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Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes

Michel Mollard*

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Abstract

The *Fibonacci cube* of dimension n , denoted as Γ_n , is the subgraph of n -cube Q_n induced by vertices with no consecutive 1's. In this short note we prove that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k , answering an open problem proposed in [2].

Keywords: Fibonacci cube, Fibonacci numbers.

AMS Subj. Class. (2010):

1 Introduction

Let n be a positive integer and denote $[n] = \{1, \dots, n\}$, and $[n]_0 = \{0, \dots, n-1\}$. The n -cube, denoted by Q_n , is the graph with vertex set

$$V(Q_n) = \{x_1x_2 \dots x_n \mid x_i \in [2]_0 \text{ for } i \in [n]\},$$

where two vertices are adjacent in Q_n if the corresponding strings differ in exactly one position. The *Fibonacci n -cube*, denoted by Γ_n , is the subgraph of Q_n induced by vertices with no consecutive 1's. Let $\{F_n\}$ be the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The number of vertices of Γ_n is $|\Gamma_n| = F_{n+2}$. Fibonacci cubes have been investigated from many points of view, we refer to the survey[1] for more information about them. Let $q_k(n)$ be the maximum number of disjoint subgraphs isomorphic to Q_k in Γ_n . This number is studied in a recent paper[2]. The authors obtained the following recursive formula

Theorem 1.1 *For every $k \geq 1$ and $n \geq 3$ $q_k(n) = q_{k-1}(n-2) + q_k(n-3)$.*

They also proposed the following problem

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Question 1.2 For $k > 1$ it will be interesting to study, asymptotically, the ratio of vertices of Γ_n covered by a maximum set of disjoint subgraphs isomorphic to Q_k , thus compare $\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|}$ and $\frac{1}{2^k}$.

The purpose of this short note is to deduce from Theorem 1.1 a recursive formula for the number of non covered vertices by a maximum set of disjoint hypercube. We obtain as a consequence, that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k thus

Theorem 1.3 For every $k \geq 1$, $\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$.

2 Proof of Theorem 1.3

Definition 2.1 Let $\{P_k(n)\}_{k=1}^{\infty}$ be the family of sequences of integers defined by

(i) $P_k(n+3) = P_k(n) + 2P_{k-1}(n+1)$ for $k \geq 2$ and $n \geq 0$

(ii) $P_k(0) = 1, P_k(1) = 2, P_k(2) = 3$, for $k \geq 2$

(iii) $P_1(n) = 0$ if $n \equiv 1[3]$ and $P_1(n) = 1$ if $n \equiv 0[3]$ or $n \equiv 2[3]$.

In the next table, we summarize the first values of $P_k(n)$.

$n \bmod 3$	0	1	2
$P_1(n)$	1	0	1
$P_2(n)$	1	$\frac{2}{3}n + \frac{4}{3}$	$\frac{2}{3}n + \frac{5}{3}$
$P_3(n)$	$\frac{2}{9}n^2 + \frac{2}{3}n + 1$	$\frac{2}{9}n^2 + \frac{8}{9}n + \frac{8}{9}$	$\frac{2}{9}n^2 + \frac{10}{9}n + \frac{10}{9}$
$P_4(n)$	$\frac{4}{81}n^3 + \frac{2}{9}n^2 + \frac{2}{9}n + 1$	$\frac{4}{81}n^3 + \frac{8}{27}n^2 + \frac{8}{27}n + \frac{8}{27}$	$\frac{4}{81}n^3 + \frac{4}{27}n^2 + \frac{10}{27}n + \frac{103}{81}$

Proposition 2.2 Let $n = 3p+r$ with $r = 0, 1$ or 2 . For a fixed r , $P_k(n)$ is a polynomial in n of degree at most $k-1$.

Proof. From (i) we can write

$$P_k(n) = 2 \sum_{i=0}^{p-1} P_{k-1}(n-2-3i) + P_k(r).$$

For any integer d the classical Faulhaber's formula express the sum $\sum_{m=0}^n m^d$ as a polynomial in n of degree $d+1$. Thus if $Q(n)$ is a polynomial of degree at most d then $\sum_{m=0}^n Q(m)$ is a polynomial in n of degree at most $d+1$. Let $Q'(m) = Q(m)$ if $m \equiv 0[3]$ and 0 otherwise. Applying this to Q' we obtain that $\sum_{m=0, m \equiv 0[3]}^n Q(m)$ is also a polynomial in n of degree at most $d+1$. Thus if $P_{k-1}(n)$ is a polynomial in n of degree at most $k-2$ then $\sum_{i=0}^{p-1} P_{k-1}(n-2-3i)$ is a polynomial of degree at most

$k - 1$. Since for a fixed r $P_1(n)$ is a constant, by induction on k , $P_k(n)$ is a polynomial in n of degree at most $k - 1$. \square

Theorem 2.3 *The number of non covered vertices of Γ_n by $q_k(n)$ disjoint Q_k is $P_k(n)$.*

Proof. This is true for $k = 1$ since the Fibonacci cube Γ_n has a perfect matching for $n \equiv 1[3]$ and a maximum matching missing a vertex otherwise.

For $k > 1$ this is true for $n = 0, 1, 2$ since the value of $P_k(n)$ are respectively 1,2,3 thus are equal to $|\Gamma_n|$ and there is no Q_k in Γ_n .

Assume the property is true for some $k \geq 1$ and any n . Then consider $k + 1$. By induction on n we can assume that the property is true for Γ_{n-3} . Let us prove it for Γ_n .

From Theorem 1.1 $q_{k+1}(n) = q_k(n - 2) + q_{k+1}(n - 3)$.

Thus the number of non covered vertices of Γ_n by $q_{k+1}(n)$ disjoint Q_{k+1} is

$$|\Gamma_n| - 2^{k+1}q_{k+1}(n) = F_{n+2} - 2^{k+1}[q_k(n-2) + q_{k+1}(n-3)] = F_{n+2} - 2 \cdot 2^k q_k(n-2) - 2^{k+1} q_{k+1}(n-3).$$

Using equalities $P_k(n - 2) = F_n - 2^k q_k(n - 2)$ and $P_{k+1}(n - 3) = F_{n-1} - 2^{k+1} q_{k+1}(n - 3)$ we obtain

$$|\Gamma_n| - 2^{k+1}q_{k+1}(n) = F_{n+2} + 2(P_k(n - 2) - F_n) + P_{k+1}(n - 3) - F_{n-1}.$$

From $F_{n+2} - 2F_n - F_{n-1} = 0$ and $2P_k(n - 2) + P_{k+1}(n - 3) = P_{k+1}(n)$ the number of non covered vertices is $P_{k+1}(n)$. So the theorem is proved.

\square

For any k , since the number of non covered vertices is polynomial in n and $|V(\Gamma_n)| = F_{n+2} \sim \frac{1}{\sqrt{5}} \left(\frac{3+\sqrt{5}}{2}\right)^n$ we obtain that

$$\lim_{n \rightarrow \infty} \frac{P_k(n)}{|V(\Gamma_n)|} = 0$$

thus

$$\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}.$$

References

- [1] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2011) 1–18.

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