

CLASSIFICATION OF EXTINCTION PROFILES FOR A ONE-DIMENSIONAL DIFFUSIVE HAMILTON-JACOBI EQUATION WITH CRITICAL ABSORPTION

Razvan Gabriel Iagar, Philippe Laurençot

▶ To cite this version:

Razvan Gabriel Iagar, Philippe Laurençot. CLASSIFICATION OF EXTINCTION PROFILES FOR A ONE-DIMENSIONAL DIFFUSIVE HAMILTON-JACOBI EQUATION WITH CRITICAL AB-SORPTION. Proceedings of the Royal Society of Edinburgh: Section A, Mathematics, 2018, 148, pp.559–574. 10.1017/S030821051700049X. hal-01324506

HAL Id: hal-01324506 https://hal.science/hal-01324506

Submitted on 1 Jun2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CLASSIFICATION OF EXTINCTION PROFILES FOR A ONE-DIMENSIONAL DIFFUSIVE HAMILTON-JACOBI EQUATION WITH CRITICAL ABSORPTION

RAZVAN GABRIEL IAGAR AND PHILIPPE LAURENÇOT

ABSTRACT. A classification of the behavior of the solutions $f(\cdot, a)$ to the ordinary differential equation $(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0$ in $(0, \infty)$ with initial condition f(0, a) = a and f'(0, a) = 0 is provided, according to the value of the parameter a > 0, the exponent p ranging in (1, 2). There is a threshold value a_* which separates different behaviors of $f(\cdot, a)$: if $a > a_*$ then $f(\cdot, a)$ vanishes at least once in $(0, \infty)$ and takes negative values while $f(\cdot, a)$ is positive in $(0, \infty)$ and decays algebraically to zero as $r \to \infty$ if $a \in (0, a_*)$. At the threshold value, $f(\cdot, a_*)$ is also positive in $(0, \infty)$ but decays exponentially fast to zero as $r \to \infty$. The proof of these results relies on a transformation to a first-order ordinary differential equation and a monotonicity property with respect to a > 0. This classification is one step in the description of the dynamics near the extinction time of a diffusive Hamilton-Jacobi equation with critical gradient absorption and fast diffusion.

1. INTRODUCTION

Let $p \in (1,2)$. Owing to its scale invariance, the diffusive Hamilton-Jacobi equation

$$\partial_t u - \partial_x \left(|\partial_x u|^{p-2} \partial_x u \right) + |\partial_x u|^{p-1} = 0 , \qquad (t, x) \in (0, \infty) \times \mathbb{R} , \qquad (1.1)$$

is expected to have self-similar solutions with separate variables, that is, solutions of the form

$$u_s(t,x) = \left((2-p)(T-t)_+\right)^{1/(2-p)} f(|x|) , \qquad (t,x) \in (0,\infty) \times \mathbb{R} , \qquad (1.2)$$

which vanish identically after a finite time T > 0. Inserting this ansatz in (1.1) leads us to the ordinary differential equation

$$(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0, \qquad r \in (0,\infty),$$
(1.3)

along with the boundary condition f'(0) = 0 stemming from the assumed symmetry and the expected smoothness of u_s with respect to the space variable. It is then natural to investigate the behavior of solutions to (1.3) according to the initial value f(0). The main motivation for such an analysis is that non-negative self-similar solutions of the form (1.2) are expected to provide an accurate description of the behavior near the extinction time of non-negative solutions to (1.1) which enjoy the finite time extinction property. Indeed, it follows from [4, Theorem 1.2] that there are many non-negative solutions to (1.1) satisfying the latter property. The classification of solutions to (1.3) performed

Date: June 1, 2016.

¹⁹⁹¹ Mathematics Subject Classification. 35C06 - 34D05 - 35B33 - 35K92 - 35K67.

 $Key \ words \ and \ phrases.$ self-similar solutions - singular diffusion - diffusive Hamilton-Jacobi equation - decay at infinity - comparison.

below allows us to identify the behavior at the extinction time of non-negative solutions to (1.1) in the companion paper [7], the initial data being even in \mathbb{R} , non-increasing on $(0, \infty)$ and decaying sufficiently rapidly as $x \to \infty$.

More precisely the main result of this paper is the following classification:

Theorem 1.1. Given a > 0 there is a unique solution $f(\cdot, a)$ to the initial value problem

$$(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0, \qquad r \in (0,\infty),$$
(1.4)

$$f(0,a) = a$$
, $f'(0,a) = 0$, (1.5)

and

$$R(a) := \inf \{r > 0 : f(r, a) = 0\} \in (0, \infty] .$$
(1.6)

Furthermore there is $a_* > 0$ with the following properties:

- (a) if $a > a_*$ then $R(a) < \infty$, f(R(a), a) = 0, and f'(R(a), a) < 0.
- (b) if $a = a_*$ then $R(a_*) = \infty$ and there is $\ell_* > 0$ such that

$$\lim_{r \to \infty} e^{r/(p-1)} f(r, a_*) = \ell_*$$

(c) if $a \in (0, a_*)$ then $R(a) = \infty$ and

$$\lim_{r \to \infty} r^{(2-p)/(p-1)} f(r,a) = \left(\frac{p-1}{2-p}\right)^{(2-p)/(p-1)}$$

Before giving a rough account of the proof of Theorem 1.1, let us complete the discussion started before the statement of Theorem 1.1 on the role of self-similar solutions to (1.1) of the form (1.2) in the description of the dynamics of non-negative solutions to (1.1) near their extinction time. According to Theorem 1.1 we have infinitely many non-negative self-similar solutions of the form (1.2) (corresponding to $a \in (0, a_*]$), but it turns out that only one is selected by the dynamics of (1.1) as the behavior near the extinction time. More precisely, as shown in [7], if u is a solution to (1.1) emanating from a non-negative even initial condition which is non-increasing on $(0, \infty)$ and decays sufficiently rapidly as $x \to \infty$ and if T_e denotes its extinction time, then u(t, x) behaves as $((2-p)(T_e-t)_+)^{1/(2-p)}f(|x|, a_*)$ as $t \to T_e$. Let us point out that this universal behavior is also true in higher space dimensions $N \ge 2$ for $p \in (2N/(N+1), 2)$, but the identification of the corresponding self-similar profile is more involved and requires completely different arguments [7]. We also point out that a similar dynamics as the one described above is observed for the fast diffusion equation

$$\partial_t v - \Delta v^m + v^m = 0$$
, $(t, x) \in (0, \infty) \times \mathbb{R}^N$,

when $m \in ((N-2)_+/N, 1)$ [2,3].

Let us now describe more precisely the proof of Theorem 1.1. Given a > 0, classical results guarantee the well-posedness of (1.4)-(1.5), see Section 2. In addition, there is $R(a) \in (0, \infty]$ such that $f(\cdot, a)$ is a decreasing one-to-one function from [0, R(a)) onto (0, a]. This property allows us to introduce $\psi(\cdot, a)$ defined on (0, 1) by

$$\psi\left(1 - \frac{f(r,a)}{a}, a\right) := \frac{|f'(r,a)|^p}{a^p} , \qquad r \in [0, R(a)) .$$
(1.7)

Thanks to (1.4)-(1.5), the function $\psi(\cdot, a)$ solves

$$\psi'(y) + \frac{p}{p-1}\psi(y)^{(p-1)/p}(y) = \frac{p}{p-1}a^{2-p}(1-y) , \qquad y \in (0,1) ,$$

$$\psi(0) = 0 .$$
(1.8)

The transformation (1.7) thus reduces the second-order differential equation (1.4) to the first-order differential equation (1.8), which is already a valuable feature, but it also has the very interesting property that $\psi(\cdot, a)$ is monotone with respect to a. The latter is in particular of utmost importance to investigate uniqueness issues, see [1,5,6,10,13] for instance, where monotonicity with respect to the shooting parameter is used to establish uniqueness of the "fast orbit" for related problems. In addition, the finiteness of R(a) as well as the behavior of f(r, a) as $r \to \infty$ when $R(a) = \infty$ are directly connected to the behavior of $\psi(y, a)$ as $y \to 1$. The core of the analysis is actually the identification of the behavior of $\psi(y, a)$ as $y \to 1$ according to the value of a and is performed in Section 3. Interpreting the results obtained in Section 3 in terms of $f(\cdot, a)$ is done in Section 4, where we prove Theorem 1.1.

We end this introduction with a couple of remarks: on the one hand, the approach developed in this paper does not seem to extend to the study and classification of self-similar solutions to (1.1) of the form (1.2) in several space dimensions, the main reason being that the variable r = |x| remains in the equation satisfied by ψ . Indeed, it seems that no transformation similar to (1.7) is available in dimension $N \ge 2$. Still, it is possible to establish a result similar to Theorem 1.1 in higher space dimensions but completely different arguments are used [7]. On the other hand, there is a *striking difference* between (1.3) and

$$(|f'|^{p-2}f')' + f - |f|^{p-2}f = 0, \qquad r \in (0,\infty),$$
(1.9)

which involves only zero order reaction terms. Indeed, in general, (1.9) and its generalizations have only one non-negative C^1 -smooth solution which is defined on $(0, \infty)$ and converges to zero as $r \to \infty$, the so-called ground state solution, see [8,9,11,12] and the references therein. This is in sharp contrast with (1.3) for which infinitely many ground states exist, see Theorem 1.1, but a single one features a faster decay as $r \to \infty$. This multiplicity of course complicates the analysis, as it requires not only to identify the possible decay rates as $r \to \infty$, but also the corresponding ranges of the parameter a.

2. Well-posedness of (1.4)-(1.5)

We begin with the well-posedness of (1.4)-(1.5) and basic properties of its solutions.

Lemma 2.1. Given a > 0, there is a unique solution $f(\cdot, a) \in C^1([0, \infty))$ to (1.4)-(1.5) such that $|f'|^{p-2}f' \in C^1([0, \infty))$. Furthermore,

$$R(a) = \inf \{r > 0 : f(r, a) = 0\} \in (0, \infty]$$

and $f(\cdot, a)$ enjoys the following properties:

$$0 < f(r,a) < a \quad and \quad -\left(a(1-e^{-r})\right)^{1/(p-1)} < f'(r,a) < 0 \ , \qquad r \in (0,R(a)) \ , \tag{2.1}$$

and

$$\frac{d}{dr}\left(e^r|f'(r,a)|^{p-2}f'(r,a)\right) = -e^rf(r,a) , \qquad r \in (0,R(a)) .$$
(2.2)

Proof. Since $p \in (1,2)$, the Cauchy-Lipschitz theorem ensures the existence and uniqueness of a solution $(f,g) \in C^1([0,\mathcal{R}(a));\mathbb{R}^2)$ to the initial value problem

$$f'(r) = -|g(r)|^{(2-p)/(p-1)}g(r) , \qquad g'(r) = -|g(r)| + f(r) , \qquad r \in (0, \mathcal{R}(a)) ,$$

$$f(0) = a , g(0) = 0 ,$$
(2.3)

where $\mathcal{R}(a) \in (0, \infty]$ is such that either $\mathcal{R}(a) = \infty$ or

$$\mathcal{R}(a) < \infty$$
 and $\limsup_{r \to \mathcal{R}(a)} \left(|f(r)| + |g(r)| \right) = \infty$. (2.4)

Since $g(r) = -|f'(r)|^{p-2}f'(r)$ by (2.3) for $r \in [0, \mathcal{R}(a))$, it readily follows from (2.3) that f solves (1.4)-(1.5). A further consequence of (2.3) is that

$$\begin{aligned} \frac{d}{dr} \left[\frac{p-1}{p} |g|^{p/(p-1)} + \frac{1}{2} f^2 \right] &= |g|^{(2-p)/(p-1)} g(f - |g|) - |g|^{(2-p)/(p-1)} gf \\ &= -|g|^{1/(p-1)} g \\ &\leq \frac{p}{p-1} \left[\frac{p-1}{p} |g|^{p/(p-1)} + \frac{1}{2} f^2 \right] \,, \end{aligned}$$

which excludes the occurrence of (2.4). Therefore $\mathcal{R}(a) = \infty$ and the positivity of a along with the continuity of f guarantee that R(a) > 0.

We next infer from (1.4)-(1.5) that

$$\lim_{r \to 0} (|f'|^{p-2} f')'(r) = -a < 0 ,$$

which implies that f' is negative in a right neighborhood of r = 0 as f'(0) = 0. Using again (1.4) we note that

$$\frac{d}{dr}\left(e^r|f'(r)|^{p-2}f'(r)\right) = e^r\left[|f'(r)|^{p-2}f'(r) + |f'(r)|^{p-1} - f(r)\right] , \qquad r > 0 .$$
(2.5)

Consequently, as long as f'(r) is negative and $r \in (0, R(a))$, there holds

$$\frac{d}{dr}\left(e^r|f'(r)|^{p-2}f'(r)\right) = -e^rf(r) < 0$$

from which we deduce that f' cannot vanish in (0, R(a)). We have thus proved that f'(r) < 0 and $f(r) \in (0, a)$ for $r \in (0, R(a))$ as well as (2.2). Combining these properties gives

$$-\frac{d}{dr}\left(e^{r}|f'(r)|^{p-1}\right) \ge -ae^{r} , \qquad r \in (0, R(a)) ,$$

hence, after integration and using (1.5),

$$e^{r}|f'(r)|^{p-1} \ge -a(e^{r}-1)$$
, $r \in (0, R(a))$

This completes the proof of Lemma 2.1.

4

3. An Alternative formulation

Let a > 0 and set $f = f(\cdot, a)$. As f' < 0 in (0, R(a)) by (2.1), the function a - f is an increasing one-to-one function from [0, R(a)) onto [0, a) and we denote its inverse by F. Then F is an increasing function from [0, a) onto [0, R(a)) and we may define

$$\psi(y) = \psi(y, a) := \frac{1}{a^p} |f'(F(ay))|^p , \qquad y \in [0, 1) .$$
(3.1)

Equivalently,

$$\psi\left(1 - \frac{f(r)}{a}\right) = \frac{|f'(r)|^p}{a^p}, \qquad r \in [0, R(a)),$$
(3.2)

and

$$\psi'\left(1 - \frac{f(r)}{a}\right) = -\frac{p}{(p-1)a^{p-1}} \left(|f'|^{p-2}f'\right)'(r) , \qquad r \in [0, R(a)) . \tag{3.3}$$

We then infer from (1.4)-(1.5), (3.2), and (3.3) that ψ solves

$$\psi'(y) + \frac{p}{p-1}\psi(y)^{(p-1)/p} = \frac{pa^{2-p}}{p-1}(1-y) , \qquad y \in (0,1) , \qquad (3.4)$$

$$\psi(0) = 0 .$$
(3.5)

We also deduce from (3.4)-(3.5) that

$$\psi'(0) = \frac{pa^{2-p}}{p-1} > 0 . ag{3.6}$$

3.1. Comparison and monotonicity. Though the equation (3.4) involves the exponent (p-1)/p, which ranges in (0, 1), the following comparison principle is available:

Lemma 3.1 (Comparison principle). Let $\xi_i \in C^1([0,1))$, i = 1, 2, be two functions satisfying $\xi_1(0) \leq \xi_2(0)$ and

$$\xi_1'(y) + \frac{p}{p-1}\xi_1(y)^{(p-1)/p} \le \xi_2'(y) + \frac{p}{p-1}\xi_2(y)^{(p-1)/p} , \qquad y \in (0,1) .$$
(3.7)

Then $\xi_1(y) \le \xi_2(y)$ for $y \in [0, 1)$.

Proof. Lemma 3.1 actually follows from the monotonicity of $z \mapsto z^{(p-1)/p}$ and we recall its proof for the sake of completeness. Let $\delta > 0$ and define

$$y_{\delta} := \inf\{y \in [0,1) : \xi_1(y) = \xi_2(y) + \delta\}$$

Clearly $y_{\delta} > 0$ since $\xi_1(0) - \xi_2(0) - \delta \leq -\delta < 0$. Assume for contradiction that $y_{\delta} < 1$. Then $\xi_1 - \xi_2 - \delta < 0$ in $[0, y_{\delta})$ and $(\xi'_1 - \xi'_2)(y_{\delta}) \geq 0$, while (3.7) gives

$$\begin{aligned} (\xi_1' - \xi_2')(y_\delta) &\leq \frac{p}{p-1} \xi_2(y_\delta)^{(p-1)/p} - \frac{p}{p-1} \xi_1(y_\delta)^{(p-1)/p} \\ &= \frac{p}{p-1} \xi_2(y_\delta)^{(p-1)/p} - \frac{p}{p-1} \left(\xi_2(y_\delta) + \delta\right)^{(p-1)/p} < 0 \end{aligned}$$

and a contradiction. Consequently, $\xi_1 \leq \xi_2 + \delta$ in [0, 1) and, since this inequality is valid for any $\delta > 0$, we conclude that $\xi_1 \leq \xi_2$ in [0, 1).

The transformation (3.1) has thus reduced the second-order equation (1.4) to the first-order equation (3.4), which lowers the complexity of the problem. An additional property, which turns out to be of high interest as well, of solutions to (3.4)-(3.5) is their monotonicity with respect to a, which is obviously a simple consequence of the comparison principle established in Lemma 3.1. A more precise result is actually available.

Lemma 3.2 (Monotonicity with respect to a). Consider $0 < a_1 < a_2$. Then there exists K(p) > 0 depending only on p such that, for $y \in [0, 1)$,

$$\begin{aligned} \psi(y,a_1) &\leq \psi(y,a_2) &\leq \psi(y,a_1) + K(p)(a_2 - a_1)^{2-p} ,\\ |\psi'(y,a_1) - \psi'(y,a_2)| &\leq K(p) \left[(a_2 - a_1)^{(2-p)(p-1)/p} + (a_2 - a_1)^{2-p} \right] \end{aligned}$$

In addition, $\psi(y, a_1) < \psi(y, a_2)$ for any $y \in (0, 1)$.

Proof. Set $\psi_i = \psi(\cdot, a_i)$, i = 1, 2. Since $a_1 < a_2$, it readily follows from (1.4)-(1.5) that we can apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi_1, \psi_2)$. Consequently, $\psi_1 \leq \psi_2$ in [0, 1).

We next put $M := p \left(a_2^{2-p} - a_1^{2-p} \right) / (p-1)$ and $\xi_2(y) = \psi_1(y) + My$ for $y \in [0,1)$. Then $\xi_2(0) = 0 = \psi_2(0)$ and it follows from (3.4) that, for $y \in (0,1)$,

$$\begin{aligned} \xi_2'(y) + \frac{p}{p-1} \xi_2(y)^{(p-1)/p} &\geq \psi_1'(y) + M + \frac{p}{p-1} \psi_1(y)^{(p-1)/p} \\ &\geq M(1-y) + \frac{p a_1^{2-p}}{p-1} (1-y) = \frac{p a_2^{2-p}}{p-1} (1-y) \\ &= \psi_2'(y) + \frac{p}{p-1} \psi_2(y)^{(p-1)/p} . \end{aligned}$$

Applying Lemma 3.1 to $(\xi_1, \xi_2) = (\psi_2, \xi_2)$ entails that $\psi_2 \leq \xi_2$ in [0, 1), which completes the proof of the first statement of Lemma 3.2. We next infer from (3.4), the Hölder continuity of $z \mapsto z^{(p-1)/p}$, and the first statement of Lemma 3.2 that

$$\begin{aligned} |\psi_1'(y) - \psi_2'(y)| &\leq \frac{p}{p-1} |\psi_1(y) - \psi_2(y)|^{(p-1)/p} + \frac{p}{p-1} (a_2 - a_1)^{2-p} \\ &\leq \frac{p}{p-1} K(p)^{(p-1)/p} (a_2 - a_1)^{(2-p)(p-1)/p} + \frac{p}{p-1} (a_2 - a_1)^{2-p} \end{aligned}$$

and thus complete the proof of the continuous dependence with respect to a.

Finally, since $a_1 < a_2$, it follows that

$$\bar{y} := \sup\{y \in (0,1) : \psi_1(z) < \psi_2(z) \text{ for } z \in (0,y)\} > 0.$$

Assume for contradiction that $\bar{y} \in (0, 1)$. Then $\psi_2(\bar{y}) = \psi_1(\bar{y})$ and, since $\psi_2 \ge \psi_1$ in (0, 1), then \bar{y} is a point of minimum for $\psi_2 - \psi_1$, so that $(\psi_2 - \psi_1)'(\bar{y}) = 0$. We infer from (3.4) that

$$0 = (\psi_2 - \psi_1)'(\bar{y}) + \frac{p}{p-1} \left[\psi_2^{(p-1)/p}(\bar{y}) - \psi_1^{(p-1)/p}(\bar{y}) \right] = \frac{p}{p-1} (a_2^{2-p} - a_1^{2-p})(1-\bar{y}),$$

which leads to $a_1 = a_2$, hence a contradiction. This proves that $\bar{y} = 1$ and thereby completes the proof of Lemma 3.2.

3.2. Behavior of $\psi(y, a)$ as $y \to 1$. We next describe the shape of $\psi(\cdot, a)$.

Lemma 3.3. Given a > 0 there is $y_a \in (0, 1)$ such that

$$\psi'(y_a, a) = 0$$
, $\psi'(y, a)(y - y_a) < 0$, $y \in (0, 1) \setminus \{y_a\}$. (3.8)

Moreover there is $\ell(a) \ge 0$ such that

$$\lim_{y \to 1} \psi(y, a) = \ell(a) , \qquad (3.9)$$

and

$$\psi(y,a) \ge a^{p(2-p)/(p-1)}(1-y)^{p/(p-1)}, \qquad y \in (y_a,1).$$
 (3.10)

Proof. We define $y_a := \inf\{y \in (0,1) : \psi'(y) = 0\}$ and note that $y_a > 0$ by (3.6). Assume for contradiction that $y_a = 1$. Then $\psi' > 0$ in [0,1) and it follows from (3.1) and (3.4) that

 $0 \le \psi(y)^{(p-1)/p} \le a^{2-p}(1-y)$, $y \in (0,1)$.

Consequently, $\psi(1) = 0 = \psi(0)$ which contradicts the strict monotonicity of ψ . Therefore $y_a \in (0, 1)$ with $\psi' > 0$ in $[0, y_a), \psi'(y_a) = 0$, and

$$\psi''(y_a) = -\psi(y_a)^{-1/p}\psi'(y_a) - \frac{pa^{2-p}}{p-1} = -\frac{pa^{2-p}}{p-1} < 0$$

In particular, ψ' is negative in a right neighborhood of y_a . Assume for contradiction that there is $z \in (y_a, 1)$ such that $\psi'(y) < 0$ for $y \in (y_a, z)$ and $\psi'(z) = 0$. Then $\psi''(z) \ge 0$, while (3.4) entails that $\psi''(z) = -pa^{2-p}/(p-1) < 0$, and a contradiction. We have thus proved (3.8) which, together with (3.1), implies in particular that ψ is positive and decreasing on $(y_a, 1)$, hence (3.9).

Finally, if $y \in [y_a, 1)$, one has $\psi'(y) < 0$ by (3.8) and we infer from (3.4) that

$$\frac{p}{p-1}\psi(y)^{(p-1)/p} \ge \frac{pa^{2-p}}{p-1}(1-y) ,$$

from which (3.10) readily follows.

The next step, which is the cornerstone of the classification of the behavior of $\psi(\cdot, a)$ according to the value of a, is to elucidate the behavior of $\psi(y, a)$ as $y \to 1$. While it is obvious if $\ell(a) > 0$, more information is needed when $\ell(a) = 0$.

Lemma 3.4. Let a > 0 and assume that $\ell(a) = 0$. Then $y \mapsto \psi(y, a)(1-y)^{-p}$ has a limit as $y \to 1$ and

$$0 \le \psi(y, a) \le \kappa (1 - y)^p$$
, $y \in (0, 1)$, (3.11)

$$\lim_{y \to 1} \psi(y, a) (1 - y)^{-p} \in \{0, \kappa\} , \qquad (3.12)$$

where $\kappa := (p - 1)^{-p}$.

Proof. It readily follows from (3.4) that, for $y \in (0, 1)$,

$$\left(\psi^{1/p}\right)'(y) + \frac{1}{p-1} = \frac{1}{p}\psi(y)^{-(p-1)/p}\psi'(y) + \frac{1}{p-1}$$
$$= \frac{a^{2-p}}{p-1}(1-y)\psi(y)^{-(p-1)/p} \ge 0 .$$

Integrating the above differential inequality over (y, 1) and using $\ell(a) = 0$ lead us to

$$\frac{1}{p-1} \ge \psi(y)^{1/p} + \frac{y}{p-1} , \qquad y \in (0,1) ,$$

hence (3.11).

We next define

$$\varphi(y) = \varphi(y, a) := \psi(y, a)(1 - y)^{-p} , \qquad y \in [0, 1) , \qquad (3.13)$$

and deduce from (3.4)-(3.5) that φ solves

$$\varphi'(y) + \frac{p}{1-y}\varphi(y)^{(p-1)/p} \left(\kappa^{1/p} - \varphi(y)^{1/p}\right) = \frac{pa^{2-p}}{p-1}(1-y)^{1-p} , \qquad y \in (0,1) , \qquad (3.14)$$
$$\varphi(0) = 0 . \qquad (3.15)$$

$$o(0) = 0$$
 (3.15)

Integrating (3.16) over (0, y) and using (3.15) give

$$\varphi(y) + p \int_0^y \Phi(z) \, dz = \frac{p a^{2-p}}{(p-1)(2-p)} \left[1 - (1-y)^{2-p} \right] \tag{3.16}$$

for $y \in [0, 1)$, where

$$\Phi(y) := \varphi(y)^{(p-1)/p} \left[\frac{\kappa^{1/p} - \varphi(y)^{1/p}}{1-y} \right] , \qquad y \in [0,1)$$

We then infer from (3.11) that $\Phi \geq 0$ in (0,1), which gives, together with (3.16) and the nonnegativity of φ ,

$$0 \le p \int_0^y \Phi(z) \, dz \le \frac{p a^{2-p}}{(p-1)(2-p)} \,, \qquad y \in [0,1) \,.$$

Consequently, $\Phi \in L^1(0,1)$ and (3.16) ensures that $\varphi(y)$ has a limit L as $y \to 1$ given by

$$\lim_{y \to 1} \varphi(y) = L := \frac{pa^{2-p}}{(p-1)(2-p)} - p \int_0^1 \Phi(y) \, dy \; .$$

Recalling the definition of Φ , we realize that

$$\lim_{y \to 1} (1-y)\Phi(y) = L^{(p-1)/p} \left(\kappa^{1/p} - L^{1/p} \right) ,$$

and the integrability of Φ implies that $L \in \{0, \kappa\}$.

-	_	_	
L	_	_	_

3.3. Classification. The outcome of Lemma 3.3 and Lemma 3.4 allows us to split the range of a into three sets according to the behavior of $\psi(y, a)$ as $y \to 1$. More precisely, we define

$$\begin{aligned} \mathcal{A} &:= \left\{ a \in (0,\infty) : \ell(a) > 0 \right\} , \\ \mathcal{B} &:= \left\{ a \in (0,\infty) : \lim_{y \to 1} \psi(y,a)(1-y)^{-p} = \kappa \right\} , \\ \mathcal{C} &:= \left\{ a \in (0,\infty) : \lim_{y \to 1} \psi(y,a)(1-y)^{-p} = 0 \right\} . \end{aligned}$$

Indeed, according to Lemma 3.3 and Lemma 3.4, the sets \mathcal{A}, \mathcal{B} , and \mathcal{C} are disjoint and

$$\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}=(0,\infty)$$
 .

We now provide a more accurate description of these sets and begin with \mathcal{A} .

Lemma 3.5. There holds

$$a \in \mathcal{A}$$
 if and only if $\sup_{y \in [0,1)} \left\{ \psi(y,a)(1-y)^{-p} \right\} > \kappa$. (3.17)

Furthermore, there is $a^* > 0$ such that $\mathcal{A} = (a^*, \infty)$.

Proof. As in the proof of Lemma 3.4, see Equation (3.13), we set $\varphi(y) = \psi(y)(1-y)^{-p}$ for $y \in [0,1)$. **Step 1.** If $a \in \mathcal{A}$ then $\ell(a) > 0$, from which we readily deduce that $\varphi(y) \to \infty$ as $y \to 1$, and obviously $\sup_{y \in [0,1)} \{\varphi(y)\} > \kappa$. Conversely, if $\sup_{y \in [0,1)} \{\varphi(y)\} > \kappa$, then necessarily $\ell(a) \neq 0$ according to Lemma 3.4 and thus $a \in \mathcal{A}$.

Step 2. We claim that \mathcal{A} is non-empty. Indeed, assume for contradiction that $\mathcal{A} = \emptyset$, so that $\ell(a) = 0$ for all a > 0. We then infer from (3.11), (3.14), and the non-negativity of φ that

$$\varphi'(y) \ge \frac{pa^{2-p}}{p-1}(1-y)^{1-p} - \frac{p\kappa^{1/p}}{1-y}\varphi(y)^{(p-1)/p} \ge \frac{pa^{2-p}}{p-1}(1-y)^{1-p} - \frac{p\kappa}{1-y}$$

for $y \in (0, 1)$. Integrating over (0, 1/2) and using once more (3.11) give

$$\kappa \ge \varphi(1/2) \ge \frac{pa^{2-p}}{(p-1)(2-p)} \left(1 - 2^{p-2}\right) - p\kappa \log 2 ,$$

and a contradiction for a large enough. Consequently, \mathcal{A} is non-empty.

Step 3. We put $a^* := \inf \mathcal{A}$. A straightforward consequence of the characterization (3.17) and the monotonicity of $\psi(\cdot, a)$ with respect to a established in Lemma 3.2 and (3.17) is that $(a^*, \infty) \subset \mathcal{A}$. Furthermore, if $a \in \mathcal{A}$, then $\ell(a) > 0$ and it follows from Lemma 3.2 that, for $\delta \in (0, a)$

$$0 < \ell(a) \le \ell(a+\delta)$$
 and $0 < \ell(a) \le \ell(a-\delta) + K(p)\delta^{2-p}$.

Therefore $\ell(a+\delta) > 0$ and $\ell(a-\delta) > 0$ for δ small enough, so that $(a-\delta, a+\delta) \subset \mathcal{A}$ for $\delta > 0$ small enough. In particular, \mathcal{A} is open and thus coincides with (a^*, ∞) .

Concerning \mathcal{C} one has the following result.

Lemma 3.6. The following statements are equivalent:

- (c1) $a \in \mathcal{C}$. (c2) $\sup \{\psi(y,a)(1-y)^{-p}\} < \kappa.$
- $y \in [0,1)$
- (c3) The derivative $\varphi'(\cdot, a)$ of the function $\varphi(\cdot, a)$ defined in (3.13) vanishes at least once in (0, 1).
- (c4) There is $Y_a \in (0, 1)$ such that

$$\varphi'(Y_a, a) = 0$$
, $\varphi'(y, a)(y - Y_a) < 0$, $y \in (0, 1) \setminus \{Y_a\}$. (3.18)

Furthermore there is $a_* > 0$ such that $\mathcal{C} = (0, a_*)$.

Proof. Recall that $\varphi(y) = \psi(y)(1-y)^{-p}$ for $y \in [0,1)$, see Equation (3.13).

Step 1. Assume first that sup $\{\varphi(y)\} < \kappa$. This property readily implies that $\ell(a) = 0$ and we $y \in [0,1)$ deduce from Lemma 3.4 that the limit of $\varphi(y)$ as $y \to 1$ is necessarily zero. Therefore $a \in \mathcal{C}$ and we have proved that $(c2) \Rightarrow (c1)$.

Consider now $a \in \mathcal{C}$. Since $\varphi(0) = 0$ by (3.15) and $\varphi(y) \to 0$ as $y \to 1$, a generalization of Rolle's theorem guarantees that φ' vanishes at least once in (0, 1), and (c1) \Rightarrow (c3).

Assume next that φ' vanishes at least once in (0,1) and denote its smallest zero by $Y_a \in (0,1)$. Since $\varphi'(0) = pa^{2-p}/(p-1) > 0$ by (3.14), the function φ' is positive in $[0, Y_a)$ and it follows from (3.14) that

$$\varphi''(Y_a) = -\frac{p(2-p)a^{2-p}}{p-1}(1-Y_a)^{-p} < 0$$
.

Consequently, φ' is negative in a right neighborhood of Y_a . Assume for contradiction that there is $Y_1 \in (Y_a, 1)$ such that $\varphi'(y) < 0$ for $y \in (Y_a, Y_1)$ and $\varphi'(Y_1) = 0$. Then $\varphi''(Y_1) \ge 0$ while (3.14) implies that $\varphi''(Y_1) = -p(2-p)a^{2-p}(1-Y_1)^{-p}/(p-1) < 0$, and a contradiction. Therefore $\varphi' < 0$ in $(Y_a, 1)$ and we have shown that φ enjoys the property (3.18), that is, (c3) \Rightarrow (c4).

Finally, assume that φ satisfies (3.18). Then $\sup \{\varphi(y)\} = \varphi(Y_a)$ and we deduce from (3.14) that $y \in [0,1)$

$$\frac{p}{1-Y_a}\varphi(Y_a)^{(p-1)/p}\left[\kappa^{1/p} - \varphi(Y_a)^{1/p}\right] = \frac{pa^{2-p}}{p-1}(1-Y_a)^{1-p} > 0$$

Consequently $\varphi(Y_a) < \kappa$ and (c4) \Rightarrow (c2).

Step 2. We now check that \mathcal{C} is non-empty. To this end, consider a > 0 such that

$$a^{2-p} \le (p-1)^{p-1}/p^p = \max_{A \in (0,1)} \left\{ A^{(p-1)/p} - A \right\}$$

We fix $A \in (0,1)$ such that $A^{(p-1)/p} - A \ge a^{2-p}$ and set $\Sigma_A(y) = A(1-y)^{p/(p-1)}$ for $y \in [0,1)$. On the one hand,

$$\begin{split} \Sigma'_A(y) + \frac{p}{p-1} \Sigma_A(y)^{(p-1)/p} &= \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - A(1-y)^{(2-p)/(p-1)} \right] \\ &\geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - A \right] \\ &\geq \frac{p}{p-1} (1-y) a^{2-p} &= \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} \end{split}$$

for $y \in (0,1)$. On the other hand, $\Sigma_A(0) = A > 0 = \psi(0)$. We are then in a position to apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \Sigma_A)$ to conclude that

$$0 \le \psi(y) \le A(1-y)^{p/(p-1)}$$
, $y \in [0,1)$

Since p < p/(p-1), the above estimate ensures that $a \in C$ and we have thus shown that C is non-empty and contains the interval $(0, (p-1)^{(p-1)/(2-p)}p^{-p/(2-p)}]$.

Step 3. Introducing $a_* := \sup\{\mathcal{C}\} > 0$, we infer from the monotonicity of $\psi(\cdot, a)$ with respect to a (Lemma 3.2) that $(0, a_*) \subset \mathcal{C}$.

Assume for contradiction that $a_* \in \mathcal{C}$. Owing to (3.18) there are $\delta > 0$ and $\varepsilon > 0$ such that

$$\varphi'(Y_{a_*} + \delta, a_*) < -2\varepsilon < \varphi'(Y_{a_*}, a_*) = 0 < 2\varepsilon < \varphi'(Y_{a_*} - \delta, a_*)$$
(3.19)

and

$$\frac{Y_{a_*}}{2} \le Y_{a_*} - \delta < Y_{a_*} < Y_{a_*} + \delta \le \frac{1 + Y_{a_*}}{2} .$$
(3.20)

Thanks to Lemma 3.2, $\varphi'(\cdot, a)$ depends continuously on a on $[0, (1 + Y_{a_*})/2]$ and we infer from (3.19) and (3.20) that there is $\alpha > 0$ small enough such that

$$\varphi'(Y_{a_*}+\delta,a) < -\varepsilon < \varepsilon < \varphi'(Y_{a_*}-\delta,a) , \qquad a \in [a_*-\alpha,a_*+\alpha] .$$

In particular, for all $a \in [a_* - \alpha, a_* + \alpha]$, the function $\varphi'(\cdot, a)$ has a zero inside the interval $(Y_{a_*} - \delta, Y_{a_*} + \delta)$. According to (c3), this means that $[a_* - \alpha, a_* + \alpha] \subset \mathcal{C}$, which contradicts the definition of a_* . Therefore $a_* \notin \mathcal{C}$ and $\mathcal{C} = (0, a_*)$.

We finally turn to the description of the set \mathcal{B} and show that it is a singleton.

Proposition 3.7. There holds $a_* = a^*$ and $\mathcal{B} = \{a_*\}$, where a^* and a_* are defined in Lemma 3.5 and Lemma 3.6, respectively.

Proof. Owing to Lemma 3.5 and Lemma 3.6 there holds $\mathcal{B} = [a_*, a^*]$ and $\varphi'(\cdot, a) > 0$ in (0, 1) for $a \in \mathcal{B}$, recalling that the function $\varphi(\cdot, a)$ is defined by (3.13). Introducing $G := \varphi(\cdot, a^*) - \varphi(\cdot, a_*)$ it follows from Lemma 3.2 and (3.14) that $G \ge 0$ and

$$G'(y) + \frac{p\kappa^{1/p}}{1-y} \left[\varphi(y, a^*)^{(p-1)/p} - \varphi(y, a_*)^{(p-1)/p} \right]$$

= $p \frac{G(y)}{1-y} + \frac{p}{p-1} \left[(a^*)^{2-p} - (a_*)^{2-p} \right] (1-y)^{1-p}$ (3.21)

for $y \in (0,1)$. Since $a_* \in \mathcal{B}$ we deduce from the definition of \mathcal{B} that there is $Y \in (0,1)$ such that

$$\varphi(y, a_*) \ge \left(p - \frac{1}{2}\right)^{-p}, \qquad y \in [Y, 1)$$

Therefore, for $y \in [Y, 1)$,

$$\varphi(y, a^*)^{(p-1)/p} - \varphi(y, a_*)^{(p-1)/p} = \frac{p-1}{p} \int_{\varphi(y, a_*)}^{\varphi(y, a^*)} z^{-1/p} dz$$
$$\leq \frac{p-1}{p} \varphi(y, a_*)^{-1/p} G(y)$$
$$\leq \frac{(p-1)(2p-1)}{2p} G(y) .$$

Combining the above estimate with (3.21) gives, for $y \in [Y, 1)$,

$$\begin{aligned} G'(y) + \frac{p}{(p-1)(1-y)} \frac{(p-1)(2p-1)}{2p} G(y) &\geq \frac{pG(y)}{1-y} + \frac{p}{p-1} \left[(a^*)^{2-p} - (a_*)^{2-p} \right] (1-y)^{1-p} \\ &\geq \frac{pG(y)}{1-y}, \end{aligned}$$

whence, after easy manipulations,

$$G'(y) \ge \frac{G(y)}{2(1-y)}, \quad y \in [Y,1).$$

Integrating the above differential inequality on [Y, y) for some $y \in (Y, 1)$, we find

$$G(y) \ge G(Y) \sqrt{\frac{1-Y}{1-y}}, \qquad y \in (Y,1).$$
 (3.22)

Assume now for contradiction that $a^* > a_*$. We deduce from Lemma 3.2 and the fact that $Y \in (0, 1)$ that $\varphi(Y, a^*) > \varphi(Y, a_*)$, that is, G(Y) > 0. It then follows from (3.22) that $G(y) \to \infty$ as $y \to 1$. However, the definition of \mathcal{B} entails that $G(y) \to 0$ as $y \to 1$, clearly in contradiction with the previous assertion. Therefore $a_* = a^*$ and the proof of Proposition 3.7 is complete.

3.4. Refined asymptotics as $y \to 1$ for $a \in C$. The final step is to identify the behavior of $\psi(y, a)$ as $y \to 1$ for $a \in C$.

Lemma 3.8. If $a \in C$ then

$$\lim_{y \to 1} \psi(y, a) (1 - y)^{-p/(p-1)} = a^{p(2-p)/(p-1)} .$$

Proof. Let $a \in \mathcal{C}$.

Step 1. We first prove that there exists $M > a^{p(2-p)/(p-1)}$ such that

$$\psi(y) \le M(1-y)^{p/(p-1)}$$
, $y \in [0,1)$. (3.23)

Indeed, let $\varepsilon \in (0, 1)$ to be determined later and define

$$\sigma_{\varepsilon}(y) := \frac{1}{2\varepsilon^{p(2-p)/(p-1)}} (1-y)^{p/(p-1)} , \qquad y \in (0,1) .$$

Owing to the definition of \mathcal{C} , there is $\bar{\varepsilon} \in (0,1)$ such that $\psi(y) \leq (1-y)^p/2$ for $y \in (1-\bar{\varepsilon},1)$. On the one hand, if $\varepsilon \in (0,\bar{\varepsilon})$, there holds

$$\sigma_{\varepsilon}(1-\varepsilon) = \frac{\varepsilon^p}{2} \ge \psi(1-\varepsilon) \; .$$

On the other hand, for $y \in (1 - \varepsilon, 1)$,

$$\begin{aligned} \sigma_{\varepsilon}'(y) + \frac{p}{p-1} \sigma_{\varepsilon}(y)^{(p-1)/p} &= \frac{p}{p-1} (1-y) \left[\frac{1}{2^{(p-1)/p} \varepsilon^{2-p}} - \frac{(1-y)^{(2-p)/(p-1)}}{2\varepsilon^{p(2-p)/(p-1)}} \right] \\ &\geq \frac{p}{p-1} (1-y) \frac{2^{1/p} - 1}{2\varepsilon^{2-p}} \\ &\geq \frac{pa^{2-p}}{p-1} (1-y) = \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} , \end{aligned}$$

as soon as

$$\frac{2^{1/p} - 1}{2\varepsilon^{2-p}} \ge a^{2-p} . \tag{3.24}$$

We next choose $\varepsilon \in (0, \overline{\varepsilon})$ satisfying (3.24). This allows us to apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \sigma_{\varepsilon})$ in order to obtain that $\psi(y) \leq \sigma_{\varepsilon}(y)$ for $y \in (0, 1 - \varepsilon)$. This inequality extends to the whole interval (0, 1), possibly taking a smaller value of ε .

Step 2. The goal of this step is to improve (3.23). To this end, fix $A \in (a^{p(2-p)/(p-1)}, M)$ and $\varepsilon \in (0, 1)$ such that

$$\varepsilon^{(2-p)/(p-1)} < \frac{A^{(p-1)/p} - a^{2-p}}{2M}$$
 (3.25)

We define

$$\tau(y) := \left(A + \frac{M - A}{\varepsilon} (1 - y)\right) (1 - y)^{p/(p-1)} , \qquad y \in (0, 1) ,$$

and deduce from (3.23) that

$$\tau(1-\varepsilon) = M\varepsilon^{p/(p-1)} \ge \psi(1-\varepsilon)$$

In addition, we infer from (3.4) and (3.25) that, for $y \in (1 - \varepsilon, 1)$,

$$\begin{split} \tau'(y) &+ \frac{p}{p-1} \tau(y)^{(p-1)/p} \\ &\geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - A(1-y)^{(2-p)/(p-1)} - \frac{2p-1}{p} \frac{M-A}{\varepsilon} (1-y)^{1/(p-1)} \right] \\ &\geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - \left(A + \frac{2p-1}{p} (M-A) \right) \varepsilon^{(2-p)/(p-1)} \right] \\ &\geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - 2M \varepsilon^{(2-p)/(p-1)} \right] \\ &\geq \frac{pa^{2-p}}{p-1} (1-y) = \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} \,. \end{split}$$

Applying Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \tau)$ implies that $\psi(y) \leq \tau(y)$ for $y \in (1 - \varepsilon, 1)$. Consequently,

$$\frac{\psi(y)}{(1-y)^{p/(p-1)}} \le A + \frac{M-A}{\varepsilon} (1-y) , \qquad y \in (1-\varepsilon, 1) ,$$

from which we deduce that

$$\limsup_{y \to 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \le A \; .$$

As A is arbitrarily chosen in $(a^{p(2-p)/(p-1)}, M)$, we end up with

$$\limsup_{y \to 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \le a^{p(2-p)/(p-1)}$$

Since

$$\liminf_{y \to 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \ge a^{p(2-p)/(p-1)}$$

by (3.10), the claimed result follows.

4. Proof of Theorem 1.1

We now undo the transformation (3.1) and interpret the outcome of Section 3 in terms of $f(\cdot, a)$. Let $a \in (0, \infty)$. It follows from (2.1) and (3.2) that

$$f'(r) = -a\psi\left(1 - \frac{f(r)}{a}\right)^{1/p}$$
, $r \in [0, R(a))$. (4.1)

Since $\psi(y) \sim pa^{2-p}y/(p-1)$ as $y \to 0$ and p > 1, the function $z \mapsto \psi(1-z)^{-1/p}$ defined on (0,1) belongs to $L^1(z_0,1)$ for all $z_0 > 0$. We may thus integrate (4.1) and find

$$\int_{f(r)/a}^{1} \frac{dz}{\psi(1-z)^{1/p}} = r , \qquad r \in [0, R(a)) .$$
(4.2)

Case 1: $a \in \mathcal{A}$. According to the definition of \mathcal{A} , $\psi(y)$ has a positive limit $\ell(a) > 0$ as $y \to 1$ and the function $z \mapsto \psi(1-z)^{-1/p}$ actually belongs to $L^1(0,1)$. We then deduce from (4.2) that

$$\int_0^1 \frac{dz}{\psi(1-z)^{1/p}} = R(a) \; ,$$

that is, $R(a) < \infty$. Furthermore, $f'(R(a)) = -a\ell(a)^{1/p} < 0$ by (4.1) and the proof of Theorem 1.1 (a) is complete.

Case 2: $a \in \mathcal{B}$. By Proposition 3.7 there holds $a = a_*$ and the definition of \mathcal{B} ensures that $\psi(1-z)^{1/p} \sim z/(p-1)$ as $z \to 0$. Therefore $z \mapsto \psi(1-z)^{-1/p}$ does not belong to $L^1(0,1)$ and we infer from (4.2) that $R(a_*) = \infty$ and

$$r \sim -(p-1)\log(f(r))$$
 as $r \to \infty$.

In particular, there is R > 0 such that

$$-\frac{p-1}{r}\log(f(r)) \ge 1 - \frac{2-p}{2} = \frac{p}{2} , \qquad r \ge R ,$$

from which we deduce that

$$\int_{R}^{\infty} e^{r} f(r) \, dr \le \int_{R}^{\infty} e^{-(2-p)r/2(p-1)} \, dr < \infty \,, \tag{4.3}$$

since $p \in (1, 2)$. Recalling (2.1), it follows from (2.2) after integration that

$$-e^r |f'(r)|^{p-1} = -\int_0^r e^\sigma f(\sigma) \, d\sigma$$

which, together with (4.3), guarantees that $e^r |f'(r)|^{p-1}$ has a finite limit as $r \to \infty$ and

$$\lim_{r \to \infty} e^r |f'(r)|^{p-1} = I := \int_0^\infty e^r f(r) \, dr \; .$$

We then infer from the above property, (4.1), and the behavior of $\psi(y)$ as $y \to 1$ that

$$f'(r) \sim -I^{1/(p-1)}e^{-r/(p-1)}$$
 and $f'(r) \sim -\frac{f(r)}{p-1}$ as $r \to \infty$,

so that $f(r) \sim (p-1)I^{1/(p-1)}e^{-r/(p-1)}$ as $r \to \infty$. We have thus proved Theorem 1.1 (b).

Case 3: $a \in \mathcal{C}$. In that case, $\psi(1-z)^{1/p} \sim a^{(2-p)/(p-1)}z^{1/(p-1)}$ as $z \to 0$ by Lemma 3.8. Since $p \in (1,2)$ the function $z \mapsto \psi(1-z)^{-1/p}$ does not belong to $L^1(0,1)$ and we infer from (4.2) that $R(a) = \infty$ and

$$\frac{p-1}{2-p} \left(\frac{f(r)}{a}\right)^{-(2-p)/(p-1)} \sim a^{(2-p)/(p-1)}r \text{ as } r \to \infty ,$$

hence Theorem 1.1 (c).

R.G. IAGAR & PH. LAURENÇOT

Acknowledgments

R. G. I. is supported by the Severo Ochoa Excellence project SEV-2015-0554 (MINECO, Spain). Part of this work has been completed while R. G. I. was enjoying a one-month "Invited Professor" stay at the Institut de Mathématiques de Toulouse, and he thanks for the hospitality and the support.

References

- X. CHEN, Y. QI, AND M. WANG, Self-similar singular solutions of a p-Laplacian evolution equation with absorption, J. Differential Equations, 190 (2003), pp. 1–15.
- M. DEL PINO AND M. SÁEZ, Asymptotic description of vanishing in a fast-diffusion equation with absorption, Differential Integral Equations, 15 (2002), pp. 1009–1023.
- [3] R. FERREIRA AND J. L. VÁZQUEZ, Extinction behaviour for fast diffusion equations with absorption, Nonlinear Anal., 43 (2001), pp. 943–985.
- [4] R. G. IAGAR AND PH. LAURENÇOT, Positivity, decay, and extinction for a singular diffusion equation with gradient absorption, J. Funct. Anal., 262 (2012), pp. 3186–3239.
- [5] —, Eternal solutions to a singular diffusion equation with critical gradient absorption, Nonlinearity, 26 (2013), pp. 3169–3195.
- [6] —, Existence and uniqueness of very singular solutions for a fast diffusion equation with gradient absorption,
 J. Lond. Math. Soc. (2), 87 (2013), pp. 509–529.
- [7] —, Self-similar extinction for a diffusive Hamilton-Jacobi equation with critical absorption. preprint, 2016.
- [8] M. K. KWONG, Uniqueness of positive solutions of $\Delta u u + u^p = 0$ in \mathbb{R}^n , Arch. Rational Mech. Anal., 105 (1989), pp. 243–266.
- [9] J. SERRIN AND M. TANG, Uniqueness of ground states for quasilinear elliptic equations, Indiana Univ. Math. J., 49 (2000), pp. 897–923.
- [10] P. SHI, Self-similar very singular solution of a p-Laplacian equation with gradient absorption: existence and uniqueness, J. Southeast Univ. (English Ed.), 20 (2004), pp. 381–386.
- [11] N. SHIOJI AND K. WATANABE, Uniqueness and nondegeneracy of positive radial solutions of $\operatorname{div}(\rho \nabla u) + \rho(-gu + hu^p) = 0$, Calc. Var. Partial Differential Equations, 55 (2016), p. 55:32.
- [12] E. YANAGIDA, Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in \mathbb{R}^n , Arch. Rational Mech. Anal., 115 (1991), pp. 257–274.
- [13] H. YE AND J. YIN, Uniqueness of self-similar very singular solution for non-Newtonian polytropic filtration equations with gradient absorption, Electron. J. Differential Equations, (2015), pp. No. 83, 9.

Instituto de Ciencias Matemáticas (ICMAT), Nicolas Cabrera 13-15, Campus de Cantoblanco, E-28049, Madrid, Spain

E-mail address: razvan.iagar@icmat.es

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, RO-014700, BUCHAREST, RO-MANIA.

Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, F-31062 Toulouse Cedex 9, France

 $E\text{-}mail\ address: \texttt{laurencoQmath.univ-toulouse.fr}$