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# STRICT $\omega$ -CATEGORIES ARE MONADIC OVER POLYGRAPHS

FRANÇOIS MÉTAYER

ABSTRACT. We give a direct proof that the category of strict  $\omega$ -categories is monadic over the category of polygraphs.

## INTRODUCTION

This short note presents a proof of monadicity for the adjunction between the category  $\mathbf{Cat}_\omega$  of strict  $\omega$ -categories and the category  $\mathbf{Pol}_\omega$  of polygraphs (or computads, as first introduced by Street in [Str76]). Here we follow the presentation and terminology of [Bur93, Mét03]. The reader may consult [Mét08] for a detailed description of the categories and functors referred to in this particular case, or [Bat98] for a broader perspective including generalized “ $A$ -computads” for a monad  $A$  on globular sets. The latter paper rightly asserts the monadicity theorem, but some parts of the proof rely on the fact that the category of  $A$ -computads is a presheaf category, which is precisely not true in the present case, where  $A$  is the monad of strict  $\omega$ -categories [MZ08, Che13]. Since then, the status of monadicity for  $\mathbf{Cat}_\omega$  has remained somewhat unclear (see e.g the entry “computad” on the  $n$ Lab [nLa]). Our proof is based on the same ideas as developed in [Bat98], except that we avoid the presheaf argument and establish instead a lifting result (Lemma 2), possibly of independent interest.

As for notations, whenever a functor  $F$  is a right-adjoint, we denote its left-adjoint by  $F^*$ . Let us finally mention a small point about terminology. Given a functor  $F : \mathbf{A} \rightarrow \mathbf{B}$ , with left-adjoint  $F^*$ , and  $T = FF^*$  the associated monad on  $\mathbf{B}$ , there is a comparison functor  $K$  from  $\mathbf{A}$  to the category  $\mathbf{B}^T$  of  $T$ -algebras: we call  $F$  *monadic* if  $K$  is an equivalence of categories, and *strictly monadic* if  $K$  is an isomorphism. We refer to [ML71, VI.7] for corresponding variants of Beck’s monadicity criterion.

## 1. THREE ADJUNCTIONS

In this section, we briefly describe three pairs of adjoint functors between categories  $\mathbf{Glob}_\omega$  of globular sets,  $\mathbf{Cat}_\omega$  of strict  $\omega$ -categories and  $\mathbf{Pol}_\omega$  of polygraphs.

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As  $\omega$ -categories are globular sets with extra structure, there is an obvious forgetful functor

$$U : \mathbf{Cat}_\omega \rightarrow \mathbf{Glob}_\omega$$

This functor  $U$  has a left-adjoint  $U^*$  taking a globular set  $X$  to the  $\omega$ -category  $U^*X$  it generates. Moreover this adjunction is strictly monadic.

A second adjunction involves functors  $V, V^*$  between  $\mathbf{Cat}_\omega$  and  $\mathbf{Pol}_\omega$ . Unlike  $U$ , the right adjoint  $V$  is not quite obvious. Thus, let  $C$  be an  $\omega$ -category, the polygraph  $P = V(C)$  is defined by induction, together with a morphism  $\epsilon^C : V^*(P) \rightarrow C$ :

- For  $n = 0$ ,  $P_0 = C_0$  and  $\epsilon_0^C$  is the identity.
- Suppose  $n > 0$ , and  $P, \epsilon^C$  have been defined up to dimension  $n-1$ . The set of  $n$ -generators of  $P$  is then the set  $P_n$  of triples  $p = (z, x, y)$  where  $z \in C_n$ ,  $x, y$  are parallel cells in  $P_{n-1}^*$  and  $z : \epsilon_{n-1}^C(x) \rightarrow \epsilon_{n-1}^C(y)$ . The source and target of  $p$  in  $P_{n-1}^*$  are  $x = s_{n-1}(p)$  and  $y = t_{n-1}(p)$  respectively, and  $\epsilon_n^C(p) = z$ . By the universal property of polygraphs,  $\epsilon_n^C$  extends uniquely to a map from  $P_n^*$  to  $C_n$  preserving compositions and identities. Functoriality of  $V$  is immediate and  $V$  is in fact right-adjoint to  $V^*$  (see [Bat98, Mét03]).

Note that

$$\epsilon^C : V^*V(C) \rightarrow C$$

is the counit of this adjunction and determines the standard polygraphic resolution of  $C$ .

We finally describe a functor

$$G : \mathbf{Pol}_\omega \rightarrow \mathbf{Glob}_\omega$$

Let  $P$  be a polygraph. Let us denote by  $j_n : P_n \rightarrow P_n^*$  the canonical inclusion of the set of  $n$ -generators of  $P$  into the set of  $n$ -cells of  $P^* = V^*(P)$ . We define the globular set  $X = G(P)$  dimensionwise, so that for each  $n \in \mathbb{N}$ ,  $X_n \subset P_n$ :

- For  $n = 0$ ,  $X_0 = P_0$ .
- Let  $n > 0$  and suppose we have defined  $X_k \subset P_k$  for all  $k < n$ , together with source and target maps building an  $(n-1)$ -globular set. Let  $X_n \subset P_n$  be the set of  $n$ -generators  $a$  of  $P$  such that  $s_{n-1}(a)$  and  $t_{n-1}(a)$  belong to  $j_{n-1}(X_{n-1})$  and define source and target maps  $s_{n-1}^X, t_{n-1}^X : X_n \rightarrow X_{n-1}$  as the unique maps such that  $j_{n-1}s_{n-1}^X(a) = s_{n-1}(a)$  and  $j_{n-1}t_{n-1}^X(a) = t_{n-1}(a)$  for each  $a \in X_n$ . This extends  $X$  to an  $n$ -globular set.

$$(1) \quad \begin{array}{ccccc} X_n & \hookrightarrow & P_n & & \\ & & \downarrow t_{n-1} & \searrow & \\ & & & s_{n-1} & \\ & & & & P_{n-1}^* \\ & & & \nearrow j_{n-1} & \\ X_{n-1} & \hookrightarrow & P_{n-1} & \longrightarrow & P_{n-1}^* \\ & & \downarrow s_{n-1}^X & & \\ & & \downarrow t_{n-1}^X & & \end{array}$$

The previous construction is clearly functorial and defines the required functor  $G$ . Remark that  $G$  admits a left adjoint  $G^* : \mathbf{Glob}_\omega \rightarrow \mathbf{Pol}_\omega$  which takes the globular set  $X$  to a polygraph  $P$  such that  $P_n = X_n$ , in other words  $G^*$  defines a natural inclusion of  $\mathbf{Glob}_\omega$  into  $\mathbf{Pol}_\omega$ .

Note that  $G$  forgets all generators of  $P$  that are not “hereditary globular”, so that for instance  $G(P)$  may have no cells at all beyond dimension 1. However, the following result shows that the functor  $G$  is not always trivial.

**Lemma 1.** *There is a natural isomorphism  $\phi : GV \rightarrow U$ , that is, the following diagram commutes up to a natural isomorphism*

$$(2) \quad \begin{array}{ccc} \mathbf{Cat}_\omega & \xrightarrow{V} & \mathbf{Pol}_\omega \\ U \downarrow & \swarrow G & \\ \mathbf{Glob}_\omega & & \end{array}$$

*Proof.* Let  $C$  be an  $\omega$ -category, and  $X = GV(C)$ . For each  $n \in \mathbb{N}$ , let  $\phi_n^C : X_n \rightarrow C_n$  be the composition of the following maps

$$X_n \hookrightarrow V(C)_n \xrightarrow{j_n} V^*V(C)_n \xrightarrow{\epsilon_n^C} C_n$$

As  $\epsilon^C$  is an  $\omega$ -morphism and (1) commutes, the family  $(\phi_n^C)_{n \in \mathbb{N}}$  defines a globular morphism  $\phi^C : GV(C) \rightarrow U(C)$ , natural in  $C$ . Thus we get a natural transformation  $\phi : GV \rightarrow U$ .

Let us now define  $\chi_n^C : C_n \rightarrow X_n$  by induction on  $n$  such that  $\phi_n^C \circ \chi_n^C = 1_{C_n}$ :

- For  $n = 0$ ,  $X_0 = C_0$  and  $\phi_0^C = 1_{C_0} = 1_{X_0}$ , so that  $\chi_0^C : C_0 \rightarrow X_0$  is also  $1_{C_0} = 1_{X_0}$ .
- Suppose  $n > 0$  and  $\chi_k^C$  has been defined up to  $k = n-1$ , and let  $z \in C_n$ . Let  $u = s_{n-1}(z)$  and  $v = t_{n-1}(z)$  in  $C_{n-1}$ . By induction hypothesis,  $\chi_{n-1}^C(u)$  and  $\chi_{n-1}^C(v)$  belong to  $X_{n-1}$ . Let  $x = j_{n-1}\chi_{n-1}^C(u)$ ,  $y = j_{n-1}\chi_{n-1}^C(v)$  in  $V^*V(C)_{n-1}$  and define  $a = \chi_n^C(z) = (z, x, y)$ . By construction  $a \in X_n$  and  $\phi_n^C(a) = z$ .

It remains to prove that  $\phi_n^C$  is injective. We reason again by induction on  $n$ :

- For  $n = 0$ ,  $\phi_0^C$  is an identity, hence injective.
- Suppose  $n > 0$  and  $\phi_{n-1}^C$  injective. Let  $a_i = (z_i, x_i, y_i) \in X_n$  for  $i = 0, 1$  such that  $\phi_n^C(a_0) = \phi_n^C(a_1)$ . Thus  $z_0 = z_1$ . Also

$$\begin{aligned} \phi_{n-1}^C(s_{n-1}^X(a_0)) &= s_{n-1}(\phi_n^C(a_0)) \\ &= s_{n-1}(\phi_n^C(a_1)) \\ &= \phi_{n-1}^C(s_{n-1}^X(a_1)) \end{aligned}$$

and because  $\phi_{n-1}^C$  is injective,

$$s_{n-1}^X(a_0) = s_{n-1}^X(a_1)$$

Now

$$\begin{aligned}
x_0 &= s_{n-1}(a_0) \\
&= j_{n-1} s_{n-1}^X(a_0) \\
&= j_{n-1} s_{n-1}^X(a_1) \\
&= s_{n-1}(a_1) \\
&= x_1
\end{aligned}$$

Likewise  $y_0 = y_1$ , and we get  $a_0 = a_1$ . Hence  $\phi_n^C$  is injective and we are done. □

## 2. LIFTING LEMMA

The forgetful functor  $U : \mathbf{Cat}_\omega \rightarrow \mathbf{Glob}_\omega$  is faithful, but clearly not full. However, globular morphisms lift to  $\omega$ -morphisms in the sense of the following result:

**Lemma 2.** *Let  $C, D$  be  $\omega$ -categories and  $\alpha : U(C) \rightarrow U(D)$  be a globular morphism. Then there is a unique morphism  $\bar{\alpha} : V(C) \rightarrow V(D)$  in  $\mathbf{Pol}_\omega$  such that the following square commutes:*

$$(3) \quad \begin{array}{ccc}
UV^*V(C) & \xrightarrow{UV^*(\bar{\alpha})} & UV^*V(D) \\
U(\epsilon^C) \downarrow & & \downarrow U(\epsilon^D) \\
U(C) & \xrightarrow{\alpha} & U(D)
\end{array}$$

*Proof.* We build the required morphism  $\bar{\alpha} : V(C) \rightarrow V(D)$  by induction on the dimension. Note that diagram (3) yields a diagram in  $\mathbf{Sets}$  at any given dimension  $n$ . We may therefore drop the letter  $U$  in the following computations. Also  $\bar{\alpha}^*$  is short for  $V^*(\bar{\alpha})$ .

- For  $n = 0$ , we have  $V(C)_0 = C_0$ ,  $V(D)_0 = D_0$ ; also  $\epsilon_0^C$  and  $\epsilon_0^D$  are identities, so that  $\bar{\alpha}_0 = \alpha_0$  is the unique solution.
- Suppose  $n > 0$  and we have defined  $\bar{\alpha}$  satisfying the commutation condition, up to dimension  $n-1$ . Let  $p = (z, x, y)$  be an  $n$ -generator of  $V(C)$ . Suppose  $\bar{\alpha}(p) = (z', x', y')$ : the commutation condition implies  $z' = \alpha(z)$ ,  $x' = \bar{\alpha}_{n-1}^*(x)$  and  $y' = \bar{\alpha}_{n-1}^*(y)$ , so that  $\bar{\alpha}$  extends in at most one way to dimension  $n$ , and uniqueness holds. As for the existence,  $x, y$  are parallel  $(n-1)$ -cells in  $V^*V(C)_{n-1}$ ; by induction hypothesis, their images  $x' = \bar{\alpha}_{n-1}^*(x)$  and  $y' = \bar{\alpha}_{n-1}^*(y)$  are  $(n-1)$ -parallel cells in  $V^*V(D)$ . Again, by induction hypothesis, (3)

commutes in dimension  $n-1$ ; also  $\alpha$  is a globular map, hence

$$\begin{aligned} s_{n-1}(z') &= s_{n-1}(\alpha_n(z)) \\ &= \alpha_{n-1}(s_{n-1}(z)) \\ &= \alpha_{n-1}(\epsilon_{n-1}^C(x)) \\ &= \epsilon_{n-1}^D(\bar{\alpha}_{n-1}^*(x)) \\ &= \epsilon_{n-1}^D(x') \end{aligned}$$

and likewise

$$t_{n-1}(z') = \epsilon_{n-1}^D(y')$$

Therefore  $p' = (z', x', y')$  is an  $n$ -generator of  $V(D)$ . Also  $s_{n-1}(p') = x' = \bar{\alpha}_{n-1}^*(x) = \bar{\alpha}_{n-1}^*(s_{n-1}(p))$  and  $t_{n-1}(p') = y' = \bar{\alpha}_{n-1}^*(y) = \bar{\alpha}_{n-1}^*(t_{n-1}(p))$ , so that  $\bar{\alpha}$  extends to a morphism in  $\mathbf{Pol}_\omega$  up to dimension  $n$ . Finally the diagram (3) commutes in dimension  $n$ : it is sufficient to check this on generators, but

$$\begin{aligned} \epsilon_n^D \bar{\alpha}_n^*(p) &= \epsilon_n^D(p') \\ &= z' \\ &= \alpha_n(z) \\ &= \alpha_n \epsilon_n^C(p) \end{aligned}$$

and we are done. □

### 3. MONADICITY

We now turn to the main result.

**Theorem 1.** *The functor  $V : \mathbf{Cat}_\omega \rightarrow \mathbf{Pol}_\omega$  is monadic.*

*Proof.* Recall that monadicity means here that  $\mathbf{Cat}_\omega$  is *equivalent* to the category of algebras of the monad  $VV^*$  on  $\mathbf{Pol}_\omega$ . By using the corresponding version of Beck's criterion, this amounts to show that (i)  $V$  reflects isomorphisms and (ii) if  $f, g$  is a parallel pair of  $\omega$ -morphisms such that the pair  $V(f), V(g)$  has a split coequalizer in  $\mathbf{Pol}_\omega$ , then  $f, g$  has a coequalizer in  $\mathbf{Cat}_\omega$ , and  $V$  preserves coequalizers of such pairs (see for instance [ML71, VI.7, exercises 3 and 6]).

First, if  $f : C \rightarrow D$  is an  $\omega$ -morphism such that  $V(f)$  is an isomorphism, then  $GV(f)$  is an isomorphism in  $\mathbf{Glob}_\omega$  and by Lemma 1,  $U(f)$  is an isomorphism. Now,  $U$  reflects isomorphisms, hence  $f$  is an isomorphism. Therefore  $V$  reflects isomorphisms as required.

Now, let  $f, g : C \rightarrow D$  be a pair of  $\omega$ -morphisms and suppose

$$(4) \quad \begin{array}{ccc} & \begin{array}{c} \overset{b}{\curvearrowright} \\ \downarrow \\ V(f) \\ \downarrow \\ V(g) \end{array} & \\ V(C) & \xrightarrow{\quad} & V(D) \xrightarrow{k} P \\ & \begin{array}{c} \uparrow \\ \downarrow \\ a \end{array} & \end{array}$$

is a split coequalizer in  $\mathbf{Pol}_\omega$  where  $k \circ a = 1_P$ ,  $V(f) \circ b = 1_{V(D)}$  and  $V(g) \circ b = a \circ k$ . By applying the functor  $G$  to (4), we get a split coequalizer in  $\mathbf{Glob}_\omega$ :

$$(5) \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{G(b)} \\ \curvearrowright \\ \xrightarrow{GV(f)} \\ \xrightarrow{GV(g)} \end{array} & \\ GV(C) & \xrightarrow{\quad} & GV(D) \xrightarrow{G(k)} G(P) \\ & \begin{array}{c} \xrightarrow{G(a)} \\ \curvearrowleft \\ \xrightarrow{G(k)} \end{array} & \end{array}$$

Then, by using the natural isomorphism  $\phi$  of Lemma 1, we obtain the following diagram

$$(6) \quad \begin{array}{ccccc} & & G(b) & & \\ & & \curvearrowright & & \\ & & \xrightarrow{GV(f)} & & \\ GV(C) & \xrightarrow{\quad} & GV(D) & \xleftarrow{G(a)} & \\ & & \curvearrowleft & & \\ \phi^C \downarrow & & \phi^D \downarrow & & \\ U(C) & \xrightarrow{U(f)} & U(D) & \xrightarrow{l} & G(P) \\ & & \curvearrowright & & \\ & & \beta & & \alpha \end{array}$$

where  $\alpha = \phi^D \circ G(a)$ ,  $l = G(k) \circ (\phi^D)^{-1}$  and  $\beta = \phi^C \circ G(b) \circ (\phi^D)^{-1}$ . Therefore  $l \circ \alpha = 1_{G(P)}$ ,  $U(f) \circ \beta = 1_{U(D)}$  and

$$\begin{aligned} U(g) \circ \beta &= U(g) \circ \phi^C \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ GV(g) \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ G(a) \circ G(k) \circ (\phi^D)^{-1} \\ &= \alpha \circ l \end{aligned}$$

and the bottom line of (6) is a split coequalizer diagram in  $\mathbf{Glob}_\omega$ . Now the functor  $U$  is strictly monadic, so that there is a unique  $\omega$ -morphism  $h : D \rightarrow E$  such that  $U(E) = G(P)$  and  $U(h) = l$  and moreover this unique morphism makes

$$(7) \quad C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} D \xrightarrow{h} E$$

a coequalizer diagram in  $\mathbf{Cat}_\omega$ . Note that, by construction,  $U(E) = G(P)$ .

It remains to show that  $V(h) : V(D) \rightarrow V(E)$  is a coequalizer of the pair  $V(f), V(g)$  in  $\mathbf{Pol}_\omega$ . By applying Lemma 2 to  $\alpha : U(E) \rightarrow U(D)$  and to  $\beta : U(D) \rightarrow U(C)$ , we get unique morphisms  $\bar{\alpha} : V(E) \rightarrow V(D)$  and  $\bar{\beta} : V(D) \rightarrow V(C)$  satisfying the required commutation condition. Consider the following diagram:

$$(8) \quad \begin{array}{ccccc} UV^*V(E) & \xrightarrow{UV^*(\bar{\alpha})} & UV^*V(D) & \xrightarrow{UV^*V(h)} & UV^*V(E) \\ U(\epsilon^E) \downarrow & & U(\epsilon^D) \downarrow & & \downarrow U(\epsilon^E) \\ U(E) & \xrightarrow{\quad \alpha \quad} & U(D) & \xrightarrow{U(h)} & U(E) \end{array}$$

The left-hand square commutes by hypothesis, and the right-hand square commutes by the naturality of  $\epsilon$ , whence the outer square also commutes. As  $U(h) \circ \alpha = 1_{U(E)}$ , the uniqueness of the lifting in Lemma 2 implies that  $V(h) \circ \bar{\alpha} = 1_{V(E)}$ . By the same uniqueness argument, we get  $V(f) \circ \bar{\beta} = 1_{V(D)}$  and  $V(g) \circ \bar{\beta} = \bar{\alpha} \circ V(h)$ . Therefore the following diagram is a split coequalizer in  $\mathbf{Pol}_\omega$

$$\begin{array}{ccccc} & \bar{\beta} & & \bar{\alpha} & \\ & \curvearrowright & & \curvearrowleft & \\ V(C) & \xrightarrow{V(f)} & V(D) & \xrightarrow{k} & V(E) \\ & \xrightarrow{V(g)} & & & \end{array}$$

and we are done.  $\square$

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