



**HAL**  
open science

## Strict $\omega$ -categories are monadic over polygraphs

François Métayer

► **To cite this version:**

| François Métayer. Strict  $\omega$ -categories are monadic over polygraphs. 2016. hal-01324337v1

**HAL Id: hal-01324337**

**<https://hal.science/hal-01324337v1>**

Preprint submitted on 31 May 2016 (v1), last revised 2 Jun 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STRICT $\omega$ -CATEGORIES ARE MONADIC OVER POLYGRAPHS

FRANÇOIS MÉTAYER

ABSTRACT. We give a direct proof that the category of strict  $\omega$ -categories is monadic over the category of polygraphs.

## INTRODUCTION

This short note presents a proof of monadicity for the adjunction between the category  $\mathbf{Cat}_\omega$  of strict  $\omega$ -categories and the category  $\mathbf{Pol}_\omega$  of polygraphs (or computads, as first introduced by Street in [Str76]). Here we follow the presentation and terminology of [Bur93, Mét03]. The reader may consult [Mét08] for a detailed description of the categories and functors referred to in this particular case, or [Bat98] for a broader perspective including generalized “ $A$ -computads” for a monad  $A$  on globular sets. The latter paper rightly asserts the monadicity theorem, but some parts of the proof rely on the fact that the category of  $A$ -computads is a presheaf category, which is precisely not true in the present case, where  $A$  is the monad of strict  $\omega$ -categories [MZ08, Che13]. Since then, the status of monadicity for  $\mathbf{Cat}_\omega$  has remained somewhat unclear (see e.g the entry “computad” on the  $n\text{Lab}$  [nLa]). Our proof is based on the same ideas as developed in [Bat98], except that we avoid the presheaf argument and establish instead a lifting result (Lemma 2), possibly of independent interest.

As for notations, whenever a functor  $F$  is a right-adjoint, we denote its left-adjoint by  $F^*$ . Let us finally mention a small point about terminology. Given a functor  $F : \mathbf{A} \rightarrow \mathbf{B}$ , with left-adjoint  $F^*$ , and  $T = FF^*$  the associated monad on  $\mathbf{B}$ , there is a comparison functor  $K$  from  $\mathbf{A}$  to the category  $\mathbf{B}^T$  of  $T$ -algebras: we call  $F$  *monadic* if  $K$  is an equivalence of categories, and *strictly monadic* if  $K$  is an isomorphism. We refer to [ML71, VI.7] for corresponding variants of Beck’s monadicity criterion.

## 1. THREE ADJUNCTIONS

In this section, we briefly describe three pairs of adjoint functors between categories  $\mathbf{Glob}_\omega$  of globular sets,  $\mathbf{Cat}_\omega$  of strict  $\omega$ -categories and  $\mathbf{Pol}_\omega$  of polygraphs.

---

2010 *Mathematics Subject Classification.* 18D05,18C15.

*Key words and phrases.*  $\omega$ -categories, polygraphs, monads.

Supported by Cathre project, ANR-13-BS02-0005.

As  $\omega$ -categories are globular sets with extra structure, there is an obvious forgetful functor

$$U : \mathbf{Cat}_\omega \rightarrow \mathbf{Glob}_\omega$$

This functor  $U$  has a left-adjoint  $U^*$  taking a globular set  $X$  to the  $\omega$ -category  $U^*X$  it generates. Moreover this adjunction is strictly monadic.

A second adjunction involves functors  $V, V^*$  between  $\mathbf{Cat}_\omega$  and  $\mathbf{Pol}_\omega$ . Unlike  $U$ , the right adjoint  $V$  is not quite obvious. Thus, let  $C$  be an  $\omega$ -category, the polygraph  $P = V(C)$  is defined by induction, together with a morphism  $\epsilon^C : V^*(P) \rightarrow C$ :

- For  $n = 0$ ,  $P_0 = C_0$  and  $\epsilon_0^C$  is the identity.
- Suppose  $n > 0$ , and  $P, \epsilon^C$  have been defined up to dimension  $n-1$ . The set of  $n$ -generators of  $P$  is then the set  $P_n$  of triples  $p = (z, x, y)$  where  $z \in C_n$ ,  $x, y$  are parallel cells in  $P_{n-1}^*$  and  $z : \epsilon_{n-1}^C(x) \rightarrow \epsilon_{n-1}^C(y)$ . The source and target of  $p$  in  $P_{n-1}^*$  are  $x = s_{n-1}(p)$  and  $y = t_{n-1}(p)$  respectively, and  $\epsilon_n^C(p) = z$ . By the universal property of polygraphs,  $\epsilon_n^C$  extends uniquely to a map from  $P_n^*$  to  $C_n$  preserving compositions and identities. Functoriality of  $V$  is immediate and  $V$  is in fact right-adjoint to  $V^*$  (see [Bat98, Mét03]).

Note that

$$\epsilon^C : V^*V(C) \rightarrow C$$

is the counit of this adjunction and determines the standard polygraphic resolution of  $C$ .

We finally describe a functor

$$G : \mathbf{Pol}_\omega \rightarrow \mathbf{Glob}_\omega$$

Let  $P$  be a polygraph. Let us denote by  $j_n : P_n \rightarrow P_n^*$  the canonical inclusion of the set of  $n$ -generators of  $P$  into the set of  $n$ -cells of  $P^* = V^*(P)$ . We define the globular set  $X = G(P)$  dimensionwise, so that for each  $n \in \mathbb{N}$ ,  $X_n \subset P_n$ :

- For  $n = 0$ ,  $X_0 = P_0$ .
- Let  $n > 0$  and suppose we have defined  $X_k \subset P_k$  for all  $k < n$ , together with source and target maps building an  $(n-1)$ -globular set. Let  $X_n \subset P_n$  be the set of  $n$ -generators  $a$  of  $P$  such that  $s_{n-1}(a)$  and  $t_{n-1}(a)$  belong to  $j_{n-1}(X_{n-1})$  and define source and target maps  $s_{n-1}^X, t_{n-1}^X : X_n \rightarrow X_{n-1}$  as the unique maps such that  $j_{n-1}s_{n-1}^X(a) = s_{n-1}(a)$  and  $j_{n-1}t_{n-1}^X(a) = t_{n-1}(a)$  for each  $a \in X_n$ . This extends  $X$  to an  $n$ -globular set.

$$(1) \quad \begin{array}{ccccc} X_n & \hookrightarrow & P_n & & \\ \downarrow s_{n-1}^X & & \downarrow t_{n-1}^X & & \searrow t_{n-1} \\ & & & & s_{n-1} \\ X_{n-1} & \hookrightarrow & P_{n-1} & \xrightarrow{j_{n-1}} & P_{n-1}^* \end{array}$$

The previous construction is clearly functorial and defines the required functor  $G$ . Remark that  $G$  admits a left adjoint  $G^* : \mathbf{Glob}_\omega \rightarrow \mathbf{Pol}_\omega$  which takes the globular set  $X$  to a polygraph  $P$  such that  $P_n = X_n$ , in other words  $G^*$  defines a natural inclusion of  $\mathbf{Glob}_\omega$  into  $\mathbf{Pol}_\omega$ .

Note that  $G$  forgets all generators of  $P$  that are not “hereditary globular”, so that for instance  $G(P)$  may have no cells at all beyond dimension 1. However, the following result shows that the functor  $G$  is not always trivial.

**Lemma 1.** *There is a natural isomorphism  $\phi : GV \rightarrow U$ , that is, the following diagram commutes up to a natural isomorphism*

$$(2) \quad \begin{array}{ccc} \mathbf{Cat}_\omega & \xrightarrow{V} & \mathbf{Pol}_\omega \\ U \downarrow & \swarrow G & \\ \mathbf{Glob}_\omega & & \end{array}$$

*Proof.* Let  $C$  be an  $\omega$ -category, and  $X = GV(C)$ . For each  $n \in \mathbb{N}$ , let  $\phi_n^C : X_n \rightarrow C_n$  be the composition of the following maps

$$X_n \hookrightarrow V(C)_n \xrightarrow{j_n} V^*V(C)_n \xrightarrow{\epsilon_n^C} C_n$$

As  $\epsilon^C$  is an  $\omega$ -morphism and (1) commutes, the family  $(\phi_n^C)_{n \in \mathbb{N}}$  defines a globular morphism  $\phi^C : GV(C) \rightarrow U(C)$ , natural in  $C$ . Thus we get a natural transformation  $\phi : GV \rightarrow U$ .

Let us now define  $\chi_n^C : C_n \rightarrow X_n$  by induction on  $n$  such that  $\phi_n^C \circ \chi_n^C = 1_{C_n}$ :

- For  $n = 0$ ,  $X_0 = C_0$  and  $\phi_0^C = 1_{C_0} = 1_{X_0}$ , so that  $\chi_0^C : C_0 \rightarrow X_0$  is also  $1_{C_0} = 1_{X_0}$ .
- Suppose  $n > 0$  and  $\chi_k^C$  has been defined up to  $k = n-1$ , and let  $z \in C_n$ . Let  $u = s_{n-1}(z)$  and  $v = t_{n-1}(z)$  in  $C_{n-1}$ . By induction hypothesis,  $\chi_{n-1}^C(u)$  and  $\chi_{n-1}^C(v)$  belong to  $X_{n-1}$ . Let  $x = j_{n-1}\chi_{n-1}^C(u)$ ,  $y = j_{n-1}\chi_{n-1}^C(v)$  in  $V^*V(C)_{n-1}$  and define  $a = \chi_n^C(z) = (z, x, y)$ . By construction  $a \in X_n$  and  $\phi_n^C(a) = z$ .

It remains to prove that  $\phi_n^C$  is injective. We reason again by induction on  $n$ :

- For  $n = 0$ ,  $\phi_0^C$  is an identity, hence injective.
- Suppose  $n > 0$  and  $\phi_{n-1}^C$  injective. Let  $a_i = (z_i, x_i, y_i) \in X_n$  for  $i = 0, 1$  such that  $\phi_n^C(a_0) = \phi_n^C(a_1)$ . Thus  $z_0 = z_1$ . Also

$$\begin{aligned} \phi_{n-1}^C(s_{n-1}^X(a_0)) &= s_{n-1}(\phi_n^C(a_0)) \\ &= s_{n-1}(\phi_n^C(a_1)) \\ &= \phi_{n-1}^C(s_{n-1}^X(a_1)) \end{aligned}$$

and because  $\phi_{n-1}^C$  is injective,

$$s_{n-1}^X(a_0) = s_{n-1}^X(a_1)$$

Now

$$\begin{aligned}
x_0 &= s_{n-1}(a_0) \\
&= j_{n-1} s_{n-1}^X(a_0) \\
&= j_{n-1} s_{n-1}^X(a_1) \\
&= s_{n-1}(a_1) \\
&= x_1
\end{aligned}$$

Likewise  $y_0 = y_1$ , and we get  $a_0 = a_1$ . Hence  $\phi_n^C$  is injective and we are done. □

## 2. LIFTING LEMMA

The forgetful functor  $U : \mathbf{Cat}_\omega \rightarrow \mathbf{Glob}_\omega$  is faithful, but clearly not full. However, globular morphisms lift to  $\omega$ -morphisms in the sense of the following result:

**Lemma 2.** *Let  $C, D$  be  $\omega$ -categories and  $\alpha : U(C) \rightarrow U(D)$  be a globular morphism. Then there is a unique morphism  $\bar{\alpha} : V(C) \rightarrow V(D)$  in  $\mathbf{Pol}_\omega$  such that the following square commutes:*

$$(3) \quad \begin{array}{ccc}
UV^*V(C) & \xrightarrow{UV^*(\bar{\alpha})} & UV^*V(D) \\
U(\epsilon^C) \downarrow & & \downarrow U(\epsilon^D) \\
U(C) & \xrightarrow{\alpha} & U(D)
\end{array}$$

*Proof.* We build the required morphism  $\bar{\alpha} : V(C) \rightarrow V(D)$  by induction on the dimension. Note that diagram (3) yields a diagram in  $\mathbf{Sets}$  at any given dimension  $n$ . We may therefore drop the letter  $U$  in the following computations. Also  $\bar{\alpha}^*$  is short for  $V^*(\bar{\alpha})$ .

- For  $n = 0$ , we have  $V(C)_0 = C_0$ ,  $V(D)_0 = D_0$ ; also  $\epsilon_0^C$  and  $\epsilon_0^D$  are identities, so that  $\bar{\alpha}_0 = \alpha_0$  is the unique solution.
- Suppose  $n > 0$  and we have defined  $\bar{\alpha}$  satisfying the commutation condition, up to dimension  $n-1$ . Let  $p = (z, x, y)$  be an  $n$ -generator of  $V(C)$ . Suppose  $\bar{\alpha}(p) = (z', x', y')$ : the commutation condition implies  $z' = \alpha(z)$ ,  $x' = \bar{\alpha}_{n-1}^*(x)$  and  $y' = \bar{\alpha}_{n-1}^*(y)$ , so that  $\bar{\alpha}$  extends in at most one way to dimension  $n$ , and uniqueness holds. As for the existence,  $x, y$  are parallel  $(n-1)$ -cells in  $V^*V(C)_{n-1}$ ; by induction hypothesis, their images  $x' = \bar{\alpha}_{n-1}^*(x)$  and  $y' = \bar{\alpha}_{n-1}^*(y)$  are  $(n-1)$ -parallel cells in  $V^*V(D)$ . Again, by induction hypothesis, (3)

commutes in dimension  $n-1$ ; also  $\alpha$  is a globular map, hence

$$\begin{aligned} s_{n-1}(z') &= s_{n-1}(\alpha_n(z)) \\ &= \alpha_{n-1}(s_{n-1}(z)) \\ &= \alpha_{n-1}(\epsilon_{n-1}^C(x)) \\ &= \epsilon_{n-1}^D(\bar{\alpha}_{n-1}^*(x)) \\ &= \epsilon_{n-1}^D(x') \end{aligned}$$

and likewise

$$t_{n-1}(z') = \epsilon_{n-1}^D(y')$$

Therefore  $p' = (z', x', y')$  is an  $n$ -generator of  $V(D)$ . Also  $s_{n-1}(p') = x' = \bar{\alpha}_{n-1}^*(x) = \bar{\alpha}_{n-1}^*(s_{n-1}(p))$  and  $t_{n-1}(p') = y' = \bar{\alpha}_{n-1}^*(y) = \bar{\alpha}_{n-1}^*(t_{n-1}(p))$ , so that  $\bar{\alpha}$  extends to a morphism in  $\mathbf{Pol}_\omega$  up to dimension  $n$ . Finally the diagram (3) commutes in dimension  $n$ : it is sufficient to check this on generators, but

$$\begin{aligned} \epsilon_n^D \bar{\alpha}_n^*(p) &= \epsilon_n^D(p') \\ &= z' \\ &= \alpha_n(z) \\ &= \alpha_n \epsilon_n^C(p) \end{aligned}$$

and we are done. □

### 3. MONADICITY

We now turn to the main result.

**Theorem 1.** *The functor  $V : \mathbf{Cat}_\omega \rightarrow \mathbf{Pol}_\omega$  is monadic.*

*Proof.* Recall that monadicity means here that  $\mathbf{Cat}_\omega$  is *equivalent* to the category of algebras of the monad  $VV^*$  on  $\mathbf{Pol}_\omega$ . By using the corresponding version of Beck's criterion, this amounts to show that (i)  $V$  reflects isomorphisms and (ii) if  $f, g$  is a parallel pair of  $\omega$ -morphisms such that the pair  $V(f), V(g)$  has a split coequalizer in  $\mathbf{Pol}_\omega$ , then  $f, g$  has a coequalizer in  $\mathbf{Cat}_\omega$ , and  $V$  preserves coequalizers of such pairs (see for instance [ML71, VI.7, exercises 3 and 6]).

First, if  $f : C \rightarrow D$  is an  $\omega$ -morphism such that  $V(f)$  is an isomorphism, then  $GV(f)$  is an isomorphism in  $\mathbf{Glob}_\omega$  and by Lemma 1,  $U(f)$  is an isomorphism. Now,  $U$  reflects isomorphisms, hence  $f$  is an isomorphism. Therefore  $V$  reflects isomorphisms as required.

Now, let  $f, g : C \rightarrow D$  be a pair of  $\omega$ -morphisms and suppose

$$(4) \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{b} \\ \curvearrowright \\ \xrightarrow{V(f)} \\ \xrightarrow{V(g)} \\ \curvearrowleft \end{array} & \\ V(C) & \xrightarrow{\quad} & V(D) \xrightarrow{k} P \\ & \begin{array}{c} \xrightarrow{a} \\ \curvearrowleft \\ \xrightarrow{V(f)} \\ \xrightarrow{V(g)} \\ \curvearrowright \end{array} & \end{array}$$

is a split coequalizer in  $\mathbf{Pol}_\omega$  where  $k \circ a = 1_P$ ,  $V(f) \circ b = 1_{V(D)}$  and  $V(g) \circ b = a \circ k$ . By applying the functor  $G$  to (4), we get a split coequalizer in  $\mathbf{Glob}_\omega$ :

$$(5) \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{G(b)} \\ \curvearrowright \\ \xrightarrow{GV(f)} \\ \xrightarrow{GV(g)} \end{array} & \\ GV(C) & \xrightarrow{\quad} & GV(D) \xrightarrow{G(k)} G(P) \\ & \begin{array}{c} \xrightarrow{G(a)} \\ \curvearrowleft \\ \xrightarrow{G(k)} \end{array} & \end{array}$$

Then, by using the natural isomorphism  $\phi$  of Lemma 1, we obtain the following diagram

$$(6) \quad \begin{array}{ccccc} & & G(b) & & \\ & & \curvearrowright & & \\ & & \xrightarrow{GV(f)} & & \\ GV(C) & \xrightarrow{\quad} & GV(D) & \xleftarrow{G(a)} & \\ & & \curvearrowleft & & \\ \phi^C \downarrow & & \phi^D \downarrow & & \\ & & U(f) & & \\ U(C) & \xrightarrow{\quad} & U(D) & \xrightarrow{l} & G(P) \\ & & \curvearrowright & & \\ & & \beta & & \alpha \end{array}$$

where  $\alpha = \phi^D \circ G(a)$ ,  $l = G(k) \circ (\phi^D)^{-1}$  and  $\beta = \phi^C \circ G(b) \circ (\phi^D)^{-1}$ . Therefore  $l \circ \alpha = 1_{G(P)}$ ,  $U(f) \circ \beta = 1_{U(D)}$  and

$$\begin{aligned} U(g) \circ \beta &= U(g) \circ \phi^C \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ GV(g) \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ G(a) \circ G(k) \circ (\phi^D)^{-1} \\ &= \alpha \circ l \end{aligned}$$

and the bottom line of (6) is a split coequalizer diagram in  $\mathbf{Glob}_\omega$ . Now the functor  $U$  is strictly monadic, so that there is a unique  $\omega$ -morphism  $h : D \rightarrow E$  such that  $U(E) = G(P)$  and  $U(h) = l$  and moreover this unique morphism makes

$$(7) \quad C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} D \xrightarrow{h} E$$

a coequalizer diagram in  $\mathbf{Cat}_\omega$ . Note that, by construction,  $U(E) = G(P)$ .

It remains to show that  $V(h) : V(D) \rightarrow V(E)$  is a coequalizer of the pair  $V(f), V(g)$  in  $\mathbf{Pol}_\omega$ . By applying Lemma 2 to  $\alpha : U(E) \rightarrow U(D)$  and to  $\beta : U(D) \rightarrow U(C)$ , we get unique morphisms  $\bar{\alpha} : V(E) \rightarrow V(D)$  and  $\bar{\beta} : V(D) \rightarrow V(C)$  satisfying the required commutation condition. Consider the following diagram:

$$(8) \quad \begin{array}{ccccc} UV^*V(E) & \xrightarrow{UV^*(\bar{\alpha})} & UV^*V(D) & \xrightarrow{UV^*V(h)} & UV^*V(E) \\ U(\epsilon^E) \downarrow & & U(\epsilon^D) \downarrow & & \downarrow U(\epsilon^E) \\ U(E) & \xrightarrow{\quad \alpha \quad} & U(D) & \xrightarrow{U(h)} & U(E) \end{array}$$

The left-hand square commutes by hypothesis, and the right-hand square commutes by the naturality of  $\epsilon$ , whence the outer square also commutes. As  $U(h) \circ \alpha = 1_{U(E)}$ , the uniqueness of the lifting in Lemma 2 implies that  $V(h) \circ \bar{\alpha} = 1_{V(E)}$ . By the same uniqueness argument, we get  $V(f) \circ \bar{\beta} = 1_{V(D)}$  and  $V(g) \circ \bar{\beta} = \bar{\alpha} \circ V(h)$ . Therefore the following diagram is a split coequalizer in  $\mathbf{Pol}_\omega$

$$\begin{array}{ccccc} & & \bar{\beta} & & \bar{\alpha} \\ & \curvearrowright & & \curvearrowleft & \\ V(C) & \begin{array}{c} \xrightarrow{V(f)} \\ \xrightarrow{V(g)} \end{array} & V(D) & \xrightarrow{k} & V(E) \end{array}$$

and we are done. □

#### ACKNOWLEDGEMENTS

Many thanks to Dimitri Ara and Albert Burroni for numerous helpful conversations on the subject.

#### REFERENCES

UNIVERSITÉ PARIS OUEST NANTERRE LA DÉFENSE, IRIF, UMR 8243 CNRS, UNIV  
PARIS DIDEROT, SORBONNE PARIS CITÉ, F-75205 PARIS, FRANCE  
*E-mail address:* `metayer@pps.univ-paris-diderot.fr`