

The symmetry rank of a proper cone is invariant with respect to duality and with respect orthogonal transformations, i.e.,  $\text{sr}(K^*) = \text{sr}(K)$  and  $\text{sr}(U(K)) = \text{sr}(K^*)$  for all  $K \in \Pi_n$  and all orthogonal matrix  $U$ . As shown in Seeger and Torki (2013), the family  $\mathfrak{L}_1$  includes:

$$\left\{ \begin{array}{l} \text{any ellipsoidal cone in the Stern–Wolkowicz (1991) sense,} \\ \text{any topheavy cone in the Fiedler–Haynsworth (1973) sense,} \\ \text{any permutation invariant proper cone.} \end{array} \right.$$

*Remark 6.3* According to Proposition 2.3, any equiangular simplicial cone is non-eccentric. This fact can be proven also by relying on Proposition 6.2. Indeed, if  $K$  is an equiangular simplicial cone as in Proposition 2.3, then  $K$  is symmetric with respect to any linear subspace of the form

$$\{x \in \mathbb{R}^n : \langle g_i - g_j, x \rangle = 0\}$$

with  $i, j \in \{1, \dots, n\}$ . Hence,  $\mathfrak{L}(K) = \mathbb{R}(G\mathbf{1}_n)$  and  $\text{sr}(K) = 1$ .

*Remark 6.4* Loosely speaking, the condition  $\text{sr}(K) = 1$  reflects a high degree of symmetry in  $K$ . A proper cone with symmetry rank equal to 2 has still a lot of symmetry in it, but not enough to guarantee non-eccentricity. Indeed, an element of the family

$$\mathfrak{L}_2 := \{K \in \Pi_n : \text{sr}(K) = 2\}$$

could perfectly well be eccentric. More surprisingly,  $\mathfrak{L}_2$  admits an ED sequence. To see this, consider a Cartesian product of two revolution cones as in (24). Each revolution cone has symmetry rank equal to 1, but the product has symmetry rank equal to 2.

The symmetry rank of a self-dual cone may be different from 1, so Propositions 6.1 and 6.2 are not comparable.

*Example 6.5* In  $\mathbb{R}^3$  consider the polyhedral cone  $K$  generated by the columns of the matrix

$$G = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This cone is proposed by Barker and Foran (1976) as example of self-dual polyhedral cone that is not simplicial. One can check that  $\{x \in \mathbb{R}^3 : x_2 = 0\}$  is the only linear subspace with respect to which  $K$  is symmetric. Hence,  $\text{sr}(K) = 2$  and Proposition 6.2 is not applicable. The cone  $K$  is non-eccentric thanks to Proposition 6.1.

### 6.1 A conic version of a non-eccentricity result for tetrahedra

Proposition 5.1 characterizes non-eccentricity for simplicial cones in  $\mathbb{R}^3$ . The next result is in the same vein, but it applies to simplicial cones in  $\mathbb{R}^4$ . Proposition 6.6 corresponds to a conic version of a classical non-eccentricity result for tetrahedra, according to which a tetrahedron has concentric inscribed and circumscribed spheres if and only if its opposite edges are of equal length, see for instance Edmonds et al. (2005, Theorem 2.1).

**Proposition 6.6** *Let  $K$  be a simplicial cone in  $\mathbb{R}^4$  with generators  $\{g_1, \dots, g_4\}$  of unit length. Then  $K$  is non-eccentric if and only if the following system holds*

$$\begin{cases} \langle g_1, g_2 \rangle = \langle g_3, g_4 \rangle, \\ \langle g_1, g_3 \rangle = \langle g_2, g_4 \rangle, \\ \langle g_1, g_4 \rangle = \langle g_2, g_3 \rangle. \end{cases} \quad (36)$$

*Proof* Suppose that  $K$  is non-eccentric. In order to obtain (36) one proceeds exactly as in the proof of Proposition 5.1. This time  $H$  is an hyperplane in  $\mathbb{R}^4$  and the sets  $C_1, C_2$  are concentric spheres. One can check that  $C_1$  (respectively,  $C_2$ ) is the sphere inscribed (respectively, circumscribed) in the tetrahedron  $\Delta := \text{co}\{w_1, \dots, w_4\}$ . By applying Edmonds et al. (2005, Theorem 2.1), one gets

$$\begin{aligned} \|w_1 - w_2\| &= \|w_3 - w_4\|, \\ \|w_1 - w_3\| &= \|w_2 - w_4\|, \\ \|w_1 - w_4\| &= \|w_2 - w_3\|. \end{aligned}$$

By squaring these equalities and simplifying, one arrives at (36). Conversely, suppose that the system (36) is in force. Let  $M$  be the Gramian matrix associated to the generators of  $K$ , i.e.,

$$M_{i,j} = \langle g_i, g_j \rangle$$

for  $i, j \in \{1, \dots, 4\}$ . By assumption, there are scalars  $b, c, d \in \mathbb{R}$  such that

$$M = \begin{bmatrix} 1 & b & c & d \\ b & 1 & d & c \\ c & d & 1 & b \\ d & c & b & 1 \end{bmatrix}. \quad (37)$$

We claim that  $K$  satisfies the baricentric equation (11) or, what is equivalent, that  $M\mu_F = \sigma\mu_G$  for some positive  $\sigma$ . Note that  $\mu_G = \mathbf{1}_4$ . Since  $F^T F = M^{-1}$  and  $M$  has the special structure (37), it follows that

$$\|f_k\| = \left[ \frac{1 - b^2 - c^2 - d^2 + 2bcd}{\det(M)} \right]^{1/2}$$

for all  $k \in \{1, \dots, 4\}$ . Hence,  $\mu_F = \delta\mathbf{1}_4$  for some positive  $\delta$ . One gets in this way,

$$M\mu_F = \delta M\mathbf{1}_4 = \sigma\mathbf{1}_4 = \sigma\mu_G$$

with  $\sigma := \delta(1 + b + c + d)$ . □

## 6.2 Eccentricity in infra-dual cones

As seen in Corollary 4.9, a sequence of infra-dual proper cones in  $\mathbb{R}^n$  cannot be ED. In other words,

$$\kappa_n := \sup_{\substack{K \in \Pi_n \\ K \text{ infra-dual}}} \psi(K)$$

is different from  $\pi/2$ . One can construct an infra-dual proper cone in  $\mathbb{R}^n$  with eccentricity as close as  $\pi/2$  as one wishes, but this can be done only at the price of increasing without bound the dimension  $n$  of the underlying space. The details are explained in the next proposition.

**Proposition 6.7** *For all  $n \geq 4$  there exists an infra-dual proper cone  $K$  in  $\mathbb{R}^n$  such that*

$$\langle \xi(K), \eta(K) \rangle < (n - 2)^{-1/2}. \quad (38)$$

*In particular,  $\lim_{n \rightarrow \infty} \kappa_n = \pi/2$ .*

*Proof* Take  $n \geq 4$  and define  $K$  as the Cartesian product of the proper cones

$$P_t := \{x \in \mathbb{R}^2 : x_2 \geq tx_1 \geq 0\},$$

$$Q := \mathbb{R}_+^{n-2},$$

where  $t$  is a positive parameter. Thus,  $K$  is the simplicial cone generated by the columns of the block structured matrix

$$G = \begin{bmatrix} 1 & 0 & & \\ t & 1 & & \\ & & & I_{n-2} \end{bmatrix}.$$

Note that  $K$  is infra-dual because all the entries of  $G$  are nonnegative. We shall prove that  $K$  satisfies (38) if one takes  $t$  large enough. Since  $P_t$  and  $Q$  are non-eccentric, one has

$$\langle \xi(K), \eta(K) \rangle = \gamma(P_t, Q),$$

where  $\gamma : \Pi_2 \times \Pi_{n-2} \rightarrow [0, 1]$  is defined as in Corollary 3.2. A matter of computation yields

$$\begin{aligned} \rho(Q) &= r(Q) = (n-2)^{-1/2}, \\ \rho(P_t) &= a_t := \left[ 1 + \left( \sqrt{1+t^2} + t \right)^2 \right]^{-1/2}, \\ r(P_t) &= b_t := \left[ 1 + \left( \sqrt{1+t^2} - t \right)^2 \right]^{-1/2}. \end{aligned}$$

This shows that

$$\gamma(P_t, Q) = \frac{(n-2)^{-1} + a_t b_t}{[(n-2)^{-1} + a_t^2]^{1/2} [(n-2)^{-1} + b_t^2]^{1/2}}$$

approaches  $(n-1)^{-1/2}$  as the parameter  $t$  goes to infinity. Since  $(n-1)^{-1/2} < (n-2)^{-1/2}$ , the inequality (38) holds for  $t$  large enough.  $\square$

Besides being infra-dual, the cone  $K$  constructed in the proof of Proposition 6.7 has the additional property of being simplicial. A result similar to Proposition 6.7 holds if infra-duality is changed by supra-duality. Despite appearances, Proposition 6.7 is not in conflict with Corollary 4.9. In the latter case the dimension  $n$  is fixed, whereas in the former case the dimension  $n$  grows without bound.

*Example 6.8* Suppose that one wishes to construct an infra-dual cone  $K$  with

$$\psi(K) > \frac{\pi}{2} - \frac{1}{100}.$$

Is it possible to do so? According to Proposition 6.7, the existence of such cone is guaranteed if

$$\cos\left(\frac{\pi}{2} - \frac{1}{100}\right) > \frac{1}{\sqrt{n-2}}.$$



So, we take  $n > 2 + [\sin(1/100)]^{-2} \approx 10002$ . This  $n$  is a large dimension indeed.

*Remark 6.9* In fact, the proof of Proposition 6.7 shows that

$$\frac{\pi}{2} - \kappa_n \leq \arcsin\left(\frac{1}{\sqrt{n-1}}\right) \quad (39)$$

for all  $n \geq 4$ . For large values of  $n$ , the term on the right-hand side of (39) behaves like  $1/\sqrt{n}$ . We do not know if the upper bound (39) is asymptotically optimal.

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