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STRICT $\omega$-CATEGORIES ARE MONADIC OVER POLYGRAPHS

FRANÇOIS MÉTAYER

Abstract. We give a direct proof that the category of strict $\omega$-categories is monadic over the category of polygraphs.

Introduction

This short note presents a proof of monadicity for the adjunction between the category $\text{Cat}_{\omega}$ of strict $\omega$-categories and the category $\text{Pol}_{\omega}$ of polygraphs (or computads, as first introduced by Street in [Str76]). Here we follow the presentation and terminology of [Bur93, Mêt03]. The reader may consult [Mêt08] for a detailed description of the categories and functors referred to in this particular case, or [Bat98] for a broader perspective including generalized “$A$-computads” for a monad $A$ on globular sets. The latter paper rightly asserts the monadicity theorem, but some parts of the proof rely on the fact that the category of $A$-computads is a presheaf category, which is precisely not true in the present case, where $A$ is the monad of strict $\omega$-categories [MZ08, Che13]. Since then, the status of monadicity for $\text{Cat}_{\omega}$ has remained somewhat unclear (see e.g the entry “computad” on the nLab [nLa]). Our proof is based on the same ideas as developed in [Bat98], except that we avoid the presheaf argument and establish instead a lifting result (Lemma 2), possibly of independent interest.

As for notations, whenever a functor $F$ is a right-adjoint, we denote its left-adjoint by $F^*$. Let us finally mention a small point about terminology. Given a functor $F : A \to B$, with left-adjoint $F^*$, and $T = FF^*$ the associated monad on $B$, there is a comparison functor $K$ from $A$ to the category $\mathcal{B}^T$ of $T$-algebras: we call $F$ monadic if $K$ is an equivalence of categories, and strictly monadic if $K$ is an isomorphism. We refer to [ML71, VI.7] for corresponding variants of Beck’s monadicity criterion.

1. Three adjunctions

In this section, we briefly describe three pairs of adjoint functors between categories $\text{Glob}_{\omega}$ of globular sets, $\text{Cat}_{\omega}$ of strict $\omega$-categories and $\text{Pol}_{\omega}$ of polygraphs.

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As $\omega$-categories are globular sets with extra structure, there is an obvious forgetful functor

$$U : \text{Cat}_\omega \to \text{Glob}_\omega$$

This functor $U$ has a left-adjoint $U^*$ taking a globular set $X$ to the $\omega$-category $U^*X$ it generates. Moreover this adjunction is strictly monadic.

A second adjunction involves functors $V, V^*$ between $\text{Cat}_\omega$ and $\text{Pol}_\omega$. Unlike $U$, the right adjoint $V$ is not quite obvious. Thus, let $C$ be an $\omega$-category, the polygraph $P = V(C)$ is defined by induction, together with a morphism $\epsilon^C : V^*(P) \to C$:

- For $n = 0$, $P_0 = C_0$ and $\epsilon^C_0$ is the identity.
- Suppose $n > 0$, and $P$, $\epsilon^C$ have been defined up to dimension $n-1$.

The set of $n$-generators of $P$ is then the set $P_n$ of triples $p = (z, x, y)$ where $z \in C_n$, $x, y$ are parallel cells in $P^n_{n-1}$ and $z : \epsilon^C_{n-1}(x) \to \epsilon^C_{n-1}(y)$. The source and target of $p$ in $P^n_{n-1}$ are $x = s_{n-1}(p)$ and $y = t_{n-1}(p)$ respectively, and $\epsilon^C_{n}(p) = z$. By the universal property of polygraphs, $\epsilon^P$ extends uniquely to a map from $P^n_n$ to $C_n$ preserving compositions and identities. Functoriality of $V$ is immediate and $V$ is in fact right-adjoint to $V^*$ (see [Bat98, Mét03]).

Note that

$$\epsilon^C : V^*V(C) \to C$$

is the counit of this adjunction and determines the standard polygraphic resolution of $C$.

We finally describe a functor

$$G : \text{Pol}_\omega \to \text{Glob}_\omega$$

Let $P$ be a polygraph. Let us denote by $j_n : P_n \to P^n_n$ the canonical inclusion of the set of $n$-generators of $P$ into the set of $n$-cells of $P^* = V^*(P)$. We define the globular set $X = G(P)$ dimensionwise, so that for each $n \in \mathbb{N}$, $X_n \subset P_n$:

- For $n = 0$, $X_0 = P_0$.
- Let $n > 0$ and suppose we have defined $X_k \subset P_k$ for all $k < n$, together with source and target maps building an $n-1$-globular set.

Let $X_n \subset P_n$ be the set of $n$-generators $a$ of $P$ such that $s_{n-1}(a)$ and $t_{n-1}(a)$ belong to $j_{n-1}(X_{n-1})$ and define source and target maps $s_{n-1}^X, t_{n-1}^X : X_n \to X_{n-1}$ as the unique maps such that $j_{n-1}s_{n-1}^X(a) = s_{n-1}(a)$ and $j_{n-1}t_{n-1}^X(a) = t_{n-1}(a)$ for each $a \in X_n$. This extends $X$ to an $n$-globular set.

\[ (1) \]

\[
\begin{array}{ccc}
X_n & \xrightarrow{j_{n-1}} & P_{n-1} \\
\downarrow s_{n-1}^X & & \downarrow t_{n-1} \\
X_{n-1} & \xrightarrow{j_{n-1}} & P^n_{n-1}
\end{array}
\]
The previous construction is clearly functorial and defines the required functor $G$. Remark that $G$ admits a left adjoint $G^* : \text{Glob}_\omega \to \text{Pol}_\omega$ which takes the globular set $X$ to a polygraph $P$ such that $P_n = X_n$, in other words $G^*$ defines a natural inclusion of $\text{Glob}_\omega$ into $\text{Pol}_\omega$.

Note that $G$ forgets all generators of $P$ that are not “hereditary globular”, so that for instance $G(P)$ may have no cells at all beyond dimension 1. However, the following result shows that the functor $G$ is not always trivial.

**Lemma 1.** There is a natural isomorphism $\phi : GV \to U$, that is, the following diagram commutes up to a natural isomorphism

\[
\begin{array}{ccc}
\text{Cat}_\omega & \xrightarrow{V} & \text{Pol}_\omega \\
U \downarrow & & \downarrow G \\
\text{Glob}_\omega & & \\
\end{array}
\]

Proof. Let $C$ be an $\omega$-category, and $X = GV(C)$. For each $n \in \mathbb{N}$, let $\phi^C_n : X_n \to C_n$ be the composition of the following maps

\[X_n \xrightarrow{\chi^C_n} V(C)_n \xrightarrow{j_n} V^*V(C)_n \xrightarrow{\epsilon^C_n} C_n\]

As $\epsilon^C$ is an $\omega$-morphism and (1) commutes, the family $(\phi^C_n)_{n \in \mathbb{N}}$ defines a globular morphism $\phi^C : GV(C) \to U(C)$, natural in $C$. Thus we get a natural transformation $\phi : GV \to U$.

Let us now define $\chi^C_n : C_n \to X_n$ by induction on $n$ such that $\phi^C_n \circ \chi^C_n = 1_{C_n}$:

- For $n = 0$, $X_0 = C_0$ and $\phi^C_0 = 1_{C_0} = 1_{X_0}$, so that $\chi^C_0 : C_0 \to X_0$ is also $1_{C_0} = 1_{X_0}$.
- Suppose $n > 0$ and $\chi^C_k$ has been defined up to $k = n-1$, and let $z \in C_n$. Let $u = s_{n-1}(z)$ and $v = t_{n-1}(z)$ in $C_{n-1}$. By induction hypothesis, $\chi^C_{n-1}(u)$ and $\chi^C_{n-1}(v)$ belong to $X_{n-1}$. Let $x = j_{n-1}\chi^C_{n-1}(u)$, $y = j_{n-1}\chi^C_{n-1}(v)$ in $V^*V(C)_{n-1}$ and define $a = \chi^C_n(z) = (z, x, y)$. By construction $a \in X_n$ and $\phi^C_n(a) = z$.

It remains to prove that $\phi^C_n$ is injective. We reason again by induction on $n$:

- For $n = 0$, $\phi^C_0$ is an identity, hence injective.
- Suppose $n > 0$ and $\phi^C_{n-1}$ injective. Let $a_i = (z_i, x_i, y_i) \in X_n$ for $i = 0, 1$ such that $\phi^C_n(a_0) = \phi^C_n(a_1)$. Thus $z_0 = z_1$. Also

\[
\begin{align*}
\phi^C_{n-1}(s_{n-1}(a_0)) &= s_{n-1}(\phi^C_n(a_0)) \\
&= s_{n-1}(\phi^C_n(a_1)) \\
&= \phi^C_{n-1}(s_{n-1}(a_1))
\end{align*}
\]

and because $\phi^C_{n-1}$ is injective,

\[s_{n-1}(a_0) = s_{n-1}(a_1)\]
Now

\[ x_0 = s_{n-1}(a_0) \]
\[ = j_{n-1}s_{n-1}^X(a_0) \]
\[ = j_{n-1}s_{n-1}^X(a_1) \]
\[ = s_{n-1}(a_1) \]
\[ = x_1 \]

Likewise \( y_0 = y_1 \), and we get \( a_0 = a_1 \). Hence \( \phi_n^C \) is injective and we are done.

\[ \square \]

2. Lifting lemma

The forgetful functor \( U : \text{Cat}_\omega \to \text{Glob}_\omega \) is faithful, but clearly not full. However, globular morphisms lift to \( \omega \)-morphisms in the sense of the following result:

**Lemma 2.** Let \( C, D \) be \( \omega \)-categories and \( \alpha : U(C) \to U(D) \) be a globular morphism. Then there is a unique morphism \( \overline{\alpha} : V(C) \to V(D) \) in \( \text{Pol}_\omega \) such that the following square commutes:

\[
\begin{array}{ccc}
UV^*V(C) & \xrightarrow{UV^*(\overline{\alpha})} & UV^*V(D) \\
\downarrow{U(\epsilon^C)} & & \downarrow{U(\epsilon^D)} \\
U(C) & \xrightarrow{\alpha} & U(D)
\end{array}
\] (3)

**Proof.** We build the required morphism \( \overline{\alpha} : V(C) \to V(D) \) by induction on the dimension. Note that diagram (3) yields a diagram in \( \text{Sets} \) at any given dimension \( n \). We may therefore drop the letter \( U \) in the following computations. Also \( \overline{\alpha}^* \) is short for \( V^*(\overline{\alpha}) \).

- For \( n = 0 \), we have \( V(C)_0 = C_0, V(D)_0 = D_0 \); also \( \epsilon^C_0 \) and \( \epsilon^D_0 \) are identities, so that \( \overline{\alpha}_0 = \alpha_0 \) is the unique solution.
- Suppose \( n > 0 \) and we have defined \( \overline{\alpha} \) satisfying the commutation condition, up to dimension \( n-1 \). Let \( p = (z, x, y) \) be an \( n \)-generator of \( V(C) \). Suppose \( \overline{\alpha}(p) = (z', x', y') \): the commutation condition implies \( z' = \alpha(z) \), \( x' = \overline{\alpha}^*_{n-1}(x) \) and \( y' = \overline{\alpha}^*_n(x) \), so that \( \overline{\alpha} \) extends in at most one way to dimension \( n \), and uniqueness holds. As for the existence, \( x, y \) are parallel \( (n-1) \)-cells in \( V^*V(C)_{n-1} \); by induction hypothesis, their images \( x' = \overline{\alpha}^*_{n-1}(x) \) and \( y' = \overline{\alpha}^*_n(x) \) are \( (n-1) \)-parallel cells in \( V^*V(D) \). Again, by induction hypothesis, (3)
commutes in dimension \( n-1 \); also \( \alpha \) is a globular map, hence
\[
\begin{align*}
  s_{n-1}(z') &= s_{n-1}(\alpha_n(z)) \\
  &= \alpha_{n-1}(s_{n-1}(z)) \\
  &= \alpha_{n-1}(\epsilon_{n-1}(x)) \\
  &= \epsilon^D_{n-1}((\overline{\alpha}_{n-1}(x))) \\
  &= \epsilon^D_{n-1}(x')
\end{align*}
\]
and likewise
\[
\begin{align*}
  t_{n-1}(z') &= \epsilon^D_{n-1}(y') \\
  s_{n-1}(p') &= x' = \overline{\alpha}_{n-1}(x) = \overline{\alpha}_{n-1}(s_{n-1}(p)) \\
  t_{n-1}(p') &= y' = \overline{\alpha}_{n-1}(y) = \overline{\alpha}_{n-1}(t_{n-1}(p))
\end{align*}
\]
Therefore \( p' = (z', x', y') \) is an \( n \)-generator of \( V(D) \). Also \( s_{n-1}(p') = x' = \overline{\alpha}_{n-1}(x) = \overline{\alpha}_{n-1}(s_{n-1}(p)) \) and \( t_{n-1}(p') = y' = \overline{\alpha}_{n-1}(y) = \overline{\alpha}_{n-1}(t_{n-1}(p)) \), so that \( \overline{\alpha} \) extends to a morphism in \( \text{Pol}_\omega \) up to dimension \( n \). Finally the diagram (3) commutes in dimension \( n \) : it is sufficient to check this on generators, but
\[
\begin{align*}
  \epsilon^D_n(\overline{\alpha}_n(p)) &= \epsilon^D_n(p') \\
  &= z' \\
  &= \alpha_n(z) \\
  &= \alpha_n\epsilon_n^C(p)
\end{align*}
\]
and we are done.

\[\square\]

3. Monadicity

We now turn to the main result.

**Theorem 1.** The functor \( V : \text{Cat}_\omega \to \text{Pol}_\omega \) is monadic.

**Proof.** Recall that monadicity means here that \( \text{Cat}_\omega \) is equivalent to the category of algebras of the monad \( V V^* \) on \( \text{Pol}_\omega \). By using the corresponding version of Beck’s criterion, this amounts to show that (i) \( V \) reflects isomorphisms and (ii) if \( f, g \) is a parallel pair of \( \omega \)-morphisms such that the pair \( V(f), V(g) \) has a split coequalizer in \( \text{Pol}_\omega \), then \( f, g \) has a coequalizer in \( \text{Cat}_\omega \), and \( V \) preserves coequalizers of such pairs (see for instance [ML71, VI.7, exercises 3 and 6]).

First, if \( f : C \to D \) is an \( \omega \)-morphism such that \( V(f) \) is an isomorphism, then \( GV(f) \) is an isomorphism in \( \text{Glob}_{\omega} \) and by Lemma 1, \( U(f) \) is an isomorphism. Now, \( U \) reflects isomorphisms, hence \( f \) is an isomorphism. Therefore \( V \) reflects isomorphisms as required.

Now, let \( f, g : C \to D \) be a pair of \( \omega \)-morphisms and suppose
\[
V(C) \xrightarrow{V(f)} V(D) \xrightarrow{V(g)} P
\]
is a split coequalizer in $\mathbf{Pol}_\omega$ where $k \circ a = 1_P$, $V(f) \circ b = 1_{V(D)}$ and $V(g) \circ a = b \circ k$. By applying the functor $G$ to (4), we get a split coequalizer in $\mathbf{Glob}_\omega$:

\[
\begin{array}{c}
G(V(C)) \xrightarrow{G(V(f))} G(V(D)) \xrightarrow{G(k)} G(P)
\end{array}
\]

(5)

Then, by using the natural isomorphism $\phi$ of Lemma 1, we obtain the following diagram

\[
\begin{array}{c}
G(V(C)) \xrightarrow{G(V(f))} G(V(D)) \xrightarrow{G(k)} G(P)
\end{array}
\]

\[
\begin{array}{c}
\phi^C
\end{array}
\]

\[
\begin{array}{c}
\phi^D
\end{array}
\]

\[
\begin{array}{c}
U(C) \xrightarrow{U(f)} U(D) \xrightarrow{l} G(P)
\end{array}
\]

where $\alpha = \phi^D \circ G(a)$, $l = G(k) \circ (\phi^D)^{-1}$ and $\beta = \phi^C \circ G(b) \circ (\phi^D)^{-1}$. Therefore $l \circ \alpha = 1_{G(P)}$, $U(f) \circ \beta = 1_{U(D)}$ and

\[
U(g) \circ \beta = U(g) \circ \phi^C \circ G(b) \circ (\phi^D)^{-1} = \phi^D \circ G(V(g)) \circ G(b) \circ (\phi^D)^{-1} = \phi^D \circ G(a) \circ G(k) \circ (\phi^D)^{-1} = \alpha \circ l
\]

and the bottom line of (6) is a split coequalizer diagram in $\mathbf{Glob}_\omega$. Now the functor $U$ is strictly monadic, so that there is a unique $\omega$-morphism $h : D \to E$ such that $U(E) = G(P)$ and $U(h) = l$ and moreover this unique morphism makes

\[
\begin{array}{c}
C \xrightarrow{f} D \xrightarrow{h} E
\end{array}
\]

(7)

a coequalizer diagram in $\mathbf{Cat}_\omega$. Note that, by construction, $U(E) = G(P)$.

It remains to show that $V(h) : V(D) \to V(E)$ is a coequalizer of the pair $V(f), V(g)$ in $\mathbf{Pol}_\omega$. By applying Lemma 2 to $\alpha : U(E) \to U(D)$ and to $\beta : U(D) \to U(C)$, we get unique morphisms $\overline{\alpha} : V(E) \to V(D)$ and $\overline{\beta} : V(D) \to V(C)$ satisfying the required commutation condition. Consider the following diagram:

\[
\begin{array}{c}
UV^*V(E) \xrightarrow{UV^*(\overline{\alpha})} UV^*V(D) \xrightarrow{UV^*(\overline{\beta})} UV^*V(E)
\end{array}
\]

\[
\begin{array}{c}
U(E) \xrightarrow{U(\epsilon^E)} U(D) \xrightarrow{U(h)} U(E)
\end{array}
\]

(8)
The left-hand square commutes by hypothesis, and the right-hand square commutes by the naturality of $\epsilon$, whence the outer square also commutes. As $U(h) \circ \alpha = 1_{U(E)}$, the uniqueness of the lifting in Lemma 2 implies that $V(h) \circ \overline{\alpha} = 1_{V(E)}$. By the same uniqueness argument, we get $V(f) \circ \overline{\beta} = 1_{V(D)}$ and $V(g) \circ \overline{\beta} = \overline{\alpha} \circ V(h)$. Therefore the following diagram is a split coequalizer in $\text{Pol}_\omega$:

$$
\begin{array}{ccc}
V(C) & \xrightarrow{V(f)} & V(D) \\
\downarrow{V(g)} & & \downarrow{k} \\
V(E) & \xrightarrow{\tau} & V(E)
\end{array}
$$

and we are done. \qed

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