



Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes

Michel Mollard

► To cite this version:

Michel Mollard. Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes. 2016. hal-01324319v1

HAL Id: hal-01324319

<https://hal.science/hal-01324319v1>

Preprint submitted on 31 May 2016 (v1), last revised 18 Oct 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes

Michel Mollard*

May 31, 2016

Abstract

The *Fibonacci cube* of dimension n , denoted as Γ_n , is the subgraph of n -cube Q_n induced by vertices with no consecutive 1's. In this short note we prove that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k , answering an open problem proposed in [2].

Keywords: Fibonacci cube, Fibonacci numbers.

AMS Subj. Class. (2010):

1 Introduction

Let n be a positive integer and denote $[n] = \{1, \dots, n\}$, and $[n]_0 = \{0, \dots, n-1\}$. The n -cube, denoted by Q_n , is the graph with vertex set

$$V(Q_n) = \{x_1x_2 \dots x_n \mid x_i \in [2]_0 \text{ for } i \in [n]\},$$

where two vertices are adjacent in Q_n if the corresponding strings differ in exactly one position. The *Fibonacci n -cube*, denoted by Γ_n , is the subgraph of Q_n induced by vertices with no consecutive 1's. Let $\{F_n\}$ be the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The number of vertices of Γ_n is $|\Gamma_n| = F_{n+2}$. Fibonacci cubes have been investigated from many points of view, we refer to the survey[1] for more information about them. Let $q_k(n)$ be the maximum number of disjoint subgraphs isomorphic to Q_k in Γ_n . This number is studied in a recent paper[2]. The authors obtained the following recursive formula

Theorem 1.1 *For every $k \geq 1$ and $n \geq 3$ $q_k(n) = q_{k-1}(n-2) + q_k(n-3)$.*

They also proposed the following problem

*Institut Fourier, CNRS Université Grenoble Alpes, email: michel.mollard@univ-grenoble-alpes.fr

Question 1.2 For $k > 1$ it will be interesting to study, asymptotically, the ratio of vertices of Γ_n covered by a maximum set of disjoint subgraphs isomorphic to Q_k , thus compare $\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|}$ and $\frac{1}{2^k}$.

The purpose of this short note is to deduce from Theorem 1.1 a recursive formula for the number of non covered vertices by a maximum set of disjoint hypercube. We obtain as a consequence, that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k thus

Theorem 1.3 For every $k \geq 1$, $\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$.

2 Proof of Theorem 1.3

Definition 2.1 Let $\{P_k(n)\}_{k=1}^\infty$ be the family of sequences of integers defined by

- (i) $P_k(n+3) = P_k(n) + 2P_{k-1}(n+1)$ for $k \geq 2$ and $n \geq 0$
- (ii) $P_k(0) = 1, P_k(1) = 2, P_k(2) = 3$, for $k \geq 2$
- (iii) $P_1(n) = 0$ if $n \equiv 1[3]$ and $P_1(n) = 1$ if $n \equiv 0[3]$ or $n \equiv 2[3]$.

In the next table, we summarize the first values of $P_k(n)$.

$n \bmod 3$	0	1	2
$P_1(n)$	1	0	1
$P_2(n)$	1	$\frac{2}{3}n + \frac{4}{3}$	$\frac{2}{3}n + \frac{5}{3}$
$P_3(n)$	$\frac{2}{9}n^2 + \frac{2}{3}n + 1$	$\frac{2}{9}n^2 + \frac{8}{9}n + \frac{8}{9}$	$\frac{2}{9}n^2 + \frac{10}{9}n + \frac{10}{9}$
$P_4(n)$	$\frac{4}{81}n^3 + \frac{2}{9}n^2 + \frac{2}{9}n + 1$	$\frac{4}{81}n^3 + \frac{4}{27}n^2 + \frac{10}{27}n + \frac{103}{81}$	$\frac{4}{81}n^3 + \frac{4}{27}n^2 + \frac{10}{27}n + \frac{103}{81}$

Proposition 2.2 Let $n = 3p+r$ with $r = 0, 1$ or 2 . For a fixed r , $P_k(n)$ is a polynomial in n of degree at most $k-1$.

Proof. From (i) we can write

$$P_k(n) = 2 \sum_{i=0}^{p-1} P_{k-1}(n-2-3i) + P_k(r).$$

For any integer d the classical Faulhaber's formula express the sum $\sum_{m=0}^n m^d$ as a polynomial in n of degree $d+1$. Thus if $Q(n)$ is a polynomial of degree at most d then $\sum_{m=0}^n Q(m)$ is a polynomial in n of degree at most $d+1$. Let $Q'(m) = Q(m)$ if $m \equiv 0[3]$ and 0 otherwise. Applying this to Q' we obtain that $\sum_{m=0, m \equiv 0[3]}^n Q(m)$ is also a polynomial in n of degree at most $d+1$. Thus if $P_{k-1}(n)$ is a polynomial in n of degree at most $k-2$ then $\sum_{i=0}^{p-1} P_{k-1}(n-2-3i)$ is a polynomial of degree at most

$k - 1$. Since for a fixed r $P_1(n)$ is a constant, by induction on k , $P_k(n)$ is a polynomial in n of degree at most $k - 1$. \square

Theorem 2.3 *The number of non covered vertices of Γ_n by $q_k(n)$ disjoint Q_k is $P_k(n)$.*

Proof. This is true for $k = 1$ since the Fibonacci cube Γ_n has a perfect matching for $n \equiv 1[3]$ and a maximum matching missing a vertex otherwise.

For $k > 1$ this is true for $n = 0, 1, 2$ since the value of $P_k(n)$ are respectively 1, 2, 3 thus are equal to $|\Gamma_n|$ and there is no Q_k in Γ_n .

Assume the property is true for some $k \geq 1$ and any n . Then consider $k + 1$. By induction on n we can assume that the property is true for Γ_{n-3} . Let us prove it for Γ_n .

From Theorem 1.1 $q_{k+1}(n) = q_k(n - 2) + q_{k+1}(n - 3)$.

Thus the number of non covered vertices of Γ_n by $q_{k+1}(n)$ disjoint Q_{k+1} is

$$|\Gamma_n| - 2^{k+1}q_{k+1}(n) = F_{n+2} - 2^{k+1}[q_k(n-2) + q_{k+1}(n-3)] = F_{n+2} - 2 \cdot 2^k q_k(n-2) - 2^{k+1}q_{k+1}(n-3).$$

Using equalities $P_k(n-2) = F_n - 2^k q_k(n-2)$ and $P_{k+1}(n-3) = F_{n-1} - 2^{k+1}q_{k+1}(n-3)$ we obtain

$$|\Gamma_n| - 2^{k+1}q_{k+1}(n) = F_{n+2} + 2(P_k(n-2) - F_n) + P_{k+1}(n-3) - F_{n-1}.$$

From $F_{n+2} - 2F_n - F_{n-1} = 0$ and $2P_k(n-2) + P_{k+1}(n-3) = P_{k+1}(n)$ the number of non covered vertices is $P_{k+1}(n)$. So the theorem is proved.

\square

For any k , since the number of non covered vertices is polynomial in n and $|V(\Gamma_n)| = F_{n+2} \sim \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n$ we obtain that

$$\lim_{n \rightarrow \infty} \frac{P_k(n)}{|V(\Gamma_n)|} = 0$$

thus

$$\lim_{n \rightarrow \infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}.$$

References

- [1] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2011) 1–18.

- [2] Sylvain Gravier, Michel Mollard, Simon Špacapan, Sara Sabrina Zemljič, On disjoint hypercubes in Fibonacci cubes, *Discrete Applied Mathematics*, Volumes 190–191(2015) 50-55, <http://dx.doi.org/10.1016/j.dam.2015.03.016>.