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# An edge-based scheme on polyhedral meshes for vector advection-reaction equations

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## Abstract

We devise and analyze an edge-based scheme on polyhedral meshes to approximate a vector advection-reaction problem. The well-posedness of the discrete problem is analyzed first under the classical positivity hypothesis of Friedrichs' systems that requires a lower bound on the lowest eigenvalue of some tensor depending on the model parameters. We also prove stability when the lowest eigenvalue is null or even slightly negative if the mesh size is small enough. A priori error estimates are established for solutions in  $\mathbf{W}^{1,q}(\Omega)$  with  $q \in (\frac{3}{2}, 2]$ . Numerical results are presented on three-dimensional polyhedral meshes.

**AMS Subject Classification.** 65N12, 65N15, 65Zxx, 76Dxx, 76Wxx

## 1 Introduction

Let  $\Omega$  be a polyhedral domain of  $\mathbb{R}^d$  with  $d = 3$  and consider a polyhedral mesh of  $\Omega$ . We use boldface fonts for  $\mathbb{R}^d$  or  $\mathbb{R}^{d \times d}$ -valued quantities. The purpose of this paper is to devise an approximation, using scalar degrees of freedom (dofs) attached to the edges of a mesh, of the  $\mathbb{R}^d$ -valued function  $\mathbf{u}$  solving the vector advection-reaction problem:

$$\nabla(\boldsymbol{\beta} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \boldsymbol{\beta} + \boldsymbol{\mu} \mathbf{u} = \mathbf{s} \quad \text{a.e. in } \Omega, \quad (1a)$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{a.e. on } \partial\Omega^-. \quad (1b)$$

The  $\mathbb{R}^d$ -valued advective field  $\boldsymbol{\beta}$  is assumed to be Lipschitz continuous in  $\Omega$  and the  $\mathbb{R}^{d \times d}$ -valued reaction tensor  $\boldsymbol{\mu}$  is assumed to be bounded in  $\Omega$ . The subset  $\partial\Omega^- \subset \partial\Omega$  denotes the inflow part of the boundary where  $\boldsymbol{\beta} \cdot \mathbf{n} < 0$  with  $\mathbf{n}$  the unit outward normal to  $\Omega$ .

The model problem (1) is encountered in various situations. For example, it models the static advection of a magnetic field ( $\mathbf{u}$  here) by a moving plasma of velocity  $\boldsymbol{\beta}$  and of anisotropic conductivity  $\boldsymbol{\mu}$ . In the context of differential geometry, the operator  $\nabla(\boldsymbol{\beta} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \boldsymbol{\beta}$  is the proxy of the Lie derivative of a differential 1-form (also called circulation) in  $\mathbb{R}^3$  (see Abraham *et al.* [1] or Heumann [17]). The Lie derivative describes more generally the advection along the vector field  $\boldsymbol{\beta}$  of a differential form on a manifold. The model problem (1) is also relevant to study, in the advection-dominant regime, the advection-diffusion of a  $\mathbb{R}^d$ -valued field, which is one the building blocks of the Oseen problem or of the magneto-hydrodynamic problem. Using vector calculus rules, we observe that  $\nabla(\boldsymbol{\beta} \cdot \mathbf{u}) = (\nabla \boldsymbol{\beta})^T \mathbf{u} + (\nabla \mathbf{u})^T \boldsymbol{\beta}$  and  $(\nabla \times \mathbf{u}) \times \boldsymbol{\beta} = (\nabla \mathbf{u}) \boldsymbol{\beta} - (\nabla \mathbf{u})^T \boldsymbol{\beta}$ , yielding  $\nabla(\boldsymbol{\beta} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \boldsymbol{\beta} = (\nabla \mathbf{u}) \boldsymbol{\beta} + (\nabla \boldsymbol{\beta})^T \mathbf{u}$ . Hence,

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the particular choice  $\boldsymbol{\mu} = -(\nabla\boldsymbol{\beta})^\top$  yields the pure advection problem (with the more usual writing  $(\nabla\mathbf{u})\boldsymbol{\beta} = (\boldsymbol{\beta}\cdot\nabla)\mathbf{u}$  in this context):

$$(\boldsymbol{\beta}\cdot\nabla)\mathbf{u} = \mathbf{s} \quad \text{a.e. in } \Omega, \quad (2a)$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{a.e. on } \partial\Omega^-. \quad (2b)$$

Edge-based schemes are rarely addressed in the literature; some examples for the Maxwell or the Navier-Stokes equations can be found in Zaglmayr [23] and Girault [16], respectively. Edge-based schemes are actually the natural way to discretize differential 1-forms. In this context, Heumann and Hiptmair recently studied in [18]  $\mathbf{H}(\text{curl}; \Omega)$ -conforming discretizations of arbitrary order using Nédélec edge finite elements on simplicial meshes with a stabilization term penalizing jumps of the normal component across mesh interfaces in the spirit of the discontinuous Galerkin method (see Lesaint & Raviart [20], or Johnson & Pitkäranta [19]). They also analyzed a discretization using a cell-based fully discontinuous polynomial approximation of arbitrary order, this time penalizing the full jump across mesh interfaces. In a different context and motivated by the discretization of the Lie derivative, we also mention the Ph.D. of Palha [22] approximating on square meshes a problem similar to (1) with the spectral element method (see also Gerritsma [15]). Based on the work of Bossavit [6], Mullen *et al.* also studied in [21] an approximation of (1) by extruding the edges of a simplicial mesh along the vector field  $\boldsymbol{\beta}$ .

To the authors' knowledge, edge-based schemes approximating the problem (1) on polyhedral meshes are not available in the literature. Recently, Bonelle & Ern proposed in [5] a Compatible Discrete Operator (CDO) scheme for the Stokes problem, where the velocity is attached to edges and the pressure to vertices. The present scheme could be combined with this work to devise a polyhedral discretization of the Oseen problem. The advantage of considering polyhedral meshes is multifold; it allows for more flexibility when meshing a complex geometry, it provides a natural framework to handle non-matching mesh refinement and mesh coarsening by cell agglomeration, and it may even yield lower computational costs and better accuracy compared to the case of the simplicial meshes (see Bonelle's Ph.D. [2]).

The first important contribution of this work is to provide a polyhedral edge-based scheme, with scalar degrees of freedom attached to the edges of the mesh leading to an  $\mathcal{O}(h^q)$  convergence rate as soon as the solution belongs to  $\mathbf{W}^{1,q}(\Omega)$  with  $q \in (\frac{3}{2}, 2]$ . To reach our goal, we first consider on each cell of the mesh a diamond partition composed of volumes surrounding each edge of the mesh. Our scheme is formulated using a reconstruction map defining piece-wise constant vector-valued functions on each diamond of this partition. This map was first considered by Codecasa *et al.* in [9] for electromagnetism and by Bonelle & Ern in [5, 3] for scalar elliptic problems. In the present work, we additionally establish the local  $\mathbf{L}^q$ -stability of this map.

The second important contribution of this work concerns the well-posedness of the discrete problem. We introduce the symmetric tensor

$$\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}} = (\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^\top) - (\nabla\cdot\boldsymbol{\beta})\mathbf{Id} + (\boldsymbol{\mu} + \boldsymbol{\mu}^\top),$$

and we denote  $\lambda_b$  its minimal eigenvalue over the domain  $\Omega$ , *ie.*

$$\lambda_b = \text{ess inf}_{\mathbf{x} \in \Omega} \min_{\mathbf{y} \in \mathbb{R}^d} \frac{(\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\mathbf{x})\mathbf{y}, \mathbf{y})_{\ell^2}}{|\mathbf{y}|_{\ell^2}^2},$$

where  $|\cdot|_{\ell^2}$  denotes the Euclidean norm induced by the Euclidean inner-product  $(\cdot, \cdot)_{\ell^2}$  in  $\mathbb{R}^d$ . Assuming that  $\mathbf{s} \in \mathbf{L}^2(\Omega)$ ,  $\mathbf{u}_D \in \mathbf{L}^2(|\boldsymbol{\beta}\cdot\mathbf{n}|; \partial\Omega)$  and that  $\text{dist}(\partial\Omega^-, \partial\Omega^+) > 0$  (with  $\partial\Omega^+$  the outflow part of the boundary), we infer from Ern & Guermond in [13] that the problem (1) is well-posed in the graph space  $\mathbf{V}_{\boldsymbol{\beta}}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid (\boldsymbol{\beta}\cdot\nabla)\mathbf{v} \in \mathbf{L}^2(\Omega)\}$  if the fields  $\boldsymbol{\beta}$  and  $\boldsymbol{\mu}$  satisfy the following Hypothesis:

**(H1)**  $\lambda_b > 0$ . We define the reference time  $\tau = \lambda_b^{-1}$ .

Under this hypothesis, the well-posedness of the discrete problem hinges on a coercivity argument. However, this assumption is somehow restrictive; *e.g.*, the basic case of a constant vector field  $\boldsymbol{\beta}$  with no

reaction term does not fulfill this hypothesis. Motivated by our recent work [7] related to scalar advection-diffusion problem (see also the work of Deuring *et al.* in [10] for faced-based finite volume schemes), we propose to extend the analysis so as to include a second hypothesis, where the minimal eigenvalue  $\lambda_b$  can take null or slightly negative values:

**(H2)**  $-C_\lambda < \lambda_b \leq 0$ , where  $C_\lambda > 0$  is a constant independent the mesh size, and there exists a potential  $\zeta \in W^{1,\infty}(\Omega)$  satisfying  $\zeta \geq 1$  and  $\text{ess inf}_\Omega (-\boldsymbol{\beta} \cdot \nabla \zeta) > 0$ . We define the reference time  $\tau = (\text{ess inf}_\Omega (-\boldsymbol{\beta} \cdot \nabla \zeta))^{-1}$ .

In the case of a continuously differentiable vector field  $\boldsymbol{\beta} \in \mathbf{C}^1(\Omega)$ , the existence of the potential  $\zeta$  is proved by Devinatz *et al.* in [11, Lemma 2.3] by considering the Cauchy problem  $d_t \mathbf{x}(t) = \boldsymbol{\beta}(\mathbf{x}(t))$ ,  $\mathbf{x}(0) = \mathbf{x}_0 \in \Omega$  when the solution remains in the domain  $\Omega$  for a finite time only. As a result, the hypothesis **(H2)** is satisfied if the vector field  $\boldsymbol{\beta}$  has no closed curves and no stationary points in  $\Omega$ . The analysis of the polyhedral edge-based scheme under this second hypothesis **(H2)** is more complex since the stability of the scheme now hinges on an inf-sup condition which is satisfied if the mesh size is smaller than a reference length that linearly depends on  $\|\nabla \boldsymbol{\beta}^T + \boldsymbol{\mu}\|_{L^\infty(\Omega)}^{-1}$ . Moreover, for the advective problem (2) (where  $\boldsymbol{\mu} \equiv -\nabla \boldsymbol{\beta}^T$ ), we prove the stability of the scheme without any restriction on the mesh size.

This paper is organized as follows. In Section 2, we introduce the notation and the analysis tools on polyhedral meshes. In Section 3, we introduce the edge-based reconstruction map and we present the numerical scheme with dofs attached to edges. In Section 4, we state the main analytic results, namely, stability under hypothesis **(H1)** or **(H2)**, boundedness and a priori error estimates delivering quasi-optimal decay rates for solutions in  $\mathbf{W}^{1,q}(\Omega)$  with  $q \in (\frac{3}{2}, 2]$ . The proofs are postponed to Section 6 to facilitate the reading. Finally, we present in Section 5 numerical results on three-dimensionnal polyhedral meshes.

## 2 Notation and analysis tools on polyhedral meshes

We consider a general mesh  $\mathcal{M}$  of  $\Omega \subset \mathbb{R}^d$  with  $d = 3$ , composed of polyhedral cells  $c \in \mathcal{C}$  (3-cells), planar faces  $f \in \mathcal{F}$  (2-cells), straight edges  $e \in \mathcal{E}$  (1-cells), and vertices  $v \in \mathcal{V}$  (0-cells). We collect the interior faces in the set  $\mathcal{F}^\circ = \{f = \partial c \cap \partial c' \mid c \neq c' \text{ and } c, c' \in \mathcal{C}\}$ , and we define  $\mathcal{F}^\partial = \mathcal{F} \setminus \mathcal{F}^\circ$  the set collecting boundary faces as  $\cdot$ . For any  $\mathbf{A}, \mathbf{X} \in \{\mathcal{V}, \mathcal{E}, \mathcal{F}, \mathcal{C}\}$ , we define the subset  $\mathbf{X}_a$  with  $a \in \mathbf{A}$  as  $\{x \in \mathbf{X} \mid a \subset \partial x\}$  if the dimension of  $a$  is smaller than that of the elements of  $\mathbf{X}$  and as  $\mathbf{X}_a = \{x \in \mathbf{X} \mid x \subset \partial a\}$  otherwise; for example,  $\mathcal{C}_e = \{c \in \mathcal{C} \mid e \subset \partial c\}$ ,  $\mathcal{E}_c = \{e \in \mathcal{E} \mid e \subset \partial c\}$ ,  $\mathcal{E}_f = \{e \in \mathcal{E} \mid e \subset \partial f\}$  and so on. For any geometric entity  $x$ , we denote  $|x|$  its Hausdorff measure. In this paper, we assume mesh regularity in the sense that

- The mesh  $\mathcal{M} := \{\mathcal{V}, \mathcal{E}, \mathcal{F}, \mathcal{C}\}$  defines a cellular complex (see Christiansen [8]), *ie.* the boundary of any  $k$ -cell,  $1 \leq k \leq d$  (recall  $d = 3$ ), is composed of a uniformly finite number of  $(k - 1)$ -cells in  $\mathcal{M}$ .
- Faces and cells are star-shaped with respect to their barycenters.
- Let  $\mathbf{x}_v$  denote the coordinates of  $v \in \mathcal{V}$  in  $\mathbb{R}^d$ . Let  $\mathbf{x}_f$  and  $\mathbf{x}_c$  denote the coordinates of the barycenters of  $f \in \mathcal{F}$  and  $c \in \mathcal{C}$ , respectively, in  $\mathbb{R}^d$ . Then, the simplicial sub-mesh composed of the tetrahedra  $[\mathbf{x}_v, \mathbf{x}_{v'}, \mathbf{x}_f, \mathbf{x}_c]$  (where  $[\mathbf{x}_1, \dots, \mathbf{x}_{k+1}]$  is the convex hull of the set  $\{\mathbf{x}_1, \dots, \mathbf{x}_{k+1}\}$ ) for all  $c \in \mathcal{C}$ , all  $f \in \mathcal{F}_c$  and all  $e \in \mathcal{E}_f$  with  $e = [\mathbf{x}_v, \mathbf{x}_{v'}]$  (see Figure 1, left panel) is shape-regular in the usual sense of Ciarlet.

For every cell  $c \in \mathcal{C}$ , we introduce the edge-based diamond partition  $\mathfrak{P}_c$  which plays a central role in our analysis. We define  $\mathfrak{P}_c = \cup\{\mathfrak{p}_{e,c}; e \in \mathcal{E}_c\}$  where the diamond  $\mathfrak{p}_{e,c}$  is defined by

$$\mathfrak{p}_{e,c} = \bigcup_{f \in \mathcal{F}_c \cap \mathcal{F}_e} [\mathbf{x}_v, \mathbf{x}_{v'}, \mathbf{x}_f, \mathbf{x}_c] \text{ with } e = [\mathbf{x}_v, \mathbf{x}_{v'}],$$

see Figure 1, right panel. Note that  $\mathfrak{P}_c$  is composed of  $\#\mathcal{E}_c$  diamonds and that each diamond  $\mathfrak{p}_{e,c}$  is composed of two tetrahedra, since  $\#(\mathcal{F}_e \cap \mathcal{F}_c) = 2$ , with  $\#$  the cardinal operator. Owing to the star-shaped property of faces and cells, we have  $c = \cup\{\mathfrak{p}; \mathfrak{p} \in \mathfrak{P}_c\}$ . The skeleton of the global partition

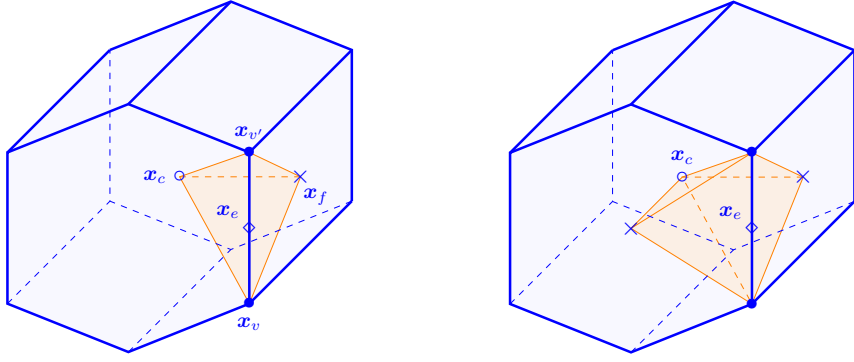


Figure 1: Left panel: tetrahedron  $[\mathbf{x}_v, \mathbf{x}_{v'}, \mathbf{x}_f, \mathbf{x}_c]$ . Right panel: local diamond  $\mathbf{p}_{e,c}$ .

$\mathfrak{P} = \cup\{\mathfrak{P}_c \mid c \in \mathbf{C}\}$  consists of the collection of all the triangular sub-faces defining the boundary of each diamond  $\mathbf{p}_{e,c}$ . There are two types of sub-faces: intra-cell sub-faces attached to a cell  $c \in \mathbf{C}$  and collected in the set  $\mathfrak{F}_c = \{\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c} \mid e \neq e' \text{ and } e, e' \in \mathbf{E}_c\}$  so that  $\mathbf{f} \not\subset \partial c$ , (see Figure 2, left panel) and inter-cell sub-faces attached to an interior face  $f \in \mathbf{F}^\circ$  and collected in the set  $\mathfrak{F}_f = \{\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c'} \mid c \neq c' \text{ and } c, c' \in \mathbf{C}_f, e \in \mathbf{E}_f\}$  (see Figure 2, right panel). All the sub-faces are oriented by a fixed unit normal

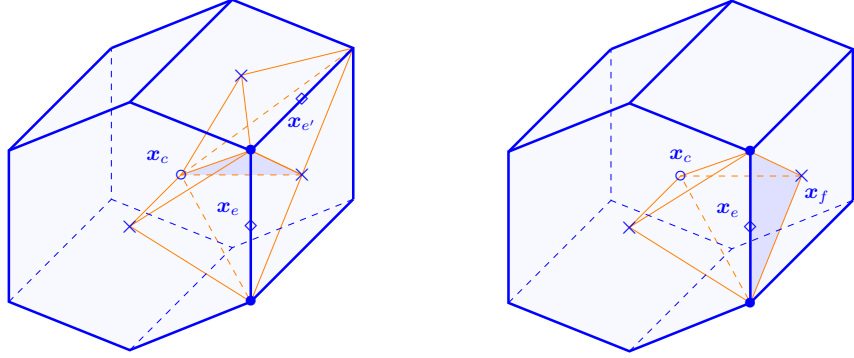


Figure 2: In blue. Left: intra-cell sub-face  $\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c} \in \mathfrak{F}_c$ . Right: inter-cell sub-face  $\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c'} \in \mathfrak{F}_f$ .

vector  $\mathbf{n}_f$ . For all  $\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c} \in \mathfrak{F}_c$  with  $e, e' \in \mathbf{E}_c$  and  $\mathbf{n}_f$  pointing from  $\mathbf{p}_{e,c}$  to  $\mathbf{p}_{e',c}$ , we define the jump and the average, respectively, as

$$[[\mathbf{v}]] = \mathbf{v}|_{\mathbf{p}_{e,c}} - \mathbf{v}|_{\mathbf{p}_{e',c}} \quad \text{and} \quad \{\{\mathbf{v}\}\} := \frac{1}{2} \left( \mathbf{v}|_{\mathbf{p}_{e,c}} + \mathbf{v}|_{\mathbf{p}_{e',c}} \right).$$

Similarly, for all  $\mathbf{f} = \partial\mathbf{p}_{e,c} \cap \partial\mathbf{p}_{e',c'} \in \mathfrak{F}_f$  with  $c, c' \in \mathbf{C}_f, e \in \mathbf{E}_f$ , and  $\mathbf{n}_f$  pointing from  $\mathbf{p}_{e,c}$  to  $\mathbf{p}_{e',c'}$ , we define

$$[[\mathbf{v}]] := \mathbf{v}|_{\mathbf{p}_{e,c}} - \mathbf{v}|_{\mathbf{p}_{e',c'}} \quad \text{and} \quad \{\{\mathbf{v}\}\} := \frac{1}{2} \left( \mathbf{v}|_{\mathbf{p}_{e,c}} + \mathbf{v}|_{\mathbf{p}_{e',c'}} \right).$$

We denote  $|\cdot|_{\ell^2}$  the Euclidean and the Frobenius norm on  $\mathbb{R}^d$  and  $\mathbb{R}^{d \times d}$ , respectively. For every set  $\omega \subset \bar{\Omega}$ , we denote  $\mathbf{L}^q(\omega)$  with  $q \in [1, \infty]$  the Banach space of  $\mathbb{R}^d$  or  $\mathbb{R}^{d \times d}$ -valued functions  $\mathbf{v}$  such that  $\|\mathbf{v}\|_{\mathbf{L}^q(\omega)} := \|\mathbf{v}|_{\ell^2}\|_{\mathbf{L}^q(\omega)} < \infty$ .

**Lemma 2.1** (Multiplicative trace inequality). *There exists  $C_T > 0$  such that*

$$\|\mathbf{v}\|_{\mathbf{L}^q(\mathbf{f})} \leq C_T \|\mathbf{v}\|_{\mathbf{L}^q(\mathbf{p})}^{1-\frac{1}{q}} \left( h_c^{-\frac{1}{q}} \|\mathbf{v}\|_{\mathbf{L}^q(\mathbf{p})}^{\frac{1}{q}} + |\mathbf{v}|_{\mathbf{W}^{1,q}(\mathbf{p})}^{\frac{1}{q}} \right), \quad (3)$$

for all  $c \in \mathbf{C}$  with  $h_c$  the diameter of  $c$ , all  $\mathbf{p} \in \mathfrak{P}_c$ , all  $\mathbf{f} \in \partial\mathbf{p}$  and all  $\mathbf{v} \in \mathbf{W}^{1,q}(\mathbf{p})$  with  $q \in [1, \infty]$ .

*Proof.* Observing that  $\mathbf{p} \in \mathfrak{P}_c$  is composed of two tetrahedra connected by a sub-face  $\mathbf{f} \in \mathfrak{F}_c$ , this result follows proceeding as in Ern & Guermond [14].  $\square$

### 3 Discrete Scheme

#### 3.1 Degrees of freedom

We consider an approximation of the continuous problem (1) with scalar dofs attached to edges. We denote  $\mathcal{E} \equiv \mathbb{R}^{\#\mathbb{E}}$  the linear space collecting these dofs and we denote  $v_e$  the entry of  $\mathbf{v} \in \mathcal{E}$  attached to the edge  $e \in \mathbb{E}$ . We additionally introduce the linear space  $\mathcal{E}_c$  collecting the dofs attached to the subset  $E_c$  for all  $c \in \mathbb{C}$ . We denote  $\mathbf{v}$  a generic element of  $\mathcal{E}$  or  $\mathcal{E}_c$ .

#### 3.2 Reconstruction map

The global reconstruction map  $\mathbf{L}_{\mathcal{E}}$  is defined locally, so that  $\mathbf{L}_{\mathcal{E}}(\mathbf{v})|_c = \mathbf{L}_{\mathcal{E}_c}(\mathbf{v})$ , for all  $c \in \mathbb{C}$ . The local reconstruction map  $\mathbf{L}_{\mathcal{E}_c} : \mathcal{E}_c \rightarrow \mathbb{P}_0(\mathfrak{P}_c; \mathbb{R}^d)$ , where  $\mathbb{P}_0(\mathfrak{P}_c; \mathbb{R}^d)$  is composed of piece-wise constant  $\mathbb{R}^d$ -valued polynomials over the diamond partition  $\mathfrak{P}_c$ , is such that

$$\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})(\mathbf{x}) := \sum_{e \in E_c} v_e \boldsymbol{\ell}_{e,c}(\mathbf{x}), \quad \forall \mathbf{v} \in \mathcal{E}_c, \quad \forall \mathbf{x} \in c, \quad (4)$$

where for all  $e \in E_c$ , the basis function  $\boldsymbol{\ell}_{e,c} \in \mathbb{P}_0(\mathfrak{P}_c; \mathbb{R}^d)$ , is defined by

$$\boldsymbol{\ell}_{e,c}|_{\mathfrak{p}_{e',c}} = \left( \mathbf{Id} - \frac{\tilde{\mathbf{f}}_c(e') \otimes \mathbf{e}'}{d|\mathfrak{p}_{e',c}|} \right) \frac{\tilde{\mathbf{f}}_c(e)}{|c|} + \frac{\tilde{\mathbf{f}}_c(e)}{d|\mathfrak{p}_{e,c}|} \delta_{e,e'}, \quad \forall e' \in E_c, \quad (5)$$

and  $\delta_{e,e'}$  is the Kronecker symbol equal to 1 if  $e = e'$  and 0 otherwise. Moreover, for all  $e \in \mathbb{E}$ ,  $\mathbf{t}_e$  is a fixed unit tangent vector to  $e$ , such that  $\mathbf{e} = |e|\mathbf{t}_e$ , and  $\tilde{\mathbf{f}}_c(e) = \int_{\tilde{f}_c(e)} \mathbf{n}_{\tilde{f}_c(e)}$  where the dual face  $\tilde{f}_c(e)$  is composed of two elementary triangles

$$\tilde{f}_c(e) = \bigcup_{f \in F_c \cap F_e} [\mathbf{x}_e, \mathbf{x}_f, \mathbf{x}_c],$$

see Figure 3, and where  $\mathbf{n}_{\tilde{f}_c(e)}$  is the unit normal vector to  $\tilde{f}_c(e)$  satisfying  $\mathbf{n}_{\tilde{f}_c(e)} \cdot \mathbf{t}_e \geq 0$ . The basis

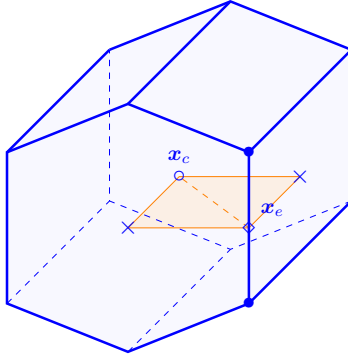


Figure 3: Local dual face  $\tilde{f}_c(e)$ .

functions  $\boldsymbol{\ell}_{e,c}$  were first considered in the context of the *Discrete Geometric Approach* by Codecasa *et al.* [9] and were recently revisited by Bonelle & Ern in [4, 5] to build Hodge operators within the CDO framework. They satisfy the following properties:

- (ℓ1) [Unisolvence] For all  $e, e' \in E_c$ ,  $\boldsymbol{\ell}_{e,c}(\mathbf{x}) \cdot \mathbf{e}' = \delta_{e,e'}$  for all  $\mathbf{x} \in \mathfrak{p}_{e',c}$ .
- (ℓ2) [Primal  $\mathbb{P}_0$ -consistency]  $\sum_{e \in E_c} \boldsymbol{\ell}_{e,c}(\mathbf{x}) \otimes \mathbf{e} = \mathbf{Id}$  for all  $\mathbf{x} \in c$ .
- (ℓ3) [Dual  $\mathbb{P}_0$ -consistency] For all  $e \in E_c$ ,  $\int_c \boldsymbol{\ell}_{e,c}(\mathbf{x}) = \tilde{\mathbf{f}}_c(e)$ .

The property (ℓ1) relies on the geometric relation  $|\mathfrak{p}_{e,c}| = \frac{1}{d} \tilde{\mathbf{f}}_c(e) \cdot \mathbf{e}$  whereas the property (ℓ2) results from the geometric relation  $\sum_{e \in E_c} \mathbf{e} \otimes \tilde{\mathbf{f}}_c(e) = \sum_{e \in E_c} \tilde{\mathbf{f}}_c(e) \otimes \mathbf{e} = |c| \mathbf{Id}$ .

### 3.3 Discrete scheme

The discrete scheme is formulated using the global bilinear form  $\mathbb{A}_{\beta,\mu} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$  such that

$$\mathbb{A}_{\beta,\mu}(\mathbf{u}, \mathbf{v}) = \mathbf{A}_{\beta,\mu}(\mathbf{u}, \mathbf{v}) + \mathbf{A}_{(\beta \cdot \mathbf{n})^-}^\vartheta(\mathbf{u}, \mathbf{v}), \quad (6)$$

where  $\mathbf{A}_{\beta,\mu}$  approximates (1a) and  $\mathbf{A}_{(\beta \cdot \mathbf{n})^-}^\vartheta$  weakly enforces the boundary condition (1b). The bilinear form  $\mathbf{A}_{\beta,\mu} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$  is composed of three bilinear forms also defined on  $\mathcal{E} \times \mathcal{E}$ :

$$\mathbf{A}_{\beta,\mu}(\mathbf{u}, \mathbf{v}) := \mathbf{g}_{\beta,\mu}(\mathbf{u}, \mathbf{v}) + \mathbf{n}_\beta(\mathbf{u}, \mathbf{v}) + \mathbf{s}_\beta(\mathbf{u}, \mathbf{v}). \quad (7)$$

The bilinear form  $\mathbf{g}_{\beta,\mu}$  is assembled cell-wise as

$$\mathbf{g}_{\beta,\mu}(\mathbf{u}, \mathbf{v}) = \sum_{c \in \mathcal{C}} \mathbf{g}_{\beta,\mu;c}(\mathbf{u}, \mathbf{v}), \quad (8)$$

and each local bilinear form  $\mathbf{g}_{\beta,\mu;c}$  results from the standard Galerkin approximation of (1a) in  $c$  using the reconstruction map  $\mathbf{L}_{\mathcal{E}_c}$ :

$$\mathbf{g}_{\beta,\mu;c}(\mathbf{u}, \mathbf{v}) = \sum_{\mathfrak{p} \in \mathfrak{P}_c} \int_{\mathfrak{p}} (\nabla(\beta \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{u})) + (\nabla \times \mathbf{L}_{\mathcal{E}_c}(\mathbf{u})) \times \beta) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + \int_c \mu \mathbf{L}_{\mathcal{E}_c}(\mathbf{u}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}). \quad (9)$$

Using identities from vector calculus and since  $\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})$  is piece-wise constant, we can reformulate this expression as

$$\mathbf{g}_{\beta,\mu;c}(\mathbf{u}, \mathbf{v}) = \int_c (\nabla \beta^T + \mu) \mathbf{L}_{\mathcal{E}_c}(\mathbf{u}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}). \quad (10)$$

Because  $\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})$  jumps across inter-cell and intra-cell sub-faces, we also consider the bilinear form  $\mathbf{n}_\beta$  such that

$$\mathbf{n}_\beta(\mathbf{u}, \mathbf{v}) = \sum_{c \in \mathcal{C}} \mathbf{n}_{\beta;c}(\mathbf{u}, \mathbf{v}) + \sum_{f \in \mathcal{F}^\circ} \mathbf{n}_{\beta;f}(\mathbf{u}, \mathbf{v}), \quad (11)$$

where the local bilinear forms  $\mathbf{n}_{\beta;x}$  with  $x = f$  or  $x = c$  are defined as

$$\mathbf{n}_{\beta;x}(\mathbf{u}, \mathbf{v}) = - \sum_{\mathfrak{f} \in \mathfrak{F}_x} \int_{\mathfrak{f}} (\beta \cdot \mathbf{n}_{\mathfrak{f}}) \llbracket \mathbf{L}_{\mathcal{E}}(\mathbf{u}) \rrbracket \cdot \{\!\! \{ \mathbf{L}_{\mathcal{E}}(\mathbf{v}) \}\!\!\}, \quad (12)$$

and the stabilization bilinear form  $\mathbf{s}_\beta$  such that

$$\mathbf{s}_\beta(\mathbf{u}, \mathbf{v}) = \sum_{c \in \mathcal{C}} \mathbf{s}_{\beta;c}(\mathbf{u}, \mathbf{v}) + \sum_{f \in \mathcal{F}^\circ} \mathbf{s}_{\beta;f}(\mathbf{u}, \mathbf{v}), \quad (13)$$

where the local bilinear forms  $\mathbf{s}_{\beta;x}$  with  $x = f$  or  $x = c$  are defined as

$$\mathbf{s}_{\beta;x}(\mathbf{u}, \mathbf{v}) = \sum_{\mathfrak{f} \in \mathfrak{F}_x} \int_{\mathfrak{f}} |\beta \cdot \mathbf{n}_{\mathfrak{f}}| \llbracket \mathbf{L}_{\mathcal{E}}(\mathbf{u}) \rrbracket \cdot \llbracket \mathbf{L}_{\mathcal{E}}(\mathbf{v}) \rrbracket. \quad (14)$$

The bilinear forms  $\mathbf{n}_\beta$  and  $\mathbf{s}_\beta$  are devised similarly to the discontinuous Galerkin method;  $\mathbf{n}_\beta$  corresponds to centered fluxes and  $\mathbf{n}_\beta + \mathbf{s}_\beta$  to upwind fluxes. Finally, the Dirichlet boundary condition is weakly enforced by means of the bilinear form  $\mathbf{A}_\alpha^\vartheta : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$  (with  $\alpha = (\beta \cdot \mathbf{n})^-$ ) such that

$$\mathbf{A}_\alpha^\vartheta(\mathbf{u}, \mathbf{v}) = \sum_{f \in \mathcal{F}^\vartheta} \mathbf{A}_{\alpha;f}^\vartheta(\mathbf{u}, \mathbf{v}). \quad (15)$$

The local bilinear form  $\mathbf{A}_{\alpha;f}^\vartheta$  is defined as

$$\mathbf{A}_{\alpha;f}^\vartheta(\mathbf{u}, \mathbf{v}) = \int_f \alpha \mathbf{L}_{\mathcal{E}_f}(\mathbf{u}) \cdot \mathbf{L}_{\mathcal{E}_f}(\mathbf{v}), \quad (16)$$

with  $c_f$  is the unique cell containing the boundary face  $f$ .

The discrete scheme consists in finding  $\mathbf{u} \in \mathcal{E}$  such that

$$\mathbb{A}_{\beta, \mu}(\mathbf{u}, \mathbf{v}) = \Sigma(\mathbf{s}, \mathbf{u}_D; \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{E}, \quad (17)$$

with the right-hand side form  $\Sigma(\mathbf{s}, \mathbf{u}_D; \cdot) : \mathcal{E} \rightarrow \mathbb{R}$  such that

$$\Sigma(\mathbf{s}, \mathbf{u}_D; \mathbf{v}) := \sum_{c \in \mathbb{C}} \int_c \mathbf{s} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + \sum_{f \in \mathbb{F}^\partial} \int_f (\beta \cdot \mathbf{n})^- \mathbf{u}_D \cdot \mathbf{L}_{\mathcal{E}_{c_f}}(\mathbf{v}). \quad (18)$$

## 4 Stability and error analysis

### 4.1 Properties of the reconstruction map

**Proposition 4.1** (Stability). *There exists  $C_{\sharp} > 0$  such that*

$$\|\mathbf{v}\|_{q,c} \leq \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)} \leq C_{\sharp} \|\mathbf{v}\|_{q,c},$$

for all  $c \in \mathbb{C}$ , all  $\mathbf{v} \in \mathcal{E}_c$ , all  $q \in [1, \infty)$  and where

$$\|\mathbf{v}\|_{q,c} = \left( \sum_{e \in \mathbb{E}_c} \frac{|\mathbf{p}_{e,c}|}{|e|^q} |\mathbf{v}_e|^q \right)^{\frac{1}{q}}. \quad (19)$$

*Remark 4.1* (Alternative definition). In lieu of (19), we could also consider the simpler discrete  $\mathbf{L}^q$ -norm given by  $\|\mathbf{v}\|_{q,c}^q = h_c^{d-q} \sum_{e \in \mathbb{E}_c} |\mathbf{v}_e|^q$ . Owing to mesh regularity, this definition is equivalent to (19) up to a uniform constant with respect to the mesh-size. We prefer to use (19) since it simplifies the proof of Proposition 4.1.

We introduce the reduction map  $\mathbf{R}_{\mathcal{E}} : \mathbf{L}^1(\Omega) \rightarrow \mathcal{E}$  such that

$$\mathbf{R}_{\mathcal{E}}(\mathbf{v})|_e := \frac{1}{|\mathbf{p}_e|} \left( \int_{\mathbf{p}_e} \mathbf{v} \cdot \mathbf{e} \right), \quad \forall e \in \mathbb{E}, \quad (20)$$

where  $\mathbf{p}_e = \cup\{\mathbf{p}_{e,c}; c \in \mathbb{C}_e\}$  is the diamond volume surrounding the edge  $e$  and  $\hat{c}$  is the local diamond patch  $\hat{c} = \cup\{\mathbf{p}_e; e \in \mathbb{E}_c\}$  surrounding the cell  $c$ ; notice that  $c \subsetneq \hat{c}$ . We also define the local reduction map  $\mathbf{R}_{\mathcal{E}_c} : \mathbf{L}^1(\hat{c}) \rightarrow \mathcal{E}_c$  from definition (20) for all  $e \in \mathbb{E}_c$ .

*Remark 4.2* (De Rham's map). Requiring more regularity, the usual de Rham's reduction map defined by  $\mathbf{R}_{\mathcal{E}}(\mathbf{v})|_e = |e|^{-1} \int_e \mathbf{v} \cdot \mathbf{e}$  for every  $e \in \mathbb{E}$  can be used as well, provided that  $\mathbf{v} \in \mathbf{H}^{1+\epsilon}(\Omega)$  or  $\mathbf{v} \in \{\mathbf{w} \in \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega), \nabla \times \mathbf{w} \in \mathbf{L}^{2+\epsilon}(\Omega)\}$  with  $\epsilon > 0$ .

For each cell  $c \in \mathbb{C}$ , we denote  $\mathcal{I}_{\mathcal{E}_c}$  the interpolation operator obtained by composing the local reconstruction map with the local reduction map, *ie.*  $\mathcal{I}_{\mathcal{E}_c} = \mathbf{L}_{\mathcal{E}_c} \circ \mathbf{R}_{\mathcal{E}_c}$ , so that  $\mathcal{I}_{\mathcal{E}_c} : \mathbf{L}^1(\hat{c}) \rightarrow \mathbb{P}_0(\mathfrak{P}_c; \mathbb{R}^d)$ .

**Proposition 4.2** (Consistency). *For all  $c \in \mathbb{C}$  and all  $\mathbf{U} \in \mathbb{P}_0(\hat{c}; \mathbb{R}^d)$  (so that  $\mathbf{U}$  is a constant function in  $\hat{c}$ ), we have  $\mathcal{I}_{\mathcal{E}_c}(\mathbf{U}) = \mathbf{U}|_c$ .*

**Lemma 4.3** (Interpolation error). *There exists  $C_{\text{INT}} > 0$  such that for all  $c \in \mathbb{C}$  and all  $\mathbf{v} \in \mathbf{W}^{1,q}(\hat{c})$  with  $q \in [1, \infty)$ ,*

$$\|\mathbf{v} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)} \leq C_{\text{INT}} h_c |\mathbf{v}|_{\mathbf{W}^{1,q}(\hat{c})}, \quad (21)$$

and for all  $\mathbf{p} \in \mathfrak{P}_c$ ,

$$\|\mathbf{v} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(\partial \mathbf{p})} \leq C_{\text{INT}} h_c^{1-\frac{1}{q}} |\mathbf{v}|_{\mathbf{W}^{1,q}(\hat{c})}. \quad (22)$$



## 4.2 Stability and well-posedness

We consider the following stability norm on the edge dof space  $\mathcal{E}$ :

$$\|\mathbf{v}\| := \left( \tau^{-1} \|\mathbf{v}\|_2^2 + |\mathbf{v}|_\partial^2 + |\mathbf{v}|_s^2 \right)^{\frac{1}{2}}, \quad (23)$$

where the reference time  $\tau > 0$  is defined by assumption **(H1)** or **(H2)**,  $\|\cdot\|_2^2 = \sum_{c \in \mathcal{C}} \|\cdot\|_{2,c}^2$  is the discrete  $L^2$ -norm with  $\|\cdot\|_{2,c}$  defined by (19),  $|\cdot|_\partial^2 = \mathbf{A}_{|\boldsymbol{\beta} \cdot \mathbf{n}|}^0(\cdot, \cdot)$  is the semi-norm induced by the bilinear form  $\mathbf{A}_{|\boldsymbol{\beta} \cdot \mathbf{n}|}$  defined by (16), and  $|\cdot|_s^2 := \mathbf{s}_\beta(\cdot, \cdot)$  is the semi-norm induced by the bilinear form  $\mathbf{s}_\beta$  defined by (13).

**Proposition 4.4** (Coercivity). *Assume that **(H1)** holds. Then,*

$$\frac{1}{2} \|\mathbf{v}\|^2 \leq \mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{E}.$$

Consequently, the discrete problem (17) is well-posed.

We now study the stability of the bilinear form  $\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}$  under the hypothesis **(H2)**. We consider the reference length  $h_0^{-1} = 4C_{\sharp}^2 L_\zeta \tau \|\boldsymbol{\mu} + \nabla \boldsymbol{\beta}^T\|_{L^\infty(\Omega)}$  where  $C_{\sharp}$  results from Proposition 4.1 and  $L_\zeta = |\zeta|_{W^{1,\infty}(\Omega)}$  is the Lipschitz constant of  $\zeta$ . If  $\boldsymbol{\mu} = -\nabla \boldsymbol{\beta}^T$ , we set conventionally  $h_0 = +\infty$ . Recalling that  $\lambda_b$  denotes the smallest eigenvalue of the tensor  $\boldsymbol{\sigma}_{\boldsymbol{\beta}, \boldsymbol{\mu}}$  over the domain  $\Omega$ , we assume that

$$1 + 2\vartheta\tau\lambda_b > 0 \quad \text{and} \quad h < h_0(1 + 2\vartheta\tau\lambda_b), \quad (24)$$

where  $\vartheta > 0$  is a constant that linearly depends on  $\|\zeta\|_{L^\infty(\Omega)} + C_T C_{\sharp} L_\zeta \max(|\Omega|^{\frac{1}{d}}, \|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\tau)$ . By convention, the second condition in (24) is void if  $\boldsymbol{\mu} = -\nabla \boldsymbol{\beta}^T$ .

**Proposition 4.5** (Inf-sup stability). *Assume that **(H2)** and (24) hold. Then, there exists  $\varrho > 0$  such that*

$$\varrho \|\mathbf{v}\| \leq \sup_{\mathbf{w} \in \mathcal{E}, \|\mathbf{w}\|=1} \mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{v} \in \mathcal{E}.$$

Consequently, the discrete problem (17) is well-posed.

Table 1 recapitulates the different situations where the discrete problem (17) is well-posed.

$\lambda_b > 0$	$-\frac{1}{2\vartheta\tau} < \lambda_b \leq 0$	
<b>(H1)</b>	<b>(H2)</b>	
	$\boldsymbol{\mu} = -\nabla \boldsymbol{\beta}^T$	$\boldsymbol{\mu} \neq -\nabla \boldsymbol{\beta}^T$
$h \in \mathbb{R}_{>0}$	$h \in \mathbb{R}_{>0}$	$h \in (0, h_0(1 + 2\vartheta\tau\lambda_b))$

Table 1: Stability of the discrete problem (17) with respect to  $\lambda_b$  and the mesh-size  $h$ .

## 4.3 Bound on consistency error and a priori error estimate

In this section, we derive an a priori error estimate by bounding the consistency error

$$\mathbb{E}(\mathbf{u}) = \sup_{\mathbf{v} \in \mathcal{E}, \|\mathbf{v}\|=1} |\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{R}_\mathcal{E}(\mathbf{u}), \mathbf{v}) - \Sigma(\mathbf{s}, \mathbf{u}_D; \mathbf{v})|.$$

In what follows, the notation  $A \lesssim B$  stands for  $A \leq CB$  where  $C$  is a positive non-dimensional constant uniform with respect to the mesh size and the model parameters.

**Lemma 4.6** (Bound on consistency error). *Assume that the exact solution satisfies  $\mathbf{u} \in \mathbf{W}^{1,q}(\Omega)$  with  $q \in [1, 2]$ . Then, the following holds:*

$$\begin{aligned} \mathbb{E}(\mathbf{u}) \lesssim & \left( \sum_{c \in \mathcal{C}} \|\nabla \boldsymbol{\beta} + \boldsymbol{\mu}^T - (\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}\|_{\mathbf{L}^\infty(c)}^q \tau^{\frac{q}{2}} h_c^{\frac{d}{2}(q-2)} \|\mathbf{u} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{u})\|_{\mathbf{L}^q(c)}^q \right)^{\frac{1}{q}} \\ & + \left( \sum_{c \in \mathcal{C}} \sum_{\mathfrak{p} \in \mathfrak{P}_c} \|\boldsymbol{\beta}\|_{\mathbf{L}^\infty(c)}^{\frac{q}{2}} h_c^{\frac{(d-1)}{2}(q-2)} \|\mathbf{u} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{u})\|_{\mathbf{L}^q(\partial \mathfrak{p})}^q \right)^{\frac{1}{q}}. \end{aligned}$$

We can now state the main result of this paper which follows from Lemmata 4.3 and 4.6.

**Theorem 4.7** (A priori estimate). *Assume that the assumptions stated in Table 1 hold. Assume that the exact solution of (1) satisfies  $\mathbf{u} \in \mathbf{W}^{1,q}(\Omega)$  with  $q \in \left(\frac{2d}{d+1}, 2\right]$ . Then, we have*

$$\begin{aligned} \|\mathbf{u} - \mathbf{R}_{\mathcal{E}}(\mathbf{u})\| \lesssim & \left( \sum_{c \in \mathcal{C}} \|\nabla \boldsymbol{\beta} + \boldsymbol{\mu}^T - (\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}\|_{\mathbf{L}^\infty(c)}^q \tau^{\frac{q}{2}} h_c^{\frac{d+2}{2}(q-\frac{2d}{d+2})} |\mathbf{u}|_{\mathbf{W}^{1,q}(\hat{c})}^q \right)^{\frac{1}{q}} \\ & + \left( \sum_{c \in \mathcal{C}} \sum_{\mathfrak{p} \in \mathfrak{P}_c} \|\boldsymbol{\beta}\|_{\mathbf{L}^\infty(c)}^{\frac{q}{2}} h_c^{\frac{d+1}{2}(q-\frac{2d}{d+1})} |\mathbf{u}|_{\mathbf{W}^{1,q}(\hat{c})}^q \right)^{\frac{1}{q}}. \end{aligned}$$

For  $d = 3$ , it follows that  $\|\mathbf{u} - \mathbf{R}_{\mathcal{E}}(\mathbf{u})\| = \mathcal{O}\left(h^{2-\frac{3}{q}}\right)$  for all  $q \in \left(\frac{3}{2}, 2\right]$ .

## 5 Numerical results

We investigate numerically the edge-based scheme (17) on four sequences of three-dimensional polyhedral meshes. Each mesh is obtained as a uniform refinement of an initial mesh. Meshes from the first sequence, denoted  $\mathbf{H}$ , are composed of hexahedra, those from the second one, denoted  $\mathbf{PrT}$ , are composed of prisms with a triangular basis, those from the third one, denoted  $\mathbf{PrG}$ , are composed of prisms with a hexagonal basis, and those of the last one, denoted  $\mathbf{CB}$ , are composed of hexahedra with non-matching interfaces; see Figure 4. The domain is the unit cube  $\Omega := [0, 1]^3$ . The exact solution corresponds to a Taylor–Green

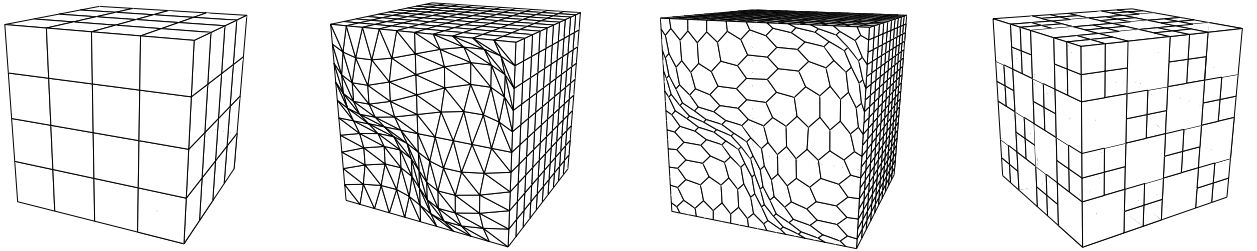


Figure 4: Examples of meshes from the four sequences. From left to right: hexahedral mesh ( $\mathbf{H}$ ), prismatic mesh with triangular basis ( $\mathbf{PrT}$ ), prismatic mesh with hexagonal basis ( $\mathbf{PrG}$ ), and Checkerboard mesh with non-matching interfaces ( $\mathbf{CB}$ ).

velocity field, the advective vector field  $\boldsymbol{\beta}$  is affine (see Figure 5, left panel) and the reaction tensor  $\boldsymbol{\mu}$  is diagonal and constant:

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x) \cos(\pi y/2) \cos(\pi z/2) \\ \cos(\pi x/2) \sin(\pi y) \cos(\pi z/2) \\ \cos(\pi x/2) \cos(\pi y/2) \sin(\pi z) \end{pmatrix}, \quad \boldsymbol{\beta} = \frac{1}{2} \begin{pmatrix} (x-2y)/2 \\ (y-2x)/2 \\ -z \end{pmatrix}, \quad \boldsymbol{\mu} = \frac{1}{2} \mathbf{Id}.$$

Note that  $\nabla \cdot \boldsymbol{\beta} = 0$  and that the eigenvalues of the tensor  $\boldsymbol{\sigma}_{\boldsymbol{\beta}, \boldsymbol{\mu}}$  are  $\{0, \frac{1}{2}, \frac{5}{2}\}$ , so that the discrete scheme (17) is well-posed owing to Proposition 4.5 if the mesh size is small enough.

We perform a convergence study by computing the relative discrete  $L^2$ -error attached to edge dofs, denoted  $\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$ , and defined by

$$\mathbf{Er}_{\mathcal{E}}(\mathbf{u}) = \frac{\|\mathbf{u} - \mathbf{R}_{\mathcal{E}}(\mathbf{u})\|_2}{\|\mathbf{R}_{\mathcal{E}}(\mathbf{u})\|_2},$$

with the norm  $\|\cdot\|_2$  on every cell of the mesh by (19). The convergence rates, shown in the left panel of

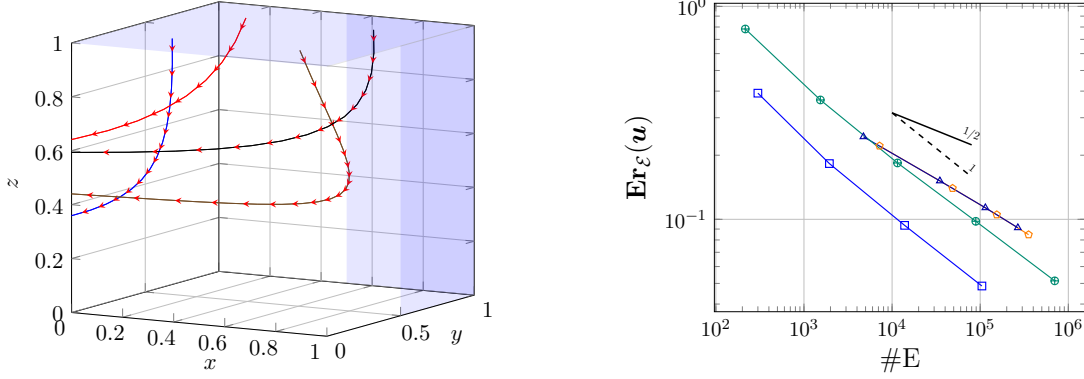


Figure 5: Left panel: inflow boundary  $\partial\Omega^-$  in blue and some streamlines of the vector field  $\boldsymbol{\beta}$ . Right panel: Discrete errors on H ( $\square$ ), PrT ( $\triangle$ ), PrG ( $\diamond$ ), and CB ( $\oplus$ ) mesh sequences.

Figure 5, lie between  $\frac{1}{2}$  and 1 for the PrT and PrG mesh sequences and are closer to 1 for the H and CB mesh sequences. Table 6 provides additional information on the computational costs by reporting the size of the linear system ( $\#E$ ), the mean stencil  $\mathbf{St}$ , the values of the discrete error  $\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$ , and the ratios  $\#E/\#V$  and  $\#E/\#C$ , indicating that the present scheme may involve less dofs than traditional Finite Volume schemes placing  $\mathbb{R}^d$ -valued unknowns at mesh vertices or at mesh cells. Note that owing to the Euler relation,  $\frac{\#V}{\#E} + \frac{\#F}{\#E} - \frac{\#C}{\#E} = \frac{2}{\#E} - 1 \approx 1$ .

$\#E$	$\mathbf{St}$	$\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$	$\frac{\#E}{\#V}$	$\frac{\#E}{\#C}$	$\#E$	$\mathbf{St}$	$\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$	$\frac{\#E}{\#V}$	$\frac{\#E}{\#C}$
3.0e+02	21	3.9e-01	2.40	4.69	4.7e+03	38	2.4e-01	3.55	2.37
1.9e+03	25	1.8e-01	2.67	3.80	3.5e+04	46	1.5e-01	3.76	2.18
1.4e+04	28	9.4e-02	2.82	3.39	1.1e+05	48	1.1e-01	3.84	2.12
1.0e+05	30	4.9e-02	2.91	3.19	2.7e+05	49	9.1e-02	3.88	2.09
$\#E$	$\mathbf{St}$	$\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$	$\frac{\#E}{\#V}$	$\frac{\#E}{\#C}$	$\#E$	$\mathbf{St}$	$\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$	$\frac{\#E}{\#V}$	$\frac{\#E}{\#C}$
7.2e+03	83	2.2e-01	2.34	5.95	1.5e+03	112	3.6e-01	2.46	5.33
4.9e+04	110	1.4e-01	2.41	5.51	1.2e+04	144	1.8e-01	2.61	5.00
1.5e+05	120	1.1e-01	2.44	5.35	8.9e+04	162	9.8e-02	2.70	4.83
3.5e+05	125	8.5e-02	2.45	5.26	7.0e+05	180	5.1e-02	2.75	4.75

Figure 6: Mean stencil  $\mathbf{St}$  and discrete error  $\mathbf{Er}_{\mathcal{E}}(\mathbf{u})$  for the H (upper left panel), PrT (upper right panel), PrG (lower left panel), and the CB (lower right panel) mesh sequences.

*Remark 5.1* (Stabilization parameter). As observed in Bonelle *et al.* [3], one can reformulate the basis functions  $\boldsymbol{\ell}_{e,c}$  as a consistent term plus a stabilization term:

$$\boldsymbol{\ell}_{e,c}|_{\mathbf{p}_{e',c}} = \underbrace{\frac{\tilde{\mathbf{f}}_c(e)}{|c|}}_{\text{Consistent term}} + \frac{1}{d} \underbrace{\left( \frac{\tilde{\mathbf{f}}_c(e)}{|\mathbf{p}_{e,c}|} \delta_{e,e'} - \frac{e' \cdot \tilde{\mathbf{f}}_c(e)}{|\mathbf{p}_{e',c}|} \frac{\tilde{\mathbf{f}}_c(e')}{|c|} \right)}_{\text{Stabilization term}}, \quad \forall e' \in E_c.$$

Numerical experiments show that it is possible to replace the parameter  $d^{-1}$  by a positive value that is reasonably close to  $d^{-1}$ ; however, in the stability analysis, this modification impacts the property  $(\ell 1)$  which is used to obtain the lower bound in Proposition 4.1.

## 6 Proofs

### 6.1 Properties of the reconstruction map

*Proof of Proposition 4.1.* Let  $c \in \mathbb{C}$ , let  $\mathbf{v} \in \mathcal{E}_c$  and let  $q \in [1, \infty)$ .

i) *Lower bound.* Owing to the definition (4) of  $\mathbf{L}_{\mathcal{E}_c}$ , we infer that

$$\|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)}^q = \sum_{e \in \mathbb{E}_c} \left\| |\mathbf{v}_e \mathbf{a}_e + \mathbf{b}_e|_{\ell^2} \right\|_{L^q(\mathfrak{p}_{e,c})}^q,$$

with

$$\mathbf{a}_e = \frac{\mathbf{e}}{|e|^2} \text{ and } \mathbf{b}_e = \left( \boldsymbol{\ell}_{e,c} - \frac{\mathbf{e}}{|e|^2} \right) \mathbf{v}_e + \sum_{e' \in \mathbb{E}_c \setminus \{e\}} \mathbf{v}_{e'} \boldsymbol{\ell}_{e',c}.$$

Using the Property  $(\ell 1)$ , we observe that  $\mathbf{a}_e \cdot \mathbf{b}_e \equiv 0$  on  $\mathfrak{p}_{e,c}$ , so that  $|\mathbf{v}_e \mathbf{a}_e + \mathbf{b}_e|_{\ell^2} \geq |\mathbf{v}_e \mathbf{a}_e|_{\ell^2}$ , whence

$$\|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)}^q \geq \sum_{e \in \mathbb{E}_c} \left\| |\mathbf{v}_e \mathbf{a}_e|_{\ell^2} \right\|_{L^q(\mathfrak{p}_{e,c})}^q = \sum_{e \in \mathbb{E}_c} |\mathbf{v}_e|^q \|\mathbf{a}_e\|_{L^q(\mathfrak{p}_{e,c})}^q.$$

Hence, the expected lower bound follows from  $\|\mathbf{a}_e\|_{L^q(\mathfrak{p}_{e,c})}^q = \frac{|\mathfrak{p}_{e,c}|}{|e|^q}$ .

ii) *Upper bound.* The discrete Hölder inequality yields

$$\|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)}^q \leq (\#\mathbb{E}_c)^{q-1} \sum_{e \in \mathbb{E}_c} |\mathbf{v}_e|^q \|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^q(c)}^q.$$

Since  $\|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^q(c)}^q \leq |c| \|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^\infty(c)}^q$ , we have  $\|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^q(c)}^q \leq C_{\sharp}^q (\#\mathbb{E}_c)^{1-q} \frac{|\mathfrak{p}_{e,c}|}{|e|^q}$  with the constant

$$C_{\sharp} = (\#\mathbb{E}_c)^{1-\frac{1}{q}} \max_{e \in \mathbb{E}_c} \left( \left( \frac{|c|}{|\mathfrak{p}_{e,c}|} \right)^{\frac{1}{q}} |e| \|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^\infty(c)} \right),$$

that is uniformly bounded owing to mesh regularity, yielding the expected upper bound. Specifically, a straightforward calculation shows that

$$\left| \boldsymbol{\ell}_{e,c}|_{\mathfrak{p}_{e,c}} \right|_{\ell^2} \leq \frac{|\tilde{f}_c(e)|}{|c|} \left( \frac{|c|}{d|\mathfrak{p}_{e,c}|} \right) \quad \text{and} \quad \left| \boldsymbol{\ell}_{e,c}|_{\mathfrak{p}_{e',c}} \right|_{\ell^2} \leq \frac{|\tilde{f}_c(e)|}{|c|} \left( 1 + \frac{1}{\cos^2(\mathbf{t}_{e'}, \mathbf{n}_{\tilde{f}_c(e')})} \right)^{\frac{1}{2}},$$

leading to

$$|e| \|\boldsymbol{\ell}_{e,c}\|_{\mathbf{L}^\infty(c)} \leq \left( \frac{|e| |\tilde{f}_c(e)|}{|c|} \right) \max \left\{ \left( \frac{|c|}{d|\mathfrak{p}_{e,c}|} \right), \max_{e' \in \mathbb{E}_c, e' \neq e} \left( 1 + \frac{1}{\cos^2(\mathbf{t}_{e'}, \mathbf{n}_{\tilde{f}_c(e')})} \right)^{\frac{1}{2}} \right\}.$$

□

*Proof of Proposition 4.2.* Let  $c \in \mathbb{C}$  and let  $e' \in \mathbb{E}_c$ . The consistency property relies on the property  $(\ell 2)$ . Indeed, given  $\mathbf{U} \in \mathbb{P}_0(\hat{c}; \mathbb{R}^d)$ , we infer that, for all  $\mathbf{x} \in \mathfrak{p}_{e',c}$ ,

$$\mathbf{L}_{\mathcal{E}_c} \mathbf{R}_{\mathcal{E}_c}(\mathbf{U})(\mathbf{x}) = \sum_{e \in \mathbb{E}_c} \mathbf{R}_{\mathcal{E}_c}(\mathbf{U})|_e \boldsymbol{\ell}_{e,c}(\mathbf{x}) = \sum_{e \in \mathbb{E}_c} (\mathbf{U} \cdot \mathbf{e}) \boldsymbol{\ell}_{e,c}(\mathbf{x}) = \left( \sum_{e \in \mathbb{E}_c} \boldsymbol{\ell}_{e,c}(\mathbf{x}) \otimes \mathbf{e} \right) \mathbf{U} = \mathbf{U}.$$

□

*Proof of Lemma 4.3.* Let  $c \in \mathbb{C}$  and let  $\mathbf{v} \in \mathbf{W}^{1,q}(\hat{c})$  with  $q \in [1, \infty)$ . Owing to the triangle inequality and the  $\mathbb{P}_0$ -consistency of the reconstruction map from Proposition 4.2, we infer that

$$\|\mathbf{v} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)} \leq \|\mathbf{v} - \mathbf{v}_{\hat{c}}\|_{\mathbf{L}^q(c)} + \|\mathcal{I}_{\mathcal{E}_c}(\mathbf{v} - \mathbf{v}_{\hat{c}})\|_{\mathbf{L}^q(c)}$$

with  $\mathbf{v}_{\hat{c}} = |\hat{c}|^{-1} \int_{\hat{c}} \mathbf{v}$ . In addition, we observe that, for all  $\mathbf{w} \in \mathbf{L}^q(\hat{c})$ ,

$$\|\mathcal{R}_{\mathcal{E}_c}(\mathbf{w})\|_{q,c}^q = \sum_{e \in \mathbb{E}_c} \frac{|\mathbf{p}_{e,c}|}{|e|^q} \left| \frac{1}{|\mathbf{p}_e|} \int_{\mathbf{p}_e} \mathbf{w} \cdot \mathbf{e} \right|^q \leq \sum_{e \in \mathbb{E}_c} \frac{|\mathbf{p}_{e,c}|}{|\mathbf{p}_e|^q} \|\mathbf{w}\|_{\mathbf{L}^1(\mathbf{p}_e)}^q \leq \sum_{e \in \mathbb{E}_c} \frac{1}{|\mathbf{p}_e|^{q-1}} \|\mathbf{w}\|_{\mathbf{L}^1(\mathbf{p}_e)}^q,$$

where we have used that  $|\mathbf{p}_{e,c}| \leq |\mathbf{p}_e|$  to infer the last inequality. Owing to the Hölder inequality, it then follows that  $\|\mathbf{w}\|_{\mathbf{L}^1(\mathbf{p}_e)}^q \leq \|\mathbf{w}\|_{\mathbf{L}^q(\mathbf{p}_e)}^q \|1\|_{\mathbf{L}^{q'}(\mathbf{p}_e)}^q$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ . Since  $\|1\|_{\mathbf{L}^{q'}(\mathbf{p}_e)}^q = |\mathbf{p}_e|^{q-1}$ , we infer that

$$\|\mathcal{R}_{\mathcal{E}_c}(\mathbf{w})\|_{q,c}^q \leq \|\mathbf{w}\|_{\mathbf{L}^q(\hat{c})}^q.$$

Using this estimate and the upper bound from Proposition 4.1, we obtain

$$\|\mathcal{I}_{\mathcal{E}_c}(\mathbf{v} - \mathbf{v}_{\hat{c}})\|_{\mathbf{L}^q(c)} \leq C_{\#} \|\mathcal{R}_{\mathcal{E}_c}(\mathbf{v} - \mathbf{v}_{\hat{c}})\|_{q,c} \leq C_{\#} \|\mathbf{v} - \mathbf{v}_{\hat{c}}\|_{\mathbf{L}^q(c)},$$

so that  $\|\mathbf{v} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)} \leq (1 + C_{\#}) \|\mathbf{v} - \mathbf{v}_{\hat{c}}\|_{\mathbf{L}^q(c)}$ . Hence,  $\|\mathbf{v} - \mathcal{I}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^q(c)} \leq (1 + C_{\#}) \phi_{\hat{c},q} h_c |\mathbf{v}|_{\mathbf{W}^{1,q}(\hat{c})}$  with

$$\phi_{\hat{c},q} = \sup_{\mathbf{w} \in \mathbf{W}^{1,q}(\hat{c})} \frac{\|\mathbf{w} - \mathbf{w}_{\hat{c}}\|_{\mathbf{L}^q(\hat{c})}}{h_c |\mathbf{w}|_{\mathbf{W}^{1,q}(\hat{c})}}.$$

Finally, we observe that the diamond  $\hat{c}$  can be decomposed as

$$\hat{c} = \bigcup_{e \in \mathbb{E}_c} \mathbf{p}_e = \bigcup_{e \in \mathbb{E}_c} \bigcup_{c \in \mathbb{C}_e} \mathbf{p}_{e,c},$$

where  $\mathbf{p}_{e,c}$  consists of two tetrahedra, so that  $\hat{c}$  is composed of  $2 \sum_{e \in \mathbb{E}_c} \#C_e$  tetrahedra connected through elements of  $\mathfrak{F}_c$  and  $\mathfrak{F}_f$  with  $f \in \mathbb{F}_c$ . Then, proceeding as Ern & Guermond in [14, Lemma 5.5], we infer that the quantity  $\phi_{\hat{c},q}$  is uniformly bounded for all  $c \in \mathbb{C}$  and all  $q \in [1, \infty)$ .  $\square$

## 6.2 Stability and well-posedness

*Proof of Proposition 4.4.* Let  $c \in \mathbb{C}$  and consider  $\mathbf{v}, \mathbf{w} \in \mathcal{E}_c$ . The definition of the bilinear form  $\mathbf{g}_{\beta,\mu;c}$  together with the definition of the tensor  $\boldsymbol{\sigma}_{\beta,\mu}$  yield

$$\mathbf{g}_{\beta,\mu;c}(\mathbf{v}, \mathbf{w}) + \mathbf{g}_{\beta,\mu;c}(\mathbf{w}, \mathbf{v}) = \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\beta,\mu} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{w}) + \sum_{\mathbf{p} \in \mathfrak{P}_c} \int_{\mathbf{p}} \nabla \cdot (\boldsymbol{\beta} \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{w})). \quad (25)$$

Choosing  $\mathbf{w} = \mathbf{v}$  in this relation leads to

$$\frac{1}{2} \sum_{\mathbf{p} \in \mathfrak{P}_c} \int_{\mathbf{p}} \nabla \cdot (\boldsymbol{\beta} |\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})|_{\ell^2}^2) - \sum_{\mathbf{f} \in \mathfrak{F}_c} \int_{\mathbf{f}} (\boldsymbol{\beta} \cdot \mathbf{n}_{\mathbf{f}}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\{\mathbf{L}_{\mathcal{E}}(\mathbf{v})\}\} = \frac{1}{2} \sum_{f \in \mathbb{F}_c} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}_c) |\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})|_{\ell^2}^2,$$

with  $\mathbf{n}_c$  the unit outward normal vector to  $c$ , so that recalling the definition (12) of  $\mathbf{n}_{\beta;c}$ , we infer that

$$\sum_{c \in \mathbb{C}} (\mathbf{g}_{\beta,\mu;c}(\mathbf{v}, \mathbf{v}) + \mathbf{n}_{\beta;c}(\mathbf{v}, \mathbf{v})) = \frac{1}{2} \sum_{c \in \mathbb{C}} \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\beta,\mu} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + \frac{1}{2} \sum_{c \in \mathbb{C}} \sum_{f \in \mathbb{F}_c} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}_c) |\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})|_{\ell^2}^2.$$

The above rightmost term is reformulated as

$$\begin{aligned} \frac{1}{2} \sum_{c \in \mathbb{C}} \sum_{f \in \mathbb{F}_c} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}_c) |\mathbf{L}_{\mathcal{E}}(\mathbf{v})|_{\ell^2}^2 &= \frac{1}{2} \sum_{f \in \mathbb{F}^{\partial}} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}) |\mathbf{L}_{\mathcal{E}_f}(\mathbf{v})|_{\ell^2}^2 + \sum_{f \in \mathbb{F}^{\circ}} \sum_{\mathbf{f} \in \mathfrak{F}_f} \int_{\mathbf{f}} (\boldsymbol{\beta} \cdot \mathbf{n}_{\mathbf{f}}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\{\mathbf{L}_{\mathcal{E}}(\mathbf{v})\}\} \\ &= \frac{1}{2} \sum_{f \in \mathbb{F}^{\partial}} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}) |\mathbf{L}_{\mathcal{E}_f}(\mathbf{v})|_{\ell^2}^2 + \sum_{f \in \mathbb{F}^{\circ}} \mathbf{n}_{\beta,f}(\mathbf{v}, \mathbf{v}) \end{aligned}$$

so that, using the definition (11) of  $\mathbf{n}_\beta$ , we arrive at

$$\mathfrak{g}_{\beta,\mu}(\mathbf{v}, \mathbf{v}) + \mathbf{n}_\beta(\mathbf{v}, \mathbf{v}) = \frac{1}{2} \sum_{c \in \mathbb{C}} \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\beta,\mu} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + \frac{1}{2} \sum_{f \in \mathbb{F}^\theta} \int_f (\boldsymbol{\beta} \cdot \mathbf{n}) |\mathbf{L}_{\mathcal{E}_f}(\mathbf{v})|_{\ell^2}^2.$$

Recalling the definition (6) of  $\mathbb{A}_{\beta,\mu}$  and combining the above relation with the bilinear forms  $\mathfrak{s}_\beta$  and  $\mathbb{A}_{(\boldsymbol{\beta} \cdot \mathbf{n})}^\theta$ , defined by (13) and (15) respectively, we obtain

$$\mathbb{A}_{\beta,\mu}(\mathbf{v}, \mathbf{v}) = \frac{1}{2} \sum_{c \in \mathbb{C}} \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\beta,\mu} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + \frac{1}{2} \mathbb{A}_{|\boldsymbol{\beta} \cdot \mathbf{n}|}^\theta(\mathbf{v}, \mathbf{v}) + \frac{1}{2} \mathfrak{s}_\beta(\mathbf{v}, \mathbf{v}). \quad (26)$$

The expected result is inferred from **(H1)**.  $\square$

To prove the inf-sup condition in Proposition 4.5, we need the two following Lemmata, where we consider the function  $\boldsymbol{\delta}(\mathbf{v})$ , such that its restriction to any mesh cell  $c \in \mathbb{C}$  is

$$\boldsymbol{\delta}(\mathbf{v})|_c = \mathbf{L}_{\mathcal{E}_c}(\zeta \mathbf{v}) - \zeta \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{E}_c, \quad (27)$$

where  $\zeta \in W^{1,\infty}(\Omega)$  is the potential defined by **(H2)** and where  $\zeta \mathbf{v} \in \mathcal{E}$  is defined by  $(\zeta \mathbf{v})_e = \zeta(\mathbf{x}_e) \mathbf{v}_e$ , for all  $e \in \mathbb{E}$ .

**Lemma 6.1** (Bounds on  $\boldsymbol{\delta}$ ). *For all  $c \in \mathbb{C}$ , we have*

$$\|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(c)} \leq 2C_\sharp L_\zeta h_c \|\mathbf{v}\|_{2,c}, \quad \forall \mathbf{v} \in \mathcal{E}_c. \quad (28a)$$

and for all  $f \in \mathbb{F}_c$ ,

$$\|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(f)} \leq 2C_T C_\sharp L_\zeta h_c^{\frac{1}{2}} \|\mathbf{v}\|_{2,c}, \quad \forall \mathbf{v} \in \mathcal{E}_c. \quad (28b)$$

*Proof.* Let  $\mathbf{v} \in \mathcal{E}$  and let  $c \in \mathbb{C}$ .

*i) Proof of (28a).* Let  $\zeta_c$  be the mean-value of  $\zeta$  over  $c$  given by  $\zeta_c = |c|^{-1} \int_c \zeta$ . Since  $\mathbf{L}_{\mathcal{E}_c}(\zeta_c \mathbf{v}) = \zeta_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v})$  because  $\zeta_c$  is constant, we have  $\boldsymbol{\delta}(\mathbf{v})|_c = (\zeta - \zeta_c) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) - \mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})$ , so that the triangle inequality, the Hölder inequality and the upper bound in Proposition 4.1 yield

$$\begin{aligned} \|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(c)} &\leq \|\zeta - \zeta_c\|_{L^\infty(c)} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(c)} + \|\mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})\|_{\mathbf{L}^2(c)} \\ &\leq C_\sharp \|\zeta - \zeta_c\|_{L^\infty(c)} \|\mathbf{v}\|_{2,c} + C_\sharp \|(\zeta - \zeta_c) \mathbf{v}\|_{2,c} \\ &\leq 2C_\sharp \|\zeta - \zeta_c\|_{L^\infty(c)} \|\mathbf{v}\|_{2,c}. \end{aligned}$$

Observing that  $\|\zeta - \zeta_c\|_{L^\infty(c)} \leq L_\zeta h_c$ , the expected result follows.

*ii). Proof of (28b).* Let  $\mathfrak{p} \in \mathfrak{P}_c$  and let  $\mathfrak{f} \subset \partial \mathfrak{p}$ . Owing to the multiplicative trace inequality (3), we have

$$\|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{f})} \leq C_T \|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}^{\frac{1}{2}} \left( h_c^{-\frac{1}{2}} \|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}^{\frac{1}{2}} + |\boldsymbol{\delta}(\mathbf{v})|_{\mathbf{H}^1(\mathfrak{p})}^{\frac{1}{2}} \right).$$

Observe that  $|\boldsymbol{\delta}(\mathbf{v})|_{\mathbf{H}^1(\mathfrak{p})} = \|\nabla \boldsymbol{\delta}(\mathbf{v})\|_{\ell^2(\mathfrak{p})}$  where  $|\nabla \boldsymbol{\delta}(\mathbf{v})|_{\ell^2}^2 = \sum_{i,j}^d |\partial_j \boldsymbol{\delta}(\mathbf{v})_i|^2$  in the Cartesian basis of  $\mathbb{R}^d$  and where  $\partial_i$  is the weak derivative in the direction  $i$ . Since  $\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})$  is piece-wise constant on  $\mathfrak{P}_c$ , it then follows that  $|\nabla \boldsymbol{\delta}(\mathbf{v})|_{\ell^2}^2 = \sum_{i,j}^d |\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})_i \partial_j \zeta|^2 = |\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})|_{\ell^2}^2 |\nabla \zeta|_{\ell^2}^2$ . As a result,  $|\boldsymbol{\delta}(\mathbf{v})|_{\mathbf{H}^1(\mathfrak{p})} \leq L_\zeta \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}$ . Moreover, proceeding as in *i)*, we infer that  $\|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})} \leq 2L_\zeta h_c \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}$ . Collecting these bounds, we infer that

$$\|\boldsymbol{\delta}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{f})} \leq 2C_T L_\zeta h_c^{\frac{1}{2}} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}.$$

Then, summing over  $\mathfrak{F}_f$  and using the upper bound of Proposition 4.1 yield the expected result.  $\square$

In what follows, we consider the non-dimensional number  $\omega_\zeta = L_\zeta \max(|\Omega|^{\frac{1}{d}}, \|\boldsymbol{\beta}\|_{\mathbf{L}^\infty(\Omega)} \tau)$ .

**Lemma 6.2** (Multiplicative stability). *There exists  $C_\zeta > 0$  independent of the mesh size and the model parameters such that*

$$\|\zeta \mathbf{v}\| \leq C_\zeta (\|\zeta\|_{L^\infty(\Omega)} + \omega_\zeta) \|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathcal{E}.$$

*Proof.* Let  $\mathbf{v} \in \mathcal{E}$  and let us rewrite  $\|\zeta \mathbf{v}\|^2$  as

$$\begin{aligned} \|\zeta \mathbf{v}\|^2 &= \sum_{c \in \mathcal{C}} \tau^{-1} \|\zeta \mathbf{v}\|_{2,c}^2 + \sum_{c \in \mathcal{C}} \mathfrak{s}_{\beta,c}(\zeta \mathbf{v}, \zeta \mathbf{v}) + \sum_{f \in \mathcal{F}^\theta} A_{|\beta \cdot \mathbf{n}_f|}(\zeta \mathbf{v}, \zeta \mathbf{v}) + \sum_{f \in \mathcal{F}^\circ} \mathfrak{s}_{\beta,f}(\zeta \mathbf{v}, \zeta \mathbf{v}) \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

We want to use the Lipschitz regularity of  $\zeta$  to bound separately these terms by  $\|\mathbf{v}\|^2$ . we recall the notation  $\zeta_c = |c|^{-1} \int_c \zeta$  from the proof of Lemma 6.1.

*i) Bound on  $T_1$ .* First, the triangle inequality implies that

$$\frac{1}{2} T_1 \leq \sum_{c \in \mathcal{C}} \tau^{-1} \|\zeta_c \mathbf{v}\|_{2,c}^2 + \sum_{c \in \mathcal{C}} \tau^{-1} \|(\zeta - \zeta_c) \mathbf{v}\|_{2,c}^2 = T_{1,1} + T_{1,2}.$$

Since  $|\zeta_c| \leq \|\zeta\|_{L^\infty(c)}$ , we infer that  $T_{1,1} \leq \sum_{c \in \mathcal{C}} \tau^{-1} \|\zeta\|_{L^\infty(c)}^2 \|\mathbf{v}\|_{2,c}^2 \leq \|\zeta\|_{L^\infty(\Omega)}^2 \|\mathbf{v}\|^2$ . The bound  $T_{1,2}$  easily follows from the Lipschitz regularity of  $\zeta$  since  $T_{1,2} \leq \sum_{c \in \mathcal{C}} \tau^{-1} L_\zeta^2 h_c^2 \|\mathbf{v}\|_{2,c}^2 \leq L_\zeta^2 h_c^2 \|\mathbf{v}\|^2$ . Combining these two bounds with  $h_c \leq |\Omega|^{\frac{1}{d}}$  the definition of  $\omega_\zeta$  yields

$$T_1 \leq 2 \left( \omega_\zeta^2 + \|\zeta\|_{L^\infty(\Omega)}^2 \right) \|\mathbf{v}\|^2.$$

*ii) Bound on  $T_2$ .* Since the bilinear form  $\mathfrak{s}_{\beta,c}$  is symmetric and positive, we infer that

$$\frac{1}{2} T_2 \leq \sum_{c \in \mathcal{C}} \mathfrak{s}_{\beta,c}(\zeta_c \mathbf{v}, \zeta_c \mathbf{v}) + \sum_{c \in \mathcal{C}} \mathfrak{s}_{\beta,c}((\zeta - \zeta_c) \mathbf{v}, (\zeta - \zeta_c) \mathbf{v}) = T_{2,1} + T_{2,2},$$

and we have directly that  $T_{2,1} \leq \sum_{c \in \mathcal{C}} \|\zeta\|_{L^\infty(c)}^2 \mathfrak{s}_{\beta,c}(\mathbf{v}, \mathbf{v}) \leq \|\zeta\|_{L^\infty(\Omega)}^2 \|\mathbf{v}\|^2$ . To bound  $T_{2,2}$ , we use the multiplicative trace inequality (3) and that  $\mathbf{L}_{\mathcal{E}_c}$  is piece-wise constant to infer that

$$\mathfrak{s}_{\beta,c}((\zeta - \zeta_c) \mathbf{v}, (\zeta - \zeta_c) \mathbf{v}) = \sum_{\mathfrak{f} \in \mathfrak{F}_c} \int_{\mathfrak{f}} |\beta \cdot \mathbf{n}_f| \|\mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})\|_{\ell^2}^2 \leq 2C_T^2 \|\beta\|_{\mathbf{L}^\infty(c)} \sum_{\mathfrak{f} \in \mathfrak{F}_c} \sum_{\mathfrak{p} \in \mathfrak{P}_c \cap \mathfrak{P}_f} h_c^{-1} \|\mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}^2,$$

where  $\mathfrak{P}_f = \{\mathfrak{p} \in \mathfrak{P} \mid \mathfrak{f} \subset \partial \mathfrak{p}\}$ . Observing that the boundary of each diamond  $\mathfrak{p}_{e,c}$  is composed of 4 sub-faces in  $\mathfrak{F}_c$ , exchanging the sums yields

$$\mathfrak{s}_{\beta,c}((\zeta - \zeta_c) \mathbf{v}, (\zeta - \zeta_c) \mathbf{v}) \leq 8C_T^2 \|\beta\|_{\mathbf{L}^\infty(c)} \sum_{\mathfrak{p} \in \mathfrak{P}_c} h_c^{-1} \|\mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})\|_{\mathbf{L}^2(\mathfrak{p})}^2 = 8C_T^2 \|\beta\|_{\mathbf{L}^\infty(c)} h_c^{-1} \|\mathbf{L}_{\mathcal{E}_c}((\zeta - \zeta_c) \mathbf{v})\|_{\mathbf{L}^2(c)}^2,$$

Owing to upper bound from Proposition 4.1, the Lipschitz regularity of  $\zeta$ , and the definition of  $\omega_\zeta$ , we infer that

$$\mathfrak{s}_{\beta,c}((\zeta - \zeta_c) \mathbf{v}, (\zeta - \zeta_c) \mathbf{v}) \leq 8C_T^2 C_\#^2 \|\beta\|_{\mathbf{L}^\infty(c)} h_c^{-1} \|(\zeta - \zeta_c) \mathbf{v}\|_{2,c}^2 \leq 8C_T^2 C_\#^2 \omega_\zeta^2 \tau^{-1} \|\mathbf{v}\|_{2,c}^2.$$

Finally, collecting these two bounds leads to

$$T_2 \leq 2 \left( \|\zeta\|_{L^\infty(\Omega)}^2 + 8C_T^2 C_\#^2 \omega_\zeta^2 \right) \|\mathbf{v}\|^2.$$

*iii) Bound on  $T_3$ .* We proceed as in the previous step *ii)* to infer that

$$T_3 \leq 2 \left( \|\zeta\|_{L^\infty(\Omega)}^2 + n_{\mathcal{F},\partial} C_T^2 C_\#^2 \omega_\zeta^2 \right) \|\mathbf{v}\|^2,$$

where  $n_{F,\partial} = (\max_{c \in C} \#(F_c \cap F^\partial))$  is the maximal number of boundary faces that a mesh cell can have. *iv) Bound on  $T_4$ .* To bound this last term, we use a different decomposition, namely

$$T_4 = \sum_{f \in F^\circ} s_{\zeta^2 \beta; f}(\mathbf{v}, \mathbf{v}) + \sum_{f \in F^\circ} \Delta_f(\mathbf{v}) = T_{4,1} + T_{4,2},$$

with  $\Delta_f(\mathbf{v}) = s_{\beta; f}(\zeta \mathbf{v}, \zeta \mathbf{v}) - s_{\zeta^2 \beta; f}(\mathbf{v}, \mathbf{v})$ . Observing that  $s_{\zeta^2 \beta; f}(\mathbf{v}, \mathbf{v}) \leq \|\zeta\|_{L^\infty(f)}^2 s_{\beta; f}(\mathbf{v}, \mathbf{v})$  for all  $f \in F^\circ$ , it follows that  $T_{4,1} \leq \|\zeta\|_{L^\infty(\Omega)}^2 \|\mathbf{v}\|^2$ . To bound the second term  $T_{4,2}$ , we recall the quantity  $\delta(\mathbf{v})$  defined by (27) and we obtain

$$\Delta_f(\mathbf{v}) = \int_f |\boldsymbol{\beta} \cdot \mathbf{n}_f| \left( \left( \llbracket \delta(\mathbf{v}) \rrbracket + \zeta \llbracket \mathbf{L}_\mathcal{E}(\mathbf{v}) \rrbracket \right)^2 - \zeta^2 \llbracket \mathbf{L}_\mathcal{E}(\mathbf{v}) \rrbracket^2 \right).$$

Then, applying Young's inequality and the trace inequality (28b) yields

$$\begin{aligned} |\Delta_f(\mathbf{v})| &\leq 2 \int_f |\boldsymbol{\beta} \cdot \mathbf{n}_f| \llbracket \delta(\mathbf{v}) \rrbracket^2 + \int_f |\boldsymbol{\beta} \cdot \mathbf{n}_f| \zeta^2 \llbracket \mathbf{L}_\mathcal{E}(\mathbf{v}) \rrbracket^2 \\ &\leq 4(2C_T C_\# L_\zeta)^2 \|\boldsymbol{\beta}\|_{L^\infty(f)} \sum_{c \in C_f} h_c \|\mathbf{v}\|_{2,c}^2 + \|\zeta\|_{L^\infty(f)}^2 s_{\beta; f}(\mathbf{v}, \mathbf{v}). \end{aligned}$$

As a result, since  $\#C_f = 2$  for all  $f \in F^\circ$  and introducing  $\omega_\zeta$ , we infer that

$$T_{4,2} \leq 32C_T^2 C_\#^2 \omega_\zeta^2 \tau^{-1} \|\mathbf{v}\|_2^2 + \|\zeta\|_{L^\infty(\Omega)}^2 \sum_{f \in F^\circ} s_{\beta; f}(\mathbf{v}, \mathbf{v}) \leq \left( 32C_T^2 C_\#^2 \omega_\zeta^2 + \|\zeta\|_{L^\infty(\Omega)}^2 \right) \|\mathbf{v}\|^2,$$

whence

$$T_4 \leq 2 \left( 16C_T^2 C_\#^2 \omega_\zeta^2 L_\zeta h + \|\zeta\|_{L^\infty(\Omega)}^2 \right) \|\mathbf{v}\|^2.$$

*v) Conclusion.* The expected inequality then follows from the above four bounds.  $\square$

*Proof of Proposition 4.5.* Let  $\mathbf{v} \in \mathcal{E}$  and define

$$\mathbb{S} = \sup_{\mathbf{w} \in \mathcal{E} \setminus \{0\}} \frac{\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|}.$$

Let us take  $\mathbf{w} = \zeta \mathbf{v} + \theta \mathbf{v}$  with  $\theta > 0$  to be chosen below. We infer from Lemma 6.2 that

$$\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{w}) \leq \mathbb{S} \|\mathbf{w}\| \leq \mathbb{S} (\theta + C_\zeta (\|\zeta\|_{L^\infty(\Omega)} + \omega_\zeta)) \|\mathbf{v}\|,$$

so that it remains to prove that  $\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{w}) \gtrsim \|\mathbf{v}\|^2$ . First, we split  $\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}$  as follows:

$$\mathbb{A}_{\boldsymbol{\beta}, \boldsymbol{\mu}}(\mathbf{v}, \mathbf{w}) = \mathbb{A}_{\boldsymbol{\beta}, -\nabla \boldsymbol{\beta}^\top + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{w}) + \mathbb{H}_{\boldsymbol{\mu} + \nabla \boldsymbol{\beta}^\top - \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{w}) = T_1 + T_2,$$

where the bilinear form  $\mathbb{H}_\alpha$  is defined on  $\mathcal{E} \times \mathcal{E}$  by

$$\mathbb{H}_\alpha(\mathbf{v}, \mathbf{w}) = \sum_{c \in C} \int_c \boldsymbol{\alpha} \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{w}),$$

for all  $\boldsymbol{\alpha} \in \mathbf{L}^\infty(\Omega)$ . Let us bound from below the two terms  $T_1$  and  $T_2$ .

*i) Bound on  $T_1$ .* We bound from below this term by considering the following decomposition

$$\begin{aligned} T_1 &= \mathbb{A}_{\boldsymbol{\beta}, -\nabla \boldsymbol{\beta}^\top + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{w}) = \mathbb{A}_{\zeta \boldsymbol{\beta}, -\nabla(\zeta \boldsymbol{\beta})^\top + \frac{1}{2} \zeta (\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{v}) \\ &\quad + \mathbb{A}_{\boldsymbol{\beta}, -\nabla \boldsymbol{\beta}^\top}(\mathbf{v}, \zeta \mathbf{v}) - \mathbb{A}_{\zeta \boldsymbol{\beta}, -\nabla(\zeta \boldsymbol{\beta})^\top}(\mathbf{v}, \mathbf{v}) \\ &\quad + \mathbb{H}_{\frac{1}{2}(\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \zeta \mathbf{v}) - \mathbb{H}_{\frac{1}{2} \zeta (\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{v}) \\ &\quad + \theta \mathbb{A}_{\boldsymbol{\beta}, -\nabla \boldsymbol{\beta}^\top + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}) \mathbf{Id}}(\mathbf{v}, \mathbf{v}) = T_{1,1} + T_{1,2} + T_{1,3} + T_{1,4}. \end{aligned}$$



Regarding  $T_{1,1}$ , we use the relation (26) to infer that

$$\begin{aligned} T_{1,1} &= \frac{1}{2} \left( \sum_{c \in \mathbf{C}} \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\zeta\boldsymbol{\beta}, -\nabla(\zeta\boldsymbol{\beta})^{\mathbf{T}} + \frac{1}{2}\zeta(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + A_{|\zeta\boldsymbol{\beta} \cdot \mathbf{n}|}^{\partial}(\mathbf{v}, \mathbf{v}) + s_{\zeta\boldsymbol{\beta}}(\mathbf{v}, \mathbf{v}) \right) \\ &\geq \frac{1}{2} \left( \sum_{c \in \mathbf{C}} \int_c \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\sigma}_{\zeta\boldsymbol{\beta}, -\nabla(\zeta\boldsymbol{\beta})^{\mathbf{T}} + \frac{1}{2}\zeta(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) + A_{|\boldsymbol{\beta} \cdot \mathbf{n}|}^{\partial}(\mathbf{v}, \mathbf{v}) + s_{\boldsymbol{\beta}}(\mathbf{v}, \mathbf{v}) \right), \end{aligned}$$

since  $\zeta \geq 1$ . Then, observing that  $\boldsymbol{\sigma}_{\zeta\boldsymbol{\beta}, -\nabla(\zeta\boldsymbol{\beta})^{\mathbf{T}} + \frac{1}{2}\zeta(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}} = -\boldsymbol{\beta} \cdot \nabla \zeta \mathbf{Id}$  and using Hypothesis **(H2)** together with the lower bound from Proposition (4.1), we infer that  $T_{1,1} \geq \frac{1}{2} \|\mathbf{v}\|^2$ . The next step consists in bounding the perturbation term  $T_{1,2}$ . To do so, we recall the identity (10) for  $\mathbf{g}_{\boldsymbol{\beta}, \boldsymbol{\mu}; c}$ , and we observe that  $\mathbf{g}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}}; c} \equiv \mathbf{0}$  and  $\mathbf{g}_{\zeta\boldsymbol{\beta}, -\nabla(\zeta\boldsymbol{\beta})^{\mathbf{T}}; c} \equiv \mathbf{0}$ , so that  $T_{1,2}$  solely consists of surfacic terms:

$$T_{1,2} = \left( n_{\boldsymbol{\beta}}(\mathbf{v}, \zeta\mathbf{v}) - n_{\zeta\boldsymbol{\beta}}(\mathbf{v}, \mathbf{v}) \right) + \left( s_{\boldsymbol{\beta}}(\mathbf{v}, \zeta\mathbf{v}) - s_{\zeta\boldsymbol{\beta}}(\mathbf{v}, \mathbf{v}) \right) + \left( A_{(\boldsymbol{\beta} \cdot \mathbf{n})^-}^{\partial}(\mathbf{v}, \zeta\mathbf{v}) - A_{(\zeta\boldsymbol{\beta} \cdot \mathbf{n})^-}^{\partial}(\mathbf{v}, \mathbf{v}) \right).$$

Now, introducing the function  $\boldsymbol{\delta}(\mathbf{v})$  locally defined by (27) and recalling that  $\boldsymbol{\beta} \in \mathbf{W}^{1,\infty}(\Omega)$ ,  $\zeta \geq 1$ , and  $\zeta \in W^{1,\infty}(\Omega)$ , so that  $\zeta \{\{\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\}\} = \{\{\zeta \mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\}\}$ , we observe that

$$\begin{aligned} n_{\boldsymbol{\beta}; x}(\mathbf{v}, \zeta\mathbf{v}) - n_{\zeta\boldsymbol{\beta}; x}(\mathbf{v}, \mathbf{v}) &= \sum_{\mathfrak{f} \in \mathfrak{F}_x} \int_{\mathfrak{f}} (\boldsymbol{\beta} \cdot \mathbf{n}_{\mathfrak{f}}) \{\{\mathbf{L}_{\mathcal{E}}(\mathbf{v})\}\} \cdot \{\{\boldsymbol{\delta}(\mathbf{v})\}\}, \\ s_{\boldsymbol{\beta}; x}(\mathbf{v}, \zeta\mathbf{v}) - s_{\zeta\boldsymbol{\beta}; x}(\mathbf{v}, \mathbf{v}) &= \sum_{\mathfrak{f} \in \mathfrak{F}_x} \int_{\mathfrak{f}} |\boldsymbol{\beta} \cdot \mathbf{n}_{\mathfrak{f}}| \{\{\mathbf{L}_{\mathcal{E}}(\mathbf{v})\}\} \cdot \{\{\boldsymbol{\delta}(\mathbf{v})\}\}, \end{aligned}$$

for all  $x \in \mathbf{F}^{\circ}$  or  $x \in \mathbf{C}$ , and

$$A_{(\boldsymbol{\beta} \cdot \mathbf{n})^-, f}^{\partial}(\mathbf{v}, \zeta\mathbf{v}) - A_{(\zeta\boldsymbol{\beta} \cdot \mathbf{n})^-, f}^{\partial}(\mathbf{v}, \mathbf{v}) = \int_f (\boldsymbol{\beta} \cdot \mathbf{n})^- \mathbf{L}_{\mathcal{E}}(\mathbf{v}) \cdot \boldsymbol{\delta}(\mathbf{v}),$$

for all  $f \in \mathbf{F}^{\partial}$ . Then, applying the Cauchy–Schwarz inequality to these three terms yields

$$T_{1,2} \leq 6 \left( |\mathbf{v}|_b^2 + |\mathbf{v}|_s^2 \right)^{\frac{1}{2}} \left( 2 \sum_{c \in \mathbf{C}} \|\boldsymbol{\beta}\|_{L^{\infty}(c)} \sum_{\mathfrak{p} \in \mathfrak{F}_c} \|\boldsymbol{\delta}(\mathbf{v})\|_{L^2(\partial\mathfrak{p})}^2 \right)^{\frac{1}{2}}.$$

In addition, observing that  $\boldsymbol{\sigma}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}} \equiv \mathbf{0}$  and using the identity (26), we have

$$\mathbb{A}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}}(\mathbf{v}, \mathbf{v}) = \frac{1}{2} \left( |\mathbf{v}|_b^2 + |\mathbf{v}|_s^2 \right),$$

so that combining this expression with the above estimate yields

$$T_{1,2} \leq 12 \left( \mathbb{A}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}} \left( \sum_{c \in \mathbf{C}} \|\boldsymbol{\beta}\|_{L^{\infty}(c)} \sum_{\mathfrak{p} \in \mathfrak{F}_c} \|\boldsymbol{\delta}(\mathbf{v})\|_{L^2(\partial\mathfrak{p})}^2 \right)^{\frac{1}{2}}.$$

Finally, we use the inequalities (28a)-(28b) together with the definition of  $\omega_{\zeta}$ , to infer that

$$T_{1,2} \leq C_{\delta} \omega_{\zeta} \left( \mathbb{A}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}} (\tau^{-1} \|\mathbf{v}\|_2^2)^{\frac{1}{2}}$$

where  $C_{\delta} > 0$  depends exclusively on the numerical constants  $C_{\mathbf{T}}$  and  $C_{\sharp}$ . Now, we collect the bounds on  $T_{1,1}$  and  $T_{1,2}$  and we apply Young's inequality to obtain

$$\mathbb{A}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}}(\mathbf{v}, \mathbf{w}) \geq \frac{1}{4} \|\mathbf{v}\|^2 + (\theta - C_{\delta}^2 \omega_{\zeta}^2) \mathbb{A}_{\boldsymbol{\beta}, -\nabla\boldsymbol{\beta}^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})\mathbf{Id}}(\mathbf{v}, \mathbf{v}) + T_{1,3}.$$

As a result, choosing  $\theta = C_{\sharp}^2 \omega_{\zeta}^2$  yields

$$\mathbb{A}_{\beta, -\nabla\beta^{\mathbf{T}} + \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \mathbf{w}) \geq \frac{1}{4} \|\mathbf{v}\|^2 + T_{1,3}. \quad (29)$$

ii) *Bound on  $T_2$ .* First, we rewrite this term as:

$$\begin{aligned} T_2 &= \theta \mathbf{H}_{\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \mathbf{v}) \\ &\quad + \mathbf{H}_{\zeta(\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id})}(\mathbf{v}, \mathbf{v}) \\ &\quad + \mathbf{H}_{\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \zeta\mathbf{v}) - \mathbf{H}_{\zeta(\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id})}(\mathbf{v}, \mathbf{v}) = T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

Concerning  $T_{2,1}$ , we have

$$T_{2,1} = \frac{\theta}{2} \sum_{c \in \mathbb{C}} \int_c \boldsymbol{\sigma}_{\beta, \mu} \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \geq \frac{\theta \lambda_b}{2} \sum_{c \in \mathbb{C}} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{L^2(c)}^2 \geq \frac{C_{\sharp}^2 \theta \lambda_b}{2} \theta \|\mathbf{v}\|_2^2,$$

where we have used Hypothesis **(H2)** (recall that  $\lambda_b \leq 0$ ) and the upper bound from Proposition 4.1. The second term  $T_{2,2}$  is treated similarly:

$$T_{2,2} = \frac{\theta}{2} \sum_{c \in \mathbb{C}} \int_c \zeta \boldsymbol{\sigma}_{\beta, \mu} \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \geq \frac{C_{\sharp}^2 \lambda_b}{2} \|\zeta\|_{L^\infty(\Omega)} \|\mathbf{v}\|_2^2.$$

Collecting these bounds yields

$$T_2 \geq \frac{\lambda_b}{2} \vartheta \|\mathbf{u}\|_2^2 + T_{2,3}, \quad (30)$$

with  $\vartheta = C_{\sharp}^2(\theta + \|\zeta\|_{L^\infty(\Omega)})$ .

iii) *Bound on  $T_1 + T_2$ .* Collecting the estimates (29) and (30), we obtain

$$\mathbb{A}_{\beta, \mu}(\mathbf{v}, \mathbf{w}) \geq \frac{1}{4} \|\mathbf{v}\|^2 + \frac{\tau \lambda_b}{2} \vartheta \tau^{-1} \|\mathbf{v}\|_2^2 + T_{1,3} + T_{2,3}.$$

We observe that

$$\begin{aligned} T_{1,3} + T_{2,3} &= \mathbf{H}_{\frac{1}{2}(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \zeta\mathbf{v}) - \mathbf{H}_{\frac{1}{2}\zeta(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \mathbf{v}) + \mathbf{H}_{\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id}}(\mathbf{v}, \zeta\mathbf{v}) - \mathbf{H}_{\zeta(\mu + \nabla\beta^{\mathbf{T}} - \frac{1}{2}(\nabla \cdot \beta)\mathbf{Id})}(\mathbf{v}, \mathbf{v}) \\ &= \mathbf{H}_{\mu + \nabla\beta^{\mathbf{T}}}(\mathbf{v}, \zeta\mathbf{v}) - \mathbf{H}_{\zeta(\mu + \nabla\beta^{\mathbf{T}})}(\mathbf{v}, \mathbf{v}) = \sum_{c \in \mathbb{C}} \int_c (\boldsymbol{\mu} + \nabla\beta^{\mathbf{T}}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \boldsymbol{\delta}(\mathbf{v}). \end{aligned}$$

Applying successively the Hölder inequality, the inequality (28a) and the upper bound from Proposition 4.1, we infer that

$$|T_{1,3} + T_{2,3}| \leq \sum_{c \in \mathbb{C}} \|\boldsymbol{\mu} + \nabla\beta^{\mathbf{T}}\|_{L^\infty(c)} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{L^2(c)} \|\boldsymbol{\delta}(\mathbf{v})\|_{L^2(c)} \leq \|\boldsymbol{\mu} + \nabla\beta^{\mathbf{T}}\|_{L^\infty(\Omega)} C_{\sharp}^2 L_{\zeta} h \|\mathbf{v}\|_2^2.$$

As a result, we obtain

$$\mathbb{A}_{\beta, \mu}(\mathbf{v}, \mathbf{w}) \geq \frac{1}{4} \|\mathbf{v}\|^2 + \left( \frac{\tau \lambda_b}{2} \vartheta - \frac{h}{4h_0} \right) \tau^{-1} \|\mathbf{v}\|_2^2,$$

with the reference length  $h_0 = \left( 4C_{\sharp}^2 \|\boldsymbol{\mu} + \nabla\beta^{\mathbf{T}}\|_{L^\infty(\Omega)} \tau L_{\zeta} \right)^{-1}$ . Hence, there exists  $\varrho' > 0$  such that  $\mathbb{A}_{\beta, \mu}(\mathbf{v}, \mathbf{w}) \geq \varrho' \|\mathbf{v}\|^2$ , as soon as  $\lambda_b$  and  $h$  satisfy (24).  $\square$

### 6.3 Bound on consistency error and a priori estimate

*Proof of Lemma 4.6.* Let  $\mathbf{y}|_c = (\mathbf{u} - \mathcal{I}_{\mathcal{E}}(\mathbf{u}))|_c$  for all  $c \in \mathbb{C}$ . Note that  $\mathbf{y}|_{\partial\mathfrak{p}} \in \mathbf{L}^q(\partial\mathfrak{p})$  for all  $\mathfrak{p} \in \mathfrak{P}_c$ . Let  $\mathbf{v} \in \mathcal{E}$ . Owing to the definitions of  $\mathbb{A}_{\beta, \mu}$  and  $\Sigma$ , it follows that  $\Sigma(\mathbf{s}, \mathbf{u}_D; \mathbf{v}) - \mathbb{A}_{\beta, \mu}(\mathbf{R}_{\mathcal{E}}(\mathbf{u}), \mathbf{v}) = T_1 + T_2 + T_3 + T_4$ , with

$$\begin{aligned} T_1 &:= \sum_{c \in \mathbb{C}} \int_c (\nabla \beta + \boldsymbol{\mu}^T - (\nabla \cdot \beta) \mathbf{Id}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y}, & T_2 &:= \sum_{X \in \{\mathbb{F}^\circ, \mathbb{C}\}} \sum_{x \in X} \sum_{\mathfrak{f} \in \tilde{\mathfrak{F}}_x} \int_{\mathfrak{f}} (\beta \cdot \mathbf{n}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\mathbf{y}\}, \\ T_3 &:= \sum_{X \in \{\mathbb{F}^\circ, \mathbb{C}\}} \sum_{x \in X} \sum_{\mathfrak{f} \in \tilde{\mathfrak{F}}_x} \int_{\mathfrak{f}} |\beta \cdot \mathbf{n}| [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\mathbf{y}\} & \text{and} & T_4 &:= \sum_{f \in \mathbb{F}^\partial} \sum_{\mathfrak{f} \in \tilde{\mathfrak{F}}_f^\partial} \int_{\mathfrak{f}} (\beta \cdot \mathbf{n})^- \mathbf{L}_{\mathcal{E}_f}(\mathbf{v}) \cdot \mathbf{y}. \end{aligned}$$

Indeed, the first term  $T_1$  is obtained using the definition (10) of  $\mathbf{g}_{\beta, \mu; c}$  together with the following integration by part formula (25) and

$$\sum_{\mathfrak{p} \in \mathfrak{P}_c} \int_{\mathfrak{p}} ((\beta \cdot \nabla) \mathbf{y}) \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) = - \sum_{\mathfrak{p} \in \mathfrak{P}_c} \int_{\mathfrak{p}} ((\beta \cdot \nabla) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v})) \cdot \mathbf{y} - \int_c (\nabla \cdot \beta) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y} + \sum_{\mathfrak{p} \in \mathfrak{P}_c} \int_{\mathfrak{p}} \nabla \cdot (\beta \mathbf{y} \cdot \mathbf{L}_{\mathcal{E}_c}(\mathbf{v})),$$

holding for all  $c \in \mathbb{C}$  and all  $\mathbf{v} \in \mathcal{E}_c$ . The terms  $T_2$  and  $T_3$  result from the rightmost term of the relation (25) and the fact that  $(\beta \cdot \mathbf{n}) [\mathbf{u}]|_{\mathfrak{f}} \equiv 0$  for all  $\mathfrak{f} \in \tilde{\mathfrak{F}}_x$ . Finally, the term  $T_4$  is inferred observing that  $\mathbf{u}_D = \mathbf{u}|_{\partial\Omega}$ . It remains to bound these four terms. First, let us consider  $T_1$ . Let  $q \in [1, 2]$  and denote  $q' \geq 2$  its conjugate number, *ie.*,  $1 = 1/q + 1/q'$ . From the Hölder inequality, we infer that

$$\left| \int_c (\nabla \beta + \boldsymbol{\mu}^T - (\nabla \cdot \beta) \mathbf{Id}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y} \right| \leq N_\infty \|\mathbf{y}\|_{\mathbf{L}^q(c)} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^{q'}(c)},$$

with  $N_\infty = \|\nabla \beta + \boldsymbol{\mu}^T - \nabla \cdot \beta \mathbf{Id}\|_{\mathbf{L}^\infty(c)}$ . Then, using a local inverse inequality (see [12, Lemma 1.138]), we infer that

$$\left| \int_c (\nabla \beta + \boldsymbol{\mu}^T - (\nabla \cdot \beta) \mathbf{Id}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y} \right| \leq N_\infty h_c^\theta \|\mathbf{y}\|_{\mathbf{L}^q(c)} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(c)},$$

with  $\theta = d \left( \frac{1}{2} - \frac{1}{q} \right)$ , so that the Hölder inequality yields

$$\left| \sum_{c \in \mathbb{C}} \int_c (\nabla \beta + \boldsymbol{\mu}^T - (\nabla \cdot \beta) \mathbf{Id}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y} \right| \leq \left( \sum_{c \in \mathbb{C}} N_\infty^q h_c^{\theta q} \|\mathbf{y}\|_{\mathbf{L}^q(c)}^q \right)^{\frac{1}{q}} \left( \sum_{c \in \mathbb{C}} \|\mathbf{L}_{\mathcal{E}_c}(\mathbf{v})\|_{\mathbf{L}^2(c)}^{q'} \right)^{\frac{1}{q'}}.$$

Moreover, recalling that  $q' \geq 2$  so that  $|\cdot|_{\ell^{q'}} \leq |\cdot|_{\ell^2}$ , and using the upper bound in Proposition 4.1 leads to

$$|T_1| = \left| \sum_{c \in \mathbb{C}} \int_c (\nabla \beta + \boldsymbol{\mu}^T - (\nabla \cdot \beta) \mathbf{Id}) \mathbf{L}_{\mathcal{E}_c}(\mathbf{v}) \cdot \mathbf{y} \right| \lesssim \left( \sum_{c \in \mathbb{C}} N_\infty^q h_c^{\theta q} \|\mathbf{y}\|_{\mathbf{L}^q(c)}^q \right)^{\frac{1}{q}} \|\mathbf{v}\|_2.$$

To bound the two terms  $T_2$  and  $T_3$ , we consider a sub-face  $\mathfrak{f} \in \tilde{\mathfrak{F}}_x$  for all  $x \in X$  with  $X \in \{\mathbb{F}^\circ, \mathbb{C}\}$ . As above, the Hölder inequality yields

$$\left| \int_{\mathfrak{f}} (\beta \cdot \mathbf{n}_{\mathfrak{f}}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\mathbf{y}\} \right| \leq \|\beta\|_{\mathbf{L}^\infty(\mathfrak{f})}^{\frac{1}{2}} \|\{\mathbf{y}\}\|_{\mathbf{L}^q(\mathfrak{f})} \| |\beta \cdot \mathbf{n}_{\mathfrak{f}}|^{\frac{1}{2}} [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \|_{\mathbf{L}^{q'}(\mathfrak{f})},$$

so that using a local inverse inequality, we obtain

$$\left| \int_{\mathfrak{f}} (\beta \cdot \mathbf{n}_{\mathfrak{f}}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\mathbf{y}\} \right| \leq h_{\mathfrak{f}}^{\theta'} \|\beta\|_{\mathbf{L}^\infty(\mathfrak{f})}^{\frac{1}{2}} \|\{\mathbf{y}\}\|_{\mathbf{L}^q(\mathfrak{f})} \| |\beta \cdot \mathbf{n}_{\mathfrak{f}}|^{\frac{1}{2}} [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \|_{\mathbf{L}^2(\mathfrak{f})},$$

with  $\theta' = (d-1)\left(\frac{1}{q} - \frac{1}{2}\right)$ . Hence, denoting  $\sum_{\mathfrak{f}} = \sum_{X \in \{\mathbb{F}^\circ, \mathbb{C}\}} \sum_{x \in X} \sum_{\mathfrak{f} \in \mathfrak{F}_x}$ , it follows from the triangle inequality, the Hölder inequality and  $q' \geq 2$  that

$$\left| \sum_{\mathfrak{f}} \int_{\mathfrak{f}} (\boldsymbol{\beta} \cdot \mathbf{n}_{\mathfrak{f}}) [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \cdot \{\mathbf{y}\} \right| \leq \left( \sum_{\mathfrak{f}} h_{\mathfrak{f}}^{\theta'q} \|\boldsymbol{\beta}\|_{\mathbf{L}^{\infty}(\mathfrak{f})}^{\frac{q}{2}} \|\{\mathbf{y}\}\|_{\mathbf{L}^q(\mathfrak{f})}^q \right)^{\frac{1}{q}} \left( \sum_{\mathfrak{f}} \|\boldsymbol{\beta} \cdot \mathbf{n}_{\mathfrak{f}}\|^{\frac{1}{2}} [\mathbf{L}_{\mathcal{E}}(\mathbf{v})] \|\mathbf{L}^2(\mathfrak{f})\|^2 \right)^{\frac{1}{2}}.$$

Next, owing to the definitions (11) and (13) of  $\mathfrak{n}_{\boldsymbol{\beta}}$  and  $\mathfrak{s}_{\boldsymbol{\beta}}$  respectively, the mesh regularity and recalling the inequality  $|a \pm b|^q \leq 2^{q-1}(|a|^q + |b|^q)$ , we infer that

$$|T_2 + T_3| \lesssim \left( \sum_{c \in \mathbb{C}} \sum_{\mathfrak{p} \in \mathfrak{P}_c} h_c^{\theta'q} \|\boldsymbol{\beta}\|_{\mathbf{L}^{\infty}(c)}^{\frac{q}{2}} \|\mathbf{y}\|_{\mathbf{L}^q(\partial \mathfrak{p})}^q \right)^{\frac{1}{q}} \mathfrak{s}_{\boldsymbol{\beta}}(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}.$$

Finally, proceeding similarly, we also infer that

$$|T_4| \lesssim \left( \sum_{f \in \mathbb{F}^\partial} h_{cf}^{\theta'q} \|\boldsymbol{\beta}\|_{\mathbf{L}^{\infty}(f)}^{\frac{q}{2}} \|\mathbf{y}\|_{\mathbf{L}^q(f)}^q \right)^{\frac{1}{q}} A_{|\boldsymbol{\beta} \cdot \mathbf{n}|}^{\partial}(\mathbf{v}, \mathbf{v})^{\frac{1}{2}},$$

and the expected result follows from the above bounds.  $\square$

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